J. Math. Sci. Univ. Tokyo
9 (2002), 257–277.

Approximation of BSDE's by Stochastic Difference Equation's

By Toshiyuki Nakayama*

Abstract. We consider a BSDE (backward stochastic differential equation)

$$\begin{cases} -dY(t) = f(B(\cdot), t, Y(t), Z(t))dt - Z(t)^* dB(t), \\ Y(1) = \xi. \end{cases}$$

We construct backward stochastic difference equations approximating the BSDE, where time and space are discrete. We show the existence and uniqueness of the solutions of the backward stochastic difference equations. Also we show a convergence result of the solutions of the backward stochastic difference equations towards that of the BSDE.

1. Introduction

Let m and d be positive integers and W be $D([0,1];\mathbf{R}^m)$ endowed with the Skorohod metric dis_W. We denote by μ the Wiener measure on W. Let $(B(t))_{t\in[0,1]}$ be the coordinate mapping process defined by $B(w,t) = w(t)$. Let $(\mathcal{F}(t))_{t\in[0,1]}$ be a filtration given by $\mathcal{F}(t) = \bigcap_{\epsilon>0} \sigma[B(s)\,;\, s \le (t+\epsilon)\wedge 1]$. Let Π be the predictable σ-field over $W \times [0,1]$.

Let $f\colon W \times [0,1] \times \mathbf{R}^d \times \mathbf{R}^{m\times d} \to \mathbf{R}^d$ be a bounded, continuous, $\Pi \otimes \mathcal{B}(\mathbf{R}^d) \otimes \mathcal{B}(\mathbf{R}^{m\times d})$-measurable mapping. Suppose that f is uniformly Lipschitz, i.e.,
there exists a positive constant C such that

$$|f(w,t,y_1,z_1) - f(w,t,y_2,z_2)| \le C(|y_1 - y_2| + |z_1 - z_2|) \tag{1}$$

for all $(w,t) \in W \times [0,1]$, $y_1, y_2 \in \mathbf{R}^d$, and $z_1, z_2 \in \mathbf{R}^{m\times d}$. Let $\xi\colon W \to \mathbf{R}^d$ be a bounded, continuous, $\mathcal{F}(1)$-measurable functional.

*The author is most grateful to Professor Kusuoka for his valuable advice and guidance.

1991 *Mathematics Subject Classification.* Primary 60B10; Secondary 60H10, 60H20, 65C30.

Now we consider the following BSDE (backward stochastic differential equation) on $(W,(\mathcal{F}(t))_{t\in[0,1]},\mu)$.

$$\left\{\begin{array}{l} -dY(t)=f(B(\cdot),t,Y(t),Z(t))dt-Z(t)^{*}dB(t),\\ Y(1)=\xi, \end{array}\right. \tag{2}$$

where $*$ denotes the transpose. A solution of the equation (2) is a pair $(Y,Z)\in H^2(\mathbf{R}^d)\times H^2(\mathbf{R}^{m\times d})$ satisfying

$$Y(t)=\xi+\int_t^1 f(B(\cdot),s,Y(s),Z(s))ds-\int_t^1 Z(s)^{*}dB(s).$$

Here $H^2(\mathbf{R}^d)$ denotes the set of all $\mathbf{R}^d$-valued predictable processes $(X(t))_{t\in[0,1]}$ on (W,μ) such that $\|X\|^2=E^{\mu}\Big[\int_0^1|X(t)|^2dt\Big]<\infty$. The existence and uniqueness of the solution is well known ([3], [7]).

The main purpose of this paper is to approximate the BSDE by backward stochastic difference equations.

Now we define a backward stochastic difference equation. Let $p_1,p_2,\dots,p_{m+1}\in(0,1)$, and $\{e_1,e_2,\dots,e_{m+1}\}\subset\mathbf{R}^m$ be a system of vectors in a general position satisfying

$$\sum_{i=1}^{m+1}p_ie_i=0,\quad \sum_{i=1}^{m+1}p_i=1,\quad \sum_{i=1}^{m+1}e_i^{l_1}e_i^{l_2}p_i=\delta_{l_1,l_2},\quad l_1,l_2=1,2,\dots,m.$$

Here e_i^l is the l'th component of e_i and δ_{l_1,l_2} is the Kronecker's symbol. Let $(\Omega_N,\mathcal{F}_N,P_N)$, $N\in\mathbf{N}$, be probability spaces. Let $\eta_N(n),n=1,2,\dots,N$, be independent $\mathbf{R}^m$-valued random variables defined on $(\Omega_N,\mathcal{F}_N,P_N)$ such that

$$P_N\{\eta_N(n)=e_i\}=p_i,\qquad i=1,2,\dots,m+1.$$

We define a random walk $(S_N(n))_{n\in\{0,1,\dots,N\}}$ by

$$\begin{aligned} S_N(n)&=\sum_{k=1}^{n}\eta_N(k),\qquad n=1,2,\dots,N,\\ S_N(0)&=0, \end{aligned}$$

and a filtration $(\mathcal{F}_N(n))_{n\in\{0,1,\dots,N\}}$ over Ω_N by

$$\begin{aligned} \mathcal{F}_N(n)&=\sigma[S_N(1),\dots,S_N(n)],\qquad n=1,2,\dots,N,\\ \mathcal{F}_N(0)&=\{\emptyset,\Omega\}. \end{aligned}$$

We define a continuous-time process B_N by

$$B_N(t) = \frac{1}{\sqrt{N}} S_N([Nt]), \qquad t \in [0,1].$$

Now we consider the following backward stochastic difference equation on (Ω_N, P_N).

$$(3) \qquad \begin{cases} -\Delta y_N(n) = \frac{1}{N} f(B_N(\cdot), \frac{n-1}{N}, y_N(n-1), z_N(n)) \\ \qquad\qquad\qquad -\frac{1}{\sqrt{N}} z_N(n)^* \Delta S_N(n), \\ \qquad\qquad\qquad n = 1, 2, \dots, N, \\ y_N(N) = \xi(B_N(\cdot)). \end{cases}$$

For each discrete-time process $(x(n))$, we denote $\Delta x(k)$ by $x(k) - x(k-1)$ for $k = 1, 2, \dots N$. Let $\mathcal{K}_N$ be the set of all $\mathbf{R}^d$-valued $(\mathcal{F}_N(n))$ -adapted processes $(y(n))_{n \in \{0,1,\dots,N\}}$ on (Ω_N, P_N) and $\mathcal{L}_N$ be the set of all $\mathbf{R}^{m \times d}$-valued $(\mathcal{F}_N(n))$-predictable processes $(z(n))_{n \in \{0,1,\dots,N\}}$ on (Ω_N, P_N) with $z(0) = 0$. A solution to Equation (3) is a pair $(y_N, z_N) \in \mathcal{K}_N \times \mathcal{L}_N$ satisfying (3). We prove the existence and uniqueness of a solution to Equation (3) for sufficiently large N in section 3.

To each process $(y_N(n), z_N(n))_{n \in \{0,1,\dots,N\}} \in \mathcal{K}_N \times \mathcal{L}_N$, we associate a continuous-time process $(\bar{y}_N(t), \bar{z}_N(t))_{t \in [0,1]}$ on (Ω_N, P_N) by

$$(\bar{y}_N(t), \bar{z}_N(t)) = (y_N(\lfloor Nt \rfloor), z_N(\lceil Nt \rceil)), \qquad t \in [0,1].$$

Here $\lfloor x \rfloor$ is the greatest integer not greater than x, and $\lceil x \rceil$ is the least integer not less than x.

Let (Y,Z) be the solution of the BSDE (2) and $(y_N, z_N) \in \mathcal{K}_N \times \mathcal{L}_N$ be that of the backward stochastic difference equation (3). In this paper, we prove the following.

Theorem 1.1. *We have the weak convergence of the distributions on* $W \times D([0,1]; \mathbf{R}^d) \times L^2([0,1]; \mathbf{R}^{m \times d})$ *such that*

$$P_N \circ (B_N, \bar{y}_N, \bar{z}_N)^{-1} \to \mu \circ (B, Y, Z)^{-1} \qquad \textit{weakly as } N \to \infty.$$

Douglas, Ma, and Protter [2] has given a numerical method for FBSDE. Their method is also found in Ma and Yong [6]. Their result is for the following FBSDE (forward-backward stochastic differential equation).

$$
(4)\qquad \begin{cases} dX(t) = b(t, X(t), Y(t), Z(t))dt + \sigma(t, X(t), Y(t))dB(t), \\ -dY(t) = \hat{b}(t, X(t), Y(t), Z(t))dt - Z(t)dB(t), \\ X(0) = x, \qquad Y(T) = g(X(T)), \end{cases}
$$

where $b, \sigma, \hat{b}, g$ are all deterministic smooth functions. In their FBSDE case, (Y, Z) is represented in terms of a PDE solution and a standard (forward) SDE solution. They solved their problem by approximating the PDE and the standard (forward) SDE. They used the combined characteristics and finite difference method for the PDE and used the first order Euler scheme for the (forward) SDE.

BSDE (2) we consider here is more general, since our drift term f is path dependent. Therefore our approach is quite different from that of their FBSDE. We construct a discrete space-time backward stochastic difference equation. Considering that BSDE with path-dependent drift term is very useful in mathematical finance, for example, its approximation is very important.

2. Representations of Martingales in Terms of a Random Walk

In this section we consider a discrete version of the martingale representation theorem. This is a preparation for constructing a backward stochastic difference equation.

In this section and the next section, we fix $N \in \mathbf{N}$, and we abbreviate Ω_N, $\mathcal{F}_N$, $(\mathcal{F}_N(n))_{n\in\{0,1,\dots,N\}}$, P_N, η_N, S_N, $\mathcal{K}_N$, and $\mathcal{L}_N$ to Ω, $\mathcal{F}$, $(\mathcal{F}(n))_{n\in\{0,1,\dots,N\}}$, P, η, S, $\mathcal{K}$, and $\mathcal{L}$, respectively for simplicity.

Let $\mathcal{H}$ be the set of all $\mathbf{R}^m$-valued $(\mathcal{F}(n))$-predictable processes $(H(n))_{n\in\{0,1,\dots,N\}}$ on Ω with $H(0) = 0$. Let $\mathcal{M}$ be the set of all $\mathbf{R}$-valued $(\mathcal{F}(n))$-martingales $(M(n))_{n\in\{0,1,\dots,N\}}$ on Ω with $M(0) = 0$.

Lemma 2.1. *Let $n \in \{1, 2, \dots, N\}$. If a function $g: (\mathbf{R}^m)^n \to \mathbf{R}$ satisfies*

$$
E[g(\eta(1), \eta(2), \dots \eta(n)) | \mathcal{F}(n-1)] = 0,
$$

then there exists a mapping $a\colon (\mathbf{R}^m)^{n-1} \to \mathbf{R}^m$ *satisfying*

$$a(\eta(1), \eta(2), \ldots, \eta(n-1))^* \eta(n) = g(\eta(1), \eta(2), \ldots, \eta(n)).$$

In the case $n = 1$*, we interpret* a *as a constant vector in* $\mathbf{R}^m$ *such that* $a^* \eta(n) = g(\eta(n))$.

Proof. Since $\{e_1, e_2, \ldots, e_m\}$ is a basis for $\mathbf{R}^m$, there exists a mapping $a\colon (\mathbf{R}^m)^{n-1} \to \mathbf{R}^m$ such that

$$a(x_1, x_2, \ldots, x_{n-1})^* e_j = g(x_1, x_2, \ldots, x_{n-1}, e_j), \quad j = 1, 2, \ldots, m, \\ x_1, x_2, \ldots, x_{n-1} \in \{e_1, e_2, \ldots, e_{m+1}\}.$$

We note that

$$\begin{aligned} &E[g(\eta(1), \eta(2), \ldots, \eta(n)) | \mathcal{F}(n-1)] \\ &= E[g(x_1, x_2, \ldots, x_{n-1}, \eta(n))] \Big|_{(x_1, x_2, \ldots, x_{n-1}) = (\eta(1), \eta(2), \ldots, \eta(n-1))} . \end{aligned}$$

Accordingly, by assumption, we have

$$E[g(x_1, x_2, \ldots, x_{n-1}, \eta(n))] = 0, \quad x_1, x_2, \ldots, x_{n-1} \in \{e_1, e_2, \ldots, e_{m+1}\}.$$

Consequently, from $\sum_{i=1}^{m+1} p_i e_i = 0$, we have

$$g(x_1, x_2, \ldots, x_{n-1}, e_{m+1}) = a(x_1, x_2, \ldots, x_{n-1})^* e_{m+1}, \\ x_1, x_2, \ldots, x_{n-1} \in \{e_1, e_2, \ldots, e_{m+1}\}.$$

This proves our Lemma. □

For $(H(n))_{n \in \{0,1,\ldots,N\}} \in \mathcal{H}$, we define $\mathbf{R}$-valued martingale $(H \cdot S(n))_{n \in \{0,1,\ldots,N\}}$

$$\begin{aligned} H \cdot S(n) &= \sum_{k=1}^{n} H(k)^* \Delta S(k) \\ &= \sum_{k=1}^{n} H(k)^* \eta(k), \qquad n = 1, \ldots, N, \\ H \cdot S(0) &= 0, \end{aligned}$$

where

$$\Delta S(k) = S(k) - S(k-1), \quad k = 1, 2, \dots N.$$

PROPOSITION 2.2. *Let $M \in \mathcal{M}$. There exists a unique $H \in \mathcal{H}$ such that*

$$M = H \cdot S.$$

PROOF. For each $n \in \{1, 2, \dots, N\}$ there exists a function $g_n: (\mathbf{R}^m)^n \to \mathbf{R}$ such that

$$M(n) - M(n-1) = g_n(\eta(1), \eta(2), \dots, \eta(n)).$$

By virtue of Lemma 2.1, for each $n \in \{1, 2, \dots, N\}$, there exists a mapping $a_n: (\mathbf{R}^m)^{n-1} \to \mathbf{R}^m$ (a_1 is a constant vector in $\mathbf{R}^m$) such that

$$g_n(\eta(1), \eta(2), \dots, \eta(n)) = a_n(\eta(1), \eta(2), \dots, \eta(n-1))^* \eta(n).$$

Setting $H(n) = a_n(\eta(1), \eta(2), \dots, \eta(n-1))$, $H(0) = 0$, we obtain $M = H \cdot S$.

Let us prove the uniqueness. Suppose that $M = H \cdot S = K \cdot S$ with $H, K \in \mathcal{H}$. Then

$$E\Big[\sum_{n=1}^{N} |H(n) - K(n)|^2\Big] = E\Big[\Big|\sum_{n=1}^{N} (H(n) - K(n))^* \eta(n)\Big|^2\Big] = 0$$

and therefore $|H(n) - K(n)| = 0$, $n \in \{1, 2, \dots, N\}$. □

3. A Difference Equation

In this section, we shall define a backward stochastic difference equation which admits a unique solution.

We fix $N \in \mathbf{N}$ and use the abbreviation as in the previous section.

Let

$$h: \Omega \times \{0, 1, \cdots, N\} \times \mathbf{R}^d \times \mathbf{R}^{m \times d} \to \mathbf{R}^d$$

be a random field such that the mapping $h(\cdot, n, \cdot, \cdot): \Omega \times \mathbf{R}^d \times \mathbf{R}^{m \times d} \to \mathbf{R}^d$ is $\mathcal{F}(n) \otimes \mathcal{B}(\mathbf{R}^d) \otimes \mathcal{B}(\mathbf{R}^{m \times d})$-measurable for each $n \in \{0, 1, \dots, N\}$. We assume that there is a constant $A > 0$ such that

$$(5) \qquad |h(\omega, n, y_1, z_1) - h(\omega, n, y_2, z_2)| \le A(|y_1 - y_2| + |z_1 - z_2|)$$

for all $y_1, y_2 \in \mathbf{R}^d$, $z_1, z_2 \in \mathbf{R}^{m\times d}$, $\omega \in \Omega$, and $n \in \{0, 1, \ldots, N\}$.

Let $\zeta : \Omega \to \mathbf{R}^d$ be $\mathcal{F}(N)$-measurable.

We consider the following backward stochastic difference equations

$$(6)\quad \begin{cases} -\Delta y(n) = \frac{1}{N} h(n-1, y(n-1), z(n)) - \frac{1}{\sqrt{N}} z(n)^* \Delta S(n), \\ \hfill n = 1, 2, \ldots, N, \\ y(N) = \zeta. \end{cases}$$

In other words,

$$(7)\quad y(n) = \zeta + \frac{1}{N} \sum_{k=n+1}^{N} h(k-1, y(k-1), z(k)) - \frac{1}{\sqrt{N}} \sum_{k=n+1}^{N} z(k)^* \Delta S(k)$$

for $n = 0, 1, \ldots, N$. A solution is a pair $(y, z) \in \mathcal{K} \times \mathcal{L}$ satisfying (7). We discuss the existence and uniqueness of a solution to this backward stochastic difference equation.

We define a mapping $\varphi : \mathcal{K}\times\mathcal{L} \to \mathcal{K}\times\mathcal{L}$ in the following. Let $(y, z) \in \mathcal{K}\times\mathcal{L}$ be given. Let $(M(n))_{n\in\{0,1,\ldots,N\}}$ be a martingale given by

$$M(n) = E\Big[\zeta + \frac{1}{N} \sum_{k=1}^{N} h(k-1, y(k-1), z(k)) \,|\mathcal{F}(n)\Big].$$

By Proposition 2.2 there exists a unique process $z' \in \mathcal{L}$ such that

$$M(n) = M(0) + \frac{1}{\sqrt{N}} \sum_{k=1}^{n} z'(k)^* \Delta S(k)$$

for all $n = 0, 1, \ldots, N$. Define the process $y' \in \mathcal{K}$ by

$$\begin{aligned} y'(n) &= M(n) - \frac{1}{N} \sum_{k=1}^{n} h(k-1, y(k-1), z(k)) \\ &= E\Big[\zeta + \frac{1}{N} \sum_{k=n+1}^{N} h(k-1, y(k-1), z(k)) \,|\mathcal{F}(n)\Big]. \end{aligned}$$

Now we define $\varphi(y, z)$ to be (y', z'). By this definition, the mapping φ maps a pair $(y, z) \in \mathcal{K} \times \mathcal{L}$ into the solution $(y', z') \in \mathcal{K} \times \mathcal{L}$ of the following backward stochastic difference equation

$$(8)\quad \begin{cases} -\Delta y'(n) = \frac{1}{N} h(n-1, y(n-1), z(n)) - \frac{1}{\sqrt{N}} z'(n)^* \Delta S(n), \\ \hfill n = 1, 2, \ldots, N, \\ y'(N) = \zeta. \end{cases}$$

LEMMA 3.1. *A pair $(y,z) \in \mathcal{K}\times\mathcal{L}$ is a solution of the backward stochastic difference equation (6) if and only if it is a fixed-point for the mapping φ.*

PROOF. From Equation (8), it is obvious that a fixed-point for φ is a solution of the backward stochastic difference equation (6). We show the converse. Let (y,z) be the solution of (6) and $(y',z') = \varphi(y,z)$. Then

$$\begin{aligned} y'(n) &= E\Big[y(n) + \frac{1}{\sqrt{N}}\sum_{k=n+1}^{N} z(k)^*\Delta S(k)\ |\mathcal{F}(n)\Big] \\ &= y(n). \end{aligned}$$

In particular, we get $y(0) = y'(0) = M(0)$. Then

$$\begin{aligned} M(n) &= E\Big[y(0) + \frac{1}{\sqrt{N}}\sum_{k=1}^{N} z(k)^*\Delta S(k)\ |\mathcal{F}(n)\Big] \\ &= M(0) + \frac{1}{\sqrt{N}}\sum_{k=1}^{n} z(k)^*\Delta S(k). \end{aligned}$$

We obtain $z = z'$ from the uniqueness in Proposition 2.2. Hence (y,z) is a fixed-point for φ. □

DEFINITION 3.2. We introduce norms $\|\cdot\|_\alpha,\ \alpha \geq 1$, in $\mathcal{K}\times\mathcal{L}$ by the following

$$\|(y,z)\|_\alpha = \left\{E\left[\sup_{n=0,1,\dots,N}(\alpha^n|y(n)|^2) + \frac{1}{N}\sum_{n=1}^{N}\alpha^n|z(n)|^2\right]\right\}^{\frac{1}{2}}.$$

THEOREM 3.3. *There is a universal constant γ such that*

$$\begin{aligned} &\|\varphi(y_1,z_1) - \varphi(y_2,z_2)\|_\alpha^2 \\ &\leq \left(1 + \frac{8\gamma^2+1}{1-\alpha/2}\right)\frac{2\alpha^2A^2}{(\alpha-1)N}\|(y_1,z_1) - (y_2,z_2)\|_\alpha^2 \end{aligned} \tag{9}$$

for all (y_1,z_1), $(y_2,z_2) \in \mathcal{K}\times\mathcal{L}$ and $\alpha \in (1,2)$.

Proof. Let $(y_i, z_i) \in \mathcal{K} \times \mathcal{L}$ and $(y_i', z_i') = \varphi(y_i, z_i)$, $i = 1, 2$. Let

$$y = y_1 - y_2 \quad z = z_1 - z_2 \quad y' = y_1' - y_2' \quad z' = z_1' - z_2'.$$

Then we have

$$(10) \quad \begin{cases} -\Delta y'(n) = \frac{1}{N}(h(n-1, y_1(n-1), z_1(n)) \\ \qquad\qquad\quad - h(n-1, y_2(n-1), z_2(n))) \\ \qquad\qquad\quad - \frac{1}{\sqrt{N}} z'(n)^* \Delta S(n), \qquad n = 1, 2, \ldots, N, \\ y'(N) = 0. \end{cases}$$

Now, we observe that

$$\begin{aligned} 0 &= \alpha^N |y'(N)|^2 \\ &= \alpha^n |y'(n)|^2 \\ &\quad + \sum_{k=n+1}^{N} \{(\alpha^k - \alpha^{k-1})|y'(k-1)|^2 + \alpha^k(|y'(k)|^2 - |y'(k-1)|^2)\} \end{aligned}$$

for all $n = 0, 1, \ldots, N$. Therefore,

$$\begin{aligned} &\alpha^n |y'(n)|^2 + \sum_{k=n+1}^{N} (\alpha^k - \alpha^{k-1})|y'(k-1)|^2 + \sum_{k=n+1}^{N} \alpha^k |\Delta y'(k)|^2 \\ &= \frac{2}{N} \sum_{k=n+1}^{N} \alpha^k y'(k-1)^* (h(k-1, y_1(k-1), z_1(k)) \\ &\qquad\qquad - h(k-1, y_2(k-1), z_2(k))) \\ &\quad - \frac{2}{\sqrt{N}} \sum_{k=n+1}^{N} \alpha^k y'(k-1)^* z'(k)^* \Delta S(k) \end{aligned}$$

for all $n = 0, 1, \ldots, N$. Setting

$$\lambda = \frac{2\alpha A^2}{(\alpha - 1)N}$$

and noting the inequality

$$\begin{aligned} &|y'(k-1)||h(k-1, y_1(k-1), z_1(k)) - h(k-1, y_2(k-1), z_2(k))| \\ &\quad \le \frac{A^2}{\lambda}|y'(k-1)|^2 + \frac{\lambda}{2}(|y(k-1)|^2 + |z(k)|^2), \end{aligned}$$

we have the following.

$$\begin{aligned}&\alpha^n|y'(n)|^2+\sum_{k=n+1}^{N}(\alpha^k-\alpha^{k-1})|y'(k-1)|^2+\sum_{k=n+1}^{N}\alpha^k|\Delta y'(k)|^2\\&\le\frac{2}{N}\sum_{k=n+1}^{N}\alpha^k(\frac{A^2}{\lambda}|y'(k-1)|^2+\frac{\lambda}{2}(|y(k-1)|^2+|z(k)|^2))\\&\quad-\frac{2}{\sqrt{N}}\sum_{k=n+1}^{N}\alpha^k y'(k-1)^*z'(k)^*\Delta S(k)\end{aligned}$$

for all $n=0,1,\dots,N$.

Furthermore, the observation $\frac{2\alpha^k A^2}{N\lambda}=\alpha^k-\alpha^{k-1}$ yields the following.

$$\begin{aligned}&\alpha^n|y'(n)|^2+\sum_{k=n+1}^{N}\alpha^k|\Delta y'(k)|^2\\&\le\lambda\left\{\alpha\sup_{k=0,1,\dots,N}(\alpha^k|y(k)|^2)+\frac{1}{N}\sum_{k=1}^{N}\alpha^k|z(k)|^2\right\}\\&\quad-\frac{2}{\sqrt{N}}\sum_{k=n+1}^{N}\alpha^k y'(k-1)^*z'(k)^*\Delta S(k)\end{aligned}\tag{11}$$

for all $n=0,1,\dots,N$. Taking expectation, we get

$$E\left[\alpha^n|y'(n)|^2+\sum_{k=n+1}^{N}\alpha^k|\Delta y'(k)|^2\right]\le\lambda\alpha\|(y,z)\|_\alpha^2$$

for all $n=0,1,\dots,N$.

Now, we have

$$\begin{aligned}E[|\Delta y'(k)|^2]&\ge\frac{1}{N}E[|z'(k)^*\Delta S(k)|^2]\\&=\frac{1}{N}E[|z'(k)|^2].\end{aligned}$$

Therefore,

$$E\left[\alpha^n|y'(n)|^2+\frac{1}{N}\sum_{k=n+1}^{N}\alpha^k|z'(k)|^2\right]\le\lambda\alpha\|(y,z)\|_\alpha^2$$

for all $n = 0, 1, \dots, N$. This implies that

$$\frac{1}{N}\sum_{k=1}^{N} \alpha^k E[|z'(k)|^2] \le \lambda\alpha \|(y, z)\|_\alpha^2. \tag{12}$$

From Davis's inequality, we obtain the following (γ is a universal constant).

$$\begin{aligned}
&E\Big[\sup_{n=0,1,\dots,N}\Big|\frac{1}{\sqrt{N}}\sum_{k=n+1}^{N} \alpha^k y'(k-1)^* z'(k)^* \Delta S(k)\Big|\Big] \\
&\le 2E\Big[\sup_{n=0,1,\dots,N}\Big|\frac{1}{\sqrt{N}}\sum_{k=1}^{n} \alpha^k y'(k-1)^* z'(k)^* \Delta S(k)\Big|\Big] \\
&\le 2\gamma E\Big[\Big(\frac{1}{N}\sum_{k=1}^{N} \alpha^{2k}|y'(k-1)|^2|z'(k)|^2\Big)^{\frac{1}{2}}\Big] \\
&\le \gamma E\Big[\frac{\alpha}{4\gamma}\sup_n(\alpha^n|y'(n)|^2) + \frac{4\gamma}{N}\sum_{k=1}^{N}\alpha^k|z'(k)|^2\Big] \\
&\le \frac{\alpha}{4}E\Big[\sup_n(\alpha^n|y'(n)|^2)\Big] + 4\gamma^2\lambda\alpha\|(y,z)\|_\alpha^2.
\end{aligned}$$

This inequality and (11) imply

$$\begin{aligned}
&E\Big[\sup_n(\alpha^n|y'(n)|^2)\Big] \\
&\qquad \le (8\gamma^2+1)\lambda\alpha\|(y,z)\|_\alpha^2 + \frac{\alpha}{2}E\Big[\sup_n(\alpha^n|y'(n)|^2)\Big],
\end{aligned}$$

and therefore

$$E\Big[\sup_n(\alpha^n|y'(n)|^2)\Big] \le \frac{8\gamma^2+1}{1-\alpha/2}\lambda\alpha\|(y,z)\|_\alpha^2. \tag{13}$$

Consequently, by (12) and (13), we have our theorem. □

By using the fixed point theorem for the contracting mapping, we have the following from Theorem 3.3.

COROLLARY 3.4. *If* $(1+\frac{8\gamma^2+1}{1-\alpha/2})\frac{2\alpha^2A^2}{(\alpha-1)N} < 1$, *then the backward stochastic difference equation (6) admits a unique solution* $(y, z) \in \mathcal{K} \times \mathcal{L}$.

4. A Connection between BSDE and Difference Equation

Here we introduce a metric space of pairs of a random variable and a probability measure. See [5] for the details.

Let $p > 1$ be a real number. Let $\mathcal{P}(M)$ be the set of all probability measures defined on a Polish space M. Let N be an arcwise connected separable metric space. Let dis_M and dis_N be distance functions on M and N respectively. We denote by $\mathcal{X}^p_{M;N}$ the set of all pairs (X,ν) which consists of a measurable map $X\colon M \to N$ and $\nu \in \mathcal{P}(M)$ such that $\int_M \mathrm{dis}_N(X(x),y)^p \nu(dx) < \infty$ for any $y \in N$. We define a function $\mathrm{Dis}^{(p)}_{M;N}$ from $\mathcal{X}^p_{M;N} \times \mathcal{X}^p_{M;N}$ into $[0,\infty)$ by

$$\begin{aligned}
&\mathrm{Dis}^{(p)}_{M;N}((X_1,\nu_1),(X_2,\nu_2)) \\
&\quad = \inf\Big\{ \Big(\int_{M\times M} ((\mathrm{dis}_M(x_1,x_2) \wedge 1) \\
&\qquad\qquad + \mathrm{dis}_N(X_1(x_1),X_2(x_2)))^p \nu(dx_1,dx_2) \Big)^{\frac{1}{p}} ; \\
&\qquad\qquad \nu \in \mathcal{P}(M\times M),\ \nu\circ\pi_1^{-1} = \nu_1,\ \nu\circ\pi_2^{-1} = \nu_2 \Big\},
\end{aligned}$$

where $\pi_i\colon M \times M \to M$ $(i = 1,2)$ are canonical projections given by

$$\pi_1(x_1,x_2) = x_1, \quad \pi_2(x_1,x_2) = x_2, \qquad x_1,x_2 \in M.$$

DEFINITION 4.1. Let $(X_n,\nu_n),(X,\nu) \in \mathcal{X}^p_{M;N}$, $n \geq 1$. We say that $(X_n,\nu_n) \to (X,\nu)$ in $\mathcal{X}^p_{M;N}$, $n \to \infty$ if $\mathrm{Dis}^{(p)}_{M;N}((X_n,\nu_n),(X,\nu)) \to 0$, $n \to \infty$.

REMARK 4.2. Note that W, $D([0,1];\mathbf{R}^d)$, and $L^2([0,1];\mathbf{R}^{m\times d})$ are arcwise connected separable metric spaces.

Now we think of the situation in Introduction. We denote by μ_N the distribution of B_N on W : $\mu_N = P_N \circ B_N^{-1}$. From Donsker's theorem ([1]), we have

$$\mu_N \to \mu \qquad \text{weakly as } N \to \infty.$$

DEFINITION 4.3. Let $N \geq 1$ and $(y, z) \in \mathcal{K}_N \times \mathcal{L}_N$. We define continuous-time process $(\bar{y}(t), \bar{z}(t))_{t\in[0,1]}$ on Ω_N by

$$(\bar{y}(t), \bar{z}(t)) = (y(\lfloor Nt \rfloor), z(\lceil Nt \rceil)), \qquad t \in [0,1],$$

and continuous-time process $(F_N(t;(y,z)))_{t\in[0,1]} = (F_N^0(t;(y,z)), F_N^1(t;(y,z)))_{t\in[0,1]}$ on W by

$$F_N(w,t;(y,z)) = \begin{cases} (y(\omega, \lfloor Nt \rfloor), z(\omega, \lceil Nt \rceil)), & w = B_N(\omega),\ t \in [0,1], \\ (0,0), & \text{otherwise.} \end{cases}$$

Let β be a real number such that $\beta > 8C^2(16\gamma^2 + 3)$. Here C is a constant in the inequality (1). Choose a positive integer N_1 such that

$$e^{\beta/N} \in (1,2) \qquad \text{for all } N \geq N_1.$$

We consider Corollary 3.4 for $\alpha = e^{\beta/N}$ $(N \geq N_1)$ and $A = C$. Note that C is independent of N. Taking into account that

$$\begin{aligned} \Big(1 + \frac{8\gamma^2+1}{1-\alpha/2}\Big)\frac{2\alpha^2C^2}{(\alpha-1)N} &= \Big(1 + \frac{8\gamma^2+1}{1-\alpha/2}\Big)\alpha^2\Big(\frac{e^{\beta/N}-1}{\beta/N}\Big)^{-1}\frac{2C^2}{\beta} \\ &\to \frac{2C^2(16\gamma^2+3)}{\beta}, \qquad N \to \infty, \end{aligned}$$

we have the following.

REMARK 4.4. There exists a positive integer $N_0 \geq N_1$ such that $(1 + \frac{8\gamma^2+1}{1-e^{\beta/N}/2})\frac{2e^{2\beta/N}C^2}{(e^{\beta/N}-1)N} \leq \frac{1}{4}$ for all $N \geq N_0$.

In this section, let N be sufficiently large such that $N \geq N_0$.

DEFINITION 4.5. Let $\varphi_N = (\varphi_N^0, \varphi_N^1)\colon \mathcal{K}_N \times \mathcal{L}_N \to \mathcal{K}_N \times \mathcal{L}_N$ be the mapping that maps $(y,z) \in \mathcal{K}_N \times \mathcal{L}_N$ into the solution $(y', z') \in \mathcal{K}_N \times \mathcal{L}_N$ of the backward stochastic difference equation

$$\text{(14)} \quad \begin{cases} -\Delta y'(n) = \frac{1}{N} f(B_N(\cdot), \frac{n-1}{N}, y(n-1), z(n)) - \frac{1}{\sqrt{N}} z'(n)^* \Delta S_N(n), \\ \qquad\qquad\qquad\qquad n = 1,2,\ldots,N,\ P_N\text{-a.s.} \\ y'(N) = \xi(B_N(\cdot)). \end{cases}$$

Let $\Phi = (\Phi^0, \Phi^1)\colon H^2(\mathbf{R}^d)\times H^2(\mathbf{R}^{m\times d}) \to H^2(\mathbf{R}^d)\times H^2(\mathbf{R}^{m\times d})$ be the mapping that maps $(Y, Z) \in H^2(\mathbf{R}^d) \times H^2(\mathbf{R}^{m\times d})$ into the solution $(Y', Z') \in H^2(\mathbf{R}^d) \times H^2(\mathbf{R}^{m\times d})$ of the following BSDE

$$\text{(15)} \quad \begin{cases} -dY'(t) = f(B(\cdot), t, Y(t), Z(t))dt - Z'(t)^* dB(t), \quad t \in [0,1],\ \mu\text{-a.s.} \\ Y'(1) = \xi. \end{cases}$$

For each $N \geq N_0$, let $(y_N, z_N) \in \mathcal{K}_N \times \mathcal{L}_N$. First we show how to obtain $\varphi_N(y_N, z_N)$ and $\Phi(Y, Z)$. We define random variables $G_N\colon W \to \mathbf{R}^d$, $N \geq N_0$ by

$$\text{(16)} \qquad G_N = \xi + \int_0^1 f(B(\cdot), \frac{[Ns]}{N}, F_N(s; (y_N, z_N)))ds, \qquad \mu_N\text{-a.s.}$$

Then it follows that

$$G_N \circ B_N = \xi \circ B_N + \frac{1}{N}\sum_{k=1}^N f(B_N(\cdot), \frac{k-1}{N}, y_N(k-1), z_N(k)), \quad P_N\text{-a.s.}$$

For each $N \geq N_0$, from Proposition 2.2, there exists a unique $z'_N \in \mathcal{L}_N$ such that

$$E^{P_N}[G_N \circ B_N | \mathcal{F}_N(n)] - E^{P_N}[G_N \circ B_N] = \frac{1}{\sqrt{N}}\sum_{k=1}^n z'_N(k)^* \Delta S_N(k), \qquad n = 0, 1, \ldots, N,\ P_N\text{-a.s.}$$

Define $y'_N \in \mathcal{K}_N$, $N \geq N_0$ by

$$y'_N(n) = E^{P_N}[G_N \circ B_N | \mathcal{F}_N(n)] - \frac{1}{N}\sum_{k=1}^n f(B_N(\cdot), \frac{k-1}{N}, y_N(k-1), z_N(k))$$

for $n = 0, 1, \ldots, N$. Note that $(y'_N, z'_N) \in \mathcal{K}_N \times \mathcal{L}_N$ is a unique solution of the following backward stochastic difference equation

$$\text{(17)} \quad \begin{cases} -\Delta y'_N(n) = \frac{1}{N} f(B_N(\cdot), \frac{n-1}{N}, y_N(n-1), z_N(n)) \\ \qquad\qquad - \frac{1}{\sqrt{N}} z'_N(n)^* \Delta S_N(n), \qquad n = 1, 2, \ldots, N,\ P_N\text{-a.s.} \\ y'_N(N) = \xi(B_N(\cdot)). \end{cases}$$

Thus we get $\varphi_N(y_N, z_N) = (y'_N, z'_N)$. Letting $(M_N(t))_{t\in[0,1]}$ be the càdlàg version of the martingale $(E^{\mu_N}[G_N|\mathcal{F}(t)])_{t\in[0,1]}$, we get the following expression

$$\begin{aligned} M_N(t) - M_N(0) &= \int_0^t F_N^1(s;\varphi_N(y_N,z_N))^* dB(s), \qquad t\in[0,1],\ \mu_N\text{-a.s.} \\ F_N^0(t;\varphi_N(y_N,z_N)) &= M_N(t) - \int_0^{\frac{[Nt]}{N}} f(B(\cdot), \frac{[Ns]}{N}, F_N(s;(y_N,z_N)))ds, \\ &\qquad\qquad t\in[0,1],\ \mu_N\text{-a.s.} \end{aligned}$$

Let $(Y, Z) \in H^2(\mathbf{R}^d)\times H^2(\mathbf{R}^{m\times d})$. We define a random variable $G\colon W \to \mathbf{R}^d$ by

$$G = \xi + \int_0^1 f(B(\cdot), s, Y(s), Z(s))ds. \tag{18}$$

Letting $(M(t))_{t\in[0,1]}$ be the continuous martingale $(E^{\mu}[G|\mathcal{F}(t)])_{t\in[0,1]}$, there exists a unique $Z' \in H^2(\mathbf{R}^{m\times d})$ such that

$$M(t) - M(0) = \int_0^t Z'(t)^* dB(s), \qquad t\in[0,1],\ \mu\text{-a.s.}$$

Define $Y' \in H^2(\mathbf{R}^d)$ by

$$Y'(t) = M(t) - \int_0^t f(B(\cdot), s, Y(s), Z(s))ds.$$

Note that $(Y', Z') \in H^2(\mathbf{R}^d)\times H^2(\mathbf{R}^{m\times d})$ is a unique solution of the following BSDE

$$\begin{cases} -dY'(t) = f(B(\cdot), t, Y(t), Z(t))dt - Z'(t)^* dB(t), \\ \qquad\qquad\qquad\qquad t\in[0,1],\ \mu\text{-a.s.} \\ Y'(1) = \xi. \end{cases} \tag{19}$$

Thus we get $\Phi(Y, Z) = (Y', Z')$.

Let p be a real number such that $p > 1$. The purpose of this section is to prove the following.

THEOREM 4.6. *Let* $(y_N, z_N) \in \mathcal{K}_N \times \mathcal{L}_N$, $N \geq N_0$ *and* $(Y, Z) \in H^2(\mathbf{R}^d) \times H^2(\mathbf{R}^{m\times d})$. *Assume that*

$$(F_N(\cdot;(y_N, z_N)), \mu_N) \to ((Y, Z), \mu) \quad \text{in } \mathcal{X}^p_{W;D([0,1];\mathbf{R}^d)\times L^2([0,1];\mathbf{R}^{m\times d})}, \qquad N \to \infty.$$

Then we have

$$(F_N(\cdot;\varphi_N(y_N, z_N)), \mu_N) \to (\Phi(Y, Z), \mu) \quad \text{in } \mathcal{X}^p_{W;D([0,1];\mathbf{R}^d)\times L^2([0,1];\mathbf{R}^{m\times d})}, \qquad N \to \infty.$$

Now we denote

$$\begin{aligned} X_N(t) &= \int_0^{\frac{[Nt]}{N}} f(B(\cdot), \frac{[Ns]}{N}, F_N(s;(y_N, z_N)))ds \\ X(t) &= \int_0^t f(B(\cdot), s, Y(s), Z(s))ds. \end{aligned}$$

We get the following.

LEMMA 4.7. *Under the assumption of Theorem 4.6,*

$$(X_N, \mu_N) \to (X, \mu) \quad \text{in } \mathcal{X}^p_{W;D([0,1];\mathbf{R}^d)}, \qquad N \to \infty.$$

PROOF. From the assumption of Theorem 4.6 and Proposition 5 in [5], we see that there exist a probability space $(\Omega, \mathcal{F}, P)$ and random variables $\Lambda_N, \Lambda\colon \Omega \to W$, $N \geq N_0$, such that the following three conditions are satisfied.

(1) $P \circ \Lambda_N^{-1} = \mu_N$, $P \circ \Lambda^{-1} = \mu$,

(2) $\lim_{N\to\infty} E^P[(\mathrm{dis}_W(\Lambda_N, \Lambda) \wedge 1)^p] = 0$,

(3) $\lim_{N\to\infty} E^P\Big[\Big(\mathrm{dis}_{D([0,1];\mathbf{R}^d)}(F^0_N(\Lambda_N, \cdot;(y_N, z_N)), Y(\Lambda, \cdot)) + \Big(\int_0^1 |F^1_N(\Lambda_N, t;(y_N, z_N)) - Z(\Lambda, t)|^2 dt\Big)^{\frac{1}{2}}\Big)^p\Big] = 0.$

Here dis_W and $\mathrm{dis}_{D([0,1];\mathbf{R}^d)}$ are Skorohod metrics.

Since f is bounded, we have

$$E^P\Big[\sup_{0\le t\le 1}|X_N(\Lambda_N,t)-\int_0^t f(\Lambda_N,\frac{[Ns]}{N},F_N(\Lambda_N,s;(y_N,z_N)))ds|^p\Big]$$
$$\le E^P\Big[\sup_{0\le t\le 1}\Big|\int_{\frac{[Nt]}{N}}^t f(\Lambda_N,\frac{[Ns]}{N},F_N(\Lambda_N,s;(y_N,z_N)))ds\Big|^p\Big]$$
$$\to 0,\quad N\to\infty.$$

We also have

$$E^P\Big[\sup_{0\le t\le 1}\Big|\int_0^t f(\Lambda_N,\frac{[Ns]}{N},F_N(\Lambda_N,s;(y_N,z_N)))ds-X(\Lambda,t)\Big|^p\Big]$$
$$\le E^P\Big[\int_0^1\Big|f(\Lambda_N,\frac{[Ns]}{N},F_N(\Lambda_N,s;(y_N,z_N)))$$
$$-f(\Lambda,s,Y(\Lambda,s),Z(\Lambda,s))\Big|^p ds\Big]\to 0,\quad N\to\infty.$$

Then we have

$$\lim_{N\to\infty}E^P[\mathrm{dis}_{D([0,1];\mathbf{R}^d)}(X_N\circ\Lambda_N,X\circ\Lambda)^p]=0. \tag{20}$$

It completes the proof. □

A consequence of Lemma 4.7 is the following.

COROLLARY 4.8. *Under the assumption of theorem 4.6,*

$$(G_N,\mu_N)\to(G,\mu)\quad \textit{in } \mathcal{X}^p_{W;\mathbf{R}^d},\qquad N\to\infty.$$

PROOF. From Lemma 4.7 and Lemma 7 in [5],

$$\inf\Big\{\limsup_{N\to\infty}E^{\mu_N}[\mathrm{dis}_{D([0,1];\mathbf{R}^d)}(X_N,\Theta)^p]$$
$$+E^{\mu}[\mathrm{dis}_{D([0,1];\mathbf{R}^d)}(X,\Theta)^p]\,;\ \Theta\in C_b(W;D([0,1];\mathbf{R}^d))\Big\}=0,$$

where $C_b(W;D([0,1];\mathbf{R}^d))$ denotes the set of continuous mappings Θ from W to $D([0,1];\mathbf{R}^d)$ such that

$$\sup_{w\in W}\mathrm{dis}_{D([0,1];\mathbf{R}^d)}(0,\Theta(w))<\infty.$$

Note that

$$\begin{aligned} |X_N(w)(1) - \Theta(w)(1)| &\leq \mathrm{dis}_{D([0,1];\mathbf{R}^d)}(X_N(w), \Theta(w)), \\ |X(w)(1) - \Theta(w)(1)| &\leq \mathrm{dis}_{D([0,1];\mathbf{R}^d)}(X(w), \Theta(w)) \end{aligned}$$

for any $w \in W$. Consequently,

$$\inf\Big\{\limsup_{N\to\infty} E^{\mu_N}[|X_N(1) - \psi|^p] + E^{\mu}[|X(1) - \psi|^p]\,;\, \psi \in C_b(W;\mathbf{R}^d)\Big\} = 0.$$

Noting that $G_N = \xi + X_N(1)$, $G = \xi + X(1)$, and $\xi \in C_b(W;\mathbf{R}^d)$, we obtain

$$\inf\Big\{\limsup_{N\to\infty} E^{\mu_N}[|G_N - \psi|^p] + E^{\mu}[|G - \psi|^p]\,;\, \psi \in C_b(W;\mathbf{R}^d)\Big\} = 0.$$

Therefore by Lemma 7 in [5], we have our Corollary. □

PROOF OF THEOREM 4.6. Combining Corollary 4.8 with Theorem 11 in [5], we obtain the following. Under the assumption of theorem 4.6, we have

$$(F_N^1(\cdot;\varphi_N(y_N, z_N)), \mu_N) \to (\Phi^1(Y,Z), \mu) \quad \text{in } \mathcal{X}^p_{W;L^2([0,1];\mathbf{R}^{m\times d})}, \qquad N \to \infty,$$

and

$$(M_N, \mu_N) \to (M, \mu) \quad \text{in } \mathcal{X}^p_{W;D([0,1];\mathbf{R}^d)}, \qquad N \to \infty.$$

From Lemma 4.7, we obtain the following. Under the assumption of theorem 4.6,

$$(F_N^0(\cdot;\varphi_N(y_N, z_N)), \mu_N) \to (\Phi^0(Y,Z), \mu) \quad \text{in } \mathcal{X}^p_{W;D([0,1];\mathbf{R}^d)}, \qquad N \to \infty.$$

Therefore we conclude Theorem 4.6. □

5. Main Result

Let N be sufficiently large such that $N \geq N_0$ (See Remark 4.4) and $p > 1$ be a real number.

We endow $W \times D([0,1];\mathbf{R}^d) \times L^2([0,1];\mathbf{R}^{m\times d})$ with the metric

$$\begin{aligned} \mathrm{dis}((w,y,z),(\tilde{w},\tilde{y},\tilde{z})) &= \mathrm{dis}_W(w,\tilde{w}) + \mathrm{dis}_{D([0,1];\mathbf{R}^d)}(y,\tilde{y}) \\ &\quad + \Big(\int_0^1 |z(t) - \tilde{z}(t)|^2 dt\Big)^{\frac{1}{2}}, \\ &(w,y,z),(\tilde{w},\tilde{y},\tilde{z}) \in W \times D([0,1];\mathbf{R}^d) \times L^2([0,1];\mathbf{R}^{m\times d}). \end{aligned}$$

Here dis_W and $\mathrm{dis}_{D([0,1];\mathbf{R}^d)}$ are Skorohod metrics.

Now let $(y_N, z_N) \in \mathcal{K}_N \times \mathcal{L}_N$ be the solution of the backward stochastic difference equation (3) on (Ω_N, P_N), $N \geq N_0$. Let $(Y, Z) \in H^2(\mathbf{R}^d) \times H^2(\mathbf{R}^{m\times d})$ be the solution of the BSDE (2). In this section, we prove the following main theorem. Theorem 1.1 is an easy consequence of Theorem 5.1.

Theorem 5.1. *It follows that*

$$((B, F_N(\cdot; (y_N, z_N))), \mu_N) \to ((B, Y, Z), \mu) \quad \text{in } \mathcal{X}^p_{W;W\times D([0,1];\mathbf{R}^d)\times L^2([0,1];\mathbf{R}^{m\times d})}, \qquad N \to \infty.$$

For each $N \geq N_0$, we define a sequence $(y_N^{(l)}, z_N^{(l)})_{l=0,1,2,\ldots}$ in $\mathcal{K}_N \times \mathcal{L}_N$ by

$$\begin{aligned} (y_N^{(0)}, z_N^{(0)}) &= (0,0) \\ (y_N^{(l)}, z_N^{(l)}) &= \varphi_N^l(y_N^{(0)}, z_N^{(0)}), \qquad l = 1, 2, \ldots. \end{aligned}$$

Here φ_N is in Definition 4.5. We can define $(\bar{y}_N^{(l)}, \bar{z}_N^{(l)})$, $(\bar{y}_N, \bar{z}_N)$, $F_N(\cdot; (y_N^{(l)}, z_N^{(l)}))$ for each N, l by Definition 4.3.

We define a sequence $(Y^{(l)}, Z^{(l)})_{l=0,1,2,\ldots}$ in $H^2(\mathbf{R}^d) \times H^2(\mathbf{R}^{m\times d})$ by

$$\begin{aligned} (Y^{(0)}, Z^{(0)}) &= (0,0) \\ (Y^{(l)}, Z^{(l)}) &= \Phi^l(Y^{(0)}, Z^{(0)}), \qquad l = 1, 2, \ldots. \end{aligned}$$

Here Φ is in Definition 4.5.

By [3] or [7], we have

$$\lim_{l\to\infty} E^\mu\left[\sup_{t\in[0,1]} |Y^{(l)}(t) - Y(t)|^2 + \int_0^1 |Z^{(l)}(t) - Z(t)|^2 dt\right] = 0.$$

Hence we have the following.

Lemma 5.2.

$$\lim_{l\to\infty} E^\mu\left[\mathrm{dis}((B, Y^{(l)}, Z^{(l)}), (B, Y, Z))^2\right] = 0.$$

PROPOSITION 5.3. *For each $l = 0, 1, 2, \ldots$, we have*

$$((B, F_N(\cdot; (y_N^{(l)}, z_N^{(l)}))), \mu_N) \to ((B, Y^{(l)}, Z^{(l)}), \mu)$$
$$\text{in } \mathcal{X}^p_{W; W\times D([0,1];\mathbf{R}^d)\times L^2([0,1];\mathbf{R}^{m\times d})}, \qquad N \to \infty.$$

PROOF. In the case $l = 0$, we have $F_N(\cdot; (y_N^{(0)}, z_N^{(0)})) = (0, 0)$ and

$$(B, \mu_N) \to (B, \mu) \qquad \text{in } \mathcal{X}^p_{W;W}, \qquad N \to \infty$$

by [5]. Using Theorem 4.6, we obtain Proposition by induction. □

LEMMA 5.4.

$$\lim_{l\to\infty} \sup_N E^{P_N}\left[\mathrm{dis}((B_N, \bar{y}_N^{(l)}, \bar{z}_N^{(l)}), (B_N, \bar{y}_N, \bar{z}_N))^2\right] = 0.$$

PROOF. From Theorem 3.3, it follows that

$$\begin{aligned}(21)\quad &\|(y_N^{(l)}, z_N^{(l)}) - (y_N^{(l-1)}, z_N^{(l-1)})\|_\alpha^2 \\ &\le \left(1 + \frac{8\gamma^2+1}{1-\alpha/2}\right)\frac{2\alpha^2C^2}{(\alpha-1)N}\|(y_N^{(l-1)}, z_N^{(l-1)}) - (y_N^{(l-2)}, z_N^{(l-2)})\|_\alpha^2\end{aligned}$$

for all $l = 2, 3, \ldots$, natural number N, and real number $\alpha \in (1, 2)$. Here C is a constant in the equation (1) and $\|\cdot\|_\alpha$ is one defined in Definition 3.2 of section 3. Recalling Remark 4.4, we have

$$\|(y_N^{(l)}, z_N^{(l)}) - (y_N^{(l-1)}, z_N^{(l-1)})\|_{e^{\beta/N}} \le \frac{1}{2}\|(y_N^{(l-1)}, z_N^{(l-1)}) - (y_N^{(l-2)}, z_N^{(l-2)})\|_{e^{\beta/N}}$$

for all $N \ge N_0$ and $l = 2, 3, \ldots$. Therefore, for every $N \ge N_0$, we obtain

$$\|(y_N^{(l)}, z_N^{(l)}) - (y_N, z_N)\|_1 \le \sum_{l'=l+1}^{\infty} \|(y_N^{(l')}, z_N^{(l')}) - (y_N^{(l'-1)}, z_N^{(l'-1)})\|_1$$
$$\le \sum_{l'=l+1}^{\infty} \|(y_N^{(l')}, z_N^{(l')}) - (y_N^{(l'-1)}, z_N^{(l'-1)})\|_{e^{\beta/N}}$$

$$\leq \sum_{l'=l+1}^{\infty} \left(\frac{1}{2}\right)^{l'-1} \|(y_N^{(1)}, z_N^{(1)})\|_{e^{\beta/N}}$$
$$= \left(\frac{1}{2}\right)^{l-1} E^{P_N}\left[\sup_{n=0,1,\ldots,N}(e^{\beta\frac{n}{N}}|y_N^{(1)}(n)|^2) + \frac{1}{N}\sum_{k=1}^{N} e^{\beta\frac{n}{N}}|z_N^{(1)}(n)|^2\right]$$
$$\leq e^{\beta}\left(\frac{1}{2}\right)^{l-1} \|(y_N^{(1)}, z_N^{(1)})\|_1.$$

Since $\sup_N \|(y_N^{(1)}, z_N^{(1)})\|_1$ is finite, the proof is complete. □

From Lemma 5.2, Proposition 5.3, and Lemma 5.4, we conclude Theorem 5.1.

References

[1] Billingsley, P., *Convergence of Probability Measures*, Wiley, New York, 1968.

[2] Douglas, J., Jr., Ma, J., and P. Protter, Numerical methods for forward-backward stochastic differential equations, Ann. Appl. Probab. **6** (1996), 940–968.

[3] El Karoui, N., Peng, S., and M. C. Quenez, Backward stochastic differential equations in finance, Mathematical Finance **7** (1997), 1–71.

[4] Jacod, J. and A. N. Shiryaev, *Limit Theorems for Stochastic Processes*, Springer, Berlin Heidelberg New York, 1987.

[5] Kusuoka, S. and T. Nakayama, On a certain metric on the space of pairs of a random variable and a probability measure, J. Math. Sci. Univ. Tokyo **8** (2001), 343–356.

[6] Ma, J. and J. Yong, *Forward backward stochastic differential equations and their applications* (Lecture notes in mathematics; 1702) Chapter 9, Springer, 1999.

[7] Yong, J. and X. Y. Zhou, *Stochastic Controls*, Springer, 1999.

(Received December 25, 2000)

University of Tokyo
Graduate School of Mathematical Science