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Real Shintani Functions on $\mathsf{U}(n,1)$ *III, Construction of Intertwining Operators*

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Abstract. We study an integral transform on $\mathsf{U}(n,1)$ (Poisson integral) in detail. As an application, we obtain a precise formula of the dimension of the space of $\mathsf{U}(n-1,1) \times \mathsf{U}(n,1)$-intertwining operators from an irreducible admissible representaion $\pi_0 \boxtimes \pi$ to the space of smooth functions on $\mathsf{U}(n,1)$ in several cases including when π_0 and π are large discrete series of $\mathsf{U}(n-1,1)$ and $\mathsf{U}(n,1)$ respectively.

0. Introduction and Basic Notations

0.1. Introduction

This is a continuation of [10]. We consider the problem to determine the space of $\mathsf{U}(n-1,1) \times \mathsf{U}(n,1)$-intertwining operators from an irreducible admissible representation $\pi_0^\vee \boxtimes \pi$ of $\mathsf{U}(n-1,1) \times \mathsf{U}(n,1)$ to the space of smooth functions on $\mathsf{U}(n,1)$. Such an intertwining operator is called a *Shintani functional* (for $\pi_0^\vee \boxtimes \pi$) and the totality of Shintani functionals is denoted by $\mathcal{I}(\pi_0|\pi)$. We give a way how to construct non-trivial Shintani functionals making use of an integral transform. As a result we obtain a multiplicity one theorem of the space of Shintani functionals in several cases (section 6). We shall explain contents of this paper. The first three sections are preliminaries. In section 4, we introduce an integral transform, whose study is a main theme of this paper. We may call our integral transform the Poisson integral in analogy with the one which appears in the theory of the affine symmetric spaces ([7], [8] and [3]). Let P_n be a minimal parabolic subgroup of $\mathsf{U}(n,1)$ and P_{n-1} that of $\mathsf{U}(n-1,1) \subset \mathsf{U}(n,1)$. It turns out that the double coset space $P_{n-1}\backslash\mathsf{U}(n,1)/P_n$ has three elements among which there exists a unique open one (Proposition 4.1.1 and Remark 4.1.1). The explicit determination of the open double coset in $\mathsf{U}(n,1)$ (Proposition 4.1.1)

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enables us to have a $\Delta\mathsf{U}(n-1,1)$-invariant distribution which belongs to a principal series representation of $\mathsf{U}(n-1,1)\times\mathsf{U}(n,1)$, when the radial parameter of the principal series is sufficiently 'positive' (Theorem 4.1.1). In 4.2 we examine the asymptotic behavior of the Poisson integrals at 'infinity' and evaluate them for the corner K-types making use of the difference-differential equations obtained in [10]. To have a non-trivial intertwining operator outside the domain of convergence of the Poisson integral, we have to obtain an analytic continuation of the intertwining operator defined by that integral. We first evaluate explicitly the integral for a corner K-type; then using the differential equations in [10], by an induction on the 'size' of the K-type, we prove that the Poisson integral for a vector with a fixed K-type is expressed in terms of the Gaussian hypergeometric series up to a polynomial factors to know its meromorphicity (Theorem 5.1.1). In section 6, using Theorem 5.1.1 and the result of [10], we give a sufficient condition for the space $\mathcal{I}(\pi_0|\pi)$ to be one dimensional (Theorem 6.1.1). Moreover we obtain a precise formula of $\dim_{\mathbb{C}}\mathcal{I}(\pi_0|\pi)$, which is 1 or 0 as we already showed in [10], when π_0 and π are principal series or discrete series (Theorems 6.2.1, 6.3.1, 6.3.2 and 6.3.4). For technical reasons, we discuss in this paper the intertwining space $\mathcal{I}(\pi_0|\pi)$ mainly which is not the one $\mathcal{I}_{\eta,\pi}$ studied in [10]. We have a theorem which ensures that these two different intertwining spaces are isomorphic. In the final section we formulate and prove that theorem (Thereom 7.1.1).

0.2. Basic notations

For a $\mathbb{C}$-vector space V, $V^\vee$ denotes the algebraic dual space of V, $\langle\ ,\ \rangle : V\times V^\vee\to\mathbb{C}$ the natural bi-linear form and $\mathtt{I}_V$ the identity map of V. For finite dimensional $\mathbb{C}$-vector spaces V and W, we identify $(V\otimes_{\mathbb{C}}W)^\vee$ with $V^\vee\otimes_{\mathbb{C}}W^\vee$, and $V^{\vee\vee}$ with V by means of the canonical isomorphisms. The Lie algebra of a Lie group G is denoted by the corresponding Germann letter $\mathfrak{g}$. For a $(\mathfrak{g},K)$-module (π,V) with K a compact subgroup of G, its contragredient is denoted by $(\pi^\vee,V^\vee)$.

For a positive integer n, let $\mathbb{C}^n=\mathrm{M}_{n,1}(\mathbb{C})$ be the space of all column vectors of degree n. We naturally identify the space $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$ with $\mathrm{M}_n(\mathbb{C})$ by letting a matrix $A=(a_{ij})_{1\leqslant i,j\leqslant n+1}\in\mathrm{M}_n(\mathbb{C})$ operate on $x=(x_i)_{1\leqslant i\leqslant n}\in\mathbb{C}^n$ as $Ax=(\sum_{j=1}^n a_{ij}x_j)_{1\leqslant i\leqslant n}\in\mathbb{C}^n$. We write $\mathtt{I}_n$ for $\mathtt{I}_{\mathbb{C}^n}$. For positive integers p and q, we write $\mathtt{O}_{p,q}$ the $p\times q$-matrix whose entries are all zero.

For a smooth function f on a Lie group G with its values in a topological vector space, put $R_X f(g) = \frac{d}{dt} f(g\exp(tX))|_{t=0}$ (resp. $L_X f(g) = \frac{d}{dt} f(\exp(-tX)g)|_{t=0}$) for $X \in \mathfrak{g}$. Extended action of the universal enveloping algebra is also denoted by R (resp. L).

For a finite dimensional $\mathbb{C}$-vector space W, $\|\cdot\|$ denotes a norm on W.

1. Preliminaries

1.1. Lie groups, Lie algebras and representations of compact groups

Let $n \geqslant 2$ be an integer. Let e_i, $1 \leqslant i \leqslant n+1$ be the standard basis of $\mathbb{C}^{n+1}$, i.e., $\mathsf{e}_i = (\delta_{ij})_{1 \leqslant j \leqslant n+1}$. Put $(\mathsf{x}|\mathsf{y}) = {}^t\bar{\mathsf{x}}\mathtt{w}_n\mathsf{y}$ for $\mathsf{x}, \mathsf{y} \in \mathbb{C}^{n+1}$ with $\mathtt{w}_n = \operatorname{diag}(\mathtt{I}_n, -1)$. Let G_n denote the Lie group consisting of all automorphisms of $\mathbb{C}^{n+1}$ preserving the Hermitian form $(\mathsf{x}|\mathsf{y})$, that is

$$G_n = \{g \in \mathsf{GL}_{n+1}(\mathbb{C}) | {}^t\bar{g}\mathtt{w}_n g = \mathtt{w}_n\}.$$

We define subgroups K_n, A_n and M_n of G_n as

$$\begin{aligned} K_n &= \{\operatorname{diag}(k_1, k_2) |\ k_1 \in \mathsf{U}(n),\ k_2 \in \mathsf{U}(1)\} \cong \mathsf{U}(n) \times \mathsf{U}(1), \\ A_n &= \left\{ \mathsf{a}_n(r) = \operatorname{diag}\left(\mathtt{I}_{n-1}, \begin{pmatrix} \operatorname{ch}(r) & \operatorname{sh}(r) \\ \operatorname{sh}(r) & \operatorname{ch}(r) \end{pmatrix}\right) \middle|\ r > 0 \right\} \cong \mathbb{R}_+^*, \\ M_n &= \{\mathtt{m}_n(x, u) = \operatorname{diag}(x, u, u) |\ x \in \mathsf{U}(n-1),\ u \in \mathsf{U}(1)\} \\ &\quad \cong \mathsf{U}(n-1) \times \mathsf{U}(1) \end{aligned}$$

with $\operatorname{sh}(r) = \frac{r - r^{-1}}{2}$, $\operatorname{ch}(r) = \frac{r + r^{-1}}{2}$. The normalizer M_n^* of M_n in K_n coincides with $M_n \cup \mathtt{w}_n M_n$ and the coset $\mathtt{w}_n M_n$ gives the non-trivial element of the little Weyl group M_n^*/M_n. Let Z_n be the center of G_n.

Let P_n be the stabilizer in G_n of the line $\mathbb{C} \cdot (\mathsf{e}_n + \mathsf{e}_{n+1})$; P_n is a minimal parabolic subgroup of G_n having $A_n M_n$ as a Levi subgroup. Let N_n denote the unipotent radical of P_n and put $\bar{N}_n = \mathtt{w}_n N_n \mathtt{w}_n^{-1}$.

The map $i_n : G_{n-1} \ni \left(\begin{smallmatrix} x_{11} & x_{13} \\ x_{31} & x_{33} \end{smallmatrix}\right) \to \begin{pmatrix} x_{11} & \mathtt{0}_{n-1,1} & x_{13} \\ \mathtt{0}_{1,n-1} & 1 & 0 \\ x_{31} & 0 & x_{33} \end{pmatrix} \in G_n$ gives an isomorphism from G_{n-1} onto the stabilizer of the vector e_n in G_n. In what follows we identify G_{n-1} with $i_n(G_{n-1})$.

We parametrize the irreducible representations of K_n and M_n as in [10, Section 3]. In particular, any irreducible finite dimensional representation of K_n (resp. M_n) is isomorphic to one of τ_λ's (resp. σ_μ's) with $\lambda = [\mathbf{l}\,;l_0] \in \mathcal{L}_n^+ = \Lambda_n^+ \times \Lambda_1$ (resp. $\mu = (\mathbf{m}\,;m_0) \in {}^\circ\mathcal{L}_n^+ = \Lambda_{n-1}^+ \times \Lambda_1^+$) in the notation of [10, subsection 3.2]. If we want to emphasize the dependence on n, we also write τ_λ^n for τ_λ and σ_μ^n for σ_μ. Let $\mathbf{l} \in \Lambda_n^+$ and $\mathbf{m} \in \Lambda_{n-1}^+$ with $\mathbf{m} \subset \mathbf{l}$ ([10, 3.1]). Then we have the $\mathsf{U}(n-1)$-projection $\mathsf{p}_{\mathbf{m}}^{\mathbf{l}} : W(\mathbf{l}) \to W(\mathbf{m})$ and the $\mathsf{U}(n-1)$-inclusion $\mathsf{j}_{\mathbf{l}}^{\mathbf{m}} : W(\mathbf{m}) \to W(\mathbf{l})$ ([10, Lemma 3.1.1, Lemma 10.1.1]). For $\mathbf{l} = (l_j)_{1 \leqslant j \leqslant n} \in \Lambda_n^+$, put $\check{\mathbf{l}} = (-l_{n+1-j})_{1 \leqslant j \leqslant n}$. We have $W(\check{\mathbf{l}}) \cong W(\mathbf{l})^\vee$ as $\mathsf{U}(n)$-module.

For $\mathbf{q} \in \Lambda_{n-2}^+$, $\mathbf{p} \in \Lambda_{n-1}^+$ and $\mathbf{l} \in \Lambda_n^+$, we put

$$\Lambda_n^+(\mathbf{p}) = \{\mathbf{k} \in \Lambda_n^+ \mid \mathbf{p} \subset \mathbf{k}\},$$
$$\Lambda_{n-1}^+(\mathbf{q}|\mathbf{l}) = \{\mathbf{m} \in \Lambda_{n-1}^+ \mid \mathbf{q} \subset \mathbf{m} \subset \mathbf{l}\}.$$

1.2. Haar measures

For an element $g \in G_n$, we can write it as

$$g = \nu_n(g)\alpha_n(g)\kappa_n(g), \quad \nu_n(g) \in N_n,\ \alpha_n(g) \in A_n,\ \kappa_n(g) \in K_n$$

uniquely along the Iwasawa decomposition $G_n = N_n A_n K_n$. Let $d_{K_n}(k)$ denote the Haar measure of the compact group K_n with total mass one. We take the Haar measure $d_{\bar{N}_n}(\bar{n})$ of $\bar{N}_n$ such that $\int_{\bar{N}_n} \delta_n(\alpha_n(\bar{n}))d_{\bar{N}_n}(\bar{n}) = 1$, where $\delta_n : A_n \to \mathbb{R}_+^*$ denotes the quasi-character defined by

$$\delta_n(\mathsf{a}_n(r)) = r^{2n}, \quad r > 0. \tag{1.2.1}$$

For $a \in A_n$ and $\nu \in \mathbb{C}$, we write $a^\nu = \delta_n(a)^{\nu/2n}$.

2. Non-Unitary Principal Series

2.1. Non-unitary principal series representations

For a finite dimensional unitary representation (σ, W) of M_n, let $\mathcal{V}_n^\sigma$ denote the Fréchet space consisting of all C^∞-functions $\varphi : K_n \to W$ such that $\varphi(mk) = \sigma(m)\varphi(k),\ k \in K_n,\ m \in M_n$. For $\nu \in \mathbb{C}$, we have the non-unitary principal series representation of G_n on $\mathcal{V}_n^\sigma$ by defining the action $\pi_n^{\sigma,\nu}$ as

$$(\pi_n^{\sigma,\nu}(g)\varphi)(k) = \varphi(\kappa_n(kg))\alpha_n(kg)^{\nu+n}, \quad \varphi \in \mathcal{V}_n^\sigma,\ g \in G_n,\ k \in K_n.$$

Then $(\pi_n^{\sigma,\nu}, \mathcal{V}_n^\sigma)$ is a smooth Fréchet representation of G_n.

The underlying $(\mathfrak{g}_n, K_n)$-module of $\pi_n^{\sigma,\nu}$ will be also denoted by $\pi_n^{\sigma,\nu}$. We write V_n^σ for the space consisting of all K_n-finite vectors in $\mathcal{V}_n^\sigma$.
We define the $\mathbb{C}$-bi-linear form $[\ |\ ] : \mathcal{V}_n^\sigma \times \mathcal{V}_n^{\sigma^\vee} \to \mathbb{C}$ by

$$[\varphi|\check{\varphi}] = \int_{K_n} \langle \varphi(k), \check{\varphi}(k)\rangle d_{K_n}(k), \quad \varphi \in \mathcal{V}_n^\sigma,\ \check{\varphi} \in \mathcal{V}_n^{\sigma^\vee}.$$

This pairing is G_n-invariant, i.e.,

$$\left[\pi_n^{\sigma,\nu}(g)\varphi \middle| \pi_n^{\sigma^\vee,-\nu}(g)\check{\varphi}\right] = [\varphi|\check{\varphi}], \quad \varphi \in \mathcal{V}_n^\sigma,\ \check{\varphi} \in \mathcal{V}_n^{\sigma^\vee},\ g \in G_n.$$

From now on we mainly consider $\pi = \pi_n^{\sigma,\nu}$ for $\sigma = \sigma^n_{(\mathbf{p};p_0)}$ with $(\mathbf{p}\ ; p_0) \in {}^\circ\mathcal{L}_n^+$. Note that the central character of π (that is denoted by $c_n(\pi)$ in [10, 2.3]) is $z = |\mathbf{p}| + p_0$. The set $\mathcal{L}_n^+(\pi_n^{\sigma,\nu})$ ([10, 4.1]) consists of those $\lambda = [\mathbf{l}\ ; l_0] \in \mathcal{L}_n^+$ such that $\mathbf{l} \in \Lambda_n^+(\mathbf{p})$, $l_0 + |\mathbf{l}| = z$ ([10, Lemma 4.2.1]).

For $\lambda = [\mathbf{l}\ ; l_0] \in \mathcal{L}_n^+(\pi)$, $w \in W(\mathbf{l})$ and $w^\vee \in W(\mathbf{l})^\vee$, we have the functions $\iota_{\mathbf{l}}^\sigma(w) \in \mathcal{V}_n^\sigma$ and $\check{\iota}_{\mathbf{l}}^\sigma(w) \in \mathcal{V}_n^{\sigma^\vee}$ such that

$$\begin{aligned} &(\iota_{\mathbf{l}}^\sigma(w))(k) = \mathsf{p}_{\mathbf{p}}^{\mathbf{l}} \circ \tau_\lambda^n(k)(w), \quad k \in K_n, \\ &(\check{\iota}_{\mathbf{l}}^\sigma(w^\vee))(k) = (\mathsf{j}_{\mathbf{l}}^{\mathbf{p}})^\vee \circ (\tau_\lambda^n)^\vee(k)(w^\vee), \quad k \in K_n. \end{aligned} \tag{2.1.1}$$

The family $\{\iota_{\mathbf{l}}^\sigma |\ \mathbf{l} \in \Lambda_n^+(\mathbf{p})\}$ so obtained is a standard system for $\pi_n^{\sigma,\nu}$ (see [10, 4.1]). Under the identification $W(\check{\mathbf{l}}) \cong W(\mathbf{l})^\vee$, the family $\{\check{\iota}_{\mathbf{l}}^\sigma |\ \mathbf{l} \in \Lambda_n^+(\mathbf{p})\}$ is regarded as a standard system for $\pi^\vee$.
Let $\nabla_{\mathbf{l}}^{\pm k} = {}^\pi\nabla_\lambda^{\pm k}$ with $\pi = \pi_n^{\sigma,\nu}$ be the Schmid operators introduced in [10, 6.1]

Proposition 2.1.1. *For an irreducible unitary representation σ of M_n with the highest weight $(\mathbf{p}\ ; p_0) \in {}^\circ\mathcal{L}_n^+$, $\mathbf{l} \in \Lambda_n^+(\mathbf{p})$ and $j \in \{1, \cdots, n\}$, there exist real numbers $A_j^\sigma(\mathbf{l})$ and $B_j^\sigma(\mathbf{l})$ such that for $\nu \in \mathbb{C}$, $\lambda = [\mathbf{l}\ ; l_0] \in \mathcal{L}_n^+(\pi_n^{\sigma,\nu})$ and $j \in \{1, \cdots, n\}$*

$$\nabla_\lambda^{+j}(\iota_{\mathbf{l}}^\sigma) = A_j^{\sigma,\nu}(\mathbf{l})\iota_{\mathbf{l}^{+j}}^\sigma, \quad \nabla_\lambda^{-j}(\iota_{\mathbf{l}}^\sigma) = B_j^{\sigma,\nu}(\mathbf{l})\iota_{\mathbf{l}^{-j}}^\sigma \tag{2.1.2}$$

with

$$\begin{aligned} A_j^{\sigma,\nu}(\mathbf{l}) &= (2l_j - 2j + 2 - p_0 + \nu + n)A_j^\sigma(\mathbf{l}), \\ B_j^{\sigma,\nu}(\mathbf{l}) &= (2l_j - 2j - p_0 - \nu + n)B_j^\sigma(\mathbf{l}). \end{aligned} \tag{2.1.3}$$

For $\mathbf{l} \in \Lambda_n^+(\mathbf{p})$, we have $A_j^\sigma(\mathbf{l}) = 0$ (resp. $B_j^\sigma(\mathbf{l}) = 0$) if and only if $\mathbf{l}^{+j} \notin \Lambda_n^+(\mathbf{p})$ (resp. $\mathbf{l}^{-j} \notin \Lambda_n^+(\mathbf{p})$).

Proof. [13, p.411, formula (9), (10)]. □

2.2. The corner K_n-types of principal series

Given $\sigma = \sigma^n_{(\mathbf{p};p_0)}$ with $(\mathbf{p}\,;p_0) \in {}^\circ\mathcal{L}^+_n$ and $\nu \in \mathbb{C}$, we consider the non-unitary principal series representation $\pi = \pi^{\sigma,\nu}_n$. Put $z = c_n(\pi)$, the central character of π.

Definition 2.2.1. $\lambda = [\mathbf{l}\,;l_0] \in \mathcal{L}^+_n(\pi)$ is said to be *cyclic* in π if the smallest sub $(\mathfrak{g}_n, K_n)$-module which contains $\mathrm{Im}(\iota^\sigma_{\mathbf{l}})$ coincides with all of V^σ_n.

Definition 2.2.2. For $h \in \{1, \ldots, n-1\}$, let $\mathbf{l}_0 = (l_k)_{1 \leqslant k \leqslant n}$ be the element of $\Lambda^+_n(\mathbf{p})$ defined by $l_k = p_k$ for $1 \leqslant k \leqslant h$ and $l_k = p_{k-1}$ for $h < k \leqslant n$; $\mathbf{l}_0$ is characterized as a unique $\mathbf{l} \in \Lambda^+_n(\mathbf{p})$ such that $\mathbf{l}^{-k} \notin \Lambda^+_n(\mathbf{p})$ for $1 \leqslant k \leqslant h$ and $\mathbf{l}^{+k} \notin \Lambda^+_n(\mathbf{p})$ for $h+1 \leqslant k \leqslant n$. Then the element $\lambda^{(h)}_\sigma = [\mathbf{l}_0\,; z - |\mathbf{l}_0|]$ of $\mathcal{L}^+_n(\pi)$ will be called the h-th corner K_n-type of π.

Note that $\lambda^{(h)}_\sigma$ is the D_h-corner in the terminology of Kraljevic [6].

Lemma 2.2.1. *Let $h \in \{1, \ldots, n-1\}$ and $\lambda^{(h)}_\sigma = [\mathbf{l}_0\,; z - |\mathbf{l}_0|]$ be the h-th corner K_n-type of π. The following conditions are equivalent.*

(1) *$\lambda^{(h)}_\sigma$ is cyclic in π.*

(2) *For $1 \leqslant i \leqslant h$ and $\mathbf{l} \in \Lambda^+_n(\mathbf{p})$ with $\mathbf{l}^{+i} \in \Lambda^+_n(\mathbf{p})$, $A^{\sigma,\nu}_i(\mathbf{l}) \neq 0$. For $h+1 \leqslant j \leqslant n$ and $\mathbf{l} \in \Lambda^+_n(\mathbf{p})$ with $\mathbf{l}^{-j} \in \Lambda^+_n(\mathbf{p})$, $B^{\sigma,\nu}_j(\mathbf{l}) \neq 0$.*

(3) *ν is not of the form*

$$\begin{aligned}
\nu &= -2p_h + 2h - 2 + p_0 - n - 2y, \\
&\qquad \exists y \in \mathbb{Z}_+ - \{p_i - p_h + h - i - 1 |\ 1 \leqslant i < h\}, \\
\nu &= 2p_h - 2h - 2 - p_0 + n - 2y, \\
&\qquad \exists y \in \mathbb{Z}_+ - \{p_h - p_j + j - h - 1 |\ h < j \leqslant n-1\}.
\end{aligned}$$

(4) *ν is not a zero of the holomorphic function*

$$(2.2.1)\quad \tilde{c}_n^{(h)}(\sigma\ ;\nu) = \prod_{1\leqslant i<h}(\nu+2p_i-2i-p_0+n) \times \prod_{h<j\leqslant n-1}(\nu-2p_j+2j+p_0-n) \times\Gamma\Big(\frac{\nu+n-p_0}{2}+p_h-h+1\Big)^{-1} \times\Gamma\Big(\frac{\nu-n+p_0}{2}-p_h+h+1\Big)^{-1}.$$

Proof. The equivalence of (3) and (4) is easy. By (2.1.3), we have the equivalence of (2) and (3). Thus it suffices to show that (1) and (2) are equivalent. Let $\nabla_{\mathbf{l}}^{\pm k}$ be the Schmid operators. Assume that the condition (2) holds. Let V denote the $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-span of $\mathrm{Im}(\iota_{\mathbf{l}_0}^{\sigma,\nu})$ in V_n^σ. Putting $\delta(\mathbf{l}) = |\mathbf{l}-\mathbf{l}_0|$ for $\mathbf{l}\in\Lambda_n^+(\mathbf{p})$, we prove that $\mathrm{Im}(\iota_{\mathbf{l}}^\sigma)$ is contained in V for all $\mathbf{l}\in\Lambda_n^+(\mathbf{p})$ by induction on the number $\delta(\mathbf{l})$. If $\delta(\mathbf{l})=0$, then we have $\mathbf{l}=\mathbf{l}_0$. Hence $\mathrm{Im}(\iota_{\mathbf{l}_0}^\sigma)\subset V$ by definition. Suppose $d>0$ and that $\mathrm{Im}(\iota_{\mathbf{l}'}^\sigma)\subset V$ for $\mathbf{l}'$ with $\delta(\mathbf{l}')<d$. Let $\mathbf{l}$ with $\delta(\mathbf{l})=d$. Since $\delta(\mathbf{l})>0$, $\mathbf{l}$ is different from $\mathbf{l}_0$. Thus we have (a) $\mathbf{l}^{-i}\in\Lambda_n^+(\mathbf{p})$ for an i with $1\leqslant i\leqslant h$, or (b) $\mathbf{l}^{+i}\in\Lambda_n^+(\mathbf{p})$ for some i with $h+1\leqslant i\leqslant n$. We first consider the case (a). We have $\nabla_{\mathbf{l}^{-i}}^{+i}(\iota_{\mathbf{l}^{-i}}^\sigma) = A_i^{\sigma,\nu}(\mathbf{l}^{-i})\iota_{\mathbf{l}}^\sigma$ and that the number $A_i^{\sigma,\nu}(\mathbf{l}^{-i})$ is non zero by assumption. Since $\delta(\mathbf{l}^{-i})=d-1$, we have $\mathrm{Im}(\iota_{\mathbf{l}^{-i}}^\sigma)\subset V$ by induction-assumption. Noting that V is a $(\mathfrak{g}_n,K_n)$-module, we then have $\mathrm{Im}(\nabla_{\mathbf{l}^{-i}}^{+i}(\iota_{\mathbf{l}^{-i}}^\sigma))\subset V$. Hence we obtain $\mathrm{Im}(\iota_{\mathbf{l}}^\sigma)\subset V$. In the same way, we can conclude $\mathrm{Im}(\iota_{\mathbf{l}}^\sigma)\subset V$ in case (b), noting $B_i^{\sigma,\nu}(\mathbf{l}^{+i})\neq 0$ in this case. Thus we have $\mathrm{Im}(\iota_{\mathbf{l}}^\sigma)\subset V$ for all $\mathbf{l}\in\Lambda_n^+(\mathbf{p})$ to get the identity $V_n^\sigma = V$. Next we show that if (2) is not true then V_n^σ has a non trivial proper $(\mathfrak{g}_{n,\mathbb{C}},K_n)$-submodule. Suppose $A_\alpha^{\sigma,\nu}(\mathbf{l})=0$ for an $\mathbf{l}=(l_j)_{1\leqslant j\leqslant n}\in\Lambda_n^+(\mathbf{p})$ and an α such that $\mathbf{l}^{+\alpha}\in\Lambda_n^+(\mathbf{p})$, $2\leqslant\alpha\leqslant h$. (The other possibilities are similarly treated.) Let V' be the $\mathbb{C}$-span of $\mathrm{Im}(\iota_{\mathbf{l}'}^\sigma)$ with $\mathbf{l}'=(l'_k)_{1\leqslant k\leqslant n}$ such that $\mathbf{l}'\in\Lambda_n^+(\mathbf{p})$, $l_\alpha\geqslant l'_\alpha\geqslant p_\alpha$. Such $\mathbf{l}'$'s form a non empty subset Γ' of $\Lambda_n^+(\mathbf{p})$, and $\Gamma'\neq\Lambda_n^+(\mathbf{p})$ because $l_\alpha<p_{\alpha-1}$. Thus the space V' is strictly smaller than V_n^σ and non-zero. We show V' is stable under the action of $\mathfrak{g}_n$ and K_n. By definition, the K_n-stability of V' is clear. So it suffices to prove that $\pi(X)f\in V'$ for $X\in\mathfrak{p}_{n,\mathbb{C}}$ ($\mathfrak{p}_n$ is the orthogonal complement of $\mathfrak{k}_n$ in

$\mathfrak{g}_n$) and $f \in V'$. We may assume that f is of the form $\iota^\sigma_{\mathbf{l}'}(w)$ with $\mathbf{l}' \in \Gamma'$ and $w \in W(\mathbf{l}')$. There exist vectors $w^{+k} \in W(\mathbf{l}'^{+k})$ and $w^{-k} \in W(\mathbf{l}'^{-k})$ for $1 \leqslant k \leqslant n$ such that

$$w \otimes X = \sum_{k=1}^{n} I_{\beta_k}(\mathbf{l}')(w^{+k}) + \sum_{k=1}^{n} I_{-\beta_k}(\mathbf{l}')(w^{-k}).$$

(For $I_{\pm\beta_k}(\mathbf{l})$ see [10, Proposition 6.1.1].) From definition and (2.1.2) we have

$$\begin{aligned}\pi(X)(f) &= \sum_{k=1}^{n} (\nabla^{+k}_{\mathbf{l}'}(\iota^\sigma_{\mathbf{l}'}))(w^{+k}) + \sum_{k=1}^{n} (\nabla^{-k}_{\mathbf{l}'}(\iota^\sigma_{\mathbf{l}'}))(w^{-k}) \\ &= \sum_{k \in J^+_{\mathbf{l}'}} A^{\sigma,\nu}_k(\mathbf{l}')\iota^\sigma_{\mathbf{l}'^{+k}}(w^{+k}) + \sum_{k \in J^-_{\mathbf{l}'}} B^{\sigma,\nu}_k(\mathbf{l}')\iota^\sigma_{\mathbf{l}'^{-k}}(w^{-k}),\end{aligned}$$

where $J^+_{\mathbf{l}'}$ (resp. $J^-_{\mathbf{l}'}$) is the set of all $1 \leqslant k \leqslant n$ such that $\mathbf{l}'^{+k} \in \Lambda^+_n(\mathbf{p})$ (resp. $\mathbf{l}'^{-k} \in \Lambda^+_n(\mathbf{p})$). Note $\alpha \in J^+_{\mathbf{l}'}$ because $l'_\alpha \leqslant l_\alpha < p_{\alpha-1}$. By definition we have that $\mathbf{l}'^{-k} \in \Gamma'$ for $k \in J^-_{\mathbf{l}'}$, and that $\mathbf{l}'^{+k} \in \Gamma'$ for $k \in J^+_{\mathbf{l}'} - \{\alpha\}$. If $l_\alpha > l'_\alpha$, then $\mathbf{l}'^{+\alpha} \in \Gamma'$ is clear. If $l'_\alpha = l_\alpha$, by (2.1.3), $A^{\sigma,\nu}_\alpha(\mathbf{l}')$ is a constant multiple of $A^{\sigma,\nu}_\alpha(\mathbf{l})$. Since $A^{\sigma,\nu}_\alpha(\mathbf{l})$ is supposed to be zero, we have $A^{\sigma,\nu}_\alpha(\mathbf{l}') = 0$. Thus we obtain $\pi(X)f \in V'$. This completes the proof. □

2.3. Knapp-Stein intertwining operators and c-functions

Let $\nu \in \mathbb{C}$ and (σ, W) a finite dimensional representation of M_n. For a function $\varphi \in \mathcal{V}^\sigma_n$, we consider the integral

$$(2.3.1) \qquad \mathcal{A}^{\sigma,\nu}_n(\varphi : k) = \int_{\bar{N}_n} \varphi(\kappa_n(\bar{n}\mathtt{w}_n k))\alpha_n(\bar{n}\mathtt{w}_n k)^{\nu+n} d_{\bar{N}_n}(\bar{n}), \quad k \in K_n.$$

The basic properties of the integral above that we need are following.

Proposition 2.3.1.

(1) *If* $\mathrm{Re}(\nu) > 0$*, then the integral* (2.3.1) *converges absolutely for every* $\varphi \in \mathcal{V}^\sigma_n$*. The function* $\mathcal{A}^{\sigma,\nu}_n(\varphi)$ *belongs to the space* $\mathcal{V}^\sigma_n$*.*

(2) *For every* $\nu \in \mathbb{C}$ *such that* $\mathrm{Re}(\nu) > 0$ *and* $\varphi \in \mathcal{V}^\sigma_n$*, the identity*

$$\mathcal{A}^{\sigma,\nu}_n(\pi^{\sigma,\nu}_n(g)\varphi) = \pi^{\sigma,-\nu}_n(g)(\mathcal{A}^{\sigma,\nu}_n(\varphi)), \quad g \in G_n$$

holds.

(3) *Let* $\nu \in \mathbb{C}$ *with* $\mathrm{Re}(\nu) > 0$. *For every* $\varphi \in \mathcal{V}_n^{\sigma}$ *and every* $\check{\varphi} \in \mathcal{V}_n^{\sigma^\vee}$ *the formula*

$$\lim_{r\to\infty} r^{n-\nu}[\pi_n^{\sigma,\nu}(\mathtt{a}_n(r))\varphi|\check{\varphi}] = \langle \mathcal{A}_n^{\sigma,\nu}(\varphi : \mathtt{w}_n), \check{\varphi}(\mathtt{I}_n)\rangle$$

holds.

Proof. For (1) and (2) see [12, p.181, Proposition 7.8]. For (3) see [12, p.198, Lemma 7.2.3]. □

Let $\nu \in \mathbb{C}$ with $\mathrm{Re}(\nu) > 0$ and $\sigma = \sigma^n_{(\mathbf{p};p_0)}$ with $(\mathbf{p}\ ; p_0) \in {}^{\circ}\mathcal{L}_n^+$. Since any irreducible K_n-module occurs in $\pi_n^{\sigma,\nu}$ with multiplicity one or zero, for every $\mathbf{l} \in \Lambda_n^+(\mathbf{p})$, there exists a complex number $c_n(\sigma, \nu\ ; \mathbf{l})$ such that

$$\mathcal{A}_n^{\sigma,\nu} \circ \iota_{\mathbf{l}}^{\sigma} = c_n(\sigma, \nu\ ; \mathbf{l})\iota_{\mathbf{l}}^{\sigma}. \tag{2.3.2}$$

The explicit formula of the numbers $c_n(\sigma, \nu\ ; \mathbf{l})$ is obtained in [2, page 976, Theorem 8.2]. From that formula we have that the functions $\nu \mapsto c_n(\sigma, \nu\ ; \mathbf{l})$ are meromorphically continued to the whole $\mathbb{C}$, and also have

Proposition 2.3.2. *Let* $h \in \{1, \ldots, n-1\}$ *and* $\lambda_\sigma^{(h)} = [\mathbf{l}_0\ ; l_0]$ *the* h*-th corner* K_n*-type of* $\pi_n^{\sigma,\nu}$. *Then we have*

$$c_n(\sigma, \nu\ ; \mathbf{l}_0) = \gamma_\sigma^{(h)} \cdot 2^{-\nu}\Gamma(\nu)\tilde{c}_n^{(h)}(\sigma\ ; \nu),$$

where $\tilde{c}_n^{(h)}(\sigma\ ; \nu)$ *is the holomorphic function given by* (2.2.1) *and* $\gamma_\sigma^{(h)}$ *is a non-zero constant independent of* ν.

3. Shintani Functions

In the first subsection we introduce the intertwining space $\mathcal{I}(\pi_0|\pi)$, whose study is our main theme of this paper. In the second subsection, we give an explicit formula of some spherical functions (Shintani functions) when $\pi_0^\vee \boxtimes \pi$ is a principal series representation.

3.1. Shintani functions

We regard G_n as a $G_n \times G_n$-space by letting $(g_1, g_2) \in G_n \times G_n$ act on G_n as $(g_1, g_2) \cdot x = g_1 x g_2^{-1}$, $x \in G_n$. Hence the space $C^\infty(G_n)$ of all C^∞-functions on G_n becomes a smooth Fréchet $G_n \times G_n$-module naturally. By restricting the action to the subgroup $G_{n-1} \times G_n$, we can regard $C^\infty(G_n)$ as a smooth $G_{n-1} \times G_n$-module. Let π_0 be an admissible $(\mathfrak{g}_{n-1,\mathbb{C}}, K_{n-1})$-module and π an admissible $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-module. We put

$$\mathcal{I}(\pi_0|\pi) = \mathrm{Hom}_{(\mathfrak{g}_{n-1,\mathbb{C}} \oplus \mathfrak{g}_{n,\mathbb{C}}, K_{n-1} \times K_n)}(\pi_0^\vee \boxtimes \pi, C^\infty(G_n)). \tag{3.1.1}$$

Theorem 3.1.1. *For an irreducible $(\mathfrak{g}_{n-1,\mathbb{C}}, K_{n-1})$-module π_0 and an irreducible $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-module π, we have*

$$\dim_{\mathbb{C}} \mathcal{I}(\pi_0|\pi) \leqslant 1.$$

Proof. Let $H_n = Z_n G_{n-1}$. We extend the representation π_0 to H_n so that the extended representation η satisfies $c_n(\eta) = c_n(\pi)$. Then by Theorem 7.1.1 we have $\dim_{\mathbb{C}} \mathcal{I}(\pi_0|\pi) = \dim_{\mathbb{C}} \mathcal{I}_{\eta,\pi}$. Since $\dim_{\mathbb{C}} \mathcal{I}_{\eta,\pi} \leqslant 1$ by [10, Theorem 8.1.1], we have the conclusion. □

3.2. Principal series Shintani functions

Let $\sigma_0 = \sigma^{n-1}_{(\mathbf{q}\,;q_0)}$ with $(\mathbf{q}\,;q_0) \in {}^\circ\mathcal{L}^+_{n-1}$ and $\sigma = \sigma^n_{(\mathbf{p}\,;p_0)}$ with $(\mathbf{p}\,;p_0) \in {}^\circ\mathcal{L}^+_n$ be irreducible unitary representations of M_{n-1} and M_n respectively and consider the principal series representations $\pi_0 = \pi^{\sigma_0,\nu_0}_{n-1}$ and $\pi = \pi^{\sigma,\nu}_n$ with $\nu_0, \nu \in \mathbb{C}$. In this case we also write $\mathcal{I}(\sigma_0, \nu_0|\sigma, \nu)$ in place of $\mathcal{I}(\pi^{\sigma_0,\nu_0}_{n-1}|\pi^{\sigma,\nu}_n)$. For $\mathcal{P} \in \mathcal{I}(\sigma_0, \nu_0|\sigma, \nu)$, $\mu = [\mathbf{m}\,; m_0] \in \mathcal{L}^+_{n-1}(\pi_0)$ and $\lambda = [\mathbf{l}\,; l_0] \in \mathcal{L}^+_n(\pi)$, we define a function $\mathcal{P}_{\mu,\lambda} : G_n \to W(\mathbf{m}) \otimes W(\mathbf{l})^\vee$ by

$$\begin{gathered}\langle \mathcal{P}_{\mu,\lambda}(g), w_0^\vee \otimes w \rangle = \mathcal{P}(\check{\iota}^{\sigma_0}_{\mathbf{m}}(w_0^\vee) \otimes \iota^{\sigma}_{\mathbf{l}}(w))(g), \\ w_0^\vee \in W(\mathbf{m})^\vee,\ w \in W(\mathbf{l}).\end{gathered} \tag{3.2.1}$$

Then we have

$$\begin{gathered}\mathcal{P}_{\mu,\lambda}(k_0 g k) = \tau^{n-1}_\mu(k_0) \otimes (\tau^n_\lambda)^\vee(k^{-1}) \mathcal{P}_{\mu,\lambda}(g), \\ k_0 \in K_{n-1},\ k \in K_n,\ g \in G_n.\end{gathered} \tag{3.2.2}$$

LEMMA 3.2.1. *We have* $G_n = K_{n-1}\mathsf{A}_n K_n$ *with*

$$\mathsf{A}_n = \{\mathsf{a}_{n-1}(r_0)\mathsf{a}_n(r) |\ r_0,\ r > 0\}.$$

PROOF. This follows from the decompositions $G_{n-1} = K_{n-1}A_{n-1}K_{n-1}$, $G_n = G_{n-1}Z_nA_nK_n$ ([3, page 108, Theorem 2.4]) and the relation $Z_nK_{n-1} = M_nZ_{n-1}$. □

LEMMA 3.2.2. *Let* $\mathcal{P} \in \mathcal{I}(\sigma_0, \nu_0|\sigma, \nu)$. *There exists a unique family* $\{f_{\mathbf{l}}(\mathbf{m}\ ; r)|\ \mathbf{l} \in \Lambda_n^+(\mathbf{p}),\ \mathbf{m} \in \Lambda_{n-1}^+(\mathbf{q}|\mathbf{l})\}$ *of* C^∞*-functions on* $r > 0$ *such that for* $\mu = [\mathbf{m}\ ; m_0] \in \mathcal{L}_{n-1}^+(\pi_0)$ *and* $\lambda = [\mathbf{l}\ ; l_0] \in \mathcal{L}_n^+(\pi)$ *the formula*

$$\begin{aligned} \text{(3.2.3)} \qquad & \langle \mathcal{P}_{\mu,\lambda}(\mathsf{a}_{n-1}(r_0)^{-1}\mathsf{a}_n(r)), w_0^\vee \otimes w\rangle \\ &= \sum_{\mathbf{n}\in\Lambda_{n-1}^+(\mathbf{q}|\mathbf{l})} f_{\mathbf{l}}(\mathbf{n}\ ; r) \\ &\quad \times \big[\iota_{\mathbf{n}}^{\sigma_0} \circ \mathsf{p}_{\mathbf{n}}^{\mathbf{l}}(w)\big|\pi_{n-1}^{\sigma_0^\vee,-\nu_0}(\mathsf{a}_{n-1}(r_0)) \circ \check{\iota}_{\mathbf{m}}^{\sigma_0}(w_0^\vee)\big], \\ &\quad w_0^\vee \in W(\mathbf{m})^\vee,\ w \in W(\mathbf{l}) \end{aligned}$$

holds.

PROOF. First of all, by Theorem 7.1.1, to $\mathcal{P} \in \mathcal{I}(\sigma_0, \nu_0|\sigma, \nu)$ there corresponds a unique $\Phi_{\mathcal{P}} \in \mathcal{I}_{\eta,\pi}$ such that

$$\big[\Phi_{\mathcal{P}}(v)(g)\big|\check{v}_0\big] = \mathcal{P}(\check{v}_0 \otimes v)(g), \quad \check{v}_0 \in V_{n-1}^{\sigma_0^\vee},\ v \in V_n^\sigma,\ g \in G_n.$$

Here η denotes the representation of $H_n = Z_nG_{n-1}$ satisfying $c_n(\eta) = c_n(\pi)$. For a fixed $r > 0$, we define two functions $\varphi_1 : \mathbb{R} \to \mathbb{C}$ and $\varphi_2 : \mathbb{R} \to \mathbb{C}$ by

$$\begin{aligned} \varphi_1(t_0) &= \langle \mathcal{P}_{\mu,\lambda}(\mathsf{a}_{n-1}(e^{t_0})^{-1}\mathsf{a}_n(r)), w_0^\vee \otimes w\rangle, \quad t_0 \in \mathbb{R}, \\ \varphi_2(t_0) &= \big[\Phi_{\mathcal{P}}(\iota_{\mathbf{l}}^\sigma(w))(\mathsf{a}_n(r))\big|\pi_{n-1}^{\sigma_0^\vee,-\nu_0}(\mathsf{a}_{n-1}(e^{t_0})) \circ \check{\iota}_{\mathbf{m}}^{\sigma_0}(w_0^\vee)\big], \quad t_0 \in \mathbb{R}. \end{aligned}$$

By [11, page 461, Lemma 2.6], the functions φ_1 and φ_2 are real analytic. We have

$$\begin{aligned} & L_{Y^p}\mathcal{P}(\check{\iota}_{\mathbf{m}}^{\sigma_0}(w_0^\vee) \otimes \iota_{\mathbf{l}}^\sigma(w))(\mathsf{a}_n(r)) \\ &= \big[\Phi_{\mathcal{P}}(\iota_{\mathbf{l}}^\sigma(w))(\mathsf{a}_n(r))\big|\pi_{n-1}^{\sigma_0^\vee,-\nu_0}(Y^p) \circ \check{\iota}_{\mathbf{m}}^{\sigma_0}(w_0^\vee)\big] \end{aligned}$$

for any $p \in \mathbb{N}$, where Y denotes the element of $\mathfrak{a}_{n-1}$ such that $\exp(\log(r_0)Y) = \mathsf{a}_{n-1}(r_0)$ for $r_0 > 0$. This in turn means that φ_1 and φ_2 have the same p-th derivative at $t_0 = 0$ for any p. Since φ_1 and φ_2 are both real analytic functions on $\mathbb{R}$ as noticed above, we have $\varphi_1(t_0) = \varphi_2(t_0)$ identically; equivalently

$$\begin{aligned}(3.2.4)\qquad &\langle \mathcal{P}_{\mu,\lambda}(\mathsf{a}_{n-1}(r_0)^{-1}\mathsf{a}_n(r)), w_0^\vee \otimes w\rangle \\ &= \left[\Phi_{\mathcal{P}}(\iota_{\mathbf{l}}^{\sigma}(w))(\mathsf{a}_n(r)) \middle| \pi_{n-1}^{\sigma_0^\vee,-\nu_0}(\mathsf{a}_{n-1}(r_0)) \circ \check{\iota}_{\mathbf{m}}^{\sigma_0}(w_0^\vee)\right]\end{aligned}$$

for all $r > 0$ and $r_0 > 0$. Let $\{f_{\mathbf{l}}(\mathbf{n}\ ; r) |\ \mathbf{l} \in \Lambda_n^+(\mathbf{p}),\ \mathbf{n} \in \Lambda_{n-1}^+(\mathbf{q}|\mathbf{l})\}$ be the standard coefficient of $\Phi_{\mathcal{P}}$ ([10, Definition 7.1.1]). By definition we have

$$\Phi_{\mathcal{P}}(\iota_{\mathbf{l}}^{\sigma}(w))(\mathsf{a}_n(r)) = \sum_{\mathbf{n}\in\Lambda_{n-1}^+(\mathbf{q}|\mathbf{l})} f_{\mathbf{l}}(\mathbf{n}\ ; r) \cdot \iota_{\mathbf{n}}^{\sigma_0} \circ \mathsf{p}_{\mathbf{n}}^{\mathbf{l}}(w).$$

Substitute this formula into (3.2.4). Then we have the formula (3.2.3). The uniqueness of $\{f_{\mathbf{l}}(\mathbf{m}\ ; r)\}$ is obvious from the formula (3.2.3). □

We call the system $\{f_{\mathbf{l}}(\mathbf{m}\ ; r) |\ \mathbf{l} \in \Lambda_n^+(\mathbf{p}),\ \mathbf{m} \in \Lambda_{n-1}^+(\mathbf{q}|\mathbf{l})\}$ the standard coefficients of $\mathcal{P}$.

Proposition 3.2.1. *Assume both $\pi_n^{\sigma,\nu}$ and $\pi_{n-1}^{\sigma_0,\nu_0}$ are irreducible. If $\mathcal{I}(\sigma_0,\nu_0|\sigma,\nu) \neq \{0\}$, then $\mathbf{q} \subset \mathbf{p}$.*

Proof. This follows from [10, Proposition 9.2.1] and Theorem 7.1.1. □

Definition 3.2.1. Let σ_0 and σ be as above with $\mathbf{q} \subset \mathbf{p}$. For every $h \in \{1, \ldots, n-1\}$, we define $\mu_{\sigma_0,\sigma}^{(h)} = [\mathbf{m}_0\ ; m_0] \in \mathcal{L}_{n-1}^+(\pi_{n-1}^{\sigma_0,\nu_0})$ by putting $\mathbf{m}_0 = (m_{0,j})_{1 \leqslant j \leqslant n-1}$ with $m_{0,j} = q_j$ for $j \in \{1, \cdots, h-1\}$, $m_{0,h} = p_h$ and $m_{0,j} = q_{j-1}$ for $j \in \{h+1, \ldots, n-1\}$.

It is easy to see that if $\lambda_\sigma^{(h)} = [\mathbf{l}_0\ ; l_0]$ is the h-th corner of $\pi_n^{\sigma,\nu}$ then $\mathbf{m}_0 \in \Lambda_{n-1}^+(\mathbf{q}|\mathbf{l}_0)$.

Proposition 3.2.2. *Let $\pi = \pi_n^{\sigma,\nu}$ and $\pi_0 = \pi_{n-1}^{\sigma_0,\nu_0}$ as above with $\mathbf{q} \subset \mathbf{p}$. Put $z = |\mathbf{p}| + p_0$, $z_0 = |\mathbf{q}| + q_0$ and $\kappa = |p_0 - q_0|$. Let $h \in \{1, \ldots, n-1\}$ and $\lambda_\sigma^{(h)} = [\mathbf{l}\ ; l_0] \in \mathcal{L}_n^+(\pi)$ the h-th corner K_n-type of π. Let $\mathcal{P} \in \mathcal{I}(\sigma_0,\nu_0|\sigma,\nu)$*

and $\{f_{\mathbf{l}}(\mathbf{m}\,;r)|\ \mathbf{m}\in\Lambda^+_{n-1}(\mathbf{q}|\mathbf{l})\}$ *the standard coefficients for* $\mathcal{P}$. *Introducing the new coordinate* $x=\mathrm{th}^2(r),\ r>1$, *we put*

$$\begin{aligned}
(3.2.5)\quad \psi(\mathbf{m}\,;x) &= (\mathrm{sh}\,r)^{-\beta^{(h)}(\mathbf{m})}(\mathrm{ch}\,r)^{-\alpha^{(h)}(\mathbf{m})}f_{\mathbf{l}}(\mathbf{m}\,;r),\\
&\qquad 0<x<1,\\
(3.2.6)\quad \beta^{(h)}(\mathbf{m}) &= -\sum_{i=1}^{h-1}m_i+\sum_{i=h+1}^{n-1}m_i+\sum_{j=1}^{h-1}q_j-\sum_{j=h}^{n-2}q_j+\kappa,\\
(3.2.7)\quad \alpha^{(h)}(\mathbf{m}) &= 2\Big(\sum_{i=1}^{h-1}m_i-\sum_{i=h+1}^{n-1}m_i\Big)-\sum_{j=1}^{h-1}q_j+\sum_{j=h}^{n-2}q_j\\
&\quad -\sum_{i=1}^{h-1}p_i+\sum_{i=h+1}^{n-1}p_i+\nu-n-\kappa
\end{aligned}$$

for $\mathbf{m}\in\Lambda^+_{n-1}(\mathbf{q}|\mathbf{l})$. *Let* $\mu_0=\mu^{(h)}_{\sigma_0,\sigma}=[\mathbf{m}_0\ ;m_0]\in\mathcal{L}^+_{n-1}(\pi_0)$ *be as in Definition 3.2.1.*

(1) *There exists a unique constant* $\gamma^{(h)}(\mathcal{P})$ *such that for* $0<x<1$

$$\begin{aligned}
(3.2.8)\quad &\psi(\mathbf{m}_0\ ;x)\\
&=\gamma^{(h)}(\mathcal{P})_2F_1\Big(\frac{\nu-\nu_0+1+\kappa}{2},\frac{-\nu-\nu_0+1+\kappa}{2}\ ;1+\kappa\ ;x\Big).
\end{aligned}$$

Moreover the family $\{\psi(\mathbf{m}\,;r)|\ \mathbf{m}\in\Lambda^+_{n-1}(\mathbf{q}|\mathbf{l})\}$ *satisfies the following recurrence relations.*

$$\begin{aligned}
(\mathsf{E}_h^{-i})_{\mathbf{m}}:\quad & -2a_i(\mathbf{l}\,;\mathbf{m}^{-i})A_i^{\sigma_0,\nu_0}(\mathbf{m}^{-i})\psi(\mathbf{m}^{-i}\ ;x)\\
&=\frac{\prod\limits_{\beta=h+1}^{n-1}(p_\beta-m_i+i-\beta)}{\prod\limits_{\beta=h+1,\beta\neq i}^{n-1}(m_\beta-m_i+i-\beta)}\Big\{2x(1-x)\frac{d}{dx}\psi(\mathbf{m}\,;x)\\
&\quad +x\Big(-2\sum_{\beta=h+1}^{n-1}m_\beta+\sum_{j=h}^{n-2}q_j-\sum_{j=1}^{h-1}q_j\\
&\qquad +z_0-\kappa-2m_i+2i+\nu-n\Big)\psi(\mathbf{m}\,;x)\\
&\quad +\Big(2\sum_{\beta=h+1}^{n-1}m_\beta-|\mathbf{p}|+\sum_{j=1}^{h-1}q_j-\sum_{j=h}^{n-2}q_j+z-z_0+\kappa\Big)\psi(\mathbf{m}\,;x)
\end{aligned}$$

$$+ x(1-x)\sum_{\beta=h+1}^{n-1}\sum_{\alpha=1}^{h-1}\frac{\prod_{j=h+1}^{n-1}(p_\beta - m_j + j - \beta)}{\prod_{j=h+1, j\neq\beta}^{n-1}(p_\beta - p_j + j - \beta)}$$
$$\times \frac{a_\alpha(\mathbf{l}_0\ ; \mathbf{m}^{-\alpha})A_\alpha^{\sigma_0,\nu_0}(\mathbf{m}^{-\alpha})}{p_\beta - m_\alpha + \alpha - \beta}\psi(\mathbf{m}^{-\alpha}\ ; x)\Big\}$$

for $\mathbf{m} \in \Lambda_{n-1}^+(\mathbf{q}|\mathbf{l})$, $h < i \leqslant n-1$ *with* $\mathbf{m}^{-i} \in \Lambda_{n-1}^+(\mathbf{q}|\mathbf{l})$.

$$(\mathsf{E}_h^{+i})_{\mathbf{m}}:\quad -2b_i(\mathbf{l}\ ; \mathbf{m}^{+i})B_i^{\sigma_0,\nu_0}(\mathbf{m}^{+i})\psi(\mathbf{m}^{+i}\ ; x)$$
$$= \frac{\prod_{\alpha=1}^{h-1}(p_\alpha - m_i + i - \alpha)}{\prod_{\alpha=1,\alpha\neq i}^{h-1}(m_\alpha - m_i + i - \alpha)}\Big\{2x(1-x)\frac{d}{dx}\psi(\mathbf{m}\ ; x)$$
$$+ x\Big(2\sum_{\alpha=1}^{h-1} m_\alpha + \sum_{j=h}^{n-2} q_j - \sum_{j=1}^{h-1} q_j$$
$$- z_0 - \kappa + 2m_i - 2i + n + \nu\Big)\psi(\mathbf{m}\ ; x)$$
$$+ \Big(-2\sum_{\alpha=1}^{h-1} m_\alpha + |\mathbf{p}| + \sum_{j=1}^{h-1} q_j - \sum_{j=h}^{n-2} q_j - z + z_0 + \kappa\Big)\psi(\mathbf{m}\ ; x)$$
$$+ x(1-x)\sum_{\beta=h+1}^{n-1}\sum_{\alpha=1}^{h-1}\frac{\prod_{j=1}^{h-1}(p_\alpha - m_j + j - \alpha)}{\prod_{j=1,j\neq\alpha}^{h-1}(p_\alpha - p_j + j - \alpha)}$$
$$\times \frac{b_\beta(\mathbf{l}_0\ ; \mathbf{m}^{+\beta})B_\beta^{\sigma_0,\nu_0}(\mathbf{m}^{+\beta})}{p_\alpha - m_\beta + \beta - \alpha}\psi(\mathbf{m}^{+\beta}\ ; x)\Big\}$$

for $\mathbf{m} \in \Lambda_{n-1}^+(\mathbf{q}|\mathbf{l})$, $1 \leqslant i < h$ *with* $\mathbf{m}^{+i} \in \Lambda_{n-1}^+(\mathbf{q}|\mathbf{l})$.

(2) *Assume* $\mathrm{Re}(\nu) > n$. *For any* $\mathbf{m} \in \Lambda_{n-1}^+(\mathbf{q}|\mathbf{l})$ *the limit* $\psi(\mathbf{m}) = \lim_{x\to 1-0}\psi(\mathbf{m}\ ; x)$ *exits. We have*

$$\psi(\mathbf{m}_0) = \gamma^{(h)}(\mathcal{P})\frac{\Gamma(\nu)\Gamma(1+\kappa)}{\Gamma\Big(\dfrac{\nu-\nu_0+1+\kappa}{2}\Big)\Gamma\Big(\dfrac{\nu+\nu_0+1+\kappa}{2}\Big)}. \tag{3.2.9}$$

Moreover the system $\{\psi(\mathbf{m})|\ \mathbf{m} \in \Lambda_{n-1}^+(\mathbf{q}|\mathbf{l})\}$ *satisfies the following recurrence relations.*

(a)

$$\begin{aligned}&(2m_i - 2i - q_0 + \nu_0 + n - 1)\psi(\mathbf{m}^{-i})\\&= C_i^-(\mathbf{m}) \cdot (2m_i - 2i - p_0 - \nu + n)\psi(\mathbf{m})\end{aligned}$$

for $\mathbf{m} \in \Lambda_{n-1}^+(\mathbf{q}|\mathbf{l})$, $h < i \leqslant n-1$ *with* $\mathbf{m}^{-i} \in \Lambda_{n-1}^+(\mathbf{q}|\mathbf{l})$ *with a system of non-zero constants* $C_i^-(\mathbf{m})$ *which depends only on* σ_0, σ *and* h.

(b)

$$\begin{aligned}&(2m_j - 2j - q_0 - \nu_0 + n + 1)\psi(\mathbf{m}^{+j})\\&= C_j^+(\mathbf{m}) \cdot (2m_j - 2j - p_0 + \nu + n)\psi(\mathbf{m})\end{aligned}$$

for $\mathbf{m} \in \Lambda_{n-1}^+(\mathbf{q}|\mathbf{l})$, $1 \leqslant j < h$ *with* $\mathbf{m}^{+j} \in \Lambda_{n-1}^+(\mathbf{q}|\mathbf{l})$ *with a system of non-zero constants* $C_j^+(\mathbf{m})$ *which depends only on* σ_0, σ *and* h.

Proof. We give a brief indication of the proof. The formula (3.2.8) follows from the same procedure described in [10, section 9]. The equations $(\mathsf{E}_h^{+i})_{\mathbf{m}}$ and $(\mathsf{E}_h^{-i})_{\mathbf{m}}$ are paraphrase of [10, (8.3.1), (8.3.2)]. The formula (3.2.9) follows from (3.2.8) combined with the formula of ${}_2F_1(a, b\,; c\,; 1)$ in [14, 14.11]. By the formula (3.2.8), the function $\psi(\mathbf{m}_0\,; x)$ on $0 < x < 1$ can be extended smoothly around $x = 1$. By induction on the number $\delta(\mathbf{m}) = |\mathbf{m} - \mathbf{m}_0|$, we can prove by using $(\mathsf{E}_h^{-i})_{\mathbf{m}}$ and $(\mathsf{E}_h^{+i})_{\mathbf{m}}$ that $\psi(\mathbf{m}\,; x)$ is continued smoothly around $x = 1$. In particular the limit $\lim_{x \to 1-0} \psi(\mathbf{m}\,; x)$ exists for any $\mathbf{m} \in \Lambda_{n-1}^+(\mathbf{q}|\mathbf{l})$. The relations (a) and (b) can be deduced from $(\mathsf{E}_h)_{\mathbf{m}}^{+i}$ and $(\mathsf{E}_h)_{\mathbf{m}}^{-i}$ by a computation. □

Corollary 3.2.1. *Retain the assumptions and the notations in Proposition 3.2.2. We have*

$$\begin{aligned}\psi(\mathbf{p}) \;=\; &\gamma^{(h)}(\mathcal{P})C_0 \prod_{\alpha=1}^{h-1} \prod_{q_\alpha \leqslant m_\alpha < p_\alpha} \frac{2m_\alpha - 2\alpha - p_0 + \nu + n}{2m_\alpha - 2\alpha - q_0 - \nu_0 + n + 1}\\&\times \prod_{\beta=h+1}^{n-1} \prod_{p_\beta < m_\beta \leqslant q_{\beta-1}} \frac{2m_\beta - 2\beta - p_0 - \nu + n}{2m_\beta - 2\beta - q_0 + \nu_0 + n - 1}\\&\times \frac{\Gamma(\nu)}{\Gamma\Big(\dfrac{\nu - \nu_0 + 1 + \kappa}{2}\Big)\Gamma\Big(\dfrac{\nu + \nu_0 + 1 + \kappa}{2}\Big)}\end{aligned} \tag{3.2.10}$$

with a non-zero constant C_0 which depends only on σ_0, σ and h.

PROOF. This follows from the formula (3.2.9) and recurrence relations (a) and (b). $\square$

4. The Poisson Integrals

In the first subsection, we introduce an integral transform using the open $P_{n-1} \times P_n$-double coset in G_n, which may be considered to be an analogue of the Poisson integrals in the representation theory of the affine symmetric spaces ([8], [3]). In 4.2, we compute the constant $\gamma^{(h)}(\mathcal{P})$ (Proposition 3.2.2 (1)) for $\mathcal{P}$ given by the Poisson integral.

4.1. Poisson integrals and their basic properties

LEMMA 4.1.1. *We have*

$$P_{n-1} \cap P_n = \mathtt{m}_{n-1}(\mathsf{U}(n-2), 1).$$

PROOF. Noting

$$\begin{aligned} (4.1.1) \qquad & \mathtt{m}_{n-1}(\mathsf{U}(n-2), 1) \\ &= \left\{ g \in G_n \,\middle|\, \begin{array}{c} g(\mathsf{e}_{n-1} + \mathsf{e}_{n+1}) = \mathsf{e}_{n-1} + \mathsf{e}_{n+1}, \quad g(\mathsf{e}_n) = \mathsf{e}_n \\ g(\mathsf{e}_n + \mathsf{e}_{n+1}) = \mathsf{e}_n + \mathsf{e}_{n+1} \end{array} \right\}, \end{aligned}$$

$P_{n-1} = \mathrm{Stab}_{G_{n-1}}(\langle \mathsf{e}_{n-1} + \mathsf{e}_{n+1} \rangle)$ and $P_n = \mathrm{Stab}_{G_n}(\langle \mathsf{e}_n + \mathsf{e}_{n+1} \rangle)$, the inclusion $\mathtt{m}_{n-1}(\mathsf{U}(n-2), 1) \subset P_{n-1} \cap P_n$ is obvious. We show the converse inclusion. Let $g \in P_{n-1} \cap P_n$. There exist scalars $\lambda \in \mathbb{C}^*$ and $\mu \in \mathbb{C}^*$ such that

$$\begin{aligned} (4.1.2) \qquad & g(\mathsf{e}_n + \mathsf{e}_{n+1}) = \lambda(\mathsf{e}_n + \mathsf{e}_{n+1}), \\ & g(\mathsf{e}_{n-1} + \mathsf{e}_{n+1}) = \mu(\mathsf{e}_{n-1} + \mathsf{e}_{n+1}), \quad g(\mathsf{e}_n) = \mathsf{e}_n. \end{aligned}$$

Substituting (4.1.2) to the equations $(g(\mathsf{e}_n) | g(\mathsf{e}_n + \mathsf{e}_{n+1})) = (\mathsf{e}_n | \mathsf{e}_n + \mathsf{e}_{n+1}) = 1$ and $(g(\mathsf{e}_n + \mathsf{e}_{n+1}) | g(\mathsf{e}_{n-1} + \mathsf{e}_{n+1})) = (\mathsf{e}_n + \mathsf{e}_{n+1} | \mathsf{e}_{n-1} + \mathsf{e}_{n+1}) = -1$ respectively, we obtain $\bar{\lambda} = 1$ and $\lambda \bar{\mu} = 1$. Hence $g \in \mathtt{m}_{n-1}(\mathsf{U}(n-2), 1)$ by (4.1.1). $\square$

Let $\Delta(P_{n-1} \cap P_n)$ be the diagonal subgroup of $P_{n-1} \times P_n$.

PROPOSITION 4.1.1.

(1) *The open subset*

$$\Omega_n = \{g \in G_n |\ (g(\mathsf{e}_n + \mathsf{e}_{n+1})|\mathsf{e}_{n-1} + \mathsf{e}_{n+1}) \neq 0,\ (g(\mathsf{e}_n + \mathsf{e}_{n+1})|\mathsf{e}_n) \neq 0\}.$$

is an orbit under the action of $P_{n-1} \times P_n$ on G_n defined by

$$(p_0, p) \cdot g = p_0 g p^{-1}, \quad (p_0, p) \in P_{n-1} \times P_n,\ g \in G_n.$$

(2) *The map $\psi : P_{n-1} \times P_n \to \Omega_n$ defined by $(p_0, p) \mapsto p_0 p^{-1}$, passing to the quotient, induces a diffeomorphism $(P_{n-1} \times P_n)/\Delta(P_{n-1} \cap P_n) \cong \Omega_n$.*

Proof. For a non-zero $\mathsf{v} \in \mathbb{C}^{n+1}$, let $\langle \mathsf{v} \rangle$ denote the line $\mathbb{C}\mathsf{v}$. The group G_n naturally acts on $\mathbf{P}^n_{\mathbb{C}}$, the n-dimensional projective manifold, and the set $\mathcal{F} = \{\langle \mathsf{v} \rangle \in \mathbf{P}^n_{\mathbb{C}} |\ (\mathsf{v}|\mathsf{v}) = 0\}$ is the G_n-orbit of $\langle \mathsf{e}_n + \mathsf{e}_{n+1} \rangle$. Since $P_n = \mathrm{Stab}(\langle \mathsf{e}_n + \mathsf{e}_{n+1} \rangle)$, we have a bijection $G_n/P_n \cong \mathcal{F}$. We first prove (1). It suffices to show that the subset

$$\tilde{\Omega}_n = \{\langle \mathsf{v} \rangle \in \mathcal{F} |\ (\mathsf{v}|\mathsf{e}_{n-1} + \mathsf{e}_{n+1}) \neq 0,\ (\mathsf{v}|\mathsf{e}_n) \neq 0\}$$

forms a single P_{n-1}-orbit. Note that $\langle \mathsf{e}_n + \mathsf{e}_{n+1} \rangle \in \tilde{\Omega}_n$. The P_{n-1}-invariance of the set $\tilde{\Omega}_n$ follows from the fact that $p_0 \in P_{n-1}$ means $p_0(\mathsf{e}_n) = \mathsf{e}_n$ and $p_0^{-1}(\mathsf{e}_{n-1} + \mathsf{e}_{n+1}) = \mu(\mathsf{e}_{n-1} + \mathsf{e}_{n+1})$ with a scalar $\mu \in \mathbb{C}^*$. Next we show that for given $\langle \mathsf{v}_1 \rangle$, $\langle \mathsf{v}_2 \rangle \in \tilde{\Omega}_n$ there exists a $p_0 \in P_{n-1}$ such that $p_0 \langle \mathsf{v}_1 \rangle = \langle \mathsf{v}_2 \rangle$. By replacing v_2 for $c\mathsf{v}_2$ with an appropriate $c \in \mathbb{C}^*$ if necessary, we may assume that $(\mathsf{v}_1|\mathsf{e}_{n-1} + \mathsf{e}_{n+1}) = (\mathsf{v}_2|\mathsf{e}_{n-1} + \mathsf{e}_{n+1})$. (This is possible because $(\mathsf{v}_i|\mathsf{e}_{n-1} + \mathsf{e}_{n+1}) \neq 0$ for $i = 1, 2$.) If we put

$$\mathsf{w}_i = \mathsf{v}_i - (\mathsf{v}_i|\mathsf{e}_n)\mathsf{e}_n, \quad i = 1,\ 2,$$

then, by Witt's theorem, we have a $p_0 \in G_n$ such that

$$\begin{aligned} &p_0(\mathsf{e}_n) = \mathsf{e}_n, \quad p_0(\mathsf{e}_{n-1} + \mathsf{e}_{n+1}) = \bar{\lambda}^{-1}(\mathsf{e}_{n-1} + \mathsf{e}_{n+1}), \\ &p_0(\mathsf{w}_1) = \lambda \mathsf{w}_2 \end{aligned}$$

with $\lambda = (\mathsf{v}_1|\mathsf{e}_n)(\mathsf{v}_2|\mathsf{e}_n)^{-1}$. The first two equations mean $p_0 \in P_{n-1}$. The last equation can be written as $p_0(\mathsf{v}_1) = \lambda \mathsf{v}_2$. Hence $p_0 \langle \mathsf{v}_1 \rangle = \langle \mathsf{v}_2 \rangle$.

Now we prove assertions in (2). By (1), the map ψ is surjective. We show that ψ is submersive. This is reduced to showing the surjectivity of the tangent map of ψ at the base point $e = (\mathtt{I}_{n+1}, \mathtt{I}_{n+1})$ mod $\Delta(P_{n-1} \cap$

P_n), or equivalently to showing $\mathfrak{g}_n = \mathfrak{p}'_{n-1} + \mathfrak{p}'_n$ with $\mathfrak{p}'_n = \mathrm{Lie}(P_n)$. (Note that the tangent space of Ω_n at $\mathtt{I}_{n+1}$ is $\mathfrak{g}_n$ since Ω_n is open in G_n.) We have $\dim_{\mathbb{R}}(\mathfrak{p}'_{n-1}) + \dim_{\mathbb{R}}(\mathfrak{p}'_n) - \dim_{\mathbb{R}}(\mathfrak{g}_n) = (n-2)^2$. On the other hand, Lemma 4.1.1 gives $\dim_{\mathbb{R}}(\mathfrak{p}'_{n-1} \cap \mathfrak{p}'_n) = (n-2)^2$. Hence we have the identity $\dim_{\mathbb{R}}(\mathfrak{g}_n) = \dim_{\mathbb{R}}(\mathfrak{p}'_{n-1}) + \dim_{\mathbb{R}}(\mathfrak{p}'_n) - \dim_{\mathbb{R}}(\mathfrak{p}'_{n-1} \cap \mathfrak{p}'_n)$, which in turn means that the map $\mathfrak{p}'_{n-1} \oplus \mathfrak{p}'_n \to \mathfrak{g}_n$ sending $(X_0, X) \in \mathfrak{p}'_{n-1} \oplus \mathfrak{p}'_n$ to $X_0 + X \in \mathfrak{g}_n$ is surjective, or equivalently $\mathfrak{g}_n = \mathfrak{p}'_{n-1} + \mathfrak{p}'_n$ as desired. This completes the proof. □

Proposition 4.1.2.

(1) *Let $\nu_0,\ \nu \in \mathbb{C}$. Let $\sigma_0 = \sigma^{n-1}_{(\mathbf{q}\,;q_0)}$ with $(\mathbf{q}\,;q_0) \in {}^\circ\mathcal{L}^+_{n-1}$ and $\sigma = \sigma^n_{(\mathbf{p};p_0)}$ with $(\mathbf{p}\,;p_0) \in {}^\circ\mathcal{L}^+_n$. Assume the condition $\mathbf{q} \subset \mathbf{p}$. Then there exists a unique function*

$$\xi^{\sigma_0,\sigma}(\nu_0,\nu) : G_n \to W(\mathbf{q}) \otimes W(\mathbf{p})^\vee$$

that satisfies the conditions listed below:

(a) $\xi^{\sigma_0,\sigma}(\nu_0,\nu)(g) = 0$ *for $g \in G_n - \Omega_n$;*

(b) *For every $p_0 = \mathsf{a}_{n-1}(r_0)m_{n-1}u_{n-1} \in P_{n-1}$ and $p = \mathsf{a}_n(r)m_n u_n \in P_n$ with $(m_{n-1}, u_{n-1}) \in M_{n-1} \times N_{n-1}$ and $(m_n, u_n) \in M_n \times N_n$, we have*

$$\begin{aligned} &\xi^{\sigma_0,\sigma}(\nu_0,\nu)(p_0 g p^{-1}) \\ &= r_0^{\nu_0+n-1} r^{-\nu+n} (\sigma_0(m_{n-1}) \otimes \sigma^\vee(m_n)) \xi^{\sigma_0,\sigma}(\nu_0,\nu)(g), \\ &\quad g \in G_n; \end{aligned}$$

(c) *we have*

$$\begin{aligned} &\langle \xi^{\sigma_0,\sigma}(\nu_0,\nu)(\mathtt{I}_{n+1}), w_0^\vee \otimes w \rangle = \langle \mathsf{p}^{\mathbf{p}}_{\mathbf{q}}(w), w_0^\vee \rangle, \\ &\quad w_0^\vee \in W(\mathbf{q})^\vee,\ w \in W(\mathbf{p}). \end{aligned}$$

(2) *The function $(\nu_0, \nu, g) \mapsto \xi^{\sigma_0,\sigma}(\nu_0,\nu)(g)$ is C^∞ on $\mathbb{C}^2 \times \Omega_n$ and is holomorphic with respect to ν_0 and ν.*

Proof. The conditions in (b) and (c) give the formula

$$\xi^{\sigma_0,\sigma}(\nu_0,\nu)(p_0 p) = r_0^{\nu_0+n-1} r^{\nu-n} (\sigma_0(m_{n-1}) \otimes \sigma^\vee(m_n^{-1})) \xi_0 \tag{4.1.3}$$

for $p_0 = \mathtt{a}_{n-1}(r_0)m_{n-1}u_{n-1} \in P_{n-1}$ and $p = \mathtt{a}_n(r)m_n u_n \in P_n$, where $\xi_0 \in W(\mathbf{q}) \otimes W(\mathbf{p})^\vee$ is a unique element satisfying

$$\langle \xi_0, w_0^\vee \otimes w\rangle = \langle \mathsf{p}_{\mathbf{q}}^{\mathbf{p}}(w), w_0^\vee\rangle, \quad w_0^\vee \in W(\mathbf{q})^\vee,\ w \in W(\mathbf{p}).$$

The identity (4.1.3) uniquely determines $\xi^{\sigma_0,\sigma}(\nu_0,\nu)|\Omega_n$. To show the existance of $\xi^{\sigma_0,\sigma}(\nu_0,\nu)$, we only have to check that the right-hand side of the identity (4.1.3) is independent of a choice of expressions $g = p_0 p,\ p_0 \in P_{n-1},\ p \in P_n$. We can confirm this by the $P_{n-1} \cap P_n$-invariance of ξ_0. This establishes (1). Next we show (2). Let $\Psi : \mathbb{R}_+^* \times \mathbb{C}^{(1)} \times N_{n-1} \times \mathsf{U}(n-1) \times N_n \times \mathbb{C}^{(1)} \times \mathbb{R}_+^* \to \Omega_n$ be the map defined by

$$\begin{aligned}
&\Psi(r_0, x_0, u_{n-1}, h, u_n, x, r)\\
&= \mathtt{a}_{n-1}(r_0)\mathtt{m}_{n-1}(\mathtt{I}_{n-2}, x_0)u_{n-1}u_n^{-1}\mathtt{m}_n(h,x)^{-1}\mathtt{a}_n(r)^{-1},\\
&\qquad r_0 > 0,\ r > 0,\ x_0 \in \mathbb{C}^{(1)},\ x \in \mathbb{C}^{(1)},\\
&\qquad h \in \mathsf{U}(n-1),\ u_{n-1} \in N_{n-1},\ u_n \in N_n.
\end{aligned}$$

By Proposition 4.1.1 (2), Ψ is a diffeomorphism. By the identity

$$\begin{aligned}
&\xi^{\sigma_0,\sigma}(\nu_0,\nu) \circ \Psi(r_0, x_0, u_{n-1}, h, u_n, x, r)\\
&= r_0^{\nu_0+n-1} r^{n-\nu} \cdot \sigma_0(\mathtt{m}_{n-1}(\mathtt{I}_{n-2}, x_0)) \otimes \sigma^\vee(\mathtt{m}_n(h,x))\xi_0,
\end{aligned}$$

the smoothness of $(\nu_0,\nu,g) \mapsto \xi^{\sigma_0,\sigma}(\nu_0,\nu)(g)$ on $\mathbb{C}^2 \times \Omega_n$ and the holomorphicity with respect to (ν_0,ν) follows. □

Theorem 4.1.1. *Let σ_0 and σ be as in Proposition 4.1.2.*

(1) *Put $\mathfrak{X}_n = \{(\nu_0,\nu) \in \mathbb{C}^2 |\ \mathrm{Re}(\nu_0) + \mathrm{Re}(\nu) > 1,\ \mathrm{Re}(\nu_0) < 1-n\}$. The function $(\nu_0,\nu,g) \to \xi^{\sigma_0,\sigma}(\nu_0,\nu)(g)$ is continuous on $\mathfrak{X}_n \times G_n$. For $(\nu_0,\nu) \in \mathfrak{X}_n$, we have a linear map $\mathcal{P}^{\sigma_0,\sigma}(\nu_0,\nu)$ from $\mathcal{V}_{n-1}^{\sigma_0^\vee} \otimes \mathcal{V}_n^\sigma$ to $C^\infty(G_n)$ such that*

$$\begin{aligned}
(4.1.4)\qquad &\mathcal{P}^{\sigma_0,\sigma}(\nu_0,\nu\ ;\varphi_0 \otimes \varphi)(g)\\
&= \int_{K_{n-1}\times K_n} \langle \xi^{\sigma_0,\sigma}(\nu_0,\nu)(k_0 g k^{-1}), \varphi_0(k_0) \otimes \varphi(k)\rangle\\
&\qquad \times d_{K_{n-1}}(k_0) d_{K_n}(k),\\
&\quad \varphi_0 \in \mathcal{V}_{n-1}^{\sigma_0^\vee},\ \varphi \in \mathcal{V}_n^\sigma.
\end{aligned}$$

The function $(\nu_0,\nu) \mapsto \mathcal{P}^{\sigma_0,\sigma}(\nu_0,\nu\ ;\varphi_0 \otimes \varphi)(g)$ on $\mathfrak{X}_n$ is holomorphic for every $\varphi_0 \in \mathcal{V}_{n-1}^{\sigma_0^\vee}$, $\varphi \in \mathcal{V}_n^\sigma$ and $g \in G_n$.

(2) *Suppose* $(\nu_0,\nu)\in\mathfrak{X}_n$. *Then* $\mathcal{P}^{\sigma_0,\sigma}(\nu_0,\nu)$ *is a non-zero element of the space* $\mathcal{I}(\sigma_0,\nu_0|\sigma,\nu)$.

Proof. By Proposition 4.1.2 (2), we have that the function $(\nu_0,\nu,g)\mapsto\xi^{\sigma_0,\sigma}(\nu_0,\nu)(g)$ is C^∞ on the open dense subset $\mathfrak{X}_n\times\Omega_n$ of $\mathfrak{X}_n\times G_n$, and that $\xi^{\sigma_0,\sigma}(\nu_0,\nu)(g)=0$ for $g\in G_n-\Omega_n$. Hence to prove the continuity of $(\nu_0,\nu,g)\mapsto\xi^{\sigma_0,\sigma}(\nu_0,\nu)(g)$ on $\mathfrak{X}_n\times G_n$, it suffices to show that

$$\lim_{(\nu_0,\nu,g)\to(\nu_0',\nu',g'),\ g\in\Omega_n}\xi^{\sigma_0,\sigma}(\nu_0,\nu)(g)=0 \tag{4.1.5}$$

for $g'\in G_n-\Omega_n$ and $(\nu_0',\nu')\in\mathfrak{X}_n$. Take a g in Ω_n and express it as

$$\begin{aligned}&g=\Psi(r_0,x_0,u_{n-1},h,u_n,x,r),\\&\quad r_0,\ r>0,\ x_0,\ x\in\mathbb{C}^{(1)},\ h\in\mathsf{U}(n-2),\ u_{n-1}\in N_{n-1},\ u_n\in N_n,\end{aligned}$$

where Ψ is the diffeomorphism introduced in the proof of Proposition 4.1.2 (2). We have

$$\begin{aligned}r&=|(g(\mathsf{e}_n+\mathsf{e}_{n+1})|\mathsf{e}_n)|^{-1},\\r_0&=|(g(\mathsf{e}_n+\mathsf{e}_{n+1})|\mathsf{e}_n)(g(\mathsf{e}_n+\mathsf{e}_{n+1})|\mathsf{e}_{n-1}+\mathsf{e}_{n+1}))^{-1}|.\end{aligned}$$

Since $\sigma_0(M_{n-1})$ and $\sigma^\vee(M_n)$ are compact, using Proposition 4.1.2 (b) and the formulas above, we have the estimate

$$\begin{aligned}&\|\xi^{\sigma_0,\sigma}(\nu_0,\nu)(g)\|\\&\leqslant C|(g(\mathsf{e}_n+\mathsf{e}_{n+1})|\mathsf{e}_{n-1}+\mathsf{e}_{n+1})|^{-\operatorname{Re}(\nu_0)-n+1}|\\&\times(g(\mathsf{e}_n+\mathsf{e}_{n+1})|\mathsf{e}_n)|^{\operatorname{Re}(\nu_0)+\operatorname{Re}(\nu)-1},\quad g\in\Omega_n,\end{aligned}\tag{4.1.6}$$

with $C>0$ a constant independent of g. Now we let $(\nu_0,\nu,g)\in\mathfrak{X}_n\times\Omega_n$ go to $(\nu_0',\nu',g')\in\mathfrak{X}_n\times(G_n-\Omega_n)$. By the definition of Ω_n, the right-hand side of the inequality (4.1.6) tends to 0 as $(\nu_0,\nu,g)\to(\nu_0',\nu',g')$. This proves (4.1.5) and establishes the first part of (1). By the continuity just established, the integral (4.1.4) converges absolutely as long as (ν_0,ν) is in $\mathfrak{X}_n$. The holomorphy with respect to (ν_0,ν) is clear.

We have the identity

$$\begin{aligned}&\mathcal{P}^{\sigma_0,\sigma}(\nu_0,\nu\,;\pi_{n-1}^{\sigma_0^\vee,-\nu_0}(g_0)\varphi_0\otimes\pi_n^{\sigma,\nu}(g')\varphi)(g)\\&=\mathcal{P}^{\sigma_0,\sigma}(\nu_0,\nu\,;\varphi_0\otimes\varphi)(g_0^{-1}gg')\end{aligned}\tag{4.1.7}$$

for $\varphi_0 \in \mathcal{V}_{n-1}^{\sigma_0^\vee}$, $\varphi \in \mathcal{V}_n^\sigma$, $g,\ g' \in G_n$ and $g_0 \in G_{n-1}$ by applying the integration formula [12, (7.4), page 170] to the left $M_{n-1} \times M_n$-invariant function $f : K_{n-1} \times K_n \to \mathbb{C}$ defined by

$$f(k_0, k) = \langle \xi^{\sigma_0,\sigma}(\nu_0, \nu)(k_0 g_0 g g'^{-1} k^{-1}), \varphi_0(k_0) \otimes \varphi(k)\rangle,$$
$$(k_0, k) \in K_{n-1} \times K_n.$$

For $g \in G_n$, the bi-linear form $\Lambda_g : \mathcal{V}_{n-1}^{\sigma_0^\vee} \times \mathcal{V}_n^\sigma \to \mathbb{C}$ given by

$$\Lambda_g(\varphi_0, \varphi) = \mathcal{P}^{\sigma_0,\sigma}(\nu_0, \nu\ ; \varphi_0 \otimes \varphi)(g), \quad \varphi_0 \in \mathcal{V}_{n-1}^{\sigma_0^\vee},\ \varphi \in \mathcal{V}_n^\sigma.$$

is continuous with respect to the C^∞-topology by the estimate

$$\left|\Lambda_g(\varphi_0, \varphi)\right| \leqslant C_g \sup_{k_{n-1} \in K_{n-1}} \|\varphi_0(k_{n-1})\| \sup_{k_n \in K_n} \|\varphi(k_n)\|$$

with $C_g = \sup_{(k_{n-1},k_n) \in K_{n-1} \times K_n} \|\xi(\sigma_0, \sigma\ ; \nu_0, \nu)(k_{n-1} g k_n)\|$. The formula (4.1.7) gives

$$\mathcal{P}^{\sigma_0,\sigma}(\nu_0, \nu\ ; \varphi_0 \otimes \varphi)(g_0^{-1} g g') = \Lambda_g(\pi_{n-1}^{\sigma_0^\vee, -\nu_0}(g_0)\varphi_0, \pi_n^{\sigma,\nu}(g')\varphi),$$
$$g_0 \in G_{n-1},\ g,\ g' \in G_n.$$

Since $\pi_{n-1}^{\sigma_0^\vee,-\nu_0}$ and $\pi_n^{\sigma,\nu}$ are smooth, this shows the function $(g_0, g') \to \mathcal{P}^{\sigma_0,\sigma}(\nu_0, \nu\ ; \varphi_0 \otimes \varphi)(g_0^{-1} g g')$ is C^∞. Finally we show that there exist $\varphi_0 \in \mathcal{V}_{n-1}^{\sigma_0^\vee}$ and $\varphi \in \mathcal{V}_n^\sigma$ such that $\mathcal{P}^{\sigma_0,\sigma}(\nu_0, \nu\ ; \varphi_0 \otimes \varphi)(\mathtt{I}_{n+1}) \neq 0$. By Proposition 4.1.2 (c), the function $\xi^{\sigma_0,\sigma}(\nu_0, \nu)(k_0 k^{-1})$ on $K_{n-1} \times K_n$ is not identically zero. Hence we can find C^∞-functions $f_0 : K_{n-1} \to W(\mathbf{q})^\vee$ and $f : K_n \to W(\mathbf{p})$ with

$$\int_{K_{n-1} \times K_n} \langle \xi^{\sigma_0,\sigma}(\nu_0, \nu)(k_0 k^{-1}), f_0(k_0) \otimes f(k)\rangle d_{K_{n-1}}(k_0) d_{K_n}(k) \neq 0.$$

The integral in the left-hand side equals $\mathcal{P}^{\sigma_0,\sigma}(\nu_0, \nu\ ; \varphi_0 \otimes \varphi)(\mathtt{I}_{n+1})$ with $\varphi_0 \in \mathcal{V}_{n-1}^{\sigma_0^\vee}$ and $\varphi \in \mathcal{V}_n^\sigma$ given by $\varphi_0(k_0) = \int_{M_{n-1}} \sigma_0^\vee(m_0)^{-1} f_0(m_0 k_0) d_{M_{n-1}}(m_0)$, $\varphi(k) = \int_{M_n} \sigma(m)^{-1} f(mk) d_{M_n}(m)$. This completes the proof. $\square$

REMARK 4.1.1.

(1) As we proved in Proposition 4.1.1, $\Omega_n = P_{n-1}P_n$ is a unique open $P_{n-1} \times P_n$-double coset in G_n. The set $G_n - \Omega_n$ decomposes into two distinct $P_{n-1} \times P_n$-double cosets Ω'_n and Ω''_n given as

$$\begin{aligned}\Omega'_n &= \{g \in G_n | (g(\mathsf{e}_n + \mathsf{e}_{n+1}) | \mathsf{e}_{n-1} + \mathsf{e}_{n+1}) \neq 0, \\ &\qquad (g(\mathsf{e}_n + \mathsf{e}_{n+1}) | \mathsf{e}_n) = 0\}, \\ \Omega''_n &= \{g \in G_n | (g(\mathsf{e}_n + \mathsf{e}_{n+1}) | \mathsf{e}_{n-1} + \mathsf{e}_{n+1}) = (g(\mathsf{e}_n + \mathsf{e}_{n+1}) | \mathsf{e}_n) = 0\}.\end{aligned}$$

(2) In the p-adic situation, analogous integrals as (4.1.4) were used in [4] to construct a class-one Shintani function.

4.2. Evaluation of the Poisson integrals for corner K_n-types

Let $\sigma_0 = \sigma^{n-1}_{(\mathbf{q};q_0)}$ and $\sigma = \sigma^{n}_{(\mathbf{p};p_0)}$ be as in Proposition 4.1.2. In particular we assume $\mathbf{q} \subset \mathbf{p}$. Put $z_0 = |\mathbf{q}| + q_0$ and $z = |\mathbf{p}| + p_0$. Our goal of this subsection is to obtain an explicit formula of the constant $\gamma^{(h)}(\mathcal{P})$ for $\mathcal{P} = \mathcal{P}^{\sigma_0,\sigma}(\nu_0, \nu)$ with $(\nu_0, \nu) \in \mathfrak{X}_n$. The result is in Theorem 4.2.1.

PROPOSITION 4.2.1. *Let* $(\nu_0, \nu) \in \mathfrak{X}_n$ *and* $\varphi_0 \in \mathcal{V}^{\sigma_0^\vee}_{n-1}$ *and* $\varphi \in \mathcal{V}^{\sigma}_n$. *Then we have*

$$\begin{aligned}(4.2.1)\quad &\lim_{r\to\infty} \lim_{r_0\to\infty} r_0^{n-1+\nu_0} r^{n-\nu} \mathcal{P}^{\sigma_0,\sigma}(\nu_0, \nu\ ; \varphi_0 \otimes \varphi)(\mathsf{a}_{n-1}(r_0)^{-1}\mathsf{a}_n(r)) \\ &= \langle \mathsf{p}^{\mathbf{p}}_{\mathbf{q}}(\mathcal{A}^{\sigma,\nu}_n(\varphi\ ; \mathtt{w}_n)), \mathcal{A}^{\sigma_0^\vee, -\nu_0}_{n-1}(\varphi_0\ ; \mathtt{w}_{n-1})\rangle.\end{aligned}$$

PROOF. Put $\xi = \xi^{\sigma_0,\sigma}(\nu_0, \nu)$ and $\mathcal{P} = \mathcal{P}^{\sigma_0,\sigma}(\nu_0, \nu)$. We extend the function φ_0 (resp. φ) to all of G_{n-1} (resp. G_n) by putting

$$\begin{aligned}(4.2.2)\quad &\varphi_0(g_0) = \varphi_0(\kappa_{n-1}(g_0))\alpha_{n-1}(g_0)^{-\nu_0+n-1}, \quad g_0 \in G_{n-1}, \\ &\varphi(g) = \varphi(\kappa_n(g))\alpha_n(g)^{\nu+n}, \quad g \in G_n.\end{aligned}$$

To begin with we prove that there exists a positive constant C and δ such that

$$\begin{aligned}(4.2.3)\quad &\left|\langle \xi(\mathsf{a}_{n-1}(r_0)\bar{n}_0\mathsf{a}_{n-1}(r_0)^{-1}\mathsf{a}_n(r)\bar{n}^{-1}\mathsf{a}_n(r)^{-1}), \varphi_0(\bar{n}_0) \otimes \varphi(\bar{n})\rangle\right| \\ &\leqslant C\alpha_{n-1}(\bar{n}_0)^{2(n-1)}\alpha_n(\bar{n})^{n+\delta}, \\ &\quad \bar{n}_0 \in \bar{N}_{n-1},\ \bar{n} \in \bar{N}_n\end{aligned}$$

holds. By the estimate (4.1.6) and the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&\big|\langle \xi(\mathsf{a}_{n-1}(r_0)\bar{n}_0\mathsf{a}_{n-1}(r_0)^{-1}\mathsf{a}_n(r)\bar{n}\mathsf{a}_n(r)^{-1}), \varphi_0(\bar{n}_0)\otimes\varphi(\bar{n})\rangle\big| \\
&\leqslant C_0\|\mathsf{a}_n(r)\bar{n}^{-1}\mathsf{a}_n(r)^{-1}(\mathsf{e}_n+\mathsf{e}_{n+1})\|^{\mathrm{Re}(\nu_0)+\mathrm{Re}(\nu)-1} \\
&\quad\times \|\mathsf{a}_{n-1}(r_0)\bar{n}_0^{-1}\mathsf{a}_{n-1}(r_0)^{-1}(\mathsf{e}_{n-1}+\mathsf{e}_{n+1}))\|^{-\mathrm{Re}(\nu_0)-n+1}\|\varphi_0(\bar{n}_0)\|\|\varphi(\bar{n})\|
\end{aligned}
$$

with a constant C_0. By the Iwasawa decomposition we obtain the estimates

$$
\begin{aligned}
&\|\mathsf{a}_n(r)\bar{n}^{-1}\mathsf{a}_n(r)^{-1}(\mathsf{e}_n+\mathsf{e}_{n+1})\| \leqslant C'\alpha_n(\mathsf{a}_n(r)\bar{n}\mathsf{a}_n(r)^{-1})^{-1}, \\
&\|\varphi(\bar{n})\| \leqslant C'\alpha_n(\bar{n})^{\mathrm{Re}(\nu)+n}, \quad \bar{n}\in\bar{N}_n, \\
&\|\mathsf{a}_{n-1}(r)\bar{n}_0^{-1}\mathsf{a}_{n-1}(r_0)^{-1}(\mathsf{e}_{n-1}+\mathsf{e}_{n+1})\| \\
&\ \leqslant C'\alpha_{n-1}(\mathsf{a}_{n-1}(r_0)\bar{n}_0\mathsf{a}_{n-1}(r_0)^{-1})^{-1}, \\
&\|\varphi_0(\bar{n}_0)\| \leqslant C'\alpha_{n-1}(\bar{n}_0)^{-\mathrm{Re}(\nu_0)+n-1}, \quad \bar{n}_0\in\bar{N}_{n-1}
\end{aligned}
$$

with a positive constant C'. Since $(\nu_0,\nu)\in\mathfrak{X}_n$, we consequently have

$$
\begin{aligned}
&\big|\langle \xi(\mathsf{a}_{n-1}(r_0)\bar{n}_0\mathsf{a}_{n-1}(r_0)^{-1}\mathsf{a}_n(r)\bar{n}\mathsf{a}_n(r)^{-1}), \varphi_0(\bar{n}_0)\otimes\varphi(\bar{n})\rangle\big| \\
&\leqslant C_1\alpha_n(\mathsf{a}_n(r)\bar{n}\mathsf{a}_n(r)^{-1})^{-\mathrm{Re}(\nu_0)-\mathrm{Re}(\nu)+1}\alpha_n(\bar{n})^{\mathrm{Re}(\nu)+n} \\
&\quad\times \alpha_{n-1}(\mathsf{a}_{n-1}(r_0)\bar{n}_0\mathsf{a}_{n-1}(r_0)^{-1})^{\mathrm{Re}(\nu_0)+n-1}\alpha_{n-1}(\bar{n}_0)^{-\mathrm{Re}(\nu_0)+n-1}.
\end{aligned}
$$

Finally we use the estimates $\alpha_n(\mathsf{a}_n(r)\bar{n}\mathsf{a}_n(r)^{-1}) \geqslant \alpha_n(\bar{n}),\ \bar{n}\in\bar{N}_n,\ r\geqslant 1$ ([12, page 188, Lemma 7.16]) to have the desired estimate (4.2.3) with $\delta = 1-\mathrm{Re}(\nu_0) > 0$. Now we compute the left-hand side of (4.2.1). First we apply the integration formula [12, (5.25) page 140], and then use (4.2.2) and Proposition 4.1.2 (b). We have

$$
\begin{aligned}
(4.2.4)\quad & r_0^{n-1+\nu_0}r^{n-\nu}\mathcal{P}(\varphi_0\otimes\varphi)(a_{n-1}(r_0)^{-1}a_n(r)) \\
&= r_0^{n-1+\nu_0}r^{n-\nu}\int_{\bar{N}_{n-1}\times\bar{N}_n}\langle\xi(\kappa_{n-1}(\bar{n}_0)\mathsf{a}_{n-1}(r_0)^{-1}\mathsf{a}_n(r)\kappa_n(\bar{n})^{-1}), \\
&\qquad\qquad \varphi_0(\kappa_{n-1}(\bar{n}_0))\otimes\varphi(\kappa_n(\bar{n}))\rangle \\
&\quad\times\alpha_{n-1}(\bar{n}_0)^{2(n-1)}\alpha_n(\bar{n})^{2n}d_{\bar{N}_{n-1}}(\bar{n}_0)d_{\bar{N}_n}(\bar{n}) \\
&= \int_{\bar{N}_{n-1}\times\bar{N}_n}\langle\xi(\mathsf{a}_{n-1}(r_0)\bar{n}_0\mathsf{a}_{n-1}(r_0)^{-1}\mathsf{a}_n(r)\bar{n}^{-1}\mathsf{a}_n(r)^{-1}), \\
&\qquad \varphi_0(\bar{n}_0)\otimes\varphi(\bar{n})\rangle d_{\bar{N}_{n-1}}(\bar{n}_0)d_{\bar{N}_n}(\bar{n}).
\end{aligned}
$$

Now let $r \to +\infty$ and $r_0 \to +\infty$. Since $(\nu_0, \nu) \in \mathfrak{X}_n$, the function ξ is continuous by Theorem 4.1.1. Hence

$$\lim_{r_0\to+\infty}\lim_{r\to+\infty} \xi(\mathtt{a}_{n-1}(r_0)\bar{n}_0\mathtt{a}_{n-1}(r_0)^{-1}\mathtt{a}_n(r)\bar{n}^{-1}\mathtt{a}_n(r)^{-1}) = \xi(\mathtt{I}_{n+1}),$$
$$\bar{n}_0 \in \bar{N}_{n-1},\ \bar{n} \in \bar{N}_n$$

because $\lim_{r\to+\infty} \mathtt{a}_{n-1}(r_0)\bar{n}_0\mathtt{a}_{n-1}(r_0)^{-1} = \mathtt{I}_n$ and $\lim_{r\to+\infty} \mathtt{a}_n(r)\bar{n}^{-1} \cdot \mathtt{a}_n(r)^{-1} = \mathtt{I}_{n+1}$. Interchanging integration and limit, we have

$$\begin{aligned}(4.2.5)\qquad &\lim_{r_0\to+\infty}\lim_{r\to+\infty} r_0^{n-1+\nu_0} r^{n-\nu}\mathcal{P}(\varphi_0\otimes\varphi)(\mathtt{a}_{n-1}(r_0)^{-1}\mathtt{a}_n(r))\\ &= \int_{\bar{N}_{n-1}\times\bar{N}_n} \langle \xi(\mathtt{I}_{n+1}), \varphi_0(\bar{n}_0)\otimes\varphi(\bar{n})\rangle d_{\bar{N}_{n-1}}(\bar{n}_0) d_{\bar{N}_n}(\bar{n})\\ &= \langle \xi(\mathtt{I}_{n+1}), \mathcal{A}_{n-1}^{\sigma^\vee,-\nu_0}(\varphi_0 : \mathtt{w}_{n-1})\otimes \mathcal{A}_n^{\sigma,\nu}(\varphi : \mathtt{w}_n)\rangle.\end{aligned}$$

Here one should note that the condition $(\nu_0, \nu) \in \mathfrak{X}_n$ implies $\mathrm{Re}(\nu) > 0$, $\mathrm{Re}(\nu_0) < 0$, which ensures the convergence of the integrals $\mathcal{A}_{n-1}^{\sigma_0^\vee,-\nu_0}(\varphi_0 : \mathtt{w}_{n-1})$ and $\mathcal{A}_n^{\sigma,\nu}(\varphi : \mathtt{w}_n)$ (Proposition 2.3.1 (1)). To obtain the first equality in (4.2.5) above we used dominated convergence theorem, noting the estimate (4.2.3) and the fact that the function $\bar{n} \to \alpha_n(\bar{n})^{n+\delta}$ with $\delta > 0$ is integrable on $\bar{N}_n$([12, page 181, Corollary 7.7]). By Proposition 4.1.2 (c), we have the conclusion. □

Corollary 4.2.1. *Let $\mathcal{P} = \mathcal{P}^{\sigma_0,\sigma}(\nu_0, \nu)$ with $(\nu_0, \nu) \in \mathfrak{X}_n$. For $\mu = [\mathbf{m}\ ; m_0] \in \mathcal{L}_{n-1}^+(\pi_{n-1}^{\sigma_0,\nu_0})$ and $\lambda = [\mathbf{l}\ ; l_0] \in \mathcal{L}_n^+(\pi_n^{\sigma,\nu})$, we have*

$$\begin{aligned}(4.2.6)\qquad &\lim_{r\to\infty}\lim_{r_0\to\infty} r_0^{n-1+\nu_0} r^{n-\nu}\langle \mathcal{P}_{\mu,\lambda}(\mathtt{a}_{n-1}(r_0)^{-1}\mathtt{a}_n(r)), w_0^\vee\otimes w\rangle\\ &= (-1)^{l_0-m_0} c_{n-1}(\sigma_0^\vee, -\nu_0\ ; \check{\mathbf{m}}) c_n(\sigma,\nu\ ;\mathbf{l})\langle \mathsf{j}_{\mathbf{m}}^{\mathbf{q}}\circ \mathsf{p}_{\mathbf{q}}^{\mathbf{p}}\circ\mathsf{p}_{\mathbf{p}}^{\mathbf{l}}(w), w_0^\vee\rangle,\\ &\quad w_0^\vee \in W(\mathbf{m})^\vee,\ w\in W(\mathbf{l}).\end{aligned}$$

Proof. By (2.3.2), we have

$$\mathcal{A}_n^{\sigma,\nu}\circ\iota_{\mathbf{l}}^{\sigma}(w) = c_n(\sigma,\nu\ ;\mathbf{l})\iota_{\mathbf{l}}^{\sigma}(w),$$
$$\mathcal{A}_{n-1}^{\sigma_0^\vee,-\nu_0}\circ\check{\iota}_{\mathbf{m}}^{\sigma_0}(w_0^\vee) = c_{n-1}(\sigma_0^\vee, -\nu_0\ ;\check{\mathbf{m}})\check{\iota}_{\mathbf{m}}^{\sigma_0}(w_0^\vee).$$

Since $\mathtt{w}_n = \mathrm{diag}(\mathtt{I}_n, -1)$, by (2.1.1), we have $\check{\iota}^{\sigma_0}_{\mathbf{m}}(w_0^\vee)(\mathtt{w}_{n-1}) = (-1)^{m_0}(\mathtt{j}^{\mathbf{q}}_{\mathbf{m}})^\vee$ and $\iota^\sigma_{\mathbf{l}}(w)(\mathtt{w}_n) = (-1)^{l_0}\mathtt{p}^{\mathbf{l}}_{\mathbf{p}}$. Noting these remarks, we get the identity (4.2.6) from (4.2.1). □

Here is the main theorem of this section.

THEOREM 4.2.1. *Let $(\nu_0, \nu) \in \mathfrak{X}_n$, and put $\pi_0 = \pi^{\sigma_0,\nu_0}_{n-1}$ and $\pi = \pi^{\sigma,\nu}_n$. Let $h \in \{1, \cdots, n-1\}$. Put $\gamma^{(h)}(\sigma_0, \sigma\ ; \nu_0, \nu) = \gamma^{(h)}(\mathcal{P}^{\sigma_0,\sigma}(\nu_0, \nu))$. Then we have*

$$(4.2.7)\qquad \begin{aligned}\gamma^{(h)}(\sigma_0, \sigma\ ; \nu_0, \nu) &= \gamma^{(h)}_{\sigma_0,\sigma} \cdot \tilde{d}^{(h)}_n(\sigma_0, \sigma\ ; \nu)^{-1}\\ &\times\ \tilde{c}^{(h)}_n(\sigma\ ; \nu) d^{(h)}_n(\sigma_0, \sigma\ ; \nu_0)\\ &\times\ \Gamma\Big(\frac{\nu - \nu_0 + 1 + \kappa}{2}\Big)\Gamma\Big(\frac{\nu + \nu_0 + 1 + \kappa}{2}\Big)\end{aligned}$$

with a non zero real number $\gamma^{(h)}_{\sigma_0,\sigma}$ which depends only on σ_0, σ and h. Here $\tilde{c}^{(h)}_n(\sigma\ ; \nu)$ is given by (2.2.1) and

$$(4.2.8)\quad \begin{aligned}\tilde{d}^{(h)}_n(\sigma_0, \sigma\ ; \nu) = &\prod_{\alpha=1}^{h-1}\ \prod_{q_\alpha \leqslant i_\alpha < p_\alpha} (\nu - p_0 + n + 2i_\alpha - 2\alpha)\\ &\times \prod_{\beta=h+1}^{n-1}\ \prod_{p_\beta < i_\beta \leqslant q_{\beta-1}} (\nu + p_0 - n - 2i_\beta + 2\beta),\end{aligned}$$

$$(4.2.9)\quad \begin{aligned}d^{(h)}_n(\sigma, \sigma_0\ ; \nu_0) = &\prod_{\alpha=1}^{h-1}\ \prod_{q_\alpha < j_\alpha \leqslant p_\alpha} (\nu_0 - 2j_\alpha + 2\alpha + q_0 - n + 1)\\ &\times \prod_{\beta=h+1}^{n-1}\ \prod_{p_\beta \leqslant j_\beta < q_{\beta-1}} (\nu_0 + 2j_\beta - 2\beta - 2q_0 + n + 1).\end{aligned}$$

PROOF. Let $L_{\mathbf{m}}(w_0^\vee, w)$ with $\mathbf{l} \in \Lambda^+_n(\mathbf{p})$ such that $\lambda = \lambda^{(h)}_\sigma = [\mathbf{l}\ ; l_0]$ denotes the limit in the left-hand side of (4.2.6). We compute it in a way different from Corollary 4.2.1. From (3.2.6) and (3.2.7), we have

$$(4.2.10)\qquad \alpha^{(h)}(\mathbf{n}) + \beta^{(h)}(\mathbf{n}) + n - \nu = \sum_{i=1}^{h-1}(n_i - p_i) + \sum_{i=h+1}^{n-1}(p_i - n_i)$$

for $\mathbf{n}=(n_j)_{1\leqslant j\leqslant n-1}\in\Lambda^+_{n-1}(\mathbf{q}|\mathbf{l})$. Since $p_i\geqslant n_i$ for $1\leqslant i<h$, $n_i\geqslant p_i$ for $h<i\leqslant n-1$ and $n_h=p_h$, the number (4.2.10) is non-positive and it is zero if and only if $\mathbf{n}=\mathbf{p}$. By Proposition 3.2.2 (2), we have

$$\begin{aligned}(4.2.11)\qquad \lim_{r\to\infty} r^{n-\nu} f_{\mathbf{l}}(\mathbf{n}\,;r) &= \lim_{r\to\infty} r^{\alpha^{(h)}(\mathbf{n})+\beta^{(h)}(\mathbf{n})+n-\nu}\\ &\quad\times\left(\frac{1+r^{-2}}{2}\right)^{\alpha^{(h)}(\mathbf{n})}\left(\frac{1-r^{-2}}{2}\right)^{\beta^{(h)}(\mathbf{n})}\psi(\mathbf{n})\\ &= \begin{cases}0 & \mathbf{n}\in\Lambda^+_{n-1}(\mathbf{q}|\mathbf{l})-\{\mathbf{p}\},\\ 2^{n-\nu}\psi(\mathbf{p}) & \mathbf{n}=\mathbf{p}\end{cases}.\end{aligned}$$

Now by the formula (3.2.3), we have

$$\begin{aligned}(4.2.12)\qquad L_{\mathbf{m}}(w_0^\vee,w) &= \sum_{\mathbf{n}\in\Lambda^+_{n-1}(\mathbf{q}|\mathbf{l})} \lim_{r\to\infty} r^{n-\nu} f_{\mathbf{l}}(\mathbf{n}\,;r)\lim_{r_0\to\infty} r_0^{n-1+\nu_0}\\ &\quad\times\left[\iota^{\sigma_0}_{\mathbf{n}}\circ \mathsf{p}^{\mathbf{l}}_{\mathbf{n}}(w)\middle|\pi^{\sigma_0^\vee,-\nu_0}_{n-1}(\mathsf{a}_{n-1}(r_0))\circ\check{\iota}^{\sigma_0}_{\mathbf{m}}(w_0^\vee)\right]\\ &= 2^{n-\nu}\psi(\mathbf{p})\lim_{r_0\to\infty} r_0^{n-1+\nu_0}\\ &\quad\times\left[\iota^{\sigma_0}_{\mathbf{p}}\circ \mathsf{p}^{\mathbf{l}}_{\mathbf{p}}(w)\middle|\pi^{\sigma_0^\vee,-\nu_0}_{n-1}(\mathsf{a}_{n-1}(r_0))\check{\iota}^{\sigma_0}_{\mathbf{m}}(w_0^\vee)\right]\\ &= 2^{n-\nu}\psi(\mathbf{p})c_{n-1}(\sigma_0^\vee,-\nu_0\,;\check{\mathbf{m}})\\ &\quad\times\langle(\iota^{\sigma_0}_{\mathbf{p}}\circ\mathsf{p}^{\mathbf{l}}_{\mathbf{p}}(w))(\mathtt{I}_{n-1}),\check{\iota}^{\sigma_0}_{\mathbf{m}}(w_0^\vee)(\mathtt{w}_{n-1})\rangle.\end{aligned}$$

The second equality follows from (4.2.11) and the last one follows from Proposition 2.3.1 (3). By (2.1.1), we easily have

$$(\iota^{\sigma_0}_{\mathbf{p}}\circ\mathsf{p}^{\mathbf{l}}_{\mathbf{p}}(w))(\mathtt{I}_{n-1})=\mathsf{p}^{\mathbf{p}}_{\mathbf{q}}\circ\mathsf{p}^{\mathbf{l}}_{\mathbf{p}}(w),\quad (\check{\iota}^{\sigma_0}_{\mathbf{m}}(w_0^\vee))(\mathtt{w}_{n-1})=(-1)^{m_0}(\mathsf{j}^{\mathbf{q}}_{\mathbf{m}})^\vee(w_0^\vee).$$

Substituting these identities to (4.2.12), we finally get

$$\begin{aligned}(4.2.13)\qquad L_{\mathbf{m}}(w_0^\vee,w) &= (-1)^{z_0-|\mathbf{p}|}2^{n-\nu}\psi(\mathbf{p})c_{n-1}(\sigma_0^\vee,-\nu_0\,;\check{\mathbf{m}})\\ &\quad\times\langle\mathsf{p}^{\mathbf{p}}_{\mathbf{q}}\circ\mathsf{p}^{\mathbf{l}}_{\mathbf{p}}(w),(\mathsf{j}^{\mathbf{q}}_{\mathbf{m}})^\vee(w_0^\vee)\rangle.\end{aligned}$$

From (4.2.6) and (4.2.13), we obtain

$$\begin{aligned}&(-1)^{m_0-l_0}c_{n-1}(\sigma_0^\vee,-\nu_0\,;\check{\mathbf{m}})c_n(\sigma,\nu\,;\mathbf{l})\langle\mathsf{p}^{\mathbf{p}}_{\mathbf{q}}\circ\mathsf{p}^{\mathbf{l}}_{\mathbf{p}}(w),(\mathsf{j}^{\mathbf{q}}_{\mathbf{m}})^\vee(w_0^\vee)\rangle\\ &\quad=(-1)^{m_0}2^{n-\nu}\psi(\mathbf{p})c_{n-1}(\sigma_0^\vee,-\nu_0\,;\check{\mathbf{m}})\langle\mathsf{p}^{\mathbf{p}}_{\mathbf{q}}\circ\mathsf{p}^{\mathbf{l}}_{\mathbf{p}}(w),(\mathsf{j}^{\mathbf{q}}_{\mathbf{m}})^\vee(w_0^\vee)\rangle.\end{aligned}$$

for any $\mathbf{m} \in \Lambda^+_{n-1}(\mathbf{q})$ and any $(w_0^\vee, w) \in W(\mathbf{m})^\vee \times W(\mathbf{l}_0)$. Hence

$$(-1)^{l_0} c_n(\sigma, \nu\ ; \mathbf{l}) = 2^{n-\nu}\psi(\mathbf{p}). \tag{4.2.14}$$

From Corollary 3.2.1, Proposition 2.3.2 and (4.2.14), we have the conclusion easily. □

5. Analytic Continuation of Poisson Integrals

In this section, we show that the integral (4.1.4) with K-finite $\varphi_0 \otimes \varphi$ when multiplied by a suitable normalizing factor is continued holomorphically to a domain of $\mathbb{C}^2$ with respect to (ν_0, ν) (Theorem 5.1.1). For that purpose we use the difference-differential equations for the Shintani functions studied in [10]. The proof of Theorem 5.1.1 is given in 5.2.

5.1. Statement of the theorem

For $h \in \{1, \cdots, n-1\}$, put

$$\mathfrak{D}^{(h)}_{\sigma_0,\sigma} = \{\nu \in \mathbb{C} |\ \tilde{d}^{(h)}_n(\sigma_0, \sigma\ ; \nu) \neq 0\}$$

Let $\mathfrak{D}_{\sigma_0,\sigma}$ be the union of $\mathfrak{D}^{(h)}_{\sigma_0,\sigma}$ for $h \in \{1, \cdots, n-1\}$.

Theorem 5.1.1. *Let $\sigma_0 = \sigma^{n-1}_{(\mathbf{q};q_0)}$ and $\sigma = \sigma^n_{(\mathbf{p};p_0)}$ with $\mathbf{q} \subset \mathbf{p}$. Put $\kappa = |p_0 - q_0|$. For $\varphi_0 \in V^{\sigma_0^\vee}_{n-1}$ and $\varphi \in V^\sigma_n$ with $(\nu_0, \nu) \in \mathfrak{X}_n$ put*

$$\begin{aligned}&\mathcal{R}^{\sigma_0,\sigma}(\nu_0, \nu\ ; \varphi_0 \otimes \varphi)(g)\\ &= \Gamma\left(\frac{\nu - \nu_0 + 1 + \kappa}{2}\right)^{-1} \Gamma\left(\frac{\nu + \nu_0 + 1 + \kappa}{2}\right)^{-1} \mathcal{P}^{\sigma_0,\sigma}(\nu_0, \nu\ ; \varphi_0 \otimes \varphi)(g).\end{aligned}$$

(1) *Let $\varphi \in V^\sigma_n$, $\varphi_0 \in V^{\sigma_0^\vee}_{n-1}$, $D_0 \in U(\mathfrak{g}_{n-1,\mathbb{C}})$, $D \in U(\mathfrak{g}_{n,\mathbb{C}})$ and $g \in G_n$. Then the function $(\nu_0, \nu) \mapsto L_{D_0} R_D \mathcal{R}^{\sigma_0,\sigma}(\nu_0, \nu\ ; \varphi_0 \otimes \varphi)(g)$ on $(\nu_0, \nu) \in \mathfrak{X}_n$ is a holomorphic function and is extended holomorphically to $\mathbb{C} \times \mathfrak{D}_{\sigma_0,\sigma}$.*

(2) *For every $(\nu_0, \nu) \in \mathbb{C} \times \mathfrak{D}_{\sigma_0,\sigma}$, the function $g \mapsto \mathcal{R}^{\sigma_0,\sigma}(\nu_0, \nu\ ; \varphi_0 \otimes \varphi)(g)$ is C^∞ on G_n. The linear map $\varphi_0 \otimes \varphi \mapsto \mathcal{R}^{\sigma_0,\sigma}(\nu_0, \nu\ ; \varphi_0 \otimes \varphi)$ gives rise to an element of $\mathcal{I}(\sigma_0, \nu_0 | \sigma, \nu)$.*

(3) *For* $h \in \{1, \ldots, n-1\}$, *put* $\lambda_\sigma^{(h)} = [\mathbf{l}\ ; l_0]$ *and* $\mu_{\sigma_0,\sigma}^{(h)} = [\mathbf{m}\ ; m_0]$. *Then there exists a non-zero constant* $\gamma_{\sigma_0,\sigma}^{(h)}$ *depending only on* σ_0, σ *and* h *such that*

$$\begin{aligned}(5.1.1)\quad &\mathcal{R}^{\sigma_0,\sigma}(\nu_0, \nu\ ; \check{\iota}_{\mathbf{m}}^{\sigma_0}(w_0^\vee) \otimes \iota_{\mathbf{l}}^{\sigma}(w))(\mathsf{a}_n(r)) \\ &= \gamma_{\sigma_0,\sigma}^{(h)} \cdot \Gamma^{(h)}(\sigma_0, \sigma\ ; \nu_0, \nu) \\ &\quad \times (\operatorname{sh} r)^{\kappa} (\operatorname{ch} r)^{\sum_{j=1}^{h-1}(q_j - p_j) + \sum_{j=h}^{n-2}(p_{j+1} - q_j) - n + \nu - \kappa} \\ &\quad \times {}_2F_1\left(\frac{-\nu_0 - \nu + 1 + \kappa}{2}, \frac{\nu_0 - \nu + 1 + \kappa}{2}\ ; 1 + \kappa\ ; \operatorname{th}^2(r)\right) \\ &\quad \cdot \langle \mathsf{p}_{\mathbf{m}}^{\mathbf{l}}(w), w_0^\vee \rangle,\end{aligned}$$

holds for $r > 0$, $(w_0^\vee, w) \in W(\mathbf{m})^\vee \times W(\mathbf{l})$ *with*

$$\begin{aligned}(5.1.2)\quad &\Gamma^{(h)}(\sigma_0, \sigma\ ; \nu_0, \nu) \\ &= \tilde{d}_n^{(h)}(\sigma_0, \sigma\ ; \nu)^{-1} \tilde{c}_n^{(h)}(\sigma\ ; \nu) d_n^{(h)}(\sigma_0, \sigma\ ; \nu_0).\end{aligned}$$

Here $\tilde{d}_n^{(h)}(\sigma_0, \sigma\ ; \nu)$, $\tilde{c}_n^{(h)}(\sigma\ ; \nu)$ *and* $d_n^{(h)}(\sigma_0, \sigma\ ; \nu_0)$ *are given by* (4.2.8), (2.2.1) *and* (4.2.9) *respectively.*

We give a proof of this theorem in the next subsection.

5.2. The proof of Theorem 5.1.1

We need the following lemma on Gaussian hypergeometric function.

Lemma 5.2.1. *Let* $c \in \mathbb{Z}$ *with* $c \geqslant 1$.

(1) *For a fixed* $0 \leqslant x < 1$, *the function* $(a, b) \mapsto {}_2F_1(a, b\ ; c\ ; x)$ *is holomorphic on* $\mathbb{C}^2$.

(2) *For every* a, $b \in \mathbb{C}$ *and* $0 \leqslant x < 1$, *we have the formula*

$$\begin{aligned}&(1 - x)\frac{d}{dx}{}_2F_1(a, b\ ; c\ ; x) \\ &= \frac{(c-a)(c-b)}{c}{}_2F_1(a, b\ ; c+1\ ; x) + (a + b - c){}_2F_1(a, b\ ; c\ ; x)\end{aligned}$$

Proof. The Taylor series of ${}_2F_1(a, b\ ; c\ ; x)$ at $x = 0$ converges on $|x| < 1$ locally uniformly with respect to $(a, b) \in \mathbb{C}^2$. From this remark (1) follows. By comparing the Taylor series expansion of the both sides of the identity at $x = 0$, we get the formula of (2). $\square$

Let $h \in \{1, \cdots, n-1\}$. For every $\mathbf{l} \in \Lambda_n^+(\mathbf{p})$ and $\mathbf{m} \in \Lambda_{n-1}^+(\mathbf{q}|\mathbf{l})$, we define polynomial functions $Q^{(h)}(\mathbf{m}\,; \nu_0)$ and $P^{(h)}(\mathbf{l}\,; \nu)$ by

$$\begin{aligned}(5.2.1)\quad & Q^{(h)}(\mathbf{m}\,; \nu_0)\\ &= \prod_{\alpha=1}^{h-1} \prod_{q_\alpha < j_\alpha \leqslant \inf(p_\alpha, m_\alpha)} (\nu_0 - 2j_\alpha + 2\alpha + q_0 - n + 1)\\ &\quad \times \prod_{\beta=h+1}^{n-1} \prod_{\sup(p_\beta, m_\beta) \leqslant j_\beta < q_{\beta-1}} (\nu_0 + 2j_\beta - 2\beta - q_0 + n + 1),\end{aligned}$$

$$\begin{aligned}(5.2.2)\quad & P^{(h)}(\mathbf{l}\,; \nu)\\ &= \prod_{\alpha=1}^{h} \prod_{p_\alpha \leqslant j_\alpha < l_\alpha} (\nu + 2j_\alpha - 2\alpha + 2 - p_0 + n)\\ &\quad \times \prod_{\beta=h}^{n-1} \prod_{l_{\beta+1} < j_{\beta+1} \leqslant p_\beta} (\nu - 2j_{\beta+1} + 2\beta + 2 + p_0 - n).\end{aligned}$$

LEMMA 5.2.2. *For $h \in \{1, \cdots, n-1\}$, $\mathbf{l} \in \Lambda_n^+(\mathbf{p})$ and $\mathbf{m} \in \Lambda_{n-1}^+(\mathbf{q}|\mathbf{l})$, put*

$$\begin{aligned}&\Gamma^{(h)}_{\mathbf{l},\mathbf{m}}(\sigma_0, \sigma\,; \nu_0, \nu)\\ &= \frac{\gamma^{(h)}(\sigma_0, \sigma\,; \nu_0, \nu)}{P^{(h)}(\mathbf{l}\,; \nu) Q^{(h)}(\mathbf{m}\,; \nu_0)} \Gamma\left(\frac{\nu - \nu_0 + 1 + \kappa}{2}\right)^{-1} \Gamma\left(\frac{\nu + \nu_0 + 1 + \kappa}{2}\right)^{-1}\end{aligned}$$

with $\gamma^{(h)}(\sigma_0, \sigma\,; \nu_0, \nu)$ given by (4.2.7). Then the function $(\nu_0, \nu) \mapsto \Gamma^{(h)}_{\mathbf{l},\mathbf{m}}(\sigma_0, \sigma\,; \nu_0, \nu)$ is holomorphic on $(\nu_0, \nu) \in \mathbb{C} \times \mathfrak{D}^{(h)}_{\sigma_0,\sigma}$.

PROOF. The functions $\nu \mapsto \tilde{c}_n^{(h)}(\sigma\,; \nu) P^{(h)}(\mathbf{l}\,; \nu)^{-1}$ and $\nu_0 \mapsto d_n^{(h)}(\sigma_0, \sigma\,; \nu_0) Q^{(h)}(\mathbf{m}\,; \nu_0)^{-1}$ are holomorphic on all of $\mathbb{C}$. From Theorem 4.2.1 and this remark, the result follows. $\square$

LEMMA 5.2.3. *For $(\nu_0, \nu) \in \mathfrak{X}_n$, let $\{f_{\mathbf{l}}(\mathbf{m}\,; \nu_0, \nu\,; r) |\ \mathbf{l} \in \Lambda_n^+(\mathbf{p}), \mathbf{m} \in \Lambda_{n-1}^+(\mathbf{q}|\mathbf{l})\}$ be the standard coefficients of $\mathcal{R}^{\sigma_0,\sigma}(\nu_0, \nu)$. Then for every $\mathbf{l} \in$*

$\Lambda_n^+(\mathbf{p})$ *and* $\mathbf{m} \in \Lambda_{n-1}^+(\mathbf{q}|\mathbf{l})$, *the function* $(\nu_0, \nu, x) \mapsto \Psi_{\mathbf{l}}(\mathbf{m}\ ;\nu_0, \nu\ ; x) = (\operatorname{sh} r)^{-\beta^{(h)}(\mathbf{m})} (\operatorname{ch} r)^{-\alpha^{(h)}(\mathbf{m})} f_{\mathbf{l}}(\mathbf{m}\ ;\nu_0, \nu\ ; r)$ *with* $x = \operatorname{th}^2(r)$ *defined on* $(\mathfrak{X}_n \cap (\mathbb{C} \times \mathfrak{D}_{\sigma_0, \sigma}^{(h)})) \times (0,1)$ *is a finite* $\mathbb{C}$*-linear combination of functions of the form*

$$
\begin{aligned}
(5.2.3) \qquad & R(\nu_0, \nu) \Gamma_{\mathbf{l},\mathbf{m}}^{(h)}(\sigma_0, \sigma\ ; \nu_0, \nu) \\
& \times x^{r_1} (1-x)^{r_2} {}_2F_1\left(\frac{-\nu_0 - \nu + 1 + \kappa}{2}, \frac{\nu_0 - \nu + 1 + \kappa}{2}\ ; r_3\ ; x \right),
\end{aligned}
$$

where $R(\nu_0, \nu)$ *is a polynomial function,* $r_1, r_2, r_3 \in \mathbb{Z}$ *with* $r_3 \geqslant \kappa + 1$.

Proof. The functions $f_{\mathbf{l}}(\mathbf{m}\ ;\nu_0, \nu\ ; r)$ satisfy the system of difference-differential equations $(\mathsf{S}^{\pm k})_{\mathbf{l},\mathbf{m}}$ and $(\mathsf{T}^{\pm k})_{\mathbf{l},\mathbf{m}}$ given in [10, Theorem 7.4.1]. By changing variables, we can obtain the equations among $\Psi_{\mathbf{l}}(\mathbf{m}\ ;\nu_0, \nu\ ; x)$'s correspondingly, which we also refer to $(\mathsf{S}^{\pm k})_{\mathbf{l},\mathbf{m}}$ and $(\mathsf{T}^{\pm k})_{\mathbf{l},\mathbf{m}}$. Put $\lambda_\sigma^{(h)} = [\mathbf{l}_0\ ; l_0]$ (Definition 2.2.2). For a given $\mathbf{l} \in \Lambda_n^+(\mathbf{p})$, put $\delta(\mathbf{l}) = |\mathbf{l} - \mathbf{l}_0|$. Then $\delta(\mathbf{l})$ is a non-negative integer and it is zero if and only if $\mathbf{l} = \mathbf{l}_0$. For a given $\mathbf{l}$ and $\mathbf{m}$, let $(\mathrm{P})_{\mathbf{l},\mathbf{m}}$ be the statement that the function $(\nu_0, \nu, x) \mapsto \Psi_{\mathbf{l}}(\mathbf{m}\ ;\nu_0, \nu\ ; x)$ is a finite $\mathbb{C}$-linear combination of functions of the form (5.2.3). We prove the statement $(\mathrm{P})_{\mathbf{l},\mathbf{m}}$ by induction on the integer $\delta(\mathbf{l})$.

If $\delta(\mathbf{l}) = 0$, then we have $\mathbf{l} = \mathbf{l}_0$. Let $\mathbf{m}_0 = (m_{0,i})_{1 \leqslant i \leqslant n-1} \in \Lambda_{n-1}^+(\mathbf{q}|\mathbf{l}_0)$ be such that $\mu_{\sigma_0,\sigma}^{(h)} = [\mathbf{m}_0\ ; m_0]$ (Definition 3.1.1). Then by the explicit formula (3.2.8), the statement $(\mathrm{P})_{\mathbf{l}_0,\mathbf{m}_0}$ is true. We proceed by induction on the number $|\mathbf{m} - \mathbf{m}_0| \geqslant 0$ to prove the statement $(\mathrm{P})_{\mathbf{l}_0,\mathbf{m}}$ for $\mathbf{m} \in \Lambda_{n-1}^+(\mathbf{q}|\mathbf{l}_0)$. Assuming the validity of the statements $(\mathrm{P})_{\mathbf{l}_0,\mathbf{m}'}$ for $\mathbf{m}' \in \Lambda_{n-1}^+(\mathbf{q}|\mathbf{l}_0)$ with $0 < |\mathbf{m}' - \mathbf{m}_0| < e$, we take an $\mathbf{m} \in \Lambda_{n-1}^+(\mathbf{q}|\mathbf{l}_0)$ with $|\mathbf{m} - \mathbf{m}_0| = e$. Since $|\mathbf{m} - \mathbf{m}_0| = e > 0$, $\mathbf{m}$ is different from $\mathbf{m}_0$. Thus we have two possbilities; (a) there exists an α with $1 \leqslant \alpha < h$ such that $\mathbf{m}^{-\alpha} \in \Lambda_{n-1}^+(\mathbf{q}|\mathbf{l}_0)$, or (b) there exists an β with $h < \beta \leqslant n-1$ such that $\mathbf{m}^{+\beta} \in \Lambda_{n-1}^+(\mathbf{q}|\mathbf{l}_0)$. (Note that $\mathbf{m}_0$ is a unique element of $\Lambda_{n-1}^+(\mathbf{q}|\mathbf{l}_0)$ such that $\mathbf{m}^{-i} \notin \Lambda_{n-1}^+(\mathbf{q}|\mathbf{l}_0)$ for all $1 \leqslant i < h$ and $\mathbf{m}^{+i} \notin \Lambda_{n-1}^+(\mathbf{q}|\mathbf{l}_0)$ for all $h < i \leqslant n-1$.) By applying the equation $(\mathsf{E}_h^{+\alpha})_{\mathbf{m}^{-\alpha}}$ in case (a) and $(\mathsf{E}_h^{-\beta})_{\mathbf{l}_0,\mathbf{m}^{+\beta}}$ in case (b) with noting the statements $(\mathrm{P})_{\mathbf{l}_0,\mathbf{m}^{-\alpha}}$ and $(\mathrm{P})_{\mathbf{l}_0,\mathbf{m}^{+\beta}}$ are assumed to be true, we have that the function $(\nu_0, \nu, x) \mapsto \Psi_{\mathbf{l}_0}(\mathbf{m}\ ;\nu_0, \nu\ ; x)$ is a finite $\mathbb{C}$-linear combination of those of the form

$$\frac{\gamma^{(h)}(\sigma_0,\sigma\ ;\nu_0,\nu)R(\nu_0,\nu)}{Q(\nu_0)}\Gamma\biggl(\frac{\nu-\nu_0+1+\kappa}{2}\biggr)^{-1}\Gamma\biggl(\frac{\nu+\nu_0+1+\kappa}{2}\biggr)^{-1}$$
$$\times D_x\biggl\{x^\lambda(1-x)^\mu{}_2F_1\biggl(\frac{-\nu_0-\nu+1+\kappa}{2},\frac{\nu_0-\nu+1+\kappa}{2}\ ;\lambda'\ ;x\biggr)\biggr\},$$

where D_x is a differential operator of the form

$$\{c_1x(1-x)+c_2\}\frac{d}{dx}+(a_1x+a_2)(1-x)^\epsilon,\quad c_1,c_2,a_1,a_2\in\mathbb{C},\ \epsilon=0,1,$$

$R(\nu_0,\nu)$ is a polynomial function and $Q(\nu_0)$ represent $Q^{(h)}(\mathbf{m}^{-\alpha}\ ;\nu_0)\cdot B_\alpha^{\sigma_0,\nu_0}(\mathbf{m})$ or $Q^{(h)}(\mathbf{m}^{+\beta}\ ;\nu_0)A_\beta^{\sigma_0,\nu_0}(\mathbf{m})$ according to the case (a) or case (b). Note that $P^{(h)}(\mathbf{l}_0\ ;\nu)$ is identically 1 by definition. From Lemma 5.2.1, the statement $(\mathrm{P})_{\mathbf{l}_0,\mathbf{m}}$ follows since $Q^{(h)}(\mathbf{m};\nu_0)$ coincides with $Q(\nu_0)$ up to a non-zero constant factor. The statements $(\mathrm{P})_{\mathbf{l}_0,\mathbf{m}}$ for all $\mathbf{m}\in\Lambda_{n-1}^+(\mathbf{q}|\mathbf{l}_0)$ are now established.

Let d be a positive integer and assume that the statements $(\mathrm{P})_{\mathbf{l},\mathbf{m}}$ are true for all $\mathbf{l}\in\Lambda_n^+(\mathbf{p})$ and all $\mathbf{m}\in\Lambda_{n-1}^+(\mathbf{q}|\mathbf{l})$ with $\delta(\mathbf{l})<d$. Let $\mathbf{l}\in\Lambda_n^+(\mathbf{p})$ be any element with $\delta(\mathbf{l})=d$. Since $\delta(\mathbf{l})>0$, we have $\mathbf{l}\neq\mathbf{l}_0$, and hence have two possibilities: (a') there exists an α with $1\leqslant\alpha\leqslant h$ such that $\mathbf{l}^{-\alpha}\notin\Lambda_n^+(\mathbf{p})$, or (b') there exists a β with $h\leqslant\beta\leqslant n-1$ such that $\mathbf{l}^{+(\beta+1)}\notin\Lambda_n^+(\mathbf{p})$. We first examine the case (a'). Take an arbitrary $\mathbf{m}=(m_i)_{1\leqslant i\leqslant n-1}\in\Lambda_{n-1}^+(\mathbf{q}|\mathbf{l})$. If $l_\alpha>m_\alpha$, then we have $\mathbf{m}\in\Lambda_{n-1}^+(\mathbf{q}|\mathbf{l}^{-\alpha})$ and can apply the equation $(\mathsf{S}^{+\alpha})_{\mathbf{l}^{-\alpha},\mathbf{m}}$. In view of Lemma 5.2.1, we can get the statement $(\mathrm{P})_{\mathbf{l},\mathbf{m}}$ from $(\mathrm{P})_{\mathbf{l}^{-\alpha},\mathbf{m}}$, noting that $P^{(h)}(\mathbf{l};\nu)$ coincides with $P^{(h)}(\mathbf{l}^{-\alpha}\ ;\nu)A_\alpha^{\sigma,\nu}(\mathbf{l}^{-\alpha})$ up to a non-zero factor. If $l_\alpha=m_\alpha$, then we see easily that $\mathbf{m}^{-\alpha}\in\Lambda_{n-1}^+(\mathbf{q}|\mathbf{l}^{-\alpha})$. Hence we can apply the equation $(\mathsf{T}^{+\alpha})_{\mathbf{l}^{-\alpha},\mathbf{m}^{-\alpha}}$. By Lemma 5.2.1, we can deduce $(\mathrm{P})_{\mathbf{l},\mathbf{m}}$ from $(\mathrm{P})_{\mathbf{l}^{-\alpha},\mathbf{m}^{-\alpha}}$ with noting that $Q^{(h)}(\mathbf{m}^{-\alpha}\ ;\nu_0)=Q^{(h)}(\mathbf{m}\ ;\nu_0)$ and that $P^{(h)}(\mathbf{l}^{-\alpha}\ ;\nu)$ equals $P^{(h)}(\mathbf{l}\ ;\nu)A_\alpha^{\sigma,\nu}(\mathbf{l}^{-\alpha})$ up to a non-zero constant factor. This settles the consideration for the case (a'). Next we examine the case (b'). Take an arbitrary $\mathbf{m}\in\Lambda_{n-1}^+(\mathbf{q}|\mathbf{l})$. If $m_\beta>l_{\beta+1}$, then we clearly have $\mathbf{m}\in\Lambda_{n-1}^+(\mathbf{q}|\mathbf{l}^{+(\beta+1)})$ and can apply the equation $(\mathsf{S}^{-(\beta+1)})_{\mathbf{l}^{+(\beta+1)},\mathbf{m}}$. Noting that $P^{(h)}(\mathbf{l}\ ;\nu)$ equals $P^{(h)}(\mathbf{l}^{+(\beta+1)}\ ;\nu)B_{\beta+1}^{\sigma,\nu}(\mathbf{l}^{+(\beta+1)})$ up to a non-zero constant factor, we have $(\mathrm{P})_{\mathbf{l},\mathbf{m}}$ from $(\mathrm{P})_{\mathbf{l}^{+(\beta+1)},\mathbf{m}}$ using Lemma 5.2.1. If $m_\beta=l_{\beta+1}$, then we can easily have $\mathbf{m}^{+\beta}\in\Lambda_{n-1}^+(\mathbf{q}|\mathbf{l}^{+(\beta+1)})$. Hence we can apply the equation $(\mathsf{T}^{-(\beta+1)})_{\mathbf{l}^{+(\beta+1)},\mathbf{m}^{+\beta}}$. Noting $Q^{(h)}(\mathbf{m}^{+\beta}\ ;\nu_0) = Q^{(h)}(\mathbf{m}\ ;\nu_0)$ and that $P^{(h)}(\mathbf{l}\ ;\nu)$ equals $P^{(h)}(\mathbf{l}^{+(\beta+1)}\ ;\nu)B^{\sigma,\nu}_{\beta+1}(\mathbf{l}^{+(\beta+1)})$ up to a non-zero constant factor, we obtain the validity of the statement $(\mathrm{P})_{\mathbf{l},\mathbf{m}}$ from that of $(\mathrm{P})_{\mathbf{l}^{+(\beta+1)},\mathbf{m}^{+\beta}}$ by using Lemma 5.2.1. This completes the proof. □

We prove (1) in Theorem 5.1.1. Let $\varphi_0 \in V^{\sigma_0^\vee}_{n-1}$, $\varphi \in V^{\sigma}_{n}$, $D_0 \in U(\mathfrak{g}_{n-1,\mathbb{C}})$, $D \in U(\mathfrak{g}_{n,\mathbb{C}})$ and $g \in G_n$. We can write g as $g = k_0^{-1}\mathsf{a}_{n-1}(r_0)^{-1}\mathsf{a}_n(r)k$ with $k_0 \in K_{n-1}$, $k \in K_n$ and $r_0,\ r > 0$ by Lemma 3.2.1. For $(\nu_0,\nu) \in \mathfrak{X}_n$, we have

$$\begin{aligned}
&L_{D_0}R_D\mathcal{R}^{\sigma_0,\sigma}(\nu_0,\nu\ ;\varphi_0\otimes\varphi)(g)\\
&= \mathcal{R}^{\sigma_0,\sigma}(\nu_0,\nu\ ;\pi^{\sigma_0^\vee,-\nu_0}_{n-1}(D_0)\varphi_0\otimes\pi^{\sigma,\nu}_{n}(D)\varphi)(g)\\
&= \mathcal{R}^{\sigma_0,\sigma}(\nu_0,\nu\ ;F(\nu_0,\nu))(\mathsf{a}_{n-1}(r_0)^{-1}\mathsf{a}_n(r)).
\end{aligned}$$

with $F(\nu_0,\nu) = (\pi^{\sigma_0^\vee,-\nu_0}_{n-1}(k_0)\pi^{\sigma_0^\vee,-\nu_0}_{n-1}(D_0)\varphi_0)\otimes(\pi^{\sigma,\nu}_{n}(k)\pi^{\sigma,\nu}_{n}(D)\varphi)$. Since φ_0 is K_{n-1}-finite and φ is K_n-finite, there exists a finite family of functions $\{\psi_i\}_{i\in I}$ and $\{T_i(\nu_0,\nu)\}_{i\in I}$ such that ψ_i is of the form $\check{\iota}^{\sigma_0}_{\mathbf{m}}(w_0^\vee)\otimes\iota^{\sigma}_{\mathbf{l}}(w)$ with $\mathbf{l}\in\Lambda^+_n(\mathbf{p})$, $\mathbf{m}\in\Lambda^+_{n-1}(\mathbf{q})$, $w_0^\vee\in W(\mathbf{m})^\vee$ and $w\in W(\mathbf{l})$, such that $T_i(\nu_0,\nu)$ is a polynomial function of $\nu_0,\ \nu$ and such that

$$F(\nu_0,\nu) = \sum_{i\in I} T_i(\nu_0,\nu)\cdot\psi_i.$$

Hence we may assume that $F(\nu_0,\nu)$ is of the form $\check{\iota}^{\sigma_0}_{\mathbf{m}}(w_0^\vee)\otimes\iota^{\sigma}_{\mathbf{l}}(w)$ without loss of generality. Furthermore, in view of the formula (3.2.3), in order to show that the function

$$(\nu_0,\nu)\mapsto\mathcal{R}^{\sigma_0,\sigma}(\nu_0,\nu\ ;\check{\iota}^{\sigma_0}_{\mathbf{m}}(w_0^\vee)\otimes\iota^{\sigma}_{\mathbf{l}}(w))(\mathsf{a}_{n-1}(r_0)^{-1}\mathsf{a}_n(r))$$

is holomorphically continued to all of $\mathbb{C}\times\mathfrak{D}^{(h)}_{\sigma_0,\sigma}$, it suffices to prove the corresponding statement for $(\nu_0,\nu)\mapsto\Psi_{\mathbf{l}}(\mathbf{m}\ ;\nu_0,\nu\ ;x)$ for all $\mathbf{l}\in\Lambda^+_n(\mathbf{p})$ and all $\mathbf{m}\in\Lambda^+_{n-1}(\mathbf{q}|\mathbf{l})$. (We note that matrix coefficients of $\pi^{\sigma_0^\vee,-\nu_0}_{n-1}$ are holomorphic with respect to ν_0 on all of $\mathbb{C}$.) By lemma 5.2.3, the function $(\nu_0,\nu)\mapsto\Psi_{\mathbf{l}}(\mathbf{m}\ ;\nu_0,\nu\ ;x)$ defined for $(\nu_0,\nu,x)\in\mathfrak{X}_n\times(0,1)$, $\nu\in\mathfrak{D}^{(h)}_{\sigma_0,\sigma}$ is a finite $\mathbb{C}$-linear combination of those of the form (5.2.3). By Lemma 5.1.1 (1) and Lemma 5.2.2, the function (5.2.3) is holomorphic with respect to $(\nu_0,\nu)\in\mathbb{C}\times\mathfrak{D}^{(h)}_{\sigma_0,\sigma}$. Hence $\Psi_{\mathbf{l}}(\mathbf{m}\ ;\nu_0,\nu\ ;x)$ is also holomorphic on $(\nu_0,\nu)\in\mathbb{C}\times\mathfrak{D}^{(h)}_{\sigma_0,\sigma}$.

The first assertion of (2) is a consequence of the following lemma.

LEMMA 5.2.4. *Let $F : (\mathbb{C} \times \mathfrak{D}_{\sigma_0,\sigma}) \times G_n \to \mathbb{C}$ be a function satisfying the following:*

(a) *For any $(\nu_0, \nu) \in \mathfrak{X}_n$ the function $g \mapsto F(\nu_0, \nu, g)$ is C^∞ on G_n.*

(b) *For any $D \in U(\mathfrak{g}_{n,\mathbb{C}})$ and any $g \in G_n$ the function $(\nu_0, \nu) \mapsto R_{D,g}F(\nu_0, \nu, g)$ is holomorphic on $\mathfrak{X}_n$ and is extended holomorphically to $\mathbb{C} \times \mathfrak{D}_{\sigma_0,\sigma}$; the resulting function $R_{D,g}F(\nu_0, \nu, g)$ is locally bounded on $\mathbb{C} \times \mathfrak{D}_{\sigma_0,\sigma} \times G_n$.*

Then for any $(\nu_0, \nu) \in \mathbb{C} \times \mathfrak{D}_{\sigma_0,\sigma}$ the function $g \mapsto F(\nu_0, \nu, g)$ is C^∞ on G_n.

PROOF. Let U be the set of all $(\nu_0, \nu) \in \mathbb{C} \times \mathfrak{D}_{\sigma_0,\sigma}$ such that the function $F(x_0, x, g)$ is C^∞ with respect to $g \in G_n$ for all (x_0, x) in a neighborhood of (ν_0, ν). From the assumption, we have $\mathfrak{X}_n \subset U$. It suffices to show that U is open and closed in $\mathbb{C} \times \mathfrak{D}_{\sigma_0,\sigma}$; indeed if this is so, U must be $\mathbb{C} \times \mathfrak{D}_{\sigma_0,\sigma}$ itself because $\mathbb{C} \times \mathfrak{D}_{\sigma_0,\sigma}$ is connected. It is clear from definition that U is open. Let (ν_0, ν) be a point in $\mathbb{C} \times \mathfrak{D}_{\sigma_0,\sigma}$ lying on the closure of U. Take $\epsilon > 0$ small enough so that the polydisc centered at (ν_0, ν) with radius ϵ is in $\mathbb{C} \times \mathfrak{D}_{\sigma_0,\sigma}$. Since (ν_0, ν) is in the closure of U, there exists a point, say (z_0, z), in U such that $|\nu_0 - z_0| < 4^{-1}\epsilon$ and $|\nu - z| < 4^{-1}\epsilon$. For any $\delta > 0$ let Δ_δ be the polydisc centered at (z_0, z) with radius δ. Then Δ_δ is contained in $\mathbb{C} \times \mathfrak{D}_{\sigma_0,\sigma}$ for $\delta < 2^{-1}\epsilon$ and is in U if δ is small enough. Since $F(\nu_0, \nu, g)$ is holomorphic on $\mathbb{C} \times \mathfrak{D}_{\sigma_0,\sigma}$ by assumption (b), it is expanded as a Taylor seires at (z_0, z):

$$(5.2.6) \quad F(x_0, x, g) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\psi_{n,m}(g)}{n!m!}(x_0 - z_0)^n (x - z)^m, \quad (x_0, x) \in \Delta_\delta$$

with

$$\psi_{n,m}(g) = \frac{n!m!}{(2\pi i)^2} \int_{C_\delta(z_0)} \int_{C_\delta(z)} \frac{F(\zeta_0, \zeta, g)}{(\zeta_0 - z_0)^{n+1}(\zeta - z)^{m+1}} d\zeta_0 d\zeta.$$

Here δ is an arbitrary positive number smaller than $2^{-1}\epsilon$ and $C_\delta(z)$ denotes the path $|\zeta - z| = \delta$ with the counter-clockwise orientation. If we take the δ

small enough, then $F(\zeta_0, \zeta, g)$ is smooth with respect to g for all $(\zeta_0, \zeta) \in \Delta_\delta$; hence $\psi_{n,m}(g)$ is C^∞ on G_n and

$$(5.2.7) \quad R_D\psi_{n,m}(g) = \frac{n!m!}{(2\pi i)^2}\int_{C_\delta(z_0)}\int_{C_\delta(z)} \frac{R_{D,g}F(\zeta_0, \zeta, g)}{(\zeta_0 - z_0)^{n+1}(\zeta - z)^{m+1}} d\zeta_0 d\zeta.$$

By the assumption (b), for a given compact set W of G_n, there exists a positive C such that $|R_{D,g}F(\zeta_0, \zeta, g)| \leqslant C$ for $(\zeta_0, \zeta, g) \in \Delta_{2^{-1}\epsilon} \times W$; hence we have the estimate

$$(5.2.8) \qquad |R_D\psi_{n,m}(g)| \leqslant n!m!(2^{-1}\epsilon)^{n+m}C, \quad g \in W$$

The fact that $R_{D,g}F(x_0, x, g)$ is holomorphic on $\mathbb{C} \times \mathfrak{D}_{\sigma_0,\sigma}$, a fortiori on Δ_δ, with respect to (x_0, x), combined with the formula (5.2.7) means that the power series

$$\sum_{n,m} \frac{R_D\psi_{n,m}(g)}{n!m!}(x_0 - z_0)^n(x - z)^m$$

is convergent on Δ_δ for all $\delta < 2^{-1}\epsilon$. Furthermore, by the estimate (5.2.8), the convergence is locally uniform with respect to g. Thus $F(x_0, x, g)$ is C^∞ with respect to g for all $(x_0, x) \in \Delta_{2^{-1}\epsilon}$. Since $\Delta_{2^{-1}\epsilon}$ is a neighborhood of (ν_0, ν) in $\mathbb{C} \times \mathfrak{D}_{\sigma_0,\sigma}$, we indeed have $(\nu_0, \nu) \in U$. Thus U is closed in $\mathbb{C} \times \mathfrak{D}_{\sigma_0,\sigma}$. $\square$

We show the last part of (2). Let φ_0 and φ be as before. It suffices to prove that for every $D_0 \in \mathfrak{g}_{n-1}$, $D \in \mathfrak{g}_n$ and $g \in G_n$ the identity

$$(5.2.9) \qquad \begin{aligned} &\mathcal{R}^{\sigma_0,\sigma}(\nu_0, \nu\,; \pi_{n-1}^{\sigma_0^\vee, -\nu_0}(D_0)\varphi_0 \otimes \pi_n^{\sigma,\nu}(D)\varphi)(g) \\ &= (L_{D_0}R_D\mathcal{R}^{\sigma_0,\sigma}(\nu_0, \nu\,; \varphi_0 \otimes \varphi)(g) \end{aligned}$$

holds. If $(\nu_0, \nu) \in \mathfrak{X}_n$, then (5.2.9) is a consequence of Theorem 4.1.1 (2). Since both sides of (5.2.9) are holomorphic with respect to (ν_0, ν) on $\mathbb{C} \times \mathfrak{D}_{\sigma_0,\sigma}$ as proved above, it is true on all of $\mathbb{C} \times \mathfrak{D}_{\sigma_0,\sigma}$ by the uniqueness of the analytic continuation. Finally the formula (5.1.1) follows from (3.2.3), (3.2.8) and (4.2.7). This completes the proof of Theorem 5.1.1.

6. Multiplicity One Theorem for the Space of Shintani Functionals

In this section, we have the multiplicity formula for the space $\mathcal{I}(\pi_0|\pi)$ for some representations π_0 and π. Since we already have multiplicity freeness of the space $\mathcal{I}(\pi_0|\pi)$ in Theorem 3.1.1, we have only to construct a non-trivial element in the space $\mathcal{I}(\pi_0|\pi)$ to ensure $\dim_{\mathbb{C}}\mathcal{I}(\pi_0|\pi) = 1$. Such an element is provided by the analytic continuation of the (normalized) Poisson integral obtained in Theorem 5.1.1.

We collect notations used in this section. For given vectors $(x_1, \dots, x_m) \in \mathbb{R}^m$ and $(y_1, \dots, y_{m+1}) \in \mathbb{R}^{m+1}$,

$$(x_1, \dots, x_m) \subset (y_1, \dots, y_{m+1})$$

means that the inequality $y_1 \geqslant x_1 \geqslant y_2 \geqslant \dots \geqslant x_m \geqslant y_{m+1}$ holds. For $\mathbf{l} = (l_j)_{1 \leqslant j \leqslant n} \in \Lambda_n^+$ and $h \in \{1, \dots, n\}$, put $\mathbf{l}[h] = (l_1, \dots, \hat{l}_h, \dots, l_n)$.

6.1. Main theorem

THEOREM 6.1.1. *Let π be an irreducible $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-module and π_0 an irreducible $(\mathfrak{g}_{n-1,\mathbb{C}}, K_{n-1})$-module. We assume that there exist an integer $h \in \{1, \dots, n-1\}$, a principal series $\pi_n^{\sigma,s}$ of G_n and a principal series $\pi_{n-1}^{\sigma_0,s_0}$ of G_{n-1} satisfying the following.*

(a) $\mathrm{Hom}_{P_{n-1}\cap P_n}(\sigma_0, \sigma) \neq \{0\}$ *and* $s \in \mathfrak{D}_{\sigma_0,\sigma}$.

(b) π_0 *is isomorphic to a quotient of* $\pi_{n-1}^{\sigma_0,s_0}$ *and* π *is isomorphic to a submodule of* $\pi_n^{\sigma,s}$.

(c) *The* $K_{n-1} \times K_n$*-module* $(\tau_\mu^{n-1})^\vee \boxtimes \tau_\lambda^n$ *with* $\lambda = \lambda_\sigma^{(h)}$ *and* $\mu = \mu_{\sigma_0,\sigma}^{(h)}$ *occurs in* $\pi_0^\vee \boxtimes \pi$.

(d) *One of the following three conditions is fulfilled.*

   (i) $\tilde{c}_n^{(h)}(\sigma\ ; s) d_n^{(h)}(\sigma_0, \sigma\ ; s_0) \neq 0$.

   (ii) $\nu = s$ *is a simple zero of* $\tilde{c}_n^{(h)}(\sigma\ ; \nu)$; $\tilde{d}_n^{(h)}(\sigma_0, \sigma\ ; s) = 0$; $d_n^{(h)}(\sigma_0, \sigma\ ; s_0) \neq 0$.

   (iii) $\nu = s$ *is a simple zero of* $\tilde{c}_n^{(h)}(\sigma\ ; \nu)$; $\tilde{d}_n^{(h)}(\sigma_0, \sigma\ ; s) = 0$; $\nu_0 = s_0$ *is a simple zero of* $d_n^{(h)}(\sigma_0, \sigma\ ; s_0)$ *and* $\mathcal{I}(\pi_{n-1}^{\sigma_0,\nu_0}|\pi) = \{0\}$ *for all* $\nu_0 \neq s_0$ *in a neighborhood of* s_0.

Then $\dim_{\mathbb{C}}\mathcal{I}(\pi_0|\pi) = 1$.

Proof. We first treat the case when one of the conditions (i) and (ii) of (d) is fulfilled. By (a), we can consider $\mathcal{R}^{\sigma_0,\sigma}(\nu_0,\nu)$ at $(\nu_0,\nu)=(s_0,s)$ (Theorem 5.1.1). By the formula (5.1.2) and the assumptions (i) or (ii) of (d), the number $\Gamma^{(h)}(\sigma_0,\sigma\,;s_0,s)\neq 0$, which combined with the formula (5.1.1) means the composite of $\mathcal{R}^{\sigma_0,\sigma}(s_0,s)$ and $\check{\iota}^{\sigma_0}_{\mathbf{m}}\otimes\iota^{\sigma}_{\mathbf{l}}$ is not zero, where $\lambda^{(h)}_{\sigma}=[\mathbf{l}\,;l_0]$ and $\mu^{(h)}_{\sigma_0\sigma}=[\mathbf{m}\,;m_0]$. By (b) we have an inclusion $i:\pi_0^{\vee}\boxtimes\pi\hookrightarrow(\pi^{\sigma_0,s_0}_{n-1})^{\vee}\boxtimes\pi^{\sigma,s}_{n}$. Since $(\pi^{\sigma_0,s_0}_{n-1})^{\vee}\boxtimes\pi^{\sigma,s}_{n}$ has multiplicity free $K_{n-1}\times K_n$-spectrum, (c) means that the map $\check{\iota}^{\sigma_0}_{\mathbf{m}}\otimes\iota^{\sigma}_{\mathbf{l}}$ factors through i. Consequently we have a non-zero intertwining operator $\mathcal{R}^{\sigma_0,\sigma}(s_0,s)\circ i\in\mathcal{I}(\pi_0|\pi)$ to get the inequality $\dim_{\mathbb{C}}\mathcal{I}(\pi_0|\pi)\geqslant 1$. Using Theorem 3.1.1, we obtain $\dim_{\mathbb{C}}\mathcal{I}(\pi_0|\pi)=1$. Next we assume the condition (iii) of (d). For $\varphi_0\in V^{\sigma_0^{\vee}}_{n-1}$ and $\varphi\in V^{\sigma}_{n}$, put

$$\tilde{\mathcal{R}}(\varphi_0\otimes\varphi)=\partial_0\mathcal{R}^{\sigma_0,\sigma}(s_0,s\,;\varphi_0\otimes\varphi)$$

with ∂_0 denoting the partial derivative with respect to ν_0 at $\nu_0=s_0$. The map $\tilde{\mathcal{R}}\circ i$ belongs to $\in\mathcal{I}(\pi_0|\pi)$. Indeed, for $D_0\in U(\mathfrak{g}_{n-1,\mathbb{C}})$ and $D\in U(\mathfrak{g}_{n,\mathbb{C}})$ by differentiating the equation (5.2.9), we have

$$\begin{aligned}(6.1.1)\qquad &\tilde{\mathcal{R}}(\pi^{\sigma_0^{\vee},-s_0}_{n-1}(D_0)\varphi_0\otimes\pi^{\sigma,s}_{n}(D)\varphi)(g)\\ &\quad+\mathcal{R}^{\sigma_0,\sigma}(s_0,s\,;\partial_0\pi^{\sigma_0^{\vee},-\nu_0}_{n-1}(D_0)\varphi_0\otimes\pi^{\sigma,s}_{n}(D)\varphi)(g)\\ &=L_{D_0}R_D\tilde{\mathcal{R}}(\varphi_0\otimes\varphi)(g).\end{aligned}$$

We show that the last condition in (iii) of (d) means the second term in the left-hand side of (6.1.1) with $\varphi_0\otimes\varphi\in\mathrm{Im}(i)$ vanishes. We may assume π and $\pi_0^{\vee}$ are submodules of $\pi^{\sigma,s}_{n}$ and $\pi^{\sigma_0^{\vee},-s_0}_{n-1}$ respectively. Let V and $V_0^{\vee}$ be the corresponding subspaces of V^{σ}_{n} and $V^{\sigma_0^{\vee}}_{n-1}$ respectively. Then $\mathrm{Im}(i)=V_0^{\vee}\otimes V$. Take $\varphi_0\otimes\varphi\in\mathrm{Im}(i)$ and put $\varphi'=\pi^{\sigma,s}_{n}(D)\varphi$, $\varphi_0'=\partial_0\pi^{\sigma_0^{\vee},-\nu_0}_{n-1}(D_0)\varphi_0$. Then we have $\varphi'\in V$ and $\varphi_0'\in V^{\sigma_0^{\vee}}_{n-1}$. By the last condition of (d-iii), there is an $\epsilon>0$ such that $\mathcal{I}(\pi^{\sigma_0,\nu_0}_{n-1}|\pi)=\{0\}$ for all $0<|\nu_0-s_0|<\epsilon$. Since $\mathcal{R}^{\sigma_0,\sigma}(\nu_0,s)|V^{\sigma_0^{\vee}}_{n-1}\otimes V$ gives an element of $\mathcal{I}(\pi^{\sigma_0,\nu_0}_{n-1}|\pi)$, it should be zero if $0<|\nu_0-s_0|<\epsilon$; hence

$$\mathcal{R}^{\sigma_0,\sigma}(\nu_0,s\,;\varphi_0'\otimes\varphi')=0.$$

Let $\nu_0\to s_0$. Then this shows that the second term of the left-hand side of (6.1.1) is zero as long as $\varphi_0\otimes\varphi\in\mathrm{Im}(i)$.

By (6.1.1), combined with the claim just established, we have $\tilde{\mathcal{R}} \circ i \in \mathcal{I}(\pi_0|\pi)$. The map $\tilde{\mathcal{R}} \circ (\check{\iota}^{\sigma_0}_{\mathbf{m}} \otimes \iota^{\sigma}_{\mathbf{l}})$ is not zero. Indeed, by differentiating (5.1.1), we see that the first term of the Taylor series of $(\operatorname{sh} r)^{-\kappa}\tilde{\mathcal{R}}(\check{\iota}^{\sigma_0}_{\mathbf{m}}(w_0^\vee)\otimes \iota^{\sigma}_{\mathbf{l}}(w))(\mathsf{a}_n(r))$ with respect to $x = \operatorname{th}^2(r)$ at $x = 0$ is given by

$$\gamma^{(h)}_{\sigma_0,\sigma}\partial_0\Gamma^{(h)}(\sigma_0, \sigma\ ; s_0, s) \cdot \langle \mathsf{p}^{\mathbf{l}}_{\mathbf{m}}(w), w_0^\vee\rangle.$$

The first and the second conditions in (iii) of (d) ensure $\partial_0\Gamma^{(h)}(\sigma_0, \sigma\ ; s_0, s) \neq 0$.

By the same reasoning as above, we then have $\tilde{\mathcal{R}} \circ i \in \mathcal{I}(\pi_0|\pi)$ and $\tilde{\mathcal{R}} \circ i \neq 0$ to obtain $\dim_{\mathbb{C}}\mathcal{I}(\pi_0|\pi) = 1$. $\square$

By Theorem 6.1.1 we can determine the number $\dim_{\mathbb{C}}\mathcal{I}(\pi_0|\pi)$ explicitly in several cases.

6.2. The case of principal series

We consider the case when π_0 and π are principal series representations.

Theorem 6.2.1. *Let σ_0 (resp. σ) be an irreducible unitary representation of M_{n-1} (resp. M_n). Let $(\nu_0, \nu) \in \mathbb{C}^2$ be such that $\pi^{\sigma_0,\nu_0}_{n-1}$ and $\pi^{\sigma,\nu}_n$ are irreducible. Then we have*

$$\dim_{\mathbb{C}}\mathcal{I}(\sigma_0, \nu_0|\sigma, \nu) = \dim_{\mathbb{C}}\operatorname{Hom}_{P_{n-1}\cap P_n}(\sigma_0, \sigma).$$

Proof. Let $\sigma = \sigma^n_{(\mathbf{p};p_0)}$ and $\sigma_0 = \sigma^{n-1}_{(\mathbf{q};q_0)}$ as in Theorem 5.1.1. Then the dimension of the space $\operatorname{Hom}_{P_{n-1}\cap P_n}(\sigma_0, \sigma)$ equals 1 or 0 according to $\mathbf{q} \subset \mathbf{p}$ or not. By Proposition 3.2.1, it suffices to show $\dim_{\mathbb{C}}\mathcal{I}(\sigma_0, \nu_0|\sigma, \nu) = 1$ when $\mathbf{q} \subset \mathbf{p}$. The irreducibility of $\pi^{\sigma_0,\nu_0}_{n-1}$ and $\pi^{\sigma,\nu}_n$ implies $\tilde{c}^{(h)}_n(\sigma\ ; \nu)d^{(h)}_n(\sigma_0, \sigma\ ; \nu_0) \neq 0$ for every h (Lemma 2.2.1). By Theorem 6.1.1 we have $\dim_{\mathbb{C}}\mathcal{I}(\sigma_0, \nu_0|\sigma, \nu) = 1$. $\square$

Remark 6.2.1. Actually, even if $\pi^{\sigma_0,\nu_0}_{n-1}$ or $\pi^{\sigma,\nu}_n$ is not necessarily irreducible, we have $\dim_{\mathbb{C}}\mathcal{I}(\sigma_0, \nu_0|\sigma, \nu) = 1$ only assuming $\mathbf{q} \subset \mathbf{p}$ and $\tilde{c}^{(h)}_n(\sigma\ ; \nu)d^{(h)}_n(\sigma_0, \sigma\ ; \nu_0) \neq 0$.

6.3. The case involving discrete series and principal series

We consider the case when one of π and π_0 is not a principal series representation but a discrete series representation. We recall the Harish-Chandra parametrization of the discrete series of G_n. For each integer $h \in \{0, \cdots, n\}$, let $\Xi^n_{(h)}$ be the set of $n+1$-tuples of real numbers $\lambda = [(\lambda_j)_{1 \leqslant j \leqslant n}\,; \lambda_{n+1}]$ with $\lambda_j + 2^{-1}n \in \mathbb{Z}$, $1 \leqslant j \leqslant n+1$ such that

$$\lambda_1 > \cdots > \lambda_h > \lambda_{n+1} > \lambda_{h+1} > \cdots > \lambda_n.$$

Let Ξ^n be the union of the sets $\Xi^n_{(h)}$ for all $h \in \{0, \dots, n\}$. Then it is known that to each $\lambda = [(\lambda_j)_{1 \leqslant j \leqslant n}\,; \lambda_{n+1}] \in \Xi^n_{(h)}$ there corresponds an irreducible $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-module π^n_λ (unique up to isomorphism) satisfying the following.

(i) The Casimir element Ω_{G_n} of G_n corresponding to the G_n-invariant form $\mathrm{tr}(XY)$ of $\mathfrak{g}_n$ acts on π^n_λ by the scalar

$$\pi^n_\lambda(\Omega_{G_n}) = \sum_{j=1}^{n+1} \lambda_j^2 - \frac{n^2}{4} - \sum_{j=1}^{n} \left(\frac{n-2j}{2}\right)^2.$$

(ii) The K_n-type $\tau^n_{[\mathbf{l};l_0]}$ with $\mathbf{l} = (l_j)_{1 \leqslant j \leqslant n}$ such that

$$\begin{aligned} &l_i = \lambda_i + 2^{-1}(2i - n), \quad i \in \{1, \dots, h\}, \\ &l_j = \lambda_j + 2^{-1}(2j - 2 - n), \quad j \in \{h+1, \dots, n\}, \\ &l_0 = \lambda_{n+1} - 2^{-1}(2h - n) \end{aligned}$$

occurs in π^n_λ with multiplicity one.

The representation π^n_λ is the discrete series with Harish-Chandra parameter λ and Blattner parameter $[\mathbf{l}\,; l_0]$. (See [12, page 310, Theorem 9.20] and [5, page 57, Theorem 9.2].) The explicit description of the set $\mathcal{L}^+_n(\pi^n_\lambda)$ is useful in the following discussion. Here we recall it.

Lemma 6.3.1. *Let π^n_λ be the discrete series of G_n with Harish Chandra parameter $\lambda \in \Xi^n_{(h)}$ and the Blattner parameter $[\mathbf{l}\,; l_0]$. Then $\mathcal{L}^+_n(\pi^n_\lambda)$ consists of all the $[\mathbf{x}\,; x_0] \in \mathcal{L}^+_n$ satisfying $x_0 = l_0 + |\mathbf{l}|$ and $\mathbf{l} \subset (x_1, \dots, x_h, l_0 + 2h - n, x_{h+1}, \dots, x_n)$.*

Proof. This follows from [6, page 436, Theorem 6 (i) and page 440, (2)]. $\square$

To apply Theorem 6.1.1 we need to know into which principal series a given representation π^n_λ can be embedded.

LEMMA 6.3.2. *Let $\pi = \pi^n_\lambda$ be a discrete series representation of G_n with Harish-Chandra parameter $\lambda = [(\lambda_j)_{1\leqslant j\leqslant n}\ ;\lambda_{n+1}] \in \Xi^n_{(h)}$, $h \in \{0,\dots,n\}$ and the Blattner parameter $[\mathbf{l}\ ;l_0]$.*

(i) *Let $0 < h$. Put*

$$\mathbf{p} = \mathbf{l}[h], \quad p_0 = \lambda_h + \lambda_{n+1}, \quad s = \lambda_h - \lambda_{n+1}$$

and $\sigma = \sigma^n_{(\mathbf{p};p_0)}$. Then π is isomorphic to a submodule of $\pi^{\sigma,\nu}_n$. Moreover the j-th corner K_n-type $\lambda^{(j)}_\sigma$ of $\pi^{\sigma,\nu}_n$ belongs to $\mathcal{L}^+_n(\pi)$ if and only if $1 \leqslant j < h$. For $1 \leqslant j < h$, the function $\tilde{c}^{(j)}_n(\sigma\ ;\nu)$ has a simple zero at $\nu = s$.

(ii) *Let $h < n$. Put*

$$\mathbf{p} = \mathbf{l}[h+1], \quad p_0 = \lambda_{n+1} + \lambda_{h+1}, \quad s = \lambda_{n+1} - \lambda_{h+1}$$

and $\sigma = \sigma^n_{(\mathbf{p};p_0)}$. Then π is isomorphic to a submodule of $\pi^{\sigma,\nu}_n$. Moreover the j-th corner K_n-type $\lambda^{(j)}_\sigma$ of $\pi^{\sigma,\nu}_n$ belongs to $\mathcal{L}^+_n(\pi)$ if and only if $h < j \leqslant n-1$. For $h < j \leqslant n-1$, the function $\tilde{c}^{(j)}_n(\sigma\ ;\nu)$ has a simple zero at $\nu = s$.

PROOF. The first assertions in (1) and (2) follow from [6, page 445, Proposition 3]. The remaining parts follow from definitions immediately. □

The following auxiliary lemma is logically unnecessary but is helpful in practice when one checks whether the polynomial function (4.2.8) or (4.2.9) has a zero at a certain point.

LEMMA 6.3.3. *Let $\mathbf{l} = (l_j)_{1\leqslant j\leqslant n} \in \Lambda^+_n$, $\mathbf{m} = (m_i)_{1\leqslant i\leqslant n-1} \in \Lambda^+_{n-1}$, $\mathbf{p} = (p_j)_{1\leqslant j\leqslant n-1} \in \Lambda^+_{n-1}$, $\mathbf{q} = (q_i)_{1\leqslant i\leqslant n-2} \in \Lambda^+_{n-2}$ and $l_0,\ m_0 \in \mathbb{Z}$. Let $j \in \{1,\dots,n-1\}$, $h \in \{0,\dots,n-1\}$ and $k \in \{0,\dots,n\}$. Put $\tilde{l}_0 = l_0+2k-n$, $\tilde{m}_0 = m_0 + 2h - n + 1$, $\sigma_0 = \sigma^{n-1}_{(\mathbf{q};q_0)}$ and $\sigma = \sigma^n_{(\mathbf{p};p_0)}$.*

(i) *Let* $0 < k$ *and* $\mathbf{q} \subset \mathbf{l}[k] = (p'_i)_{1 \leqslant i \leqslant n-1}$. *Put* $\sigma' = \sigma^n_{(\mathbf{l}[k];l_0+l_k)}$. *Then* $\tilde{d}^{(j)}_n(\sigma_0, \sigma' \,; l_k - \tilde{l}_0) = 0$ *if and only if* $q_\alpha \leqslant l_0 + k - n < p'_\alpha$ *with an* $\alpha \in \{1, \ldots, j-1\}$ *or* $p'_\beta < l_k - k + \beta \leqslant q_{\beta-1}$ *with a* $\beta \in \{j+1, \ldots, n-1\}$.

(ii) *Let* $k < n$ *and* $\mathbf{q} \subset \mathbf{l}[k+1] = (p''_i)_{1 \leqslant i \leqslant n-1}$. *Put* $\sigma'' = \sigma^n_{(\mathbf{l}[k+1];l_{k+1}+l_0)}$. *Then* $\tilde{d}^{(j)}_n(\sigma_0, \sigma'' \,; \tilde{l}_0 - l_{k+1}) = 0$ *if and only if* $q_\alpha \leqslant l_{k+1} - k + \alpha < p'_\alpha$ *with an* $\alpha \in \{1, \ldots, j-1\}$ *or* $p'_\beta < l_0 + k - n + \beta \leqslant q_{\beta-1}$ *with a* $\beta \in \{j+1, \ldots, n-1\}$.

(iii) *Let* $0 < h$ *and* $\mathbf{m}[h] = (q'_i)_{1 \leqslant i \leqslant n-2} \subset \mathbf{p}$. *Put* $\sigma'_0 = \sigma^{n-1}_{(\mathbf{m}[h];m_0+m_h)}$. *Then* $d^{(j)}_n(\sigma'_0, \sigma \,; \tilde{m}_0 - m_h) = 0$ *if and only if* $q'_\alpha < m_0 - (n-1) + h + \alpha \leqslant p_\alpha$ *with an* $\alpha \in \{1, \ldots, j-1\}$ *or* $p_\beta \leqslant m_h - h - 1 - \beta < q'_{\beta-1}$ *with a* $\beta \in \{j+1, \ldots, n-1\}$.

(iv) *Let* $h < n-1$ *and* $\mathbf{m}[h+1] = (q''_i)_{1 \leqslant i \leqslant n-1} \subset \mathbf{p}$. *Put* $\sigma''_0 = \sigma^{n-1}_{(\mathbf{m}[h+1];m_{h+1}+m_0)}$. *Then* $d^{(j)}_n(\sigma''_0, \sigma \,; m_{h+1} - \tilde{m}_0) = 0$ *if and only if* $q''_\alpha < m_{h+1} - h + \alpha \leqslant p_\alpha$ *with an* $\alpha \in \{1, \ldots, j-1\}$ *or* $p_\beta \leqslant m_0 + h - \beta - n < q''_{\beta-1}$ *with a* $\beta \in \{j+1, \ldots, n-1\}$.

Proof. This follows from (4.2.8) and (4.2.9) immediately. □

We first consider the case when π is a discrete series.

Theorem 6.3.1. *Let* $\pi = \pi^n_\lambda$ *with* $\lambda \in \Xi^n_{(h)}$ *be a discrete series representation of* G_n *and* $\pi_0 = \pi^{\sigma_0,\nu_0}_{n-1}$ *an irreducible principal series of* G_{n-1} *with* $\sigma_0 = \sigma^{n-1}_{(\mathbf{q};q_0)}$, $\mathbf{q} = (q_j)_{1 \leqslant j \leqslant n-2} \in \Lambda^+_{n-2}$. *Let* $[\mathbf{l}\,; l_0]$ *be the Blattner parameter of* π^n_λ.

(1) *Assume* $h = 0$ *or* $h = n$. *Then we have* $\mathcal{I}(\pi_0|\pi) = \{0\}$.

(2) *Assume* $0 < h < n$. *Then we have* $\dim_{\mathbb{C}} \mathcal{I}(\pi_0|\pi) = 1$ *if and only if the condition*

$$(6.3.1) \qquad (q_1, \ldots, q_{h-1}, l_0 + 2h - n, q_h, \ldots, q_{n-2}) \subset \mathbf{l}$$

holds, otherwise $\mathcal{I}(\pi_0|\pi) = \{0\}$.

Proof. Let η be the $(\mathfrak{h}_{n,\mathbb{C}}, K_n \cap H_n)$-module such that $\eta|Z_n = \pi^n_\lambda|Z_n$ with $H_n = Z_n G_{n-1}$. Then by Theorem 7.1.1, we have $\dim_{\mathbb{C}} \mathcal{I}(\pi_0|\pi) = \dim_{\mathbb{C}} \mathcal{I}_{\eta,\pi}$. We have $m^+_1(\eta) = +\infty$, $m^+_j(\eta) = q_{j-1}$ for $j \in \{2, \ldots, n-1\}$ and $m^-_{n-1}(\eta) = -\infty$, $m^-_i(\eta) = q_i$ for $i \in \{1, \ldots, n-2\}$ in the notation of [10, 8.1]. When $h = 0$, the condition (c) of [10, Theorem 8.1.1] is not satisfied because $m^+_1(\eta) = +\infty$. Hence $\dim_{\mathbb{C}} \mathcal{I}_{\eta,\pi} = 0$ by [10, Theorem 8.1.1]. In the same way we have $\dim_{\mathbb{C}} \mathcal{I}_{\eta,\pi} = 0$ when $h = n$. We consider the case $0 < h < n$. We show that $\dim_{\mathbb{C}} \mathcal{I}(\pi_0|\pi) = 1$ assuming the condition (6.3.1). We take $\sigma = \sigma^n_{(\mathbf{p};p_0)}$ as in (i) of Lemma 6.3.2 if $1 < h$ and as in (ii) of Lemma 6.3.2 if $h < n-1$. In either cases we have $\mathbf{q} \subset \mathbf{p}$. By Lemma 6.3.2, π becomes a $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-submodule of the principal series $\pi^{\sigma,s}_n$. When $1 < h$, it turns out that $\lambda^{(h-1)}_\sigma \in \mathcal{L}^+_n(\pi)$, $\tilde{d}^{(h-1)}_n(\sigma_0, \sigma\ ; s) = 0$, $\tilde{d}^{(h)}_n(\sigma_0, \sigma\ ; s) \neq 0$ and that the function $\tilde{c}^{(h-1)}_n(\sigma\ ; \nu)$ has a simple zero at $\nu = s$. When $h < n-1$, it turns out that $\lambda^{(h+1)}_\sigma \in \mathcal{L}^+_n(\pi)$, $\tilde{d}^{(h+1)}_n(\sigma_0, \sigma\ ; s) = 0$, $\tilde{d}^{(h)}_n(\sigma_0, \sigma\ ; s) \neq 0$ and that the function $\tilde{c}^{(h+1)}_n(\sigma\ ; \nu)$ has a simple zero at $\nu = s$. Since $\pi_0 = \pi^{\sigma_0,\nu_0}_{n-1}$ is irreducible, we have $d^{(j)}_n(\sigma_0, \sigma\ ; \nu_0) \neq 0$ for all $j \in \{1, \ldots, n-1\}$. Thus all the assumptions in Theorem 6.1.1 are fulfilled. Hence we have $\dim_{\mathbb{C}} \mathcal{I}(\pi_0|\pi) = 1$. Conversely if $\mathcal{I}(\pi_0|\pi) \neq \{0\}$ or equivalently $\dim_{\mathbb{C}} \mathcal{I}_{\eta,\pi} = 1$ by Theorem 7.1.1, then [10, Theorem 8.1.1] (with $\mathbf{l}$ as above) yields the inequality (6.3.1). $\square$

Next we consider the case when π_0 is a discrete series.

Theorem 6.3.2. *Let $\pi_0 = \pi^{n-1}_\mu$ be a discrete series representation of G_{n-1} with Harish-Chandra parameter $\mu \in \Xi^{n-1}_{(h)}$ and $\pi = \pi^{\sigma,\nu}_n$ be an irreducible principal series representation of G_n. Let $[\mathbf{m}\ ; m_0]$ be the Blattner parameter of π_0 and $\sigma = \sigma^n_{(\mathbf{p};p_0)}$ with $\mathbf{p} \in \Lambda^+_{n-1}$. Put $h^+ = \sup(1, h)$. Then $\dim_{\mathbb{C}} \mathcal{I}(\pi_0|\pi) = 1$ if and only if $\mathbf{m}[h^+] \subset \mathbf{p}$, otherwise $\mathcal{I}(\pi_0|\pi) = \{0\}$.*

Proof. Assume the condition $\mathbf{m}[h^+] \subset \mathbf{p}$ holds. We consider the case $0 < h < n-1$. Let $\sigma_0 = \sigma^{n-1}_{(\mathbf{m}[h];\mu_n+\mu_h)}$ and $s_0 = \mu_n - \mu_h$. Then from $\mathbf{m}[h^+] \subset \mathbf{p}$ we have $\mathbf{q} \subset \mathbf{p}$. Since π is irreducible, the number $\tilde{c}^{(j)}_n(\sigma\ ; \nu)$ is not zero for all j. Hence the condition (a) of Theorem 6.1.1 is satisfied. By Lemma 6.3.2 (1), $\pi_0^\vee$ is embedded into $(\pi^{\sigma_0,s_0}_{n-1})^\vee$. Using Lemma 6.3.1 we can check $\mu^{(h)}_{\sigma_0,\sigma} \in \mathcal{L}^+_{n-1}(\pi_0)$ to know (c) of Theorem 6.1.1 is fulfilled. With the aid of Lemma 6.3.3 we see that the condition $\mathbf{m}[h^+] \subset \mathbf{p}$ implies $d^{(h)}_n(\sigma_0, \sigma\ ; s_0) \neq 0$. Since $\tilde{c}^{(h)}_n(\sigma\ ; \nu) \neq 0$ as noticed above, the condition (d-i) of Theorem 6.1.1 is attained. Now we apply Theorem 6.1.1 to obtain $\dim_{\mathbb{C}}\mathcal{I}(\pi_0|\pi)=1$. Conversely, if we assume $\dim_{\mathbb{C}}\mathcal{I}(\pi_0|\pi)\neq 0$, then we have $\mathcal{I}_{\eta,\pi}\neq\{0\}$ by Theorem 7.1.1. Hence by applying [10, Theorem 8.1.1] (with $\lambda=\lambda_\sigma^{(n-1)}$), we obtain the condition $\mathbf{m}[h^+]\subset\mathbf{p}$. The cases $h=0,n-1$ can be treated similarly. □

6.4. The case of discrete series representation

In this subsection we treat the case when both of π_0 and π are large discrete series. Let $\pi_0=\pi_\mu^{n-1}$ with $\mu=[(\mu_j)_{1\leqslant j\leqslant n-1}\ ;\mu_n]\in\Xi_{(h)}^{n-1}$, $h\in\{0,\dots,n-1\}$ be a discrete series representation of G_{n-1} and $\pi=\pi_\lambda^n$ with $\lambda=[(\lambda_i)_{1\leqslant i\leqslant n}\ ;\lambda_{n+1}]\in\Xi_{(k)}^n$, $k\in\{0,\cdots,n\}$ be a discrete series representation of G_n. Let $[\mathbf{m}\ ;m_0]\in\mathcal{L}_{n-1}^+(\pi_0)$ and $[\mathbf{l}\ ;l_0]\in\mathcal{L}_n^+(\pi)$ be the Blattner parameters of π_0 and π respectively. Put $\mathbf{m}=(m_j)_{1\leqslant j\leqslant n-1}$ and $\mathbf{l}=(l_i)_{1\leqslant i\leqslant n}$.

Theorem 6.4.1. *Let π_0 and π be as above. We assume $0<h<n-1$ and $0<k<n$. Then $\dim_{\mathbb{C}}\mathcal{I}(\pi_0|\pi)=1$ if and only if one of the following conditions is satisfied, otherwise it is zero.*

(i) $k<h$ *and*

$$(\mu_1,\dots,\mu_{k-1},\lambda_{n+1},\mu_k,\dots,\mu_{n-1})\subset(\lambda_1,\dots,\lambda_{h+1},\mu_n,\lambda_{h+2},\dots,\lambda_n).$$

(ii) $k=h$, *and*

(a)
$$(\mu_1,\dots,\mu_{h-1},\lambda_{n+1},\mu_h,\dots,\mu_{n-1})\subset(\lambda_1,\dots,\lambda_{h+1},\mu_n,\lambda_{h+2},\dots,\lambda_n)$$

or

(b)
$$(\mu_1,\dots,\mu_h,\lambda_{n+1},\mu_{h+1},\dots,\mu_{n-1})\subset(\lambda_1,\dots,\lambda_h,\mu_n,\lambda_{h+1},\dots,\lambda_n).$$

(iii) $k=h+1$, *and*

(a)
$$(\mu_1,\dots,\mu_h,\lambda_{n+1},\mu_{h+1},\dots,\mu_{n-1})\subset(\lambda_1,\dots,\lambda_{h+1},\mu_n,\lambda_{h+2},\dots,\lambda_n)$$

or

(b)
$$(\mu_1,\dots,\mu_{h+1},\lambda_{n+1},\mu_{h+2},\dots,\mu_{n-1})\subset(\lambda_1,\dots,\lambda_h,\mu_n,\lambda_{h+1},\dots,\lambda_n).$$

(iv) $h+1<k$ *and*

$$(\mu_1,\dots,\mu_k,\lambda_{n+1},\mu_{k+1},\dots,\mu_{n-1}) \subset (\lambda_1,\dots,\lambda_h,\mu_n,\lambda_{h+1},\dots,\lambda_n).$$

PROOF. We show that $\mathcal{I}(\pi_0|\pi) \neq \{0\}$ assuming one of the conditions (i) to (iv). We take σ_0, σ, s_0 and s in each cases:

(i) $\sigma_0 = \sigma^{n-1}_{(\mathbf{m}[h+1];\mu_{h+1}+\mu_n)}$, $s_0 = \mu_{h+1}-\mu_n$; $\quad \sigma = \sigma^{n}_{(\mathbf{l}[k+1];\lambda_{n+1}+\lambda_{k+1})}$, $s = \lambda_{n+1}-\lambda_{k+1}$.

(ii) $\sigma_0 = \sigma^{n-1}_{(\mathbf{m}[h+1];\mu_{h+1}+\mu_n)}$, $s_0 = \mu_{h+1}-\mu_n$; $\quad \sigma = \sigma^{n}_{(\mathbf{l}[h+1];\lambda_{n+1}+\lambda_{h+1})}$, $s = \lambda_{n+1}-\lambda_{h+1}$.

(iii) $\sigma_0 = \sigma^{n-1}_{(\mathbf{m}[h];\mu_n+\mu_h)}$, $s_0 = \mu_n-\mu_h$; $\quad \sigma = \sigma^{n}_{(\mathbf{l}[h+1];\lambda_{n+1}+\lambda_{h+1})}$, $s = \lambda_{n+1}-\lambda_{h+1}$.

(iv) $\sigma_0 = \sigma^{n-1}_{(\mathbf{m}[h];\mu_n+\mu_h)}$, $s_0 = \mu_n-\mu_h$; $\quad \sigma = \sigma^{n}_{(\mathbf{l}[k];\lambda_k+\lambda_{n+1})}$, $s = \lambda_k-\lambda_{n+1}$.

In each cases it turns out that the first condition in (a) of Theorem 6.1.1 is satisfied. With the aid of Lemma 6.3.1 we can confirm that the $K_{n-1}\times K_n$-module $(\tau^{n-1}_\mu)^\vee \boxtimes \tau^n_\lambda$ occurs in $(\pi_0)^\vee \boxtimes \pi$ with $\mu = \mu^{(h+1)}_{\sigma_0,\sigma}$, $\lambda = \lambda^{(h+1)}_\sigma$ in case (i) or (ii) and with $\mu = \mu^{(h)}_{\sigma_0,\sigma}$, $\lambda = \lambda^{(h)}_\sigma$ in case (iii) or (iv); hence the condition (c) of Theorem 6.1.1 is fulfilled. We consider the case (i) or (ii-a). By using Lemma 6.3.3 we can confirm $d^{(h+1)}_n(\sigma_0,\sigma\ ;s_0) \neq 0$, $\tilde{d}^{(h+1)}_n(\sigma_0,\sigma\ ;s) = 0$. This combined with Lemma 6.3.2 (ii) shows the condition (d-ii) of Theorem 6.1.1 is satisfied. Applying Theorem 6.1.1 we have $\dim_{\mathbb{C}}\mathcal{I}(\pi_0|\pi) = 1$.

In the case (iv) or (iii-a), we also have $\dim_{\mathbb{C}}\mathcal{I}(\pi_0|\pi) = 1$ by the same reasoning as in the case (i) or (ii-a) but with $h+1$ replaced by h in the discussion.

We consider the case (ii-b). We can see that $\nu_0 = s_0$ is a simple zero of $d^{(h+1)}(\sigma_0,\sigma\ ;\nu)$ and $\tilde{d}^{(h+1)}_n(\sigma_0,\sigma\ ;s) = 0$ with the aid of Lemma 6.3.3. This combined with Lemma 6.3.2 (ii) shows that the first three conditions in (d-iii) of Theorem 6.1.1 is satisfied. The last condition in (d-iii) of Theorem 6.1.1 follows from Theorem 6.3.1 (2). Indeed, the condition (6.3.1) is not attained by our choice of $\mathbf{q}$. Since $\pi^{\sigma_0,\nu_0}_{n-1}$ is irreducible for all $\nu_0 \neq s_0$ in a neighborhood of s_0 we have $\mathcal{I}(\pi^{\sigma_0,\nu_0}_{n-1}|\pi) = \{0\}$ for such a ν_0. Now we apply Theorem 6.1.1 to obtain $\dim_{\mathbb{C}}\mathcal{I}(\pi_0|\pi) = 1$.

In the case (iii-b), by the same reasoning as in the case (ii-b) but with $h+1$ replaced by h in the discussion, we also have $\dim_{\mathbb{C}} \mathcal{I}(\pi_0|\pi) = 1$.

To complete the proof, we have to show that $\dim_{\mathbb{C}} \mathcal{I}(\pi_0|\pi) \neq 0$ implies one of the conditions (i) to (iv). Since $\dim_{\mathbb{C}} \mathcal{I}(\pi_0|\pi) \neq 0$ means $\dim_{\mathbb{C}} \mathcal{I}_{\eta,\pi} \neq 0$ by Theorem 7.1.1, we apply [10, Theorem 8.3.1] to get the conditions (i) to (vi). □

7. Frobenius Reciprocity

The aim of this section is to prove Theorem 7.1.1, which is used in the proof of Theorem 3.1.1 and Lemma 3.2.1.

7.1. Formulation of the theorem

Let (π_0, V_0) be an admissible $(\mathfrak{g}_{n-1,\mathbb{C}}, K_{n-1})$-module and (π, V) an admissible $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-module. We assume that both of them have central characters with $c_{n-1}(\pi_0) = c_0$ and $c_n(\pi) = c$ ([10, 2.3]). Put $H_n = Z_n G_{n-1}$. Given an admissible $(\mathfrak{g}_{n-1,\mathbb{C}}, K_{n-1})$-module π_0 we can extend π_0 to a $(\mathfrak{h}_{n,\mathbb{C}}, K_n \cap H_n)$-module uniquely so that the extended representation η satisfies that $\eta|Z_n$ and $\pi|Z_n$ correspond to the same character of Z_n. Then we can consider the intertwining space $\mathcal{I}_{\eta,\pi} = \mathrm{Hom}_{(\mathfrak{g}_{n,\mathbb{C}}, K_n)}(\pi, \mathrm{Ind}_{H_n}^{G_n}(\eta))$ as in [10, 2.4].

Theorem 7.1.1. *Let π_0, η and π be as above. Then there exists a unique linear bijection $\mathcal{P} \mapsto \Phi_{\mathcal{P}}$ from $\mathcal{I}(\pi_0|\pi)$ to $\mathcal{I}_{\eta,\pi}$ such that*

$$\mathcal{P}(v_0^\vee \otimes v)(g) = \langle \Phi_{\mathcal{P}}(v)(g), v_0^\vee \rangle, \quad v_0^\vee \in V_0^\vee,\ v \in V,\ g \in G_n. \tag{7.1.1}$$

Before we begin the proof of Theorem 7.1.1, it may be useful to give a main point of the discussion, because the argument is technically a little complicated. To compare two spaces $\mathcal{I}(\pi_0|\pi)$ and $\mathcal{I}_{\eta,\pi}$, we first introduce an auxiliary $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-module ρ^η, whose nature is best understood if we write it as $\mathrm{Hom}_{(\mathfrak{h}_{n,\mathbb{C}}, K_n \cap H_n)}(V_0^\vee, C^\infty(G_n))$. It is natural to expect that ρ^η is 'the same' as the induced module $\mathrm{Ind}_{H_n}^{G_n}(\eta)$. Actually to make precise this statement and to establish it (Theorem 7.1.2) is the main point of the whole argument. Indeed, if $\rho^\eta \cong \mathrm{Ind}_{H_n}^{G_n}(\eta)$ is true, then the identification

$$\mathcal{I}(\pi_0|\pi) \cong \mathrm{Hom}_{(\mathfrak{g}_{n,\mathbb{C}}, K_n)}(\pi, \mathrm{Hom}_{(\mathfrak{h}_{n,\mathbb{C}}, K_n \cap H_n)}(V_0^\vee, C^\infty(G_n))) \cong \mathcal{I}_{\eta,\pi}$$

is valid and Theorem 7.1.1 follows.

To formulate precisely the statement stated above we need some notations. Let $\mathcal{F} = V_0^\infty$ and $\mathcal{F}^\vee = (V_0^\vee)^\infty$ be the smooth Fréchet H_n-modules that are the canonical globalizations of η and $\eta^\vee$ respectively, and $\langle \ , \rangle : \mathcal{F} \times \mathcal{F}^\vee \to \mathbb{C}$ the canonical H_n-invariant pairing ([1]). The actions of H_n on $\mathcal{F}$ and on $\mathcal{F}^\vee$ are also denoted by η and $\eta^\vee$ respectively. Let $\mathcal{H}^\eta$ be the space of all maps $\psi : V_0^\vee \times G_n \to \mathbb{C}$ such that

(a) $\psi(v^\vee, g)$ is linear with respect to $v^\vee \in V_0^\vee$ for a fixed g, and is C^∞ with respect to $g \in G_n$ for a fixed $v^\vee$;

(b) for $k_0 \in K_n \cap H_n$, $X_0 \in \mathfrak{h}_n$, $v^\vee \in V_0^\vee$ and $g \in G_n$, we have

$$\psi(\eta^\vee(X_0)v^\vee, g) = L_{X_0,g}\psi(v^\vee, g), \quad \psi(\eta^\vee(k_0)v^\vee, g) = \psi(v^\vee, k_0^{-1}g);$$

(c) for $v^\vee \in V_0^\vee$ the function $g \mapsto \psi(v^\vee, g)$ is right K_n-finite.

By defining the operators $\rho^\eta(X)$ and $\rho^\eta(k)$ with $X \in \mathfrak{g}_n$ and $k \in K_n$ as

$$(\rho^\eta(X)\psi)(v^\vee, g) = R_{X,g}\psi(v^\vee, g), \quad (\rho^\eta(k)\psi)(v^\vee, g) = \psi(v^\vee, gk)$$

for $(v^\vee, g) \in V_0^\vee \times G_n$, we have a $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-module $(\rho^\eta, \mathcal{H}^\eta)$. Let $C_\eta^\infty(H_n\backslash G_n)^0$ be the space of right K_n-finite functions $F : G_n \to \mathcal{F}$ such that $F(hg) = \eta(h)F(g)$ for all $(h, g) \in H_n \times G_n$. By the right translation we have the induced representation $\mathrm{Ind}_{H_n}^{G_n}(\eta)$ on that space ([10, 2.4]).

Theorem 7.1.2. *There exists a unique linear map $F \mapsto \psi_F$ from $C_\eta^\infty(H_n\backslash G_n)^0$ to $\mathcal{H}^\eta$ such that*

$$\psi_F(v^\vee, g) = \langle v^\vee, F(g)\rangle, \quad v^\vee \in V_0^\vee,\ g \in G_n, \tag{7.1.2}$$

that gives a $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-isomorphism $\mathrm{Ind}_{H_n}^{G_n}(\eta) \cong \rho^\eta$.

Now we give the proof of Theorem 7.1.1 using Theorem 7.1.2 whose proof is given in the next subsection. Let $\mathcal{P} \in \mathcal{I}(\pi_0|\pi)$. Given a vector $v \in V$, we have an element $\psi_v \in \mathcal{H}^\eta$ such that

$$\psi_v(v^\vee, g) = \mathcal{P}(v^\vee \otimes v)(g), \quad (v^\vee, g) \in V_0^\vee \times G_n.$$

By Theorem 7.1.2, we have a function $\Phi_{\mathcal{P}}(v) \in C_\eta^\infty(H_n\backslash G_n)^0$ such that

$$\psi_v(v^\vee, g) = \langle v^\vee, \Phi_{\mathcal{P}}(v)(g)\rangle, \quad v^\vee \in V_0^\vee,\ g \in G_n.$$

By a routine argument, we can prove that the map $\Phi_{\mathcal{P}} : V \to C_\eta^\infty(H_n\backslash G_n)^0$ so obtained is a $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-homomorphism satisfying the formula (7.1.1). Conversely given $\Phi \in \mathcal{I}_{\eta,\pi}$, the formula (7.1.1) with $\Phi_{\mathcal{P}}$ replaced by Φ defines the map $\mathcal{P} : V_0^\vee \otimes V \to C^\infty(G_n)$, which turns out to be in $\mathcal{I}(\pi_0|\pi)$. The map $\mathcal{I}_{\eta,\pi} \to \mathcal{I}(\pi_0|\pi)$ so defined provides the inverse map of $\mathcal{P} \mapsto \Phi_{\mathcal{P}}$. This completes the proof of Theorem 7.1.1.

7.2. Proof of Theorem 7.1.2

Let $\mathfrak{p}_n$ (resp. $\mathfrak{q}_n$) be the orthogonal complement of $\mathfrak{k}_n$ (resp. $\mathfrak{h}_n$) in $\mathfrak{g}_n$ with respect to the Killing form of $\mathfrak{g}_n$.

For $\lambda \in \mathcal{L}_n^+$, let $C_{\eta,\tau_\lambda}^\infty(\mathfrak{p}_n \cap \mathfrak{q}_n)$ be the space of all C^∞-functions $\varphi : \mathfrak{p}_n \cap \mathfrak{q}_n \to \mathrm{Hom}(W_\lambda, \mathcal{F})$ such that

$$\varphi(\mathrm{Ad}(k_0)X) = \eta(k_0) \circ \varphi(X) \circ \tau_\lambda(k_0^{-1}), \quad k_0 \in K_n \cap H_n. \tag{7.2.1}$$

The space $C_{\eta,\tau_\lambda}^\infty(H_n\backslash G_n/K_n) = \mathrm{Hom}_{K_n}(W_\lambda, C_\eta^\infty(H_n\backslash G_n))$ consists of all C^∞-functions $F : G_n \to \mathrm{Hom}(W_\lambda, \mathcal{F})$ such that $F(hgk) = \eta(h)\circ F(g)\circ\tau_\lambda(k)$ for $(h, g, k) \in H_n \times G_n \times K_n$.

Lemma 7.2.1. *The map*

$$C_{\eta,\tau_\lambda}^\infty(H_n\backslash G_n/K_n) \to C_{\eta,\tau_\lambda}^\infty(\mathfrak{p}_n \cap \mathfrak{q}_n)$$

which sends a fuction $F \in C_{\eta,\tau_\lambda}^\infty(H_n\backslash G_n/K_n)$ to the function $\varphi = (F \circ \exp)|(\mathfrak{p}_n \cap \mathfrak{q}_n)$ is a linear bijection.

Proof. It is known that the map $(Y, X, k) \mapsto \exp(Y)\exp(X)k$ is a diffeomorphism from $(\mathfrak{h}_n \cap \mathfrak{p}_n) \times (\mathfrak{p}_n \cap \mathfrak{q}_n) \times K_n$ onto G_n ([3, page 106, Proposition 2.2]). Hence, given $\varphi \in C_{\eta,\tau_\lambda}^\infty(\mathfrak{p}_n \cap \mathfrak{q}_n)$, if we put

$$F_\varphi(\exp(Y)\exp(X)k) = \eta(\exp(Y)) \circ \varphi(X) \circ \tau_\lambda(k),$$
$$(Y, X, k) \in (\mathfrak{h}_n \cap \mathfrak{p}_n) \times (\mathfrak{p}_n \cap \mathfrak{q}_n) \times K_n,$$

then we have a C^∞-function $F_\varphi : G_n \to \mathrm{Hom}(W_\lambda, \mathcal{F})$. Let $g \in G_n$, $k \in K_n$, $h \in H_n$. By the decomposition $G_n = \exp(\mathfrak{h}_n\cap\mathfrak{p}_n)\exp(\mathfrak{p}_n\cap\mathfrak{q}_n)K_n$ mentioned above and the Cartan decomposition $H_n = \exp(\mathfrak{h}_n \cap \mathfrak{p}_n)(K_n \cap H_n)$, we can write g and h as $g = \exp(Y)\exp(X)k_1$ and $h = \exp(Y_0)k_0$ with $Y,\ Y_0 \in \mathfrak{h}_n \cap \mathfrak{p}_n$, $X \in \mathfrak{p}_n \cap \mathfrak{q}_n$, $k_0 \in K_n \cap H_n$ and $k_1 \in K_n$. Using the Cartan decomposition for H_n again, we have $\exp(Y_0)\exp(\mathrm{Ad}(k_0)Y) = \exp(Y_0')k_0'$ with $k_0' \in K_n \cap H_n$ and $Y_0' \in \mathfrak{h}_n \cap \mathfrak{p}_n$. Then since $hgk = \exp(Y_0')\exp(\mathrm{Ad}(k_0'k_0)X)k_0'k_0k_1k$, using (7.2.1), we have

$$\begin{aligned}
F_\varphi(hgk) &= \eta(\exp(Y_0')) \circ \varphi(\mathrm{Ad}(k_0'k_0)X) \circ \tau_\lambda(k_0'k_0k_1k) \\
&= \eta(\exp(Y_0)\exp(\mathrm{Ad}(k_0)Y){k_0'}^{-1}) \circ \eta(k_0') \circ \varphi(\mathrm{Ad}(k_0)X) \\
&\quad \circ \tau_\lambda(k_0')^{-1} \circ \tau_\lambda(k_0'k_0k_1k) \\
&= \eta(\exp(Y_0)\exp(\mathrm{Ad}(k_0)Y)) \circ \eta(k_0) \circ \varphi(X) \circ \tau_\lambda(k_0^{-1}) \circ \tau_\lambda(k_0k_1k) \\
&= \eta(\exp(Y_0)k_0\exp(Y)) \circ \varphi(X) \circ \tau_\lambda(k_1k) \\
&= \eta(h) \circ \eta(\exp(Y)) \circ \varphi(X) \circ \tau_\lambda(k_1) \circ \tau_\lambda(k) \\
&= \eta(h) \circ F_\varphi(g) \circ \tau_\lambda(k).
\end{aligned}$$

Thus $F_\varphi \in C^\infty_{\eta,\tau_\lambda}(H_n\backslash G_n/K_n)$. It is easy to check that the map $\varphi \mapsto F_\varphi$ just constructed gives the inverse map of $F \mapsto (F \circ \exp)|(\mathfrak{p}_n \cap \mathfrak{q}_n)$. $\square$

Let $\lambda \in \mathcal{L}_n^+$. The space $\mathcal{H}^\eta[\lambda] = \mathrm{Hom}_{K_n}(\tau_\lambda, \rho^\eta|K_n)$ is identified with the totality of maps $f : V_0^\vee \times G_n \times W_\lambda \to \mathbb{C}$ such that

(i) $f(v^\vee, g, w)$, as a function of $(v^\vee, g)$, satisfies the conditions (a), (b) and (c) in 7.1, and it is linear with respect to $w \in W_\lambda$;

(ii) $f(v^\vee, g, \tau_\lambda(k)w) = f(v^\vee, gk, w)$, $\quad v^\vee \in V_0^\vee$, $g \in G_n$, $w \in W_\lambda$, $k \in K_n$.

Lemma 7.2.2. *Let $f \in \mathcal{H}^\eta[\lambda]$.*

(1) *For any $(X, w) \in (\mathfrak{p}_n \cap \mathfrak{q}_n) \times W_\lambda$, the linear form*

$$v^\vee \mapsto f(v^\vee, \exp(X), w), \quad v^\vee \in V_0^\vee$$

on $V_0^\vee$ is $K_n \cap H_n$-finite.

(2) *For $(X, w) \in (\mathfrak{p}_n \cap \mathfrak{q}_n) \times W_\lambda$, there exists a unique $K_n \cap H_n$-finite vector $v_f(X\ ; w) \in \mathcal{F}_\eta$ such that*

$$\langle v^\vee, v_f(X\ ; w)\rangle = f(v^\vee, \exp(X), w), \quad v^\vee \in V_0^\vee. \tag{7.2.2}$$

For a fixed $X \in \mathfrak{p}_n \cap \mathfrak{q}_n$, the vector $v_f(X\ ; w)$ depends on $w \in W_\lambda$ linearly.

Proof. (1) Let $x = \exp(X)$ with $X \in \mathfrak{p}_n \cap \mathfrak{q}_n$. For $w \in W_\lambda$ and $k_0 \in K_n \cap H_n$, let $l(k_0\ ; w)$ be the linear form on $V_0^\vee$ defined by

$$\langle l(k_0\ ; w), v^\vee \rangle = f(\eta^\vee(k_0)v^\vee, x, w), \quad v^\vee \in V_0^\vee.$$

We show that for a fixed $w \in W_\lambda$, the $\mathbb{C}$-span of the linear forms $l(k_0\ ; w)$ with $k_0 \in K_n \cap H_n$ is of finite dimension. Take a $k_0 \in K_n \cap H_n$; we can write it of the form $k_0 = \mathrm{diag}(u, t_0, t)$ with $u \in \mathsf{U}(n-1)$, $t_0,\ t \in \mathsf{U}(1)$. Put $m = \mathrm{diag}(t^{-1}u, 1, 1)$ and $z_0 = \mathrm{diag}(t\mathtt{I}_{n-1}, t_0, t)$; then $m \in M_n$ and $k_0 = mz_0$. Since z_0 is in the center of H_n, we have $\eta^\vee(z_0)v^\vee = t_0^{c_0-c}t^{-c_0}v^\vee$. Hence using the property (i) and (ii) of f above and noting that m is commutative with x, we have

$$\begin{aligned}\langle l(k_0\ ; w), v^\vee \rangle &= f(\eta^\vee(m)\eta^\vee(z_0)v^\vee, x, w)\\ &= t_0^{c_0-c}t^{-c_0}f(\eta^\vee(m)v^\vee, x, w)\\ &= t_0^{c_0-c}t^{-c_0}f(v^\vee, m^{-1}x, w)\\ &= t_0^{c_0-c}t^{-c_0}f(v^\vee, xm^{-1}, w)\\ &= t_0^{c_0-c}t^{-c_0}f(v^\vee, x, \tau_\lambda(m^{-1})w)\\ &= t_0^{c_0-c}t^{-c_0}\langle l(\mathtt{I}_{n+1}\ ; \tau_\lambda(m^{-1})w), v^\vee \rangle\end{aligned}$$

for $v^\vee \in V_0^\vee$. From this computation, we have $l(k_0\ ; w) = t_0^{c_0-c}t^{-c_0} \cdot l(\mathtt{I}_{n+1} : \tau_\lambda(m^{-1})w)$. Hence the $\mathbb{C}$-span of linear forms $l(k_0\ ; w)$ for $k_0 \in K_n \cap H_n$ is contained in that of linear forms $l(\mathtt{I}_{n+1}\ ; \tau_\lambda(m)w)$ for $m \in M_n$. Hence

$$\begin{aligned}\dim \langle l(k_0\ ; w) |\ k_0 \in K_n \cap H_n \rangle_{\mathbb{C}} &\leqslant \dim \langle l(\mathtt{I}_{n+1}\ ; \tau_\lambda(m)w) |\ m \in M_n \rangle_{\mathbb{C}}\\ &\leqslant \dim_{\mathbb{C}}(W_\lambda).\end{aligned}$$

Since W_λ is finite dimensional, this completes the proof.
(2) Let $V_0^{\vee\vee}$ be the space of $K_n \cap H_n$-finite linear forms on $V_0^\vee$. Then as a consequence of the admissibility of η, the natural map $V_0 \to V_0^{\vee\vee}$ is bijective. From this fact, combined with (1), the existance of $v_f(X\ ; w)$ follows. $\square$

By Lemma 7.2.2 (2), we have the map φ_f from $\mathfrak{p}_n \cap \mathfrak{q}_n$ to $\mathrm{Hom}(W_\lambda, \mathcal{F})$ such that

$$(\varphi_f(X))(w) = v_f(X\ ; w), \quad X \in \mathfrak{p}_n \cap \mathfrak{q}_n,\ w \in W_\lambda. \tag{7.2.3}$$

LEMMA 7.2.3. *Let $f \in \mathcal{H}^\eta[\lambda]$. Then the function $\varphi_f : \mathfrak{p}_n \cap \mathfrak{q}_n \to \mathrm{Hom}(W_\lambda, \mathcal{F})$ is C^∞. Furthermore, it belongs to the space $C^\infty_{\eta,\tau_\lambda}(\mathfrak{p}_n \cap \mathfrak{q}_n)$.*

PROOF. We use the notations in the proof of Lemma 7.2.2. Let $w \in W_\lambda$. Since $v_f(X\ ; w)$ corresponds to $l(\mathtt{I}_{n+1}\ ; w)$ by the natural $(\mathfrak{h}_{n,\mathbb{C}}, K_n \cap H_n)$-isomorphism $V_0 \to V_0^{\vee\vee}$, it is implicitly proved in the proof of the previous proposition that the smallest $K_n \cap H_n$-submodule of $\mathcal{F}$ contaning the vector $v_f(X\ ; w)$ is contained in the sum of images of $h \in \mathrm{Hom}_{M_n}(\langle \tau_\lambda(M_n)w\rangle_{\mathbb{C}}, \mathcal{F})$. From this, combined with the fact that η is M_n-admissible, we have that the function $X \mapsto (\varphi_f(X))(w)$ takes its values in a finite dimensional subspace. Since $X \mapsto \langle v^\vee, (\varphi_f(X))(w)\rangle = f(v^\vee, \exp(X), w)$ is C^∞ for any $v^\vee \in V_0^\vee$, the map $\varphi_f(X)(w)$ is C^∞ on $\mathfrak{p}_n \cap \mathfrak{q}_n$ also. In order to show that φ_f belongs to $C^\infty_{\eta,\tau_\lambda}(\mathfrak{p}_n \cap \mathfrak{q}_n)$ we have only to check that it satisfies the condition (7.2.1). Let $X \in \mathfrak{p}_n \cap \mathfrak{q}_n$, $x = \exp(X)$ and $k_0 \in K_n \cap H_n$. By using the property (i) of f, we have

$$f(v^\vee, k_0 x k_0^{-1}, w) = f(\eta^\vee(k_0^{-1})v^\vee, x, \tau_\lambda(k_0^{-1})w), \quad v^\vee \in V_0^\vee.$$

By (7.2.2), this can be written as follows.

$$\begin{aligned}\langle v^\vee, v_f(\mathrm{Ad}(k_0)X\ ; w)\rangle &= \langle \eta^\vee(k_0^{-1})v^\vee, v_f(X\ ; \tau_\lambda(k_0^{-1})w)\rangle \\ &= \langle v^\vee, \eta(k_0)\cdot v_f(X\ ; \tau_\lambda(k_0^{-1})w)\rangle.\end{aligned}$$

Thus the equation (7.2.1) is proved. □

LEMMA 7.2.4. *Let $\lambda \in \mathcal{L}_n^+$. Then there exists a linear map $f \mapsto F_f$ from $\mathcal{H}^\eta[\lambda]$ to $C^\infty_{\eta,\tau_\lambda}(H_n\backslash G_n/K_n)$ such that*

$$\begin{aligned}&\langle v^\vee, (F_f(\exp(X)))(w)\rangle = f(v^\vee, \exp(X), w),\\ &v^\vee \in V_0^\vee,\ X \in \mathfrak{p}_n \cap \mathfrak{q}_n,\ w \in W_\lambda.\end{aligned} \tag{7.2.3}$$

PROOF. Let $f \in \mathcal{H}^\eta[\lambda]$. By Lemma 7.2.1 and Lemma 7.2.3, the existance of F_f follows. The uniqueness of F_f follows from the fact that V_0 is dense in $\mathcal{F}$. □

Now we begin the proof of Theorem 7.1.2. Since $\Psi : F \mapsto \psi_F$ is a K_n-homomorphism, it is enough to prove that for any $\lambda \in \mathcal{L}_n^+$ the induced map $\Psi_\lambda : C^\infty_{\eta,\tau_\lambda}(H_n\backslash G_n/K_n) \to \mathcal{H}^\eta[\lambda]$ is bijective. We shall show that the map $f \mapsto F_f$ in Lemma 7.2.4 gives the inverse map of Ψ_λ.

For any $F \in C^\infty_{\eta,\tau_\lambda}(H_n\backslash G_n/K_n)$, put $f = \Psi_\lambda(F)$. Then from (7.1.2), we have

$$f(v^\vee, \exp(X), w) = \langle v^\vee, (F(\exp(X)))(w)\rangle, \quad v^\vee \in V_0^\vee,\ a \in A_n,\ w \in W_\lambda.$$

Hence we have $F_f = F$ by (7.2.3). For any $f \in \mathcal{H}^\eta[\lambda]$, from (7.1.2), we have

$$\begin{aligned}&\Psi_\lambda(F_f)(v^\vee, \exp(X), w) = \langle v^\vee, (F_f(\exp(X)))(w)\rangle,\\ &v^\vee \in V_0^\vee,\ a \in A_n,\ w \in W_\lambda.\end{aligned}$$

Hence by (7.2.3), we obtain $\Psi_\lambda(F_f) = f$. This completes the proof. $\square$

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