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Real Shintani Functions on $\mathsf{U}(n,1)$

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Abstract. Let $G = \mathsf{U}(n,1)$ and $H = \mathsf{U}(n-1,1) \times \mathsf{U}(1)$ with $n \geqslant 2$. We realize H as a closed subgroup of G, so that (G, H) forms a semisimple symmetric pair of rank one. For irreducible representations π and η of G and H respectively, we consider the space $\mathcal{I}_{\eta,\pi} = \mathrm{Hom}_{\mathfrak{g}_{\mathbb{C}},K}(\pi, \mathrm{Ind}_H^G(\eta))$ with K a maximal compact subgroup in G and $\mathfrak{g}_{\mathbb{C}}$ the complexified Lie algebra of G. The functions that belong to $\mathrm{Im}(\Phi)$ for some $\Phi \in \mathcal{I}_{\eta,\pi}$ will be called the *Shintani functions*. We prove that $\dim_{\mathbb{C}}\mathcal{I}_{\eta,\pi} \leqslant 1$ for any π and any η, giving an explicit formula of the Shintani functions that generate a 'corner' K-type of π in terms of Gaussian hypergeometric series. We also give an explicit formula of corner K-type matrix coefficients of π in the usual sense.

1. Introduction and Basic Notations

1.1. Introduction

The Shintani functions have their origin in an unpublished work by Shintani, where he presented a new way to get an integral representation of L-functions for automorphic forms on symplectic groups by using some type of generalized spherical functions. In that work, he gave several conjectures concerning their basic properties, which have been solved affirmatively by Murase and Sugano [13]. Furthermore they developed a general theory on integral representations of automorphic L-functions for classical groups by means of such type of special functions, refering them 'Shintani functions'([12], [13], [14], [8]).

Now we explain more precisely what the Shintani functions are. Let G be a reductive algebraic group defined over a local field k and H its spherical subgroup which is also defined over k ([1]). Let G and H stand for the associated locally compact groups of k-valued points, K_G and K_H their maximal compact subgroups respectively. For an irreducible admissible representation $\eta \boxtimes \pi$ of $H \times G$, the Shintani functions belonging to $\eta \boxtimes \pi$ are

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defined to be $K_H \times K_G$-finite functions on G which belong to the image of a $H \times G$-intertwining operator $\Phi : \eta \boxtimes \pi \to C^\infty(G)$ (Shintani funtional), where we regard G as a $H \times G$-space by letting G act on G by the right translation and H by the left translation.

In L-function theoretical point of view, it is important to know the uniqueness of Shintani functional Φ, i.e., $\dim_{\mathbb{C}} \mathrm{Hom}_{H\times G}(\eta \boxtimes \pi, C^\infty(G)) \leqslant 1$, and to have explicit formulas of the Shintani functions in some sense. By Murase, Sugano and Kato, the uniqueness of Shintani functional and explicit formulas of the Shintani functions are available in many cases when k is non-archimedian, G and H are unramified over k and $\eta \boxtimes \pi$ is of class one. But as for the groups over archimedian fields, the situation is different. They consider only those automorphic forms whose archimedian component is a special type of holomorphic or antiholomorphic discrete series representation, so that the corresponding Shintani functions are elementary in nature. In order to handle automorphic forms with more general type of archimedian component, investigations about the Shintani functions on real Lie groups are necessary.

In this paper, we discuss the following problems for $G = \mathsf{U}(n,1)$ with $n \geqslant 2$ as G and $H = \mathsf{U}(n-1,1) \times \mathsf{U}(1)$.

(1) For given irreducible Harish-Chandra modules η and π of H and G respectively, to determine dimension of the $\mathbb{C}$-vector space

$$\mathcal{I}_{\eta,\pi} = \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi, C^\infty\mathrm{Ind}_H^G(\eta)),$$

where $\mathfrak{g}_{\mathbb{C}}$ denotes the complexified Lie algebra of G, K a maximal compact subgroup such that $K \cap H$ is maximally compact in H.

(2) To get an explicit formula of K-finite functions in

$$\mathcal{S}_{\eta,\pi} = \mathrm{Image}(\mathcal{I}_{\eta,\pi} \otimes_{\mathbb{C}} \mathcal{H}_\pi \to C^\infty\mathrm{Ind}_H^G(\eta))$$

on a split torus A of G that contains a complete set of representatives of the double coset space $H\backslash G/K$. Here $\mathcal{H}_\pi$ is the representation space of π.

Our main result can be stated roughly as follows.

THEOREM 1.1.1.

(1) *For any irreducible* $(\mathfrak{g}_{\mathbb{C}}, K)$*-module* π *and any irreducible* $(\mathfrak{h}_{\mathbb{C}}, H \cap K)$*-module* η*, we have*

$$\dim_{\mathbb{C}} \mathcal{I}_{\eta,\pi} \leqslant 1.$$

(2) *Let* τ *be the corner* K*-type for* π *prescribed to each* π *as in 8.2. Then we have an explicit formula of some of the functions in* $\mathcal{S}_{\eta,\pi}$ *with* K*-type* τ *on a split torus of* G *in terms of the Gaussian hypergeometric functions. We also have a system of difference-differential equations which determines the* A*-radial parts of functions in* $\mathcal{S}_{\eta,\pi}$ *with arbitraly* K*-type recursively.*

One can find a more accurate form of the theorem above in Theorem 8.1.1, Theorem 8.3.1 and Theorem 8.2.1. We give a necessary condition for the space $\mathcal{I}_{\eta,\pi}$ to be non-zero (see Theorem 8.1.1, Theorem 8.2.1), which, in some cases, is also sufficient to ensure the existance of a non-zero functional in $\mathcal{I}_{\eta,\pi}$.

Now we give a few words on some technical points. We already considerd the problems (1) and (2) when $n = 2$ in [18]. Although the method employed in this paper is basically the same as in [18], the situation becomes considerably complicated because the maximal compact subgroup K is much bigger than before. In the computation we have to manipulate concretely various operators associated with a general finite dimensional representation of such a big compact group. For that purpose we use the Gelfand-Zetlin basis, that is also used in [17] to discusses Whittaker model of the discrete series representations of rank 1 classical groups.

Finally we remark several points not mentioned above. Though we stressed L-function theoretical aspects of Shintani functions, they play a role in various number-theoretical aspects, for example Fourier expansion of automorphic forms, trace formulas, etc. Furthermore apart from these applications, we believe that they are interesting themselves in view of the harmonic analysis on homogenous spaces.

As an application of the explicit formula obtained in this paper, we can compute a local zeta integral in the theory of Murase and Sugano, which reduces to a kind of Mellin transformation of real Shintani functions on a split torus. This will be discussed in another paper ([19]).

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1.2. Basic notations

For given matrices $X_1, X_2, \dots, X_m$ with $X_i \in \mathsf{M}_{n_i}(\mathbb{C})$, we write $\mathrm{diag}(X_1, X_2, \dots, X_m)$ for the matrix of size $n = \sum_{i=1}^m n_i$ which represents the linear endomorphism $X_1 \oplus X_2 \oplus \cdots \oplus X_m$ of $\mathbb{C}^n = \bigoplus_{i=1}^m \mathbb{C}^{n_i}$. For a positive integer p, $\mathtt{I}_p$ denotes the identity matrix of size p. For positive integers p and q, we write $\mathtt{O}_{p,q}$ for the $p \times q$-matrix whose entries are all zero.

For $\mathbf{l} = (l_i)_{1 \leqslant i \leqslant n} \in \mathbb{Z}^n$, set $|\mathbf{l}| = \sum_{i=1}^n l_i$.

For a given real Lie group, we denote its Lie algebra and its complexified Lie algebra by the corresponding German letter and that with the subscript $\mathbb{C}$ respectively; for example L, $\mathfrak{l}$, $\mathfrak{l}_{\mathbb{C}}$.

For $r > 0$, put

$$\mathrm{sh}(r) = \frac{r - r^{-1}}{2}, \quad \mathrm{ch}(r) = \frac{r + r^{-1}}{2}, \quad \mathrm{th}(r) = \frac{\mathrm{sh}(r)}{\mathrm{ch}(r)}.$$

For a C^∞-function f defined on a Lie group G with its values in some topological vector space and an $X \in \mathfrak{g}$, put

$$R_X f(g) = \lim_{t \to 0} \frac{f(g \exp(tX)) - f(g)}{t}, \quad g \in G.$$

This defines an action R of $\mathfrak{g}$ on the space of functions f; the extended action of the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$ will be also denoted by R.

2. Preliminaries

We introduce basic objects which will be used throughout this paper.

2.1. Unitary groups

Let n be a positive integer. Put

$$\mathsf{U}(n) = \{x \in \mathsf{GL}_n(\mathbb{C}) |\ {}^t\bar{x}x = \mathtt{I}_n\}.$$

Let G_n be the group of linear automorphisms of $\mathbb{C}^{n+1}$ preserving the Hermitian form $\mathtt{w}_n = \mathrm{diag}(\mathtt{I}_n, -1)$, that is

$$G_n = \{g \in \mathsf{GL}_{n+1}(\mathbb{C}) |{}^t\bar{g}\mathtt{w}_n g = \mathtt{w}_n\}.$$

The Lie algebra of G_n is realized as

$$\mathfrak{g}_n = \{X \in \mathfrak{gl}_{n+1}(\mathbb{C}) |\ {}^t\bar{X}\mathtt{w}_n + \mathtt{w}_n X = \mathtt{O}_{n+1,n+1}\},$$

considered to be a real Lie subalgebra of $\mathfrak{gl}_{n+1}(\mathbb{C}) = \mathfrak{g}_{n,\mathbb{C}}$.
For a pair of integers $i,\ j$ with $1 \leqslant i, j \leqslant n+1$, put $E_{ij} = (\delta_{ip}\delta_{jq})_{1 \leqslant p,q \leqslant n+1} \in \mathfrak{gl}_{n+1}(\mathbb{C})$; then E_{ij}'s make up a basis of $\mathfrak{gl}_{n+1}(\mathbb{C})$.

2.2. Subgroups of G_n

Let n be a positive integer. Set

$$K_n = \{\mathrm{diag}(k_1, k_2) |\ k_1 \in \mathsf{U}(n),\ k_2 \in \mathsf{U}(1)\},$$

$$A_n = \left\{ a_r = \begin{pmatrix} \mathtt{I}_{n-1} & \mathtt{O}_{n-1,1} & \mathtt{O}_{n-1,1} \\ \mathtt{O}_{1,n-1} & \mathrm{ch}(r) & \mathrm{sh}(r) \\ \mathtt{O}_{1,n-1} & \mathrm{sh}(r) & \mathrm{ch}(r) \end{pmatrix} \middle|\ r > 0 \right\},$$

$$M_n = \{\mathrm{diag}(x, u, u) |\ u \in \mathsf{U}(1),\ x \in \mathsf{U}(n-1)\}.$$

Then K_n is a maximal compact subgroup of G_n, A_n is a maximally split torus of G_n, M_n is the centralizer of A_n in K_n. Let Z_n be the center of G_n; it consists of all the scalar matrices in G_n and is isomorphic to $\mathsf{U}(1)$.
We have isomorphisms $K_n \cong \mathsf{U}(n) \times \mathsf{U}(1)$ and $M_n \cong \mathsf{U}(n-1) \times \mathsf{U}(1)$ defined by the assignments

(2.2.1) $$\mathsf{U}(n) \times \mathsf{U}(1) \ni (k_1, k_2) \longrightarrow \mathrm{diag}(k_1, k_2) \in K_n$$

and

(2.2.2) $$\mathsf{U}(n-1) \times \mathsf{U}(1) \ni (x, u) \longrightarrow \mathrm{diag}(x, u, u) \in M_n$$

respectively.
Let H_n be the fixed-point subgroup of the involution $\sigma : g \to S^{-1}gS$ with $S = \mathrm{diag}(\mathtt{I}_{n-1}, -1, 1)$ of G_n; it coincides with the image of the embedding

(2.2.3) $$G_{n-1} \times \mathsf{U}(1) \ni \left(\begin{pmatrix} h_{11} & h_{13} \\ h_{31} & h_{33} \end{pmatrix}, h_{22}\right) \to \begin{pmatrix} h_{11} & \mathtt{O}_{n-1,1} & h_{13} \\ \mathtt{O}_{1,n-1} & h_{22} & 0 \\ h_{31} & 0 & h_{33} \end{pmatrix} \in G_n$$

with $h_{11} \in \mathsf{M}_{n-1}(\mathbb{C})$, $h_{13} \in \mathsf{M}_{n-1,1}(\mathbb{C})$, $h_{31} \in \mathsf{M}_{1,n-1}(\mathbb{C})$ and $h_{33} \in \mathbb{C}$. Then the pair (G_n, H_n) is a semisimple symmetric pair of split rank one.

2.3. Central characters

Let L be a closed θ-stable subgroup of G_n containing Z_n with θ the Cartan involution of G_n corresponding to K_n. Note that Z_n is contained in the intersection of the center of L and K_n. Given an $(\mathfrak{l}_{\mathbb{C}}, K_n \cap L)$-module $(\rho, \mathcal{V})$, if there exists an integer c satisfying

$$\rho(z_x) = x^c 1_{\mathcal{V}}, \quad z_x = \mathrm{diag}(x, x, \dots, x),\ x \in \mathsf{U}(1), \tag{2.3.1}$$

we write $c_n(\rho)$ for that integer c. For example this is the case when ρ is irreducible.

2.4. Induced representations and intertwining spaces

First recall the notion of admissible representation of a Lie group ([2], [6, section 5], [22]). Let G be a reductive Lie group with compact center. A representation $(\pi, \mathcal{V})$ of G on a Fréchet space $\mathcal{V}$ is said to be admissible if it is smooth, is of moderate growth and every irreducible continuous representation of a maximal compact subgroup K of G occurs in π with finite multiplicity. For an irreducible admissible representation π of G, we define $(\pi^\vee, \mathcal{V}^\vee)$ to be the canonical Fréchet globalization of the dual of the underlying $(\mathfrak{g}, K)$-module of π. The canonical G-invariant pairing $\mathcal{V} \times \mathcal{V}^\vee \to \mathbb{C}$ is non-degenerate and π is canonically isomorphic to $(\pi^\vee)^\vee$.

For two smooth Fréchet representations $(\pi_i, \mathcal{V}_i)$ with $i = 1, 2$, the space of continuous G-homomorphisms from $\mathcal{V}_1$ to $\mathcal{V}_2$ is denoted by $\mathrm{Hom}_G(\pi_1, \pi_2)$. Let $(\eta, \mathcal{F})$ be an irreducible admissible representation of a closed subgroup H of G. We can form a smooth Fréchet representation $C^\infty\mathrm{Ind}_H^G(\eta)$ of G by inducing η from H to G. Recall that its representation space $C_\eta^\infty(H\backslash G)$ consists of C^∞-functions $F : G \to \mathcal{F}$ satisfying $F(hg) = \eta(h)F(g),\ g \in G,\ h \in H$, and G acts by the right-translation on that space.

Proposition 2.4.1. *Let $(\eta, \mathcal{F})$ and $(\pi, \mathcal{V})$ be irreducible admissible representations of H_n and G_n respectively. Let ρ be the $H_n \times G_n$-module with representation space $C^\infty(G_n)$ on which H_n acts by the left-translation and G_n by the right-translation. Then there exists a canonical isomorphism*

$$\mathrm{Hom}_{H_n \times G_n}(\eta^\vee \boxtimes \pi, \rho) \cong \mathrm{Hom}_{G_n}(\pi, C^\infty\mathrm{Ind}_{H_n}^{G_n}(\eta))$$

such that Ψ in the space of left-hand side corresponds to Φ in the space of right-hand side if and only if

$$\Psi(\check{v} \otimes w)(g) = \langle \Phi(w)(g), \check{v} \rangle, \quad \check{v} \in \mathcal{F}^\vee,\ w \in \mathcal{V},\ g \in G_n \tag{2.4.1}$$

holds. Here $\langle\ ,\ \rangle : \mathcal{F} \times \mathcal{F}^\vee \to \mathbb{C}$ *is the canonical pairing.*

Proof. Let ΔH_n be the diagonal subgroup of $H_n \times G_n$. Since $\eta \cong (\eta^\vee)^\vee$ canonically, there exists a canonical isomorphism

$$\mathrm{Hom}_{\Delta H_n}((\eta^\vee \boxtimes \pi)|\Delta H_n, 1_{\Delta H_n}) \ni \Psi' \overset{\cong}{\to} \Phi' \in \mathrm{Hom}_{H_n}(\pi|H_n, \eta) \tag{2.4.2}$$

such that

$$\langle \Phi'(w), \check{v}\rangle = \Psi'(\check{v} \otimes w), \quad \check{v} \in \mathcal{F}^\vee,\ w \in \mathcal{V}.$$

By the Frobenius reciprocity the intertwining space in the right-hand side of (2.4.2) is isomorphic to

$$\mathrm{Hom}_{G_n}(\pi, C^\infty\mathrm{Ind}_{H_n}^{G_n}(\eta)).$$

Since $\Delta H_n\backslash(H_n \times G_n) \cong G_n$ as $H_n \times G_n$-spaces, we have $\rho \cong C^\infty\mathrm{Ind}_{\Delta H_n}^{H_n\times G_n}(1_{\Delta H_n})$. Hence by Frobenius reciprocity, the intertwining space in the left-hand side of (2.4.2) is isomorphic to

$$\mathrm{Hom}_{H_n\times G_n}(\eta^\vee \boxtimes \pi, \rho).$$

This completes the proof. □

We introduce an infinitesimal version of $\mathrm{Hom}_{G_n}(\pi, C^\infty\mathrm{Ind}_{H_n}^{G_n}(\eta))$, that is our main concern in this paper. Let η be an irreducible $(\mathfrak{h}_{n,\mathbb{C}}, K_n \cap H_n)$-module and π an irreducible $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-module. We denote the canonical Fréchet globalizations of η and π by the same letters. We put

$$\mathcal{I}_{\eta,\pi} = \mathrm{Hom}_{(\mathfrak{g}_{n,\mathbb{C}},K_n)}(\pi, \mathrm{Ind}_{H_n}^{G_n}(\eta))$$

with $\mathrm{Ind}_{H_n}^{G_n}(\eta)$ the underlying $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-module of $C^\infty\mathrm{Ind}_{H_n}^{G_n}(\eta)$.

Corollary 2.4.1. *Let* η *and* π *be as in Proposition 2.4.1. Then we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{H_n\times G_n}(\eta^\vee \boxtimes \pi, \rho) = \dim_{\mathbb{C}} \mathrm{Hom}_{G_n}(\pi, C^\infty\mathrm{Ind}_{H_n}^{G_n}(\eta)) \leqslant \dim_{\mathbb{C}} \mathcal{I}_{\eta,\pi}.$$

Proof. The first equality follows from Proposition 2.4.1. The second inequality is a consequence of the fact that the K_n-finite vectors of $\mathcal{V}$ is everywhere dense. □

3. Representation Theory of Compact Unitary Groups

In this section we recall briefly the Gelfand-Zetlin basis of a finite dimensional representation of a compact unitary group. For more detailed treatment on this material, see [21] for example. Later we parametrize the unitary duals of K_n and M_n.

3.1. Gelfand-Zetlin schemes

For a positive integer n, set

$$\Lambda_n = \mathbb{Z}^n,$$
$$\Lambda_n^+ = \{\mathbf{l} = (l_i)_{1 \leqslant i \leqslant n} \in \mathbb{Z}^n |\ l_i \geqslant l_{i+1},\ 1 \leqslant i \leqslant n-1\}.$$

An element of Λ_n^+ is called a dominant weight of size n.

For given dominant weights $\mathbf{q} = (q_i)_{1 \leqslant i \leqslant n} \in \Lambda_n^+$ and $\mathbf{q}' = (q'_j)_{1 \leqslant j \leqslant n-1} \in \Lambda_{n-1}^+$ we write $\mathbf{q}' \subset \mathbf{q}$ if $q_j \geqslant q'_j \geqslant q_{j+1}$ holds for $1 \leqslant j \leqslant n-1$. A sequence of dominant weights $Q = (\mathbf{q}_i)_{1 \leqslant i \leqslant n}$ is called a *Gelfand-Zetlin scheme* if $\mathbf{q}_i \in \Lambda_i^+,\ 1 \leqslant i \leqslant n$ and $\mathbf{q}_j \subset \mathbf{q}_{j+1},\ 1 \leqslant j \leqslant n-1$; the totality of them is denoted by $GZ^{(n)}$. For every $\mathbf{q} \in \Lambda_n^+$ the subset of $GZ^{(n)}$ consisting of those schemes with $\mathbf{q}_n = \mathbf{q}$ is denoted by $GZ^{(n)}(\mathbf{q})$.

For a given dominant weight $\mathbf{q} = (q_i)_{1 \leqslant i \leqslant n} \in \Lambda_n^+$ and an integer $1 \leqslant k \leqslant n$, put $\mathbf{q}^{+k} = (q_i + \delta_{ki})_{1 \leqslant i \leqslant n}$ (resp. $\mathbf{q}^{-k} = (q_i - \delta_{ki})_{1 \leqslant i \leqslant n}$); $\mathbf{q}^{\pm k} \in \Lambda_n$ is not necessarily dominant. For a Gelfand-Zetlin scheme $Q = (\mathbf{q}_h)_{1 \leqslant h \leqslant n} \in GZ^{(n)}$, Q_j^{+i} (resp. Q_j^{-i}) stands for the sequence of weights $(\mathbf{q}'_h)_{1 \leqslant h \leqslant n}$ such that $\mathbf{q}'_h = \mathbf{q}_h$ $(h \neq j)$, $\mathbf{q}'_j = \mathbf{q}_j^{+i}$ (resp. $\mathbf{q}'_j = \mathbf{q}_j^{-i}$).

For $\mathbf{l} \in \Lambda_n^+$, $\mathbf{m} \in \Lambda_{n-1}^+$, $1 \leqslant k \leqslant n$ and $1 \leqslant i \leqslant n-1$, set

$$\gamma^+(\mathbf{m}\,;\mathbf{l},k) = \left| \frac{\prod_{h=1}^{n-1} (m_h - l_k + k - h - 1)}{\prod_{h=1,h\neq k}^{\nu} (l_h - l_k + k - h)} \right|^{1/2}, \quad \gamma^-(\mathbf{m}\,;\mathbf{l},k) = \left| \frac{\prod_{h=1}^{n-1} (m_h - l_k + k - h)}{\prod_{h=1,h\neq k}^{n} (l_h - l_k + k - h)} \right|^{1/2}$$

and

$$a_i(\mathbf{l}\,;\mathbf{m}) = \left| \frac{\prod_{h=1}^{n} (l_h - m_i + i - h)}{\prod_{h\neq i,h=1}^{n-1} (m_h - m_i + i - h - 1)} \right|^{1/2}, \quad b_i(\mathbf{l}\,;\mathbf{m}) = \left| \frac{\prod_{h=1}^{n} (l_h - m_i + i - h + 1)}{\prod_{h\neq i,h=1}^{n-1} (m_h - m_i + i - h + 1)} \right|^{1/2}.$$

PROPOSITION 3.1.1 (Gelfand-Zetlin). *For every $\mathbf{q} \in \Lambda_n^+$, let $W(\mathbf{q})$ be the $\mathbb{C}$-vector space freely generated by symbols $|Q\rangle$ with $Q = (\mathbf{q}_j)_{1 \leqslant j \leqslant n} \in GZ^{(n)}(\mathbf{q})$. There exists a unique Lie algebra homomorphism $\chi_{\mathbf{q}}^{(n)} : \mathfrak{gl}_n(\mathbb{C}) \to \mathrm{End}_{\mathbb{C}}(W(\mathbf{q}))$ such that $\chi_{\mathbf{q}}^{(n)}(E_{jj})$ with $1 \leqslant j \leqslant n$ and $\chi_{\mathbf{q}}^{(n)}(E_{j,j+1})$, $\chi_{\mathbf{q}}^{(n)}(E_{j+1,j})$ with $1 \leqslant j \leqslant n-1$ are given by the formulae* (3.1.1), (3.1.2) *and* (3.1.3) *below. Furthermore, the action $\chi_{\mathbf{q}}^{(n)}$ of $\mathfrak{u}(n)$ thus obtained can be globalized to that of $\mathsf{U}(n)$ giving an irreducible $\mathsf{U}(n)$-module $(\chi_{\mathbf{q}}^{(n)}, W(\mathbf{q}))$. The equivalence classes of $(\chi_{\mathbf{q}}^{(n)}, W(\mathbf{q}))$ for $\mathbf{q} \in \Lambda_n^+$ exhaust the set $\widehat{\mathsf{U}(n)}$.*

$$\chi_{\mathbf{q}}^{(n)}(E_{jj})|Q\rangle = (|\mathbf{q}_j| - |\mathbf{q}_{j-1}|)|Q\rangle, \tag{3.1.1}$$

$$\chi_{\mathbf{q}}^{(n)}(E_{j,j+1})|Q\rangle = \sum_{i=1}^{j} \gamma^+(\mathbf{q}_{j-1}\ ; \mathbf{q}_j, i) a_i(\mathbf{q}_{j+1}\ ; \mathbf{q}_j)|Q_j^{+i}\rangle, \tag{3.1.2}$$

$$\chi_{\mathbf{q}}^{(n)}(E_{j+1,j})|Q\rangle = \sum_{i=1}^{j} \gamma^-(\mathbf{q}_{j-1}\ ; \mathbf{q}_j, i) b_i(\mathbf{q}_{j+1}\ ; \mathbf{q}_j)|Q_j^{-i}\rangle, \tag{3.1.3}$$

for $Q = (\mathbf{q}_j)_{1 \leqslant j \leqslant n} \in GZ^{(n)}$ with $\mathbf{q}_j = (q_{ij})_{1 \leqslant i \leqslant j} \in \Lambda_j^+$, $1 \leqslant j \leqslant n$.

PROOF. [21, page 363]. □

The Gelfand-Zetlin scheme has a nice behavior under the pullback via the inclusion

$$\mathsf{U}(n-1) \ni x \longrightarrow \mathrm{diag}(x, 1) \in \mathsf{U}(n).$$

Indeed, we have

LEMMA 3.1.1.

(1) *For a dominant weight $\mathbf{l} \in \Lambda_n^+$, set*

$$\Delta(\mathbf{l}) = \{\mathbf{m} \in \Lambda_{n-1}^+ |\ \mathbf{m} \subset \mathbf{l}\}.$$

Then $\chi_{\mathbf{l}}^{(n)}|\mathsf{U}(n-1)$ decomposes into a multiplicity free direct sum of $\chi_{\mathbf{m}}^{(n-1)}$'s with $\mathbf{m} \in \Delta(\mathbf{l})$.

(2) *For every $\mathbf{m} \in \Delta(\mathbf{l})$, we define the $\mathbb{C}$-linear map*

$$\mathsf{p}_{\mathbf{m}}^{\mathbf{l}} : W(\mathbf{l}) \to W(\mathbf{m})$$

by giving its values on the basis $|Q\rangle$, $Q \in GZ^{(n)}(\mathbf{l})$ *as follows: for* $Q = (\mathbf{l}\ ; Q') \in GZ^{(n)}(\mathbf{l})$ *set* $\mathsf{p}^{\mathbf{l}}_{\mathbf{m}}|Q\rangle = |Q'\rangle$ *if* $Q' \in GZ^{(n-1)}(\mathbf{m})$ *and* $\mathsf{p}^{\mathbf{l}}_{\mathbf{m}}|Q\rangle = 0$ *otherwise. Then*

$$\mathsf{p}^{\mathbf{l}}_{\mathbf{m}} \in \mathrm{Hom}_{\mathsf{U}(n-1)}(\chi^{(n)}_{\mathbf{l}}|\mathsf{U}(n-1), \chi^{(n-1)}_{\mathbf{m}}).$$

PROOF. This is a consequence of Proposition 3.1.1. □

3.2. Representations of K_n and M_n

The torus T_n consisting of all diagonal matrices in G_n is a compact Cartan subgroup of G_n contained in K_n. Let $\sqrt{-1}\mathfrak{t}_n^*$ be the space of all $\mathbb{R}$-linear forms $\mathfrak{t}_n \to \sqrt{-1}\mathbb{R}$; it containes $\mathcal{L}_n$, the unitary character group of T_n. For $1 \leqslant k \leqslant n+1$, let $\epsilon_k \in \mathcal{L}_n$ be the character defined by

$$\epsilon_k(t) = t_k, \quad t = \mathrm{diag}(t_1, \dots, t_{n+1}) \in T_n.$$

Then the family $\{\epsilon_k\}_{k=1}^{n+1}$ gives a basis of the abelian group $\mathcal{L}_n$. The root system for (T_n, K_n) is

$$R_{\mathrm{c}} = \{\pm(\epsilon_h - \epsilon_k)|\ 1 \leqslant h < k \leqslant n\}.$$

We fix a positive system of R_{c} as

$$R_{\mathrm{c}}^+ = \{\epsilon_h - \epsilon_k|\ 1 \leqslant h < k \leqslant n\}.$$

Let $\mathcal{L}_n^+$ be the set of R_{c}^+-dominant elements in $\mathcal{L}_n$. For $\mathbf{l} = (l_k)_{1 \leqslant k \leqslant n} \in \Lambda_n^+$ and $l_0 \in \Lambda_1$, put

$$[\mathbf{l}\ ; l_0] = \sum_{k=1}^{n} l_k \epsilon_k + l_0 \epsilon_{n+1}. \tag{3.2.1}$$

Then $\mathcal{L}_n^+$ is the totality of all $\lambda = [\mathbf{l}\ ; l_0]$ with $(\mathbf{l}, l_0) \in \Lambda_n^+ \times \Lambda_1$.

Set ${}^\circ T_n = T_n \cap M_n$; ${}^\circ T_n$ is a maximal torus in M_n and consists of all elements $t = \mathrm{diag}(t_1, \dots, t_n, t_{n+1}) \in T_n$ with $t_n = t_{n+1}$. Let ${}^\circ\mathcal{L}_n$ be the unitary character group of ${}^\circ T_n$. For $1 \leqslant i \leqslant n$, let ${}^\circ\epsilon_i$ be the image of $\epsilon_i \in \mathcal{L}_n$ by the restriction map $\mathcal{L}_n \to {}^\circ\mathcal{L}_n$. Then $\{{}^\circ\epsilon_i\}_{i=1}^n$ gives a basis of the abelian group ${}^\circ\mathcal{L}_n$. Let ${}^\circ R_{\mathrm{c}}$ be the root system for $({}^\circ T_n, M_n)$, or explicitly

$${}^\circ R_{\mathrm{c}} = \{\pm({}^\circ\epsilon_i - {}^\circ\epsilon_j)|\ 1 \leqslant i < j \leqslant n-1\}.$$

We fix a positive system of ${}^\circ R_{\rm c}$ as

$$ {}^\circ R_{\rm c}^+ = \{{}^\circ\epsilon_i - {}^\circ\epsilon_j |\ 1 \leqslant i < j \leqslant n-1\}. $$

Let ${}^\circ\mathcal{L}_n^+$ be the set of ${}^\circ R_{\rm c}^+$-dominant characters in ${}^\circ\mathcal{L}_n$. For $\mathbf{m} = (m_i)_{1\leqslant i\leqslant n-1} \in \Lambda_{n-1}^+$ and $m_0 \in \Lambda_1$, put

$$ (3.2.2) \qquad (\mathbf{m}\ ; m_0) = \sum_{i=1}^{n-1} m_i {}^\circ\epsilon_i + m_0 {}^\circ\epsilon_n. $$

Then ${}^\circ\mathcal{L}_n^+$ is the totality of all $\mu = (\mathbf{m}\ ; m_0)$ with $(\mathbf{m}, m_0) \in \Lambda_{n-1}^+ \times \Lambda_1$. By the highest weight theory, we have $\widehat{K_n} \cong \mathcal{L}_n^+$ and $\widehat{M_n} \cong {}^\circ\mathcal{L}_n^+$.

Definition 3.2.1.

(1) For every $\lambda = [\mathbf{l}\ ; l_0] \in \mathcal{L}_n^+$, we denote by τ_λ the representation $\chi_{\mathbf{l}}^{(n)} \boxtimes \chi_{l_0}^{(1)}$ of $K_n = \mathsf{U}(n) \times \mathsf{U}(1)$ acting on $W(\mathbf{l})$. The representation space of τ_λ will be also denoted by W_λ in some situation.

(2) For every $\mu = (\mathbf{m}\ ; m_0) \in {}^\circ\mathcal{L}_n^+$, we denote by σ_μ the representation $\chi_{\mathbf{m}}^{(n-1)} \boxtimes \chi_{m_0}^{(1)}$ of $M_n = \mathsf{U}(n-1) \times \mathsf{U}(1)$ acting on $W(\mathbf{m})$.

Let $\lambda = [\mathbf{l}\ ; l_0] \in \mathcal{L}_n^+$ and $\mu = (\mathbf{m}\ ; m_0) \in {}^\circ\mathcal{L}_n^+$. Then the integers $c_n(\tau_\lambda)$ and $c_n(\sigma_\mu)$ defined in 2.3 are given as

$$ (3.2.3) \qquad c_n(\tau_\lambda) = |\mathbf{l}| + l_0, \quad c_n(\sigma_\mu) = |\mathbf{m}| + m_0. $$

The following proposition tells how a given irreducible representation of K_n decomposes to irreducible representations of M_n when restricted to M_n.

Proposition 3.2.1. *Let* $\lambda = [\mathbf{l}\ ; l_0] \in \mathcal{L}_n^+$.

(1) *The representation* $\tau_\lambda | M_n$ *of* M_n *is a multiplicity free direct sum of* σ_μ*'s with* $\mu = (\mathbf{m}\ ; c_n(\tau_\lambda) - |\mathbf{m}|)$, $\mathbf{m} \in \Delta(\mathbf{l})$.

(2) *Let* $\mathbf{m} \in \Delta(\mathbf{l})$ *and put* $\mu = (\mathbf{m}\ ; c_n(\tau_\lambda) - |\mathbf{m}|)$. *Then the* $\mathbb{C}$*-linear map* $\mathsf{p}_{\mathbf{m}}^{\mathbf{l}} : W(\mathbf{l}) \to W(\mathbf{m})$ *defined in Lemma 3.1.1 make up a basis of the one dimensional* $\mathbb{C}$*-vector space* $\mathrm{Hom}_{M_n}(\tau_\lambda | M_n, \sigma_\mu)$.

Proof. This follows from Lemma 3.1.1. □

4. Review of Representation Theory of G_n

We recall some facts about the representation theory of the unitary group G_n, which is necessary for our study. All of the materials in this section will be found in Kraljević [9], [10]. Though [9] and [10] deal with the semisimple group $\mathsf{SU}(n,1)$, we can translate the results into those for $\mathsf{U}(n,1)$ easily. Note that our parametrizations of $\widehat{K_n}$ and $\widehat{M_n}$ are different from Kraljević's.

4.1. K_n-spectrum

The root system of $\mathfrak{g}_{n,\mathbb{C}}$ with respect to $\mathfrak{t}_{n,\mathbb{C}}$ is

$$R = \{\pm(\epsilon_i - \epsilon_j)|\ 1 \leqslant i < j \leqslant n+1\},$$

and that for $\mathfrak{k}_{n,\mathbb{C}}$ with respect to $\mathfrak{t}_{n,\mathbb{C}}$ is

$$R_{\mathrm{c}} = \{\pm(\epsilon_i - \epsilon_j)|\ 1 \leqslant i < j \leqslant n\}.$$

Put

$$R_{\mathrm{nc}} = \{\pm(\epsilon_i - \epsilon_{n+1})|\ 1 \leqslant i \leqslant n\},$$

the set of non-compact roots in R. Let $\rho_{\mathrm{c}} = \sum_{i=1}^{n} \frac{n+1-2i}{2}\epsilon_i$ be the half sum of roots in R_{c}^{+}.

Given an admissible $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-module $(\pi, \mathcal{H}_\pi)$, we say that π is K_n-simple if every irreducible unitary representation of K_n occurs in $\pi|K_n$ with multiplicity one or zero. It is a well-known fact that an irreducible $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-module is always K_n-simple in the present situation that $\mathfrak{g}_n = \mathfrak{u}(n,1)$. Put

$$\mathcal{L}_n^{+}(\pi) = \{\lambda \in \mathcal{L}_n^{+}|\ \mathrm{Hom}_{K_n}(\tau_\lambda, \pi|K_n) \neq \{0\}\}.$$

We assume that π has a central character, i.e., there exists an integer z such that $\pi(x\mathtt{I}_{n+1})$ acts on $\mathcal{H}_\pi$ by $x^z 1_{\mathcal{H}_\pi}$ for $x \in \mathsf{U}(1)$. Then $\lambda \in \mathcal{L}_n^{+}(\pi)$ implies $c_n(\tau_\lambda) = c_n(\pi) = z$. Hence, for such a π, there exists a unique subset $\Lambda_n^{+}(\pi)$ of Λ_n^{+} such that

$$\mathcal{L}_n^{+}(\pi) = \{[\mathbf{l}\ ; z - |\mathbf{l}|] \in \mathcal{L}_n^{+}|\ \mathbf{l} \in \Lambda_n^{+}(\pi)\}.$$

For an $\mathbf{l} \in \Lambda_n^{+}(\pi)$, take a basis $\iota_{\mathbf{l}}^{\pi}$ of the space $\mathrm{Hom}_{K_n}(\tau_{[\mathbf{l}\ ;z-|\mathbf{l}|]}, \pi|K_n) \cong \mathbb{C}$ and fix the system $\{\iota_{\mathbf{l}}^{\pi}|\ \mathbf{l} \in \Lambda_n^{+}(\pi)\}$ for once and for all. The system will be called *the standard system for* π.

PROPOSITION 4.1.1. *A choice of the standard system* $\{\iota^{\pi}_{\mathbf{l}} \mid \mathbf{l} \in \Lambda^{+}_{n}(\pi)\}$ *uniquely determines* $2n$ *functions*

$$A^{\pi}_{k} : \Lambda_n \to \mathbb{C},\ B^{\pi}_{k} : \Lambda_n \to \mathbb{C}, \quad 1 \leqslant k \leqslant n$$

satisfying

$$A^{\pi}_{k}(\mathbf{l}) = B^{\pi}_{k}(\mathbf{l}^{+k}) = 0 \text{ if either } \mathbf{l} \notin \Lambda^{+}_{n}(\pi) \text{ or } \mathbf{l}^{+k} \notin \Lambda^{+}_{n}(\pi)$$

and

$$\pi(E_{n,n+1})\iota^{\pi}_{\mathbf{l}}|Q\rangle = \sum_{k=1}^{n} \gamma^{+}(\mathbf{q}_{n-1}\ ; \mathbf{l}, k) A^{\pi}_{k}(\mathbf{l}) \cdot \iota^{\pi}_{\mathbf{l}^{+k}}|Q^{+k}_{n}\rangle,$$
$$\pi(E_{n+1,n})\iota^{\pi}_{\mathbf{l}}|Q\rangle = \sum_{k=1}^{n} \gamma^{-}(\mathbf{q}_{n-1}\ ; \mathbf{l}, k) B^{\pi}_{k}(\mathbf{l}) \cdot \iota^{\pi}_{\mathbf{l}^{-k}}|Q^{-k}_{n}\rangle$$

for any $Q = (\mathbf{q}_k)_{1 \leqslant k \leqslant n} \in GZ^{(n)}(\mathbf{l})$.

PROOF. This can be found in [9, section 5]. □

If $(\pi, \mathcal{H}_\pi)$ is an irreducible $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-module, then $Z(\mathfrak{g}_{n,\mathbb{C}})$, the center of $U(\mathfrak{g}_{n,\mathbb{C}})$, acts on $\mathcal{H}_\pi$ through a character ${}^{\pi}\xi : Z(\mathfrak{g}_{n,\mathbb{C}}) \to \mathbb{C}$, the *infinitesimal character* of π.

It is known that the isomorphism class of π is determined by the two invariants $\mathcal{L}^{+}_{n}(\pi)$ and ${}^{\pi}\xi(\Omega_{G_n})$ with Ω_{G_n} the Casimir element of G_n ([9, Theorem 9.2]).

4.2. Elementary representations

Let P_n be a minimal parabolic subgroup of G_n having M_nA_n as a Levi subgroup. Given a complex number $s \in \mathbb{C}$ and an irreducible unitary representation (σ, V) of M_n, let $\mathcal{V}^{\infty}_{\sigma,s}$ denote the space of all C^∞-functions $\varphi : G_n \to V$ such that

$$\varphi(a(r)mug) = r^{n+s}\sigma(m)\varphi(g), \quad r > 0,\ u \in N_n,\ m \in M_n,\ g \in G_n,$$

with N_n the unipotent radical of P_n. Letting G_n act on $\mathcal{V}^{\infty}_{\sigma,s}$ by the right translation, we have a representaion of G_n on $\mathcal{V}^{\infty}_{\sigma,s}$. Let $\mathcal{V}_{\sigma,s}$ be the subspace of all K_n-finite functions; $\mathcal{V}_{\sigma,s}$ carries a natural $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-module structure $\pi^{\sigma,s}_{n}$.

An irreducible $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-module which is isomorphic to some $\pi_n^{\sigma,s}$ with suitable σ and s is called to be *elementary*. (We adopt Kraljevic's terminology here; $\pi_n^{\sigma,s}$ is called principal series commonly.) When $\sigma = \sigma_\mu$ with $\mu = (\mathbf{m}\ ; m_0) \in {}^\circ\mathcal{L}_n^+$ we also write $\pi_n(\mathbf{m}, m_0\ ; s)$ for $\pi_n^{\sigma,s}$.

Lemma 4.2.1. *Let π be an elementary irreducible $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-module isomorphic to $\pi_n^{\sigma,s}$, where σ is an irreducible unitary representation of M_n with highest weight $\mu = (\mathbf{m}\ ; m_0) \in {}^\circ\mathcal{L}_n^+$ and $s \in \mathbb{C}$. Then we have*

$$\mathcal{L}_n^+(\pi) = \{\lambda = [\mathbf{l}\ ; l_0] \in \mathcal{L}_n^+ |\ \mathbf{m} \subset \mathbf{l},\ c_n(\tau_\lambda) = c_n(\sigma_\mu)\},$$
$$c_n(\pi) = c_n(\sigma_\mu) = |\mathbf{m}| + m_0.$$

Proof. As a result of Frobenius' reciprocity, we know that τ_λ with $\lambda \in \mathcal{L}_n^+$ occurs in $\pi_n^{\sigma,s}|K_n$ if and only if σ occurs in $\tau_\lambda|M_n$. This fact combined with Lemma 3.1.1 gives the explicit form of $\mathcal{L}_n^+(\pi)$ as above. □

4.3. Non-elementary representations

There exist $n+1$ sets of positive roots $R^+_{(h)}$, $0 \leqslant h \leqslant n$ in R that contains R^+_{c}; the set $R^+_{(h)}$ is given by

$$R^+_{(h)} = R^+_{\mathrm{c}} \cup \{\epsilon_j - \epsilon_{n+1} |\ 1 \leqslant j \leqslant h\} \cup \{\epsilon_{n+1} - \epsilon_j |\ h+1 \leqslant j \leqslant n\}.$$

We put $R^+ = R^+_{(n)}$ and $R^- = R^+_{(0)}$. Let $\rho^{(h)} = \sum_{i=1}^{h} \frac{n+2-2i}{2}\epsilon_i + \sum_{i=h+1}^{n} \frac{n-2i}{2}\epsilon_i + \frac{n-2h}{2}\epsilon_{n+1}$ be the half the sum of roots in $R^+_{(h)}$. Let C be the open Weyl chamber in $\sqrt{-1}\mathfrak{t}_n^*$ corresponding to R^+_{c}, that is

$$C = \{\lambda = [(\lambda_j)_{1 \leqslant j \leqslant n}\ ; \lambda_{n+1}] \in \sqrt{-1}\mathfrak{t}_n^* |\ \lambda_j > \lambda_{j+1},\ 1 \leqslant j \leqslant n-1\}.$$

For each h with $0 \leqslant h \leqslant n$, let D_h be the open Weyl chamber in $\sqrt{-1}\mathfrak{t}_n^*$ corresponding to the positive root system $R^+_{(h)}$, namely

$$D_0 = \{\lambda \in C |\ \lambda_{n+1} > \lambda_1\},$$
$$D_h = \{\lambda \in C |\ \lambda_h > \lambda_{n+1} > \lambda_{h+1}\}, \quad 0 < h < n$$
$$D_n = \{\lambda \in C |\ \lambda_n > \lambda_{n+1}\}.$$

The center of the universal enveloping algebra $Z(\mathfrak{g}_{n,\mathbb{C}})$ of $\mathfrak{g}_{n,\mathbb{C}}$ can be identified with $S(\mathfrak{t}_{n,\mathbb{C}})^W$, the algebra of all W-invariant symmetric tensors over

$\mathfrak{t}_{n,\mathbb{C}}$, through the Harish-Chandra isomorphism $\varphi_{\mathfrak{t}}$. Here W is the Weyl group of R, which we identify with $\mathfrak{S}_{n+1}$, the symmetric group of degree $n+1$ letting $\sigma \in \mathfrak{S}_{n+1}$ operate on ϵ_k, $1 \leqslant k \leqslant n+1$ by the rule $\sigma(\epsilon_k) = \epsilon_{\sigma(k)}$. For any $\lambda \in \mathfrak{t}^*_{n,\mathbb{C}}$, let $\xi_\lambda : Z(\mathfrak{g}_{n,\mathbb{C}}) \to \mathbb{C}$ denote the $\mathbb{C}$-algebra homomorphism defined by

$$\xi_\lambda(X) = \varphi_{\mathfrak{t}}(X)(\lambda), \qquad X \in Z(\mathfrak{g}_{n,\mathbb{C}}).$$

We do not give the precise definition of $\varphi_{\mathfrak{t}}$ here, only remarking that if π is a finite dimensional irreducible $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-module with R^+-highest weight $\lambda \in \mathcal{L}_n$ then ${}^\pi\xi = \xi_{\lambda+\rho}$ with $\rho = \rho^{(n)}$. The following definition is due to Kraljević [10].

Definition 4.3.1. Let π be an irreducible $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-module. Let λ be a dominant weight and $0 \leqslant h \leqslant n$ an integer.

(1) λ is called a D_h-fundamental weight of π if τ_λ occurs in $\pi|K_n$ and ${}^\pi\xi = \xi_{\lambda+2\rho_{\mathrm{c}}-\rho^{(h)}}$.

(2) λ is called a D_h-corner of π if $\lambda \in \mathcal{L}_n^+(\pi)$ and $\lambda - \beta \notin \mathcal{L}_n^+(\pi)$ for all $\beta \in R_{\mathrm{nc}} \cap R^+_{(h)}$.

(3) λ is called a D_h-fundamental corner of π if it is a D_h-fundamental weight of π and a D_h-corner of π at the same time.

Let z_0 be an integer. For any pair of integers (h,k) with $0 \leqslant h \leqslant k \leqslant n$, let $S_{hk}(z_0)$ be the set of all $\mathbf{r} = (r_i)_{1 \leqslant i \leqslant n+1} \in \Lambda^+_{n+1}$ such that $|\mathbf{r}| - z_0 = h + k - n$ and either $r_h > r_{h+1}$ or $r_{k+1} > r_{k+2}$. Let $\mathcal{S}_n$ be the set of all quadratuples $(h,k,\mathbf{r},z_0)$ such that h,k and z_0 are integers with $0 \leqslant h \leqslant k \leqslant n$ and $\mathbf{r} \in S_{hk}(z_0)$. For a given $\zeta = (h,k,\mathbf{r},z_0) \in \mathcal{S}_n$, put

$$\text{(4.3.1)} \qquad \begin{aligned} \lambda^h(\zeta) &= \sum_{i=1}^{k} r_i\epsilon_i + \sum_{i=k+1}^{n} r_{i+1}\epsilon_i + (z_0 - \sum_{i=1}^{n+1} r_i + r_{k+1})\epsilon_{n+1}, \\ \lambda^k(\zeta) &= \sum_{i=1}^{h} r_i\epsilon_i + \sum_{i=h+1}^{n} r_{i+1}\epsilon_i + (z_0 - \sum_{i=1}^{n+1} r_i + r_{h+1})\epsilon_{n+1}. \end{aligned}$$

Note that $c_n(\tau_{\lambda^h(\zeta)}) = c_n(\tau_{\lambda^k(\zeta)}) = z_0$ and $\xi_{\lambda^h(\zeta)+2\rho_{\mathrm{c}}-\rho^{(h)}} = \xi_{\lambda^k(\zeta)+2\rho_{\mathrm{c}}-\rho^{(k)}}$. Indeed, we have

$$\sigma_{hk}(\lambda^k(\zeta) + 2\rho_{\mathrm{c}} - \rho^{(k)}) = \lambda^h(\zeta) + 2\rho_{\mathrm{c}} - \rho^{(h)}$$

with σ_{hk} the element of $W = \mathfrak{S}_{n+1}$ defined by

$$\begin{aligned}
&\sigma_{hk}(i) = i, \quad i \in \{1, \dots, h\} \cup \{k+1, \dots, n\},\\
&\sigma_{hk}(i) = i+1, \quad i \in \{h+1, \dots, k-1\},\\
&\sigma_{hk}(k) = n+1, \quad \sigma_{hk}(n+1) = h+1.
\end{aligned}$$

Theorem 4.3.1. *Let $\zeta = (h, k, \mathbf{r}, z_0) \in \mathcal{S}_n$.*

(1) *To ζ, there corresponds an isomorphism class Π_ζ of irreducible non-elementary $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-modules that has the following property: An irreducible $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-module π belongs to the class Π_ζ if and only if $c_n(\pi) = z_0$,*

$$\tag{4.3.2} \Lambda_n^+(\pi) = \left\{ \mathbf{l} \in \Lambda_n^+ \,\middle|\, \begin{aligned} &l_1 \geqslant r_1 \geqslant \cdots \geqslant l_h \geqslant r_h,\\ &r_{h+1} \geqslant l_{h+1} \geqslant \cdots l_k \geqslant r_{k+1},\\ &r_{k+2} \geqslant l_{k+1} \geqslant \cdots \geqslant r_{n+1} \geqslant l_n \end{aligned} \right\},$$

and ${}^{\pi}\xi = \xi_{\lambda^h(\zeta)+2\rho_c-\rho^{(h)}} = \xi_{\lambda^k(\zeta)+2\rho_c-\rho^{(k)}}$.

(2) *Let π be an irreducible $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-module in the class Π_ζ. Then $\lambda^h(\zeta)$ is the unique D_h-fundamental corner of π; $\lambda^k(\zeta)$ is the unique D_k-fundamental corner of π. For $i \in \{0, \dots, n\}$ with $i \neq h, k$, π has no D_i-fundamental corner.*

(3) *The map $\zeta \mapsto \Pi_\zeta$ gives a bijection from $\mathcal{S}_n$ onto the set of isomorphism classes of irreducible non-elementary $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-modules.*

(4) *The class Π_ζ contains a unitarizable element if and only if $\lambda^h(\zeta) = \lambda^k(\zeta)$.*

Proof. (1) and (3) follow from the matters found in [10, section 4]. (2) follows from [10, Proposition 2]. (4) follows from [10, Theorem 5 (i)]. □

THEOREM 4.3.2. *For $\zeta = (h, k, \mathbf{r}, z_0) \in \mathcal{S}_n$, let $\mathcal{I}_\zeta$ be the set of pairs of integers $\kappa = (q, p)$ such that $1 \leqslant q < p \leqslant n+1$, $q \in \{h, h+1\}$ and $p \in \{k+1, k+2\}$. For each $\kappa = (q, p) \in \mathcal{I}_\zeta$, put*

$$\begin{aligned}\mu_\kappa = {} & \sum_{i=1}^{q} r_i{}^\circ\epsilon_i + \sum_{i=q+1}^{p-1} r_{i+1}{}^\circ\epsilon_i + \sum_{i=p}^{n-1} r_{i+2}{}^\circ\epsilon_i \\ & + (n - h - k + r_q + r_p)^\circ\epsilon_n \in {}^\circ\mathcal{L}_n^+, \\ s_\kappa = {} & r_q - r_p + k - h \in \mathbb{C}.\end{aligned}$$

Then there exists a unique $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-subquotient of the elementary representation $\pi_n^{\sigma_{\mu_\kappa}, s_\kappa}$ that belongs to the class Π_ζ.

PROOF. This follows from [10, Proposition 3]. □

5. Decomposition of Tensor Products

In the first place we recall some basic facts on the affine symmetric space $H_n \backslash G_n$. We also prove some technical lemmas on decomposition of tensor products of representations of M_n, K_n and H_n.

5.1. Decomposition of Lie algebras

Recall the involution σ of G_n (see 2.2). Let $\mathfrak{q}_n$ be the (-1)-eigenspace of the involution $d\sigma$ of $\mathfrak{g}_n$, and $\mathfrak{p}_n$ that of the Cartan involution of $\mathfrak{g}_n$. We have the direct sum decompositions

$$\begin{aligned}&\mathfrak{g}_n = \mathfrak{k}_n \oplus \mathfrak{p}_n = \mathfrak{h}_n \oplus \mathfrak{q}_n, \\ &\mathfrak{k}_n = (\mathfrak{k}_n \cap \mathfrak{h}_n) \oplus (\mathfrak{k}_n \cap \mathfrak{q}_n), \qquad \mathfrak{p}_n = (\mathfrak{h}_n \cap \mathfrak{p}_n) \oplus (\mathfrak{q}_n \cap \mathfrak{p}_n)\end{aligned}$$

and

$$\begin{aligned}(\mathfrak{k}_n \cap \mathfrak{q}_n)_\mathbb{C} &= (\mathfrak{k}_n \cap \mathfrak{q}_n)^+ \oplus (\mathfrak{k}_n \cap \mathfrak{q}_n)^-, \\ (\mathfrak{h}_n \cap \mathfrak{p}_n)_\mathbb{C} &= (\mathfrak{h}_n \cap \mathfrak{p}_n)^+ \oplus (\mathfrak{h}_n \cap \mathfrak{p}_n)^-,\end{aligned} \tag{5.1.1}$$

with

$$\begin{aligned}(\mathfrak{k}_n \cap \mathfrak{q}_n)^+ &= \sum_{i=1}^{n-1} \mathbb{C}E_{i,n}, \quad (\mathfrak{k}_n \cap \mathfrak{q}_n)^- = \sum_{i=1}^{n-1} \mathbb{C}E_{n,i}, \\ (\mathfrak{h}_n \cap \mathfrak{p}_n)^+ &= \sum_{i=1}^{n-1} \mathbb{C}E_{i,n+1}, \quad (\mathfrak{h}_n \cap \mathfrak{p}_n)^- = \sum_{i=1}^{n-1} \mathbb{C}E_{n+1,i}.\end{aligned}$$

5.2. Structure theory

The Lie algebra of A_n is given by $\mathfrak{a}_n = \mathbb{R}H_1$ with $H_1 = E_{n,n+1} + E_{n+1,n}$; it is a maximally split abelian subspace of $\mathfrak{q}_n \cap \mathfrak{p}_n$. Let M_n^* denote the normalizer of A_n in $H_n \cap K_n$; then $M_n^* = M_n \cup \mathtt{w}_n M_n$ containes M_n as a normal subgroup with index two, and the quotient group M_n^*/M_n is isomorphic to the Weyl group of the root system for $(\mathfrak{g}_n, \mathfrak{a}_n)$. By the structure theory of the semisimple symmetric space G_n/H_n, we have the Cartan-Iwasawa type decomposition of G_n.

Proposition 5.2.1.

(1) *The multiplication map* $\psi : H_n \times A_n \times K_n \to G_n,\ (h,a,k) \to hak$ *is a surjective* C^∞*-map and its tangent map at* (h,a,k) *is surjective if and only if* $a \neq \mathtt{I}_{n+1}$. *For every* $a \in A_n - \{\mathtt{I}_{n+1}\}$ *we have the decomposition*

$$\mathfrak{g}_n = \mathrm{Ad}(a^{-1})\mathfrak{h}_n + \mathfrak{a}_n + \mathfrak{k}_n. \tag{5.2.1}$$

(2) *Let* $g = hak$ *with* $h \in H_n,\ a \in A_n$ *and* $k \in K_n$. *The fibre of* ψ *above* g *is given by*

$$\psi^{-1}(g) = \{(hl, l^{-1}al, l^{-1}k)|\ l \in M_n^*\ \} \text{ if } a \neq \mathtt{I}_{n+1},$$
$$\psi^{-1}(g) = \{(hl, \mathtt{I}_{n+1}, l^{-1}k)|\ l \in H_n \cap K_n\ \} \text{ if } a = \mathtt{I}_{n+1}.$$

Proof. See [15, Theorem 9, Theorem 10]. □

Let (τ, W_τ) be a finite dimensional continuous representation of K_n and (η, V_η) a Harish-Chandra $(\mathfrak{h}_{n,\mathbb{C}}, K_n \cap H_n)$-module. Let $(\eta^\infty, V_\eta^\infty)$ be a C^∞-globalization of η on a Fréchet space. We define $C_{\eta,\tau}^\infty(H_n\backslash G_n/K_n)$ to be the $\mathbb{C}$-vector space consisting of C^∞-functions $F : G_n \to \mathrm{Hom}(W_\tau, V_\eta^\infty)$ with the equivariant property

$$F(hgk) = \eta^\infty(h) \circ F(g) \circ \tau(k), \qquad h \in H_n,\ g \in G_n,\ k \in K_n. \tag{5.2.2}$$

Note that since W_τ is of finite dimension, the $\mathbb{C}$-vector space $\mathrm{Hom}(W_\tau, V_\eta^\infty)$ of all $\mathbb{C}$-linear maps from W_τ to V_η^∞ becomes a Fréchet space naturally.

Let us denote by $C_{\eta,\tau}^\infty(A_n)$ the totality of C^∞-functions $\varphi : A_n \to \mathrm{Hom}_{\mathbb{C}}(W_\tau, V_\eta^\infty)$ which is of the form $\varphi = F|A_n$ with an $F \in$

$C^{\infty}_{\eta,\tau}(H_n\backslash G_n/K_n)$. Then, as a result of Proposition 5.2.1 and the equation (5.2.2), by the application $F \mapsto F|A_n$, the space $C^{\infty}_{\eta,\tau}(H_n\backslash G_n/K_n)$ is mapped onto $C^{\infty}_{\eta,\tau}(A_n)$ isomorphically, and $\varphi \in C^{\infty}_{\eta,\tau}(A_n)$ satisfies the equations

$$\eta(m) \circ \varphi(a) \circ \tau(m^{-1}) = \varphi(a), \qquad m \in M_n,\ a \in A_n, \tag{5.2.3}$$

$$\eta(\mathtt{w}_n) \circ \varphi(a) \circ \tau(\mathtt{w}_n) = \varphi(a^{-1}), \qquad a \in A_n, \tag{5.2.4}$$

$$\eta(l) \circ \varphi(\mathtt{I}_{n+1}) \circ \tau(l) = \varphi(\mathtt{I}_{n+1}), \qquad l \in H_n \cap K_n. \tag{5.2.5}$$

5.3. Representation theory of K_n and M_n

For any $\mathrm{Ad}(M_n)$-stable subspace $\mathfrak{v}$ of $\mathfrak{g}_{n,\mathbb{C}}$, the action of M_n on $\mathfrak{v}$ is denoted by ${}^{\circ}\mathrm{Ad}_{\mathfrak{v}}$. The subspaces $(\mathfrak{k}_n \cap \mathfrak{q}_n)^{\pm}$ and $(\mathfrak{h}_n \cap \mathfrak{p}_n)^{\pm}$ are typical examples of such $\mathfrak{v}$.

Proposition 5.3.1. *Let $\epsilon \in \{+,-\}$. The two representations ${}^{\circ}\mathrm{Ad}_{(\mathfrak{k}_n\cap\mathfrak{q}_n)^{\epsilon}}$ and ${}^{\circ}\mathrm{Ad}_{(\mathfrak{h}_n\cap\mathfrak{p}_n)^{\epsilon}}$ are isomorphic. The linear map $\gamma : (\mathfrak{k}_n\cap\mathfrak{q}_n)^{\epsilon} \to (\mathfrak{h}_n \cap \mathfrak{p}_n)^{\epsilon}$ defined by*

$$\gamma(E_{n,i}) = E_{n+1,i},\ \gamma(E_{i,n}) = -E_{i,n+1}, \quad i \in \{1, \dots, n-1\} \tag{5.3.1}$$

gives an M_n-isomorphism.

Proof. This is easy. □

Proposition 5.3.2. *Let $\mu = (\mathbf{m}\ ; m_0) \in {}^{\circ}\mathcal{L}^{+}_{n}$. For $i \in \{1, \dots, n-1\}$ and $\epsilon \in \{+,-\}$, put*

$$\mu^{\epsilon i} = (\mathbf{m}^{\epsilon i}\ ; m_0 - \epsilon 1). \tag{5.3.2}$$

Then we have

$$\sigma_{\mu} \otimes {}^{\circ}\mathrm{Ad}_{(\mathfrak{k}_n\cap\mathfrak{q}_n)^{\epsilon}} \cong \bigoplus_{i=1}^{n-1} \sigma_{\mu^{\epsilon i}}, \tag{5.3.3}$$

ignoring $\sigma_{\mu^{\epsilon i}}$ if $\mu^{\epsilon i} \notin {}^{\circ}\mathcal{L}^{+}_{n}$.

Proof. Put ${}^{\circ}\mathbf{e}^{+} = (\delta_{1i})_{1\leqslant i\leqslant n-1} \in \Lambda^{+}_{n-1}$ and ${}^{\circ}\mathbf{e}^{-} = (-\delta_{n-1i})_{1\leqslant i\leqslant n-1} \in \Lambda^{+}_{n-1}$. Then we easily see that ${}^{\circ}\mathrm{Ad}_{(\mathfrak{k}_n\cap\mathfrak{q}_n)^{\pm}}$ is isomorphic to $\sigma_{({}^{\circ}\mathbf{e}^{\pm}\ ;\mp 1)}$. Hence the proposition follows from [26, page 231, Example 2]. □

Proposition 5.3.3. *Let $\mu = (\mathbf{m}\ ; m_0) \in {}^\circ\mathcal{L}_n^+$. Then the M_n-isomorphism* (5.3.3) *can be chosen uniquely so that the formulas*

$$\tilde{\epsilon}_\mu^+(E_{n,n-1}) \circ I_i^+(\mathbf{m})|Q_{n-1}^{+i}\rangle = \gamma^+(\mathbf{q}_{n-2}\ ; \mathbf{m}, i)|Q\rangle, \tag{5.3.5}$$

$$\tilde{\epsilon}_\mu^-(E_{n-1,n}) \circ I_i^-(\mathbf{m})|Q_{n-1}^{-i}\rangle = \gamma^-(\mathbf{q}_{n-2}\ ; \mathbf{m}, i)|Q\rangle \tag{5.3.6}$$

holds for any $|Q\rangle = (\mathbf{q}_j)_{1 \leqslant j \leqslant n-1} \in GZ^{(n-1)}(\mathbf{m})$, where $I_i^\pm(\mathbf{m})$ denotes the M_n-inclusion $W(\mathbf{m}^{\pm i}) \to W(\mathbf{m}) \otimes_{\mathbb{C}} (\mathfrak{k}_n \cap \mathfrak{q}_n)^\pm$ corresponding to (5.3.3) *and for $Y \in (\mathfrak{k}_n \cap \mathfrak{q}_n)^\mp$,*

$$\tilde{\epsilon}_\mu^\pm(Y) : W(\mathbf{m}) \otimes_{\mathbb{C}} (\mathfrak{k}_n \cap \mathfrak{q}_n)^\pm \to W(\mathbf{m})$$

is the map defined by

$$\tilde{\epsilon}_\mu^\pm(Y)(w \otimes X) = \langle Y, X\rangle w, \quad w \in W(\mathbf{m}),\ X \in (\mathfrak{k}_n \cap \mathfrak{q}_n)^\pm. \tag{5.3.7}$$

Proof. This follows from matters in [21, section 18.2]. □

From now on we assume that the isomorphisms (5.3.3) is chosen as in Proposition 5.3.3 for any $\mu \in {}^\circ\mathcal{L}_n^+$.

Let $\lambda = [\mathbf{l}\ ; l_0] \in \mathcal{L}_n^+$ and $\mu = (\mathbf{m}\ ; m_0) \in {}^\circ\mathcal{L}_n^+$ with σ_μ occuring in $\tau_\lambda|M_n$; thus $\mathrm{Hom}_{M_n}(\tau_\lambda|M_n, \sigma_\mu) = \mathbb{C}\mathsf{p}_\mathbf{m}^\mathbf{l}$ with $\mathsf{p}_\mathbf{m}^\mathbf{l}$ as in Lemma 3.1.1. Taking an $\mathbb{R}$-basis $\{X_j\}_{j=1}^{2n-2}$ of $\mathfrak{k}_n \cap \mathfrak{q}_n$ such that $\langle X_i, X_j\rangle = \mathrm{tr}(X_iX_j) = -\delta_{ij}$, we set

$$[\mathsf{p}_\mathbf{m}^\mathbf{l}](w) = \sum_{j=1}^{2n-2} \mathsf{p}_\mathbf{m}^\mathbf{l}(\tau_\lambda(X_j)w) \otimes X_j.$$

Then the right-hand side of the above identity is an element of $W(\mathbf{m}) \otimes (\mathfrak{k}_n \cap \mathfrak{q}_n)_{\mathbb{C}}$ independent of the choice of $\{X_j\}$, and the map $w \to [\mathsf{p}_\mathbf{m}^\mathbf{l}](w)$ defines an M_n-intertwining operator $\tau_\lambda|M_n \to \sigma_\mu \otimes {}^\circ\mathrm{Ad}_{(\mathfrak{k}_n\cap\mathfrak{q}_n)_{\mathbb{C}}}$. Let $\mathrm{pr}^\pm : (\mathfrak{k}_n\cap\mathfrak{q}_n)_{\mathbb{C}} \to (\mathfrak{k}_n\cap\mathfrak{q}_n)^\pm$ be the projection corresponding to the decomposition (5.1.1) and set

$$[\mathsf{p}_\mathbf{m}^\mathbf{l}]^\pm = (1_{W(\mathbf{m})} \otimes \mathrm{pr}^\pm) \circ [\mathsf{p}_\mathbf{m}^\mathbf{l}].$$

By taking $\{2^{-1/2}(-E_{in}+E_{ni}), (-2)^{-1/2}(E_{in}+E_{ni})\}_{i=1}^{n-1}$ for $\{X_j\}$ above, we have

$$[\mathsf{p}_{\mathbf{m}}^{\mathbf{l}}]^{+}(w) = -\sum_{i=1}^{n-1} \mathsf{p}_{\mathbf{m}}^{\mathbf{l}}(\tau_\lambda(E_{ni})w) \otimes E_{in}, \tag{5.3.8}$$

$$[\mathsf{p}_{\mathbf{m}}^{\mathbf{l}}]^{-}(w) = -\sum_{i=1}^{n-1} \mathsf{p}_{\mathbf{m}}^{\mathbf{l}}(\tau_\lambda(E_{in})w) \otimes E_{ni}. \tag{5.3.9}$$

LEMMA 5.3.1. *Let* $\lambda = [\mathbf{l}\ ; l_0] \in \mathcal{L}_n^+$ *and* $\mu = (\mathbf{m}\ ; m_0) \in {}^\circ\mathcal{L}_n^+$ *such that* σ_μ *occurs in* $\tau_\lambda|M_n$. *Then we have*

$$[\mathsf{p}_{\mathbf{m}}^{\mathbf{l}}]^{+} = -\sum_{i=1}^{n-1} a_i(\mathbf{l}\ ; \mathbf{m}) \cdot I_i^{+}(\mathbf{m}) \circ \mathsf{p}_{\mathbf{m}^{+i}}^{\mathbf{l}}, \tag{5.3.10}$$

$$[\mathsf{p}_{\mathbf{m}}^{\mathbf{l}}]^{-} = -\sum_{i=1}^{n-1} b_i(\mathbf{l}\ ; \mathbf{m}) \cdot I_i^{-}(\mathbf{m}) \circ \mathsf{p}_{\mathbf{m}^{-i}}^{\mathbf{l}}. \tag{5.3.11}$$

Here in the summation in (5.3.10) *resp.* (5.3.11) *with respect to* i*, the terms for those* i *with* $\mathbf{m}^{+i} \notin \Delta(\mathbf{l})$ *resp.* $\mathbf{m}^{-i} \notin \Delta(\mathbf{l})$ *should be ignored.*

PROOF. By the decomposition (5.3.3), the linear maps $I_i^{+}(\mathbf{m}) \circ \mathsf{p}_{\mathbf{m}^{+i}}^{\mathbf{l}}$ with $i \in \{1, \ldots, n-1\}$ such that $\mathbf{m}^{+i} \in \Delta(\mathbf{l})$ make up a basis of the $\mathbb{C}$-vector space

$$\mathrm{Hom}_{M_n}(\tau_\lambda|M_n, \sigma_\mu \otimes {}^\circ\mathrm{Ad}_{(\mathfrak{k}_n \cap \mathfrak{q}_n)^+}).$$

Since $[\mathsf{p}_{\mathbf{m}}^{\mathbf{l}}]^{+}$ belongs to this space, we can write it of the form

$$[\mathsf{p}_{\mathbf{m}}^{\mathbf{l}}]^{+} = \sum_{i=1}^{n-1} a_i \cdot I_i^{+}(\mathbf{m}) \circ \mathsf{p}_{\mathbf{m}^{+i}}^{\mathbf{l}} \tag{5.3.12}$$

with $a_i \in \mathbb{C}$, $i \in \{1, \ldots, n-1\}$. In order to determine a_j's, we use the following identity obtained from (5.3.12) by evaluating it at $|\tilde{Q}\rangle$ with $\tilde{Q} = (\mathbf{l}\ ; \mathbf{m}^{+j}\ ; Q) \in GZ^{(n)}(\mathbf{l})$ and applying $\tilde{\epsilon}_\mu^{+}(E_{n,n-1})$:

$$\tilde{\epsilon}_\mu^{+}(E_{n,n-1}) \circ [\mathsf{p}_{\mathbf{m}}^{\mathbf{l}}]^{+}|\tilde{Q}\rangle = \sum_{i=1}^{n-1} a_i \cdot \tilde{\epsilon}_\mu^{+}(E_{n,n-1}) \circ I_i^{+}(\mathbf{m}) \circ \mathsf{p}_{\mathbf{m}^{+i}}^{\mathbf{l}}|\tilde{Q}\rangle, \tag{5.3.13}$$

Hence using (5.3.8) and (3.1.3), we have

$$\begin{aligned}\tilde{\epsilon}_\mu^+(E_{n,n-1}) \circ [\mathsf{p}_{\mathbf{m}}^{\mathbf{l}}]^+ |\tilde{Q}\rangle &= -\mathsf{p}_{\mathbf{m}}^{\mathbf{l}} \circ \tau_\lambda(E_{n,n-1})|\tilde{Q}\rangle \\ &= -\sum_{i=1}^{n-1} \gamma^-(\mathbf{q}_{n-2}\ ; \mathbf{m}^{+j}, i) b_i(\mathbf{l}\ ; \mathbf{m}^{+j}) \cdot \mathsf{p}_{\mathbf{m}}^{\mathbf{l}} |\tilde{Q}_{n-1}^{-i}\rangle.\end{aligned}$$

Noting that $\mathsf{p}_{\mathbf{m}}^{\mathbf{l}}|\tilde{Q}_{n-1}^{-i}\rangle$ is 0 unless $i = j$ in which case it is $|\mathbf{m}\ ; Q\rangle$, we see that the right-hand side of (5.3.13) becomes $\gamma^-(\mathbf{q}_{n-2}\ ; \mathbf{m}^{+j}, j) b_j(\mathbf{l}\ ; \mathbf{m}^{+j})|\mathbf{m}\ ; Q\rangle$. On the other hand, the formula (5.3.5) gives

$$\tilde{\epsilon}_\mu^+(E_{n,n-1}) \circ I_j^+(\mathbf{m}) \circ \mathsf{p}_{\mathbf{m}^{+j}}^{\mathbf{l}} |\tilde{Q}\rangle = \gamma^+(\mathbf{q}_{n-2}\ ; \mathbf{m}, j)|\mathbf{m}\ ; Q\rangle.$$

Thus the right-hand side of (5.3.13) becomes

$$a_j \gamma^+(\mathbf{q}_{n-2}\ ; \mathbf{m}, j)|\mathbf{m}\ ; Q\rangle$$

since $\mathsf{p}_{\mathbf{m}^{+i}}^{\mathbf{l}}|\tilde{Q}\rangle = 0$ if $i \neq j$. Hence we get

$$a_j \gamma^+(\mathbf{q}_{n-2}\ ; \mathbf{m}, j) = -\gamma^-(\mathbf{q}_{n-2}\ ; \mathbf{m}^{+j}, j) b_j(\mathbf{l}\ ; \mathbf{m}^{+j})$$

for $1 \leqslant j \leqslant n-1$. Since $\gamma^-(\mathbf{q}_{n-2}\ ; \mathbf{m}^{+j}, j) = \gamma^+(\mathbf{q}_{n-2}\ ; \mathbf{m}, j)$ and $b_j(\mathbf{l}\ ; \mathbf{m}^{+j}) = a_j(\mathbf{l}\ ; \mathbf{m})$, we get the desired formula. □

5.4. Representation theory of H_n

Recall that $H_n = G_{n-1} \times \mathsf{U}(1)$ and $K_n \cap H_n = K_{n-1} \times \mathsf{U}(1)$ through the isomorphism defined by (2.2.3). For a Harish-Chandra $(\mathfrak{g}_{n-1,\mathbb{C}}, K_{n-1})$-module (η_0, V) and $c_0 \in \Lambda_1$, we can extend η_0 to a $(\mathfrak{h}_{n,\mathbb{C}}, K_n \cap H_n)$-module by letting $\mathsf{U}(1)$-factor of H_n act on V by χ_{c_0}, which is denoted by $\eta_0[c_0]$. Let η be an irreducible $(\mathfrak{h}_{n,\mathbb{C}}, K_n \cap H_n)$-module. Then η_0, the restriction of η to $(\mathfrak{g}_{n-1,\mathbb{C}}, K_{n-1})$, is irreducible and $\eta = \eta_0[c_0]$ with a $c_0 \in \Lambda_1$. The subset $\Lambda_{n-1}^+(\eta_0)$ of Λ_{n-1}^+ defined in 4.1 is also denoted by $\Lambda_{n-1}^+(\eta)$.
Let $\{\iota_{\mathbf{m}}^{\eta_0} |\ \mathbf{m} \in \Lambda_{n-1}^+(\eta_0)\}$ be the standard system for η_0, which we call the standard system for $\eta = \eta_0[c_0]$ and write $\iota_{\mathbf{m}}^{\eta}$ in place of $\iota_{\mathbf{m}}^{\eta_0}$ in the sequel. Then it is an easy matter to check that $\iota_{\mathbf{m}}^{\eta} \in \mathrm{Hom}_{M_n}(\sigma_{(\mathbf{m}\ ; c_n(\eta)-|\mathbf{m}|)}, \eta|M_n)$ for every $\mathbf{m} \in \Lambda_{n-1}^+(\eta)$.

Lemma 5.4.1. *Let $\eta = \eta_0[c_0]$ be an irreducible $(\mathfrak{h}_{n,\mathbb{C}}, K_n \cap H_n)$-module. Then $\eta|M_n$ decomposes to a multiplicity free direct sum of $\sigma_{(\mathbf{m}\ ; c_n(\eta)-|\mathbf{m}|)}$'s with $\mathbf{m} \in \Lambda_{n-1}^+(\eta)$.*

PROOF. Obvious. □

Let η be as in Lemma 5.4.1 and $\mu = (\mathbf{m}\ ; m_0) \in {}^\circ\mathcal{L}_n^+$ with σ_μ occuring in $\eta|M_n$. We set

$$[\iota_{\mathbf{m}}^{\eta}](w \otimes Y) = \eta(Y) \circ \iota_{\mathbf{m}}^{\eta}(w), \quad Y \in (\mathfrak{h}_n \cap \mathfrak{p}_n)_{\mathbb{C}},\ w \in W(\mathbf{m}) \tag{5.4.1}$$

Then we can easily check that the map $w \otimes Y \to [\iota_{\mathbf{m}}^{\eta}](w \otimes Y)$ defines an M_n-intertwining operator $\sigma \otimes {}^\circ\mathrm{Ad}_{(\mathfrak{h}_n \cap \mathfrak{p}_n)_{\mathbb{C}}} \to \eta|M_n$. By composing $[\iota_{\mathbf{m}}^{\eta}]$ with the natural inclusions $(\mathfrak{h}_n \cap \mathfrak{p}_n)^{\pm} \to (\mathfrak{h}_n \cap \mathfrak{p}_n)_{\mathbb{C}}$ we get M_n-intertwining operators

$$[\iota_{\mathbf{m}}^{\eta}]^{\pm} : \sigma \otimes {}^\circ\mathrm{Ad}_{(\mathfrak{h}_n \cap \mathfrak{p}_n)^{\pm}} \to \eta|M_n.$$

For $\mu = (\mathbf{m}\ ; m_0) \in {}^\circ\mathcal{L}_n^+$, let

$$P_i^{\pm}(\mathbf{m}) : W(\mathbf{m}) \otimes_{\mathbb{C}} (\mathfrak{k}_n \cap \mathfrak{q}_n)^{\pm} \to W(\mathbf{m}^{\pm i})$$

be the M_n-projections corresponding to the decompositions (5.3.3).

LEMMA 5.4.2. *Let η be as in Lemma 5.4.1 and $\{\iota_{\mathbf{m}}^{\eta} |\ \mathbf{m} \in \Lambda_{n-1}^+(\eta)\}$ the standard system for η. Let γ be the map defined in Proposition 5.3.1. Then, for every $\mathbf{m} \in \Lambda_{n-1}^+(\eta)$, we have*

$$[\iota_{\mathbf{m}}^{\eta}]^{+} = -\sum_{i=1}^{n-1} A_i^{\eta}(\mathbf{m}) \cdot \iota_{\mathbf{m}^{+i}}^{\eta} \circ P_i^{+}(\mathbf{m}) \circ (1_{W(\mathbf{m})} \otimes \gamma^{-1}), \tag{5.4.2}$$

$$[\iota_{\mathbf{m}}^{\eta}]^{-} = \sum_{i=1}^{n-1} B_i^{\eta}(\mathbf{m}) \cdot \iota_{\mathbf{m}^{-i}}^{\eta} \circ P_i^{-}(\mathbf{m}) \circ (1_{W(\mathbf{m})} \otimes \gamma^{-1}), \tag{5.4.3}$$

where $A_i^{\eta}(\mathbf{m})$, $B_i^{\eta}(\mathbf{m})$ with $i \in \{1, \ldots, n-1\}$ are the $2(n-1)$ functions for η defined in Proposition 4.1.1. In the summation of (5.4.2) *resp.* (5.4.3) *with respect to i, the terms for those i with $\mathbf{m}^{+i} \notin \Lambda_n^+(\eta)$ resp. $\mathbf{m}^{-i} \notin \Lambda_n^+(\eta)$ should be neglected.*

PROOF. Using Proposition 4.1.1 and Proposition 5.3.3, we can prove this in the same way as Lemma 5.3.1. □

6. Schmid Operators and Casimir Operators

In the first subsection we introduce the Schmid operators in an abstruct way. In the second subsection we collect several formulas on Casimir operators of subgroups of G_n for later convenience. In the final subsection we introduce a notion of the Shintani function and give a system of equations for Shintani functions using Schmid operators and Casimir operators.

6.1. Schmid operators

The space $\mathfrak{p}_n$ is identified with the tangent space of G_n/K_n at the base point and is a K_n-module via the adjoint action. The complexified space $\mathfrak{p}_{n,\mathbb{C}}$ decomposes into a direct sum of two irreducible K_n-invariant subspaces

$$\mathfrak{p}_n^+ = \sum_{k=1}^{n} \mathbb{C}E_{k,n+1}, \qquad \mathfrak{p}_n^- = \sum_{k=1}^{n} \mathbb{C}E_{n+1,k}.$$

For any K_n-stable subspace $\mathfrak{w}$ of $\mathfrak{g}_{n,\mathbb{C}}$, let $\mathrm{Ad}_{\mathfrak{w}}$ denote the action of K_n on that space.

Let $\lambda \in \mathcal{L}_n^+$. The representation $\tau_\lambda \otimes \mathrm{Ad}_{\mathfrak{p}_{n,\mathbb{C}}}$ decomposes to a direct sum of irreducible representations

$$(6.1.1) \qquad \tau_\lambda \otimes \mathrm{Ad}_{\mathfrak{p}_{n,\mathbb{C}}} \cong \bigoplus_{\beta \in R_{\mathrm{nc}}(\lambda)} \tau_{\lambda+\beta},$$

with $R_{\mathrm{nc}}(\lambda)$ the set of non-compact roots β such that $\lambda + \beta \in \mathcal{L}_n^+$ (see [26, page 231, Example 2]).

For $k \in \{1, \dots, n\}$, put $\beta_k = \epsilon_k - \epsilon_{n+1}$. Then we have $R_{\mathrm{nc}} = \{\pm\beta_k |\ k \in \{1, \dots, n\}\}$.

Proposition 6.1.1. *Let* $\lambda = [\mathbf{l}\ ; l_0] \in \mathcal{L}_n^+$. *Then we can choose the* K_n*-isomorphism* (6.1.1) *uniquely so that the formula*

$$(6.1.2) \qquad \epsilon_\lambda(E_{n+1,n}) \circ I_{\beta_k}(\lambda)|Q_n^{+k}\rangle = \gamma^+(\mathbf{q}_{n-1}\ ; \mathbf{l}, k)|Q\rangle,$$

$$(6.1.3) \qquad \epsilon_\lambda(E_{n,n+1}) \circ I_{-\beta_k}(\lambda)|Q_n^{-k}\rangle = \gamma^-(\mathbf{q}_{n-1}\ ; \mathbf{l}, k)|Q\rangle$$

holds for any $|Q\rangle = (\mathbf{q}_j)_{1 \leqslant j \leqslant n} \in GZ^{(n)}(\mathbf{l})$, *where* $I_\beta(\lambda)$ *denotes the* K_n*-inclusion* $W_{\lambda+\beta} \to W_\lambda \otimes_{\mathbb{C}} \mathfrak{p}_{n,\mathbb{C}}$ *corresponding to* (6.1.1) *and for* $Y \in \mathfrak{p}_{n,\mathbb{C}}$,

$$\epsilon_\lambda(Y) : W(\mathbf{l}) \otimes \mathfrak{p}_{n,\mathbb{C}} \to W(\mathbf{l})$$

is the map defined by

$$\epsilon_\lambda(Y)(w\otimes X) = \langle Y, X\rangle w, \quad w\in W(\mathbf{l}),\ X\in\mathfrak{p}_{n,\mathbb{C}}. \tag{6.1.4}$$

PROOF. This follows from matters in [21, section 18.2]. □

From now on, we assume that the isomorphism (6.1.1) is chosen as in Proposition 6.1.1 for any $\lambda\in\mathcal{L}_n^+$.
Let $(\rho,\mathcal{M})$ be a $(\mathfrak{g}_{n,\mathbb{C}},K_n)$-module. For any $\lambda=[\mathbf{l}\,;l_0]\in\mathcal{L}_n^+$, put

$$\mathcal{M}[\lambda] = \mathrm{Hom}_{K_n}(\tau_\lambda,\rho|K_n).$$

We define $\mathbb{C}$-linear operators

$$^\rho\nabla_\lambda^\beta : \mathcal{M}[\lambda]\to\mathcal{M}[\lambda+\beta], \quad \beta\in R_{\mathrm{nc}}(\lambda)$$

in the following manner. First putting

$$^\rho\nabla_\lambda(f)(w\otimes X) = \rho(X)f(w), \quad f\in\mathcal{M}[\lambda],\ w\in W_\lambda,\ X\in\mathfrak{p}_{n,\mathbb{C}},$$

we define the linear map $^\rho\nabla_\lambda$ from $\mathcal{M}[\lambda]$ to $\mathrm{Hom}_{K_n}(\tau_\lambda\otimes\mathrm{Ad}_{\mathfrak{p}_{n,\mathbb{C}}},\rho|K_n)$. Then for $\beta\in R_{\mathrm{nc}}(\lambda)$, we set

$$^\rho\nabla_\lambda^\beta(f) = {}^\rho\nabla_\lambda(f)\circ I_\beta(\lambda), \qquad f\in\mathcal{M}[\lambda].$$

The linear maps of the form $^\rho\nabla_\lambda^\beta$ are called *Schmid operators*. We easily have

$$\begin{aligned} {}^\rho\nabla_\lambda(f) = {} & \sum_{k=1}^n \rho(E_{k,n+1})\circ f\circ\epsilon_\lambda(E_{n+1,k}) \\ & + \sum_{k=1}^n \rho(E_{n+1,k})\circ f\circ\epsilon_\lambda(E_{k,n+1}), \quad f\in\mathcal{M}[\lambda]. \end{aligned} \tag{6.1.5}$$

Here is a convention that will be adopted throughout this paper: For a non-dominant $\lambda\in\mathcal{L}_n$, we put $\mathcal{M}[\lambda]=\{0\}$. For a given $\beta\in R_{\mathrm{nc}}$ and a given $\lambda\in\mathcal{L}_n$, when at least one of the weights λ and $\lambda+\beta$ is non-dominant, we define $^\rho\nabla_\lambda^\beta:\mathcal{M}[\lambda]\to\mathcal{M}[\lambda+\beta]$ to be the zero map.

The following lemma will be proved easily from definitions.

Lemma 6.1.1. *Let $(\rho, \mathcal{M})$ and $(\rho', \mathcal{M}')$ be $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-modules and $\Phi : \mathcal{M} \to \mathcal{M}'$ a $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-homomorphism. Then for $\lambda \in \mathcal{L}_n^+$ and $\beta \in R_{\mathrm{nc}}$, we have*

$$ {}^{\rho'}\nabla_\lambda^\beta(\Phi \circ f) = \Phi \circ {}^{\rho}\nabla_\lambda^\beta(f), \quad f \in \mathcal{M}[\lambda]. $$

Now let η be a Harish-Chandra $(\mathfrak{h}_{n,\mathbb{C}}, K_n \cap H_n)$-module and take the induced representation $\mathrm{Ind}_{H_n}^{G_n}(\eta)$ for $(\rho, \mathcal{M})$. Then the space $\mathcal{M}[\lambda]$ is naturally identified with the space of functions $C^\infty_{\eta,\tau_\lambda}(H_n\backslash G_n/K_n)$; the linear opeators ${}^{\rho}\nabla_\lambda^\beta$ give rise to the first order differential operators

$$ \nabla_{\eta,\lambda}^\beta : C^\infty_{\eta,\tau_\lambda}(H_n\backslash G_n/K_n) \to C^\infty_{\eta,\tau_{\lambda+\beta}}(H_n\backslash G_n/K_n). $$

For $k \in \{1, \ldots, n\}$, we write $\nabla_{\eta,\lambda}^{+k}$ and $\nabla_{\eta,\lambda}^{-k}$ in place of $\nabla_{\eta,\lambda}^{\beta_k}$ and $\nabla_{\eta,\lambda}^{-\beta_k}$ for simplicity.

6.2. Casimir operators

Let L be a closed subgroup of G_n stable under the Cartan involution corresponding to K_n. Let Ω_L be the Casimir element of L corresponding to the invariant form $\langle X, Y\rangle = \mathrm{trace}(XY)$, $X, Y \in \mathfrak{l}$.

Proposition 6.2.1. *We have*

$$ \Omega_{G_n} = \sum_{i=1}^{n+1} E_{i,i}^2 - \sum_{i=1}^{n+1}(n-2i+2)E_{i,i} + 2\sum_{1\leqslant i<j\leqslant n+1} E_{ij}E_{ji}, \tag{6.2.1} $$

$$ \Omega_{K_n} = \sum_{i=1}^{n+1} E_{i,i}^2 - \sum_{i=1}^{n}(n-2i+1)E_{i,i} + 2\sum_{1\leqslant i<j\leqslant n} E_{ij}E_{ji}, \tag{6.2.2} $$

$$ \begin{aligned} \Omega_{H_n} = &\sum_{i=1}^{n+1} E_{i,i}^2 - \sum_{i=1}^{n-1}(n-2i+1)E_{i,i} + (n-1)E_{n+1,n+1} \\ &+ 2\sum_{1\leqslant i<j\leqslant n-1} E_{ij}E_{ji} + 2\sum_{i=1}^{n-1} E_{i,n+1}E_{n+1,i}, \end{aligned} \tag{6.2.3} $$

$$ \Omega_{M_n} = \sum_{i=1}^{n-1} E_{i,i}^2 + \frac{1}{2}(E_{n+1,n+1}+E_{n,n})^2 \tag{6.2.4} $$

$$-\sum_{i=1}^{n-1}(n-2i)E_{i,i}+2\sum_{1\leqslant i<j\leqslant n-1}E_{ij}E_{ji}.$$

Proposition 6.2.2.

(1) *Let π be an irreducible $(\mathfrak{g}_{n,\mathbb{C}},K_n)$-module with $c_n(\pi)=z_0$, which occurs in a subquotient of $\pi_n(\mathbf{p},z_0-|\mathbf{p}|\,;s)$ with $s\in\mathbb{C}$ and $\mathbf{p}=(p_i)_{1\leqslant i\leqslant n-1}\in\Lambda^+_{n-1}$. Then Ω_{G_n} acts on π by the scalar given by*

$$\Omega_{G_n}(\pi)=\frac{1}{2}s^2-\frac{1}{2}n^2+\sum_{i=1}^{n-1}p_i^2+\frac{1}{2}(z_0-|\mathbf{p}|)^2+\sum_{i=1}^{n-1}(n-2i)p_i.$$

(2) *Let $\lambda=[\mathbf{l}\,;l_0]\in\mathcal{L}^+_n$. Then the operator $\tau_\lambda(\Omega_{K_n})$ acts on $W(\mathbf{l})$ by the scalar*

$$\Omega_{K_n}(\mathbf{l}\,;l_0)=\sum_{k=1}^{n}l_k^2+l_0^2+\sum_{k=1}^{n}(n-2k+1)l_k.$$

(3) *Let $\mu=(\mathbf{m}\,;m_0)\in{}^\circ\mathcal{L}^+_n$. Then the operator $\sigma_\mu(\Omega_{M_n})$ acts on $W(\mathbf{m})$ by the scalar*

$$\Omega_{M_n}(\mathbf{m}\,;m_0)=\sum_{i=1}^{n-1}m_i^2+\frac{1}{2}m_0^2+\sum_{i=1}^{n-1}(n-2i)m_i.$$

(4) *Let $\eta=\eta_0[c_0]$ be an irreducible $(\mathfrak{h}_{n,\mathbb{C}},H_n\cap K_n)$-module with $c_{n-1}(\eta_0)=c$, which occurs in a subquotient of $\pi_{n-1}(\mathbf{q},c-|\mathbf{q}|\,;\nu)[c_0]$ with $\nu\in\mathbb{C}$ and $\mathbf{q}=(q_j)_{1\leqslant j\leqslant n-2}\in\Lambda^+_{n-2}$. Then Ω_{H_n} acts on η by the scalar*

$$\Omega_{H_n}(\eta)=\frac{1}{2}\nu^2-\frac{1}{2}(n-1)^2+c_0^2+\sum_{j=1}^{n-2}q_j^2+\frac{1}{2}(c-|\mathbf{q}|)^2+\sum_{j=1}^{n-2}(n-1-2j)q_j.$$

Let η be an irreducible $(\mathfrak{h}_{n,\mathbb{C}},K_n\cap H_n)$-module. Since $\mathrm{Ad}(k)\Omega_{G_n}=\Omega_{G_n}$ for every $k\in K_n$, the operator $R_{\Omega_{G_n}}$ acting on $C^\infty_{\eta^\infty}(H_n\backslash G_n)$ commutes with the action of K_n. Hence it induces a linear operator $\Omega_{\eta,\lambda}$ on $C^\infty_{\eta,\tau_\lambda}(H_n\backslash G_n/K_n)$ for every $\lambda=[\mathbf{l}\,;l_0]\in\mathcal{L}^+_n$, which we call *the Casimir operator.*

6.3. Differential equations of Shintani functions

Let η be an irreducible $(\mathfrak{h}_{n,\mathbb{C}}, K_n \cap H_n)$-module and π an irreducible $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-module with $c_n(\pi) = z$.

For $\Phi \in \mathcal{I}_{\eta,\pi}$ and $\lambda = [\mathbf{l}\ ; z - |\mathbf{l}|] \in \mathcal{L}_n^+(\pi)$, we define the function $\Phi_{\mathbf{l}}$ in $C^\infty_{\eta,\tau_\lambda}(H_n\backslash G_n/K_n)$ by putting

$$(\Phi_{\mathbf{l}}(g))(w) = (\Phi \circ \iota^\pi_{\mathbf{l}}(w))(g), \quad w \in W(\mathbf{l}),\ g \in G_n. \tag{6.3.1}$$

The functions of the form $\Phi_{\mathbf{l}}$ will be called the Shintani functions. The totality of $\Phi_{\mathbf{l}}$ with Φ ranging over $\mathcal{I}_{\eta,\pi}$ coincides with $\mathcal{M}[\lambda]$ with $\mathcal{M}$ the π-isotypic part of $\mathrm{Ind}_{H_n}^{G_n}(\eta)$. For convention we put $\Phi_{\mathbf{l}} = 0$ if $\mathbf{l} \notin \Lambda_n^+(\pi)$.

Proposition 6.3.1. *Let $\Phi \in \mathcal{I}_{\eta,\pi}$ and $\{\Phi_{\mathbf{l}}|\ \mathbf{l} \in \Lambda_n^+(\pi)\}$ be as above. Then we have*

$$\Omega_{\eta,\lambda}\Phi_{\mathbf{l}}(g) = \Omega_{G_n}(\pi)\Phi_{\mathbf{l}}(g), \tag{6.3.2}$$

$$\nabla^{+k}_{\eta,\lambda}\Phi_{\mathbf{l}}(g) = A^\pi_k(\mathbf{l})\Phi_{\mathbf{l}^{+k}}(g), \tag{6.3.3}$$

$$\nabla^{-k}_{\eta,\lambda}\Phi_{\mathbf{l}}(g) = B^\pi_k(\mathbf{l})\Phi_{\mathbf{l}^{-k}}(g) \tag{6.3.4}$$

for $\lambda = [\mathbf{l}\ ; z - |\mathbf{l}|] \in \mathcal{L}_n^+(\pi)$ and $k \in \{1, \dots, n\}$.

Proof. The last two equations follow from Lemma 6.1.1 and Proposition 4.1.1. The first one is rather obvious. □

7. Radial Part of Schmid Operator and Casimir Operator

Now we begin our investigation on the Shintani functions. In the first subsection, we shall introduce the notion of *standard coefficient* (Definition 7.1.1 (2)), which will play a key role in our explicit computations. Its definition is based on the fact that the M_n-spectrums of η and τ are multiplicity free. The subsections 7.2 and 7.3 are devoted to proving Theorem 7.2.1 and Theorem 7.3.1, which give explicit formulas of the A_n-radial parts of the Casimir operators and the Schmid operators in terms of the standard coefficients; these formulas are crucial for further investigation. From now on, n denotes an integer with $n \geqslant 2$.

7.1. Standard coefficients

Let (η, V_η) be an irreducible $(\mathfrak{h}_{n,\mathbb{C}}, K_n \cap H_n)$-module and $\lambda = [\mathbf{l}\,; l_0] \in \mathcal{L}_n^+$. Let $\{\iota_{\mathbf{m}}^\eta |\ \mathbf{m} \in \Lambda_{n-1}^+(\eta)\}$ be the standard system for η. We first have

LEMMA 7.1.1.

(1) *The intertwining space* $\mathrm{Hom}_{M_n}(\tau_\lambda | M_n, \eta^\infty | M_n)$ *is zero unless* $c_n(\tau_\lambda) = c_n(\eta)$.

(2) *Suppose* $c_n(\tau_\lambda) = c_n(\eta)$. *Then for every* $\mathbf{m} \in \Lambda_{n-1}^+(\eta) \cap \Delta(\mathbf{l})$ *the* $\mathbb{C}$-*linear map* $\omega_{\eta,\lambda}(\mathbf{m}) := \iota_{\mathbf{m}}^\eta \circ \mathsf{p}_{\mathbf{m}}^{\mathbf{l}} : W(\mathbf{l}) \to V_\eta^\infty$ *is an* M_n-*intertwining operator; moreover the family*

$$\{\omega_{\eta,\lambda}(\mathbf{m})|\ \mathbf{m} \in \Lambda_{n-1}^+(\eta) \cap \Delta(\mathbf{l})\}$$

provides us with a basis of the finite dimensional $\mathbb{C}$-*vector space* $\mathrm{Hom}_{M_n}(\tau_\lambda | M_n, \eta^\infty | M_n)$.

PROOF. (1) Since $Z_n \subset M_n$, the assertion in (1) follows from the definition of the numbers $c_n(\tau_\lambda)$ and $c_n(\eta)$ (see 2.3).
(2) By using the branching formulas for $\eta | M_n$ and $\tau_\lambda | M_n$ given in Lemma 5.4.1 and Proposition 3.2.1 respectively, we have that the intertwining space $\mathrm{Hom}_{M_n}(\tau_\lambda | M_n, \eta^\infty | M_n)$ decomposes to a direct sum of

$$\mathrm{Hom}_{M_n}(\sigma_{(\mathbf{m}\ ;-|\mathbf{m}|+c_n(\tau_\lambda))}, \sigma_{(\mathbf{m}\ ;c_n(\eta)-|\mathbf{m}|)}) \tag{7.1.1}$$

with $\mathbf{m} \in \Lambda_{n-1}^+(\eta) \cap \Delta(\mathbf{l})$. The assumption $c_n(\tau_\lambda) = c_n(\eta)$ and Schur's lemma imply that the space (7.1.1) is one dimensional and coincides with $\mathbb{C}\omega_{\eta,\lambda}(\mathbf{m})$. □

PROPOSITION 7.1.1. *Let* η *and* λ *be as above.*

(1) *We have that* $C_{\eta,\tau_\lambda}^\infty(A_n) = \{0\}$ *unless* $c_n(\tau_\lambda) = c_n(\eta)$ *and* $\Lambda_{n-1}^+(\eta) \cap \Delta(\mathbf{l}) \neq \emptyset$.

(2) *Suppose* $c_n(\tau_\lambda) = c_n(\eta)$. *Let* $\varphi \in C_{\eta,\tau_\lambda}^\infty(A_n)$. *Then* φ *can be written uniquely in the form*

$$\varphi(a_r) = \sum_{\mathbf{m} \in \Lambda_{n-1}^+(\eta) \cap \Delta(\mathbf{l})} f(\mathbf{m}\ ; r) \cdot \omega_{\eta,\lambda}(\mathbf{m}), \quad r > 0 \tag{7.1.2}$$

with $\{f(\mathbf{m}\ ; r) | \mathbf{m} \in \Lambda_{n-1}^+(\eta) \cap \Delta(\mathbf{l})\}$ *a family of* C^∞-*functions on* $r > 0$.

Proof. By (5.2.3) we have $\varphi(A_n) \subset \mathrm{Hom}_{M_n}(\tau_\lambda|M_n, \eta^\infty|M_n)$ for $\varphi \in C^\infty_{\eta,\tau_\lambda}(A_n)$. Hence the proposition is a consequence of Lemma 7.1.1. □

From now on we consider representations η and τ_λ satisfying the condition

$$c_n(\tau_\lambda) = c_n(\eta), \quad \Lambda^+_{n-1}(\eta) \cap \Delta(\mathbf{l}) \neq \emptyset \tag{7.1.3}$$

since otherwise $C^\infty_{\eta,\tau_\lambda}(A_n) = \{0\}$ by Proposition 7.1.1.

Definition 7.1.1. Let $\eta = \eta_0[c_0]$ be an irreducible $(\mathfrak{h}_{n,\mathbb{C}}, K_n \cap H_n)$-module and $\lambda = [\mathbf{l}\,; l_0] \in \mathcal{L}^+_n$ such that $c_n(\eta) = c_n(\tau_\lambda)$.

(1) Put $\Lambda^+_{n-1}(\eta|\lambda) = \Lambda^+_{n-1}(\eta) \cap \Delta(\mathbf{l})$.

(2) For a given function $\varphi \in C^\infty_{\eta,\tau_\lambda}(A_n)$, the family $\{f(\mathbf{m}\,; r)|\ \mathbf{m} \in \Lambda^+_{n-1}(\eta|\lambda)\}$ consisting of C^∞-functions on $r > 0$ and satisfying (7.1.2) will be called *the standard coefficient* of φ.

7.2. A_n-radial part of the Casimir operator

Let η and $\lambda = [\mathbf{l}\,; l_0]$ be as in 7.1. We impose the condition (7.1.3) on them. There exists a unique linear operator

$$\mathrm{Rad}(\Omega_{\eta,\lambda}) : C^\infty_{\eta,\tau_\lambda}(A_n) \to C^\infty_{\eta,\tau_\lambda}(A_n)$$

such that $\mathrm{Rad}(\Omega_{\eta,\lambda})(F|A_n) = (\Omega_{\eta,\lambda}F)|A_n$ for $F \in C^\infty_{\eta,\tau_\lambda}(H_n\backslash G_n/K_n)$, which we call *the A_n-radial part of* $\Omega_{\eta,\lambda}$. In this subsection we shall present an explicit formula of A_n-radial part of the Casimir operator in terms of the standard coefficients. The final result is given in Theorem 7.2.1.

For $a = a_r \in A_n$ and $X \in \mathfrak{g}_{n,\mathbb{C}}$, let ${}^aX$ denote the element $\mathrm{Ad}(a^{-1})X \in \mathfrak{g}_{n,\mathbb{C}}$. If $a \in A_n - \{\mathrm{I}_{n+1}\}$, the decomposition (5.2.1), combined with the Poincaré-Birkhoff-Witt's theorem, implies that $U(\mathfrak{g}_{n,\mathbb{C}})$ is a $\mathbb{C}$-span of elements of the form

$$ {}^aD_1 H_1^q D_2, \quad D_1 \in U(\mathfrak{h}_{n,\mathbb{C}}), \quad q \in \mathbb{N}, \quad D_2 \in U(\mathfrak{k}_{n,\mathbb{C}}) \tag{7.2.1}$$

with $H_1 = E_{n,n+1} + E_{n+1,n}$.

Lemma 7.2.1. *Let $a \in A_n - \{\mathrm{I}_{n+1}\}$. Take $D \in U(\mathfrak{g}_{n,\mathbb{C}})$ of the form ${}^aD_1H_1^qD_2$ with $D_1 \in U(\mathfrak{h}_{n,\mathbb{C}})$, $D_2 \in U(\mathfrak{k}_{n,\mathbb{C}})$ and $q \in \mathbb{N}$. Then we have*

$$R_DF(a) = \eta(D_1) \circ R^q_{H_1}F(a) \circ \tau_\lambda(D_2), \quad F \in C^\infty_{\eta,\tau_\lambda}(H_n\backslash G_n/K_n).$$

PROOF. We omit the proof because it is easy. □

We need an expression of Ω_{G_n} as a linear combination of elements of the form (7.2.1). To get such an expression, we prove lemmas. First we have

LEMMA 7.2.2. *Let $a = a_r$ with $r > 0$, $r \neq 1$. In $\mathfrak{gl}_{n+1}(\mathbb{C})$ the identities*

$$E_{i,n+1} = \frac{1}{\mathrm{ch}(r)} {}^a E_{i,n+1} - \mathrm{th}(r) E_{i,n}, \quad i \in \{1, \dots, n-1\}, \tag{7.2.2}$$

$$E_{n+1,i} = \frac{1}{\mathrm{ch}(r)} {}^a E_{n+1,i} + \mathrm{th}(r) E_{n,i}, \quad i \in \{1, \dots, n-1\}, \tag{7.2.3}$$

$$E_{n,n+1} = \frac{1}{4\mathrm{sh}(r)\mathrm{ch}(r)} {}^a H' + \frac{1}{2} H_1 - \frac{1}{4}\left(\mathrm{th}(r) + \frac{1}{\mathrm{th}(r)}\right) H', \tag{7.2.4}$$

$$E_{n+1,n} = \frac{-1}{4\mathrm{sh}(r)\mathrm{ch}(r)} {}^a H' + \frac{1}{2} H_1 + \frac{1}{4}\left(\mathrm{th}(r) + \frac{1}{\mathrm{th}(r)}\right) H' \tag{7.2.5}$$

hold with $H' = E_{n,n} - E_{n+1,n+1}$.

PROOF. These can be proved by direct matrix computations. □

Using this, we have

LEMMA 7.2.3. *Let $a = a_r$ with $r > 0$, $r \neq 1$. In $U(\mathfrak{g}_{n,\mathbb{C}})$ we have*

$$\begin{aligned} &E_{i,n+1} E_{n+1,i} \\ &= \frac{1}{2}\mathrm{th}(r) H_1 + \frac{1}{2}\left(1 - \frac{1}{2\mathrm{ch}^2(r)}\right) H' \\ &- \frac{1}{4\mathrm{ch}^2(r)} {}^a H' - \left(1 - \frac{1}{\mathrm{ch}^2(r)}\right) E_{n,i} E_{i,n} \\ &+ \frac{1}{\mathrm{ch}^2(r)} {}^a E_{i,n+1} {}^a E_{n+1,i} + \frac{\mathrm{th}(r)}{\mathrm{ch}(r)} ({}^a E_{i,n+1} E_{n,i} - {}^a E_{n+1,i} E_{i,n}), \\ &i \in \{1, \dots, n-1\}, \end{aligned} \tag{7.2.6}$$

and

$$\begin{aligned} &E_{n,n+1} E_{n+1,n} \\ &= \frac{1}{4} H_1^2 + \frac{1}{4}\left(\mathrm{th}(r) + \frac{1}{\mathrm{th}(r)}\right) H_1 \end{aligned} \tag{7.2.7}$$

$$- \frac{1}{16\mathrm{ch}^2(r)\mathrm{sh}^2(r)}(({}^aH')^2 - 2{}^aH'H' + (H')^2)$$
$$+ \frac{1}{16\mathrm{ch}^2(r)}((H')^2 + 2{}^aH'H') - \frac{1}{4}(H')^2 + \frac{1}{2}H'.$$

Proof. These can be obtained by a direct computation with the aid of Lemma 7.2.2. □

Proposition 7.2.1. *Let $a = a_r$ with $r > 0,\ r \neq 1$. We have*

$$\begin{aligned}(7.2.8)\quad & 2\Omega_{G_n} \\ =& H_1^2 + \left(\frac{1}{\mathrm{th}(r)} + (2n-1)\mathrm{th}(r)\right)H_1 \\ & - \frac{1}{4\mathrm{ch}^2(r)\mathrm{sh}^2(r)}({}^aH'{}^aH' - 2{}^aH'H' + H'H') \\ & - \frac{1}{\mathrm{ch}^2(r)}({}^aH'{}^aH' - {}^aH'H' + H'H' - 2{}^a\Omega_{H_n} - 2\Omega_{K_n} + 4\Omega_{M_n}) \\ & + 2\Omega_{M_n} + \frac{4\mathrm{th}(r)}{\mathrm{ch}(r)}\sum_{i=1}^{n-1}(-{}^aE_{n+1,i}E_{i,n} + {}^aE_{i,n+1}E_{n,i}).\end{aligned}$$

Proof. By using (6.2.1) and (6.2.2), we have

$$(7.2.9)\qquad \Omega_{G_n} = 2\sum_{i=1}^{n} E_{i,n+1}E_{n+1,i} + \Omega_{K_n} - Z_0 + (n+1)E_{n+1,n+1},$$

where $Z_0 = \sum_{i=1}^{n+1} E_{ii} \in \mathfrak{g}_{n,\mathbb{C}}$. First substitute the formulas given in Lemma 7.2.3 for the terms $E_{i,n+1}E_{n+1,i}$ in the right-hand side of (7.2.9) and then use the identities

$$\begin{aligned}(7.2.10)\qquad 2\sum_{i=1}^{n-1} E_{n,i}E_{i,n} =& \Omega_{K_n} - \Omega_{M_n} \\ & - \frac{1}{2}(H')^2 + \frac{n-1}{2}H' + \frac{n+1}{2}\widetilde{H} - Z_0,\end{aligned}$$

$$\begin{aligned}(7.2.11)\quad 2\sum_{i=1}^{n-1} {}^aE_{i,n+1}{}^aE_{n+1,i} =& {}^a\Omega_{H_n} - {}^a\Omega_{M_n} \\ & - \frac{1}{2}({}^aH')^2 + {}^aZ_0 + \frac{n-1}{2}{}^aH' - \frac{n+1}{2}{}^a\widetilde{H}\end{aligned}$$

with $H' = E_{n,n} - E_{n+1,n+1}$ and $\widetilde{H} = E_{n,n} + E_{n+1,n+1} \in \mathfrak{m}_{n,\mathbb{C}}$, which are deduced from (6.2.2), (6.2.3) and (6.2.4). Since $\mathrm{Ad}(a)$, $a \in A_n$ acts trivially on $\mathfrak{m}_{n,\mathbb{C}}$, we have ${}^a\Omega_{M_n} = \Omega_{M_n}$, ${}^aZ_0 = Z_0$ and ${}^a\widetilde{H} = \widetilde{H}$. Noting this remark, we get the desired formula of Ω_{G_n} after some elementary computations. □

Now recall the map $\omega_{\eta,\lambda}(\mathbf{m})$ (see Lemma 7.1.1). For convention we extend the domain of the map $\mathbf{m} \to \omega_{\eta,\lambda}(\mathbf{m})$ to all of Λ_{n-1} by setting $\omega_{\eta,\lambda}(\mathbf{m}) = 0$ for $\mathbf{m} \in \Lambda_{n-1} - \Lambda^+_{n-1}(\eta|\lambda)$.
We want to know explicitly the action of the terms occuring in the right-hand side of (7.2.8) when applied to $\{\omega_{\eta,\lambda}(\mathbf{m})\}$.

Definition 7.2.1.

(1) Let $\lambda = [\mathbf{l}\,; l_0] \in \mathcal{L}^+_n$. For $\mathbf{m} \in \Delta(\mathbf{l})$, we put

$$\tau_{\lambda,\mathbf{m}} = -|\mathbf{m}| + c_n(\tau_\lambda) - 2l_0 = -|\mathbf{m}| + |\mathbf{l}| - l_0.$$

(2) Let $\eta = \eta_0[c_0]$ be an irreducible $(\mathfrak{h}_{n,\mathbb{C}}, K_n \cap H_n)$-module. For $\mathbf{m} \in \Lambda^+_{n-1}(\eta)$, we put

$$\eta_{\mathbf{m}} = |\mathbf{m}| - c_n(\eta) + 2c_0 = |\mathbf{m}| - c_{n-1}(\eta_0) + c_0.$$

Lemma 7.2.4. *For every* $\mathbf{m} \in \Lambda^+_{n-1}(\eta|\lambda)$, *we have*

$$\begin{aligned} \omega_{\eta,\lambda}(\mathbf{m}) \circ \tau_\lambda(H') &= \tau_{\lambda,\mathbf{m}} \cdot \omega_{\eta,\lambda}(\mathbf{m}), \\ \eta(H') \circ \omega_{\eta,\lambda}(\mathbf{m}) &= \eta_{\mathbf{m}} \cdot \omega_{\eta,\lambda}(\mathbf{m}) \end{aligned} \tag{7.2.12}$$

with $H' = E_{n,n} - E_{n+1,n+1}$.

Proof. Put $\mu = (\mathbf{m}\,; c_n(\tau_\lambda) - |\mathbf{m}|) \in {}^\circ\mathcal{L}^+_n$. Since $H' = \widetilde{H} - 2E_{n+1,n+1}$ with $\widetilde{H} = E_{n,n} + E_{n+1,n+1} \in \mathfrak{m}_{n,\mathbb{C}}$, using $\sigma_\mu(\widetilde{H}) = (c_n(\tau_\lambda) - |\mathbf{m}|)1_{W(\mathbf{m})}$ and $\tau_\lambda(E_{n+1,n+1}) = 2l_0 1_{W(\mathbf{l})}$, we have

$$\begin{aligned} \omega_{\eta,\lambda}(\mathbf{m}) \circ \tau_\lambda(H') &= \iota^\eta_{\mathbf{m}} \circ \sigma_\mu(\widetilde{H}) \circ \mathsf{p}^{\mathbf{l}}_{\mathbf{m}} - 2\omega_{\eta,\lambda}(\mathbf{m}) \circ \tau_\lambda(E_{n+1,n+1}) \\ &= (c_n(\tau_\lambda) - |\mathbf{m}|) \cdot \omega_{\eta,\lambda}(\mathbf{m}) - 2l_0 \cdot \omega_{\eta,\lambda}(\mathbf{m}) \\ &= \tau_{\lambda,\mathbf{m}} \cdot \omega_{\eta,\lambda}(\mathbf{m}). \end{aligned}$$

The second formula can be proved in a similar way. (Note the assumption $c_n(\eta) = c_n(\tau_\lambda)$.) $\square$

LEMMA 7.2.5. *For every* $\mathbf{m} \in \Lambda^+_{n-1}(\eta|\lambda)$, *we have*

$$\begin{aligned}\sum_{i=1}^{n-1}\Big(&-\eta(E_{n+1,i}) \circ \omega_{\eta,\lambda}(\mathbf{m}) \circ \tau_\lambda(E_{i,n}) \\ &+ \eta(E_{i,n+1}) \circ \omega_{\eta,\lambda}(\mathbf{m}) \circ \tau_\lambda(E_{n,i})\Big) \\ = \sum_{i=1}^{n-1}\Big(&a_i(\mathbf{l}\,;\mathbf{m})A_i^\eta(\mathbf{m}) \cdot \omega_{\eta,\lambda}(\mathbf{m}^{+i}) \\ &- b_i(\mathbf{l}\,;\mathbf{m})B_i^\eta(\mathbf{m}) \cdot \omega_{\eta,\lambda}(\mathbf{m}^{-i})\Big).\end{aligned} \tag{7.2.13}$$

PROOF. Let $\delta_{\mathbf{m}}$ denote the left-hand side of (7.2.13). We have

$$\delta_{\mathbf{m}} = [\iota^\eta_{\mathbf{m}}] \circ (1_{W(\mathbf{m})} \otimes \gamma) \circ [\mathsf{p}^{\mathbf{l}}_{\mathbf{m}}] \tag{7.2.14}$$

where $\gamma : (\mathfrak{k}_n \cap \mathfrak{q}_n)_{\mathbf{C}} \cong (\mathfrak{h}_n \cap \mathfrak{p}_n)_{\mathbf{C}}$ is the M_n-isomorphism in Proposition 5.3.1. Indeed, by (5.3.1), (5.3.8), (5.3.9) and (5.4.1),

$$\begin{aligned}&[\iota^\eta_{\mathbf{m}}] \circ (1_{W(\mathbf{m})} \otimes \gamma) \circ [\mathsf{p}^{\mathbf{l}}_{\mathbf{m}}](w) \\ &= \sum_{i=1}^{n-1}[\iota^\eta_{\mathbf{m}}]\Big(\mathsf{p}^{\mathbf{l}}_{\mathbf{m}}(\tau_\lambda(E_{i,n})w) \otimes \gamma(-E_{n,i}) + \mathsf{p}^{\mathbf{l}}_{\mathbf{m}}(\tau(E_{n,i})w) \otimes \gamma(-E_{i,n})\Big) \\ &= \sum_{i=1}^{n-1}\Big(-\eta(E_{n+1,i})\iota^\eta_{\mathbf{m}} \circ \mathsf{p}^{\mathbf{l}}_{\mathbf{m}}(\tau(E_{i,n})w) + \eta(E_{i,n+1})\iota^\eta_{\mathbf{m}} \circ \mathsf{p}^{\mathbf{l}}_{\mathbf{m}}(\tau(E_{n,i})w)\Big) \\ &= \delta_{\mathbf{m}}.\end{aligned}$$

On the other hand we have

$$\begin{aligned}[\iota^\eta_{\mathbf{m}}] &\circ (1_{W(\mathbf{m})} \otimes \gamma) \circ [\mathsf{p}^{\mathbf{l}}_{\mathbf{m}}] \\ &= \sum_{i=1}^{n-1}\Big(a_i(\mathbf{l}\,;\mathbf{m})A_i^\eta(\mathbf{m}) \cdot \omega_{\eta,\lambda}(\mathbf{m}^{+i}) \\ &\qquad - b_i(\mathbf{l}\,;\mathbf{m})B_i^\eta(\mathbf{m}) \cdot \omega_{\eta,\lambda}(\mathbf{m}^{-i})\Big).\end{aligned} \tag{7.2.15}$$

Indeed,

$$
\begin{aligned}
&[\iota^{\eta}_{\mathbf{m}}]\circ(1_{W(\mathbf{m})}\otimes\gamma)\circ[\mathsf{p}^{\mathbf{l}}_{\mathbf{m}}]\\
&=[\iota^{\eta}_{\mathbf{m}}]\circ(1\otimes\gamma)\circ\Bigg(-\sum_{i=1}^{n-1}a_i(\mathbf{l}\,;\mathbf{m})\cdot I_i^{+}(\mathbf{m})\circ\mathsf{p}^{\mathbf{l}}_{\mathbf{m}^{+i}}\\
&\qquad\qquad\qquad\qquad -\sum_{i=1}^{n-1}b_i(\mathbf{l}\,;\mathbf{m})\cdot I_i^{-}(\mathbf{m})\circ\mathsf{p}^{\mathbf{l}}_{\mathbf{m}^{-i}}\Bigg)\\
&=\sum_{i=1}^{n-1}\Big(a_i(\mathbf{l}\,;\mathbf{m})A_i^{\eta}(\mathbf{m})\cdot\omega_{\eta,\lambda}(\mathbf{m}^{+i})-b_i(\mathbf{l}\,;\mathbf{m})B_i^{\eta}(\mathbf{m})\cdot\omega_{\eta,\lambda}(\mathbf{m}^{-i})\Big).
\end{aligned}
$$

Here the first equality follows from Lemma 5.3.1 and the second one is a consequence of the formulas

$$
\begin{aligned}
[\iota^{\eta}_{\mathbf{m}}]\circ(1\otimes\gamma)\circ I_i^{+}(\mathbf{m})&=-A_i^{\eta}(\mathbf{m})\cdot\iota^{\eta}_{\mathbf{m}^{+i}},\\
[\iota^{\eta}_{\mathbf{m}}]\circ(1\otimes\gamma)\circ I_i^{-}(\mathbf{m})&=B_i^{\eta}(\mathbf{m})\cdot\iota^{\eta}_{\mathbf{m}^{-i}},
\end{aligned}
$$

that is a paraphrase of Lemma 5.4.2. We get the conclusion by (7.2.14) and (7.2.15). □

From the above two lemmas, using Lemma 7.2.1 and Proposition 7.2.1, we finally obtain the desired formula.

THEOREM 7.2.1. *Let $\eta=\eta_0[c_0]$ be an irreducible $(\mathfrak{h}_{n,\mathbb{C}},K_n\cap H_n)$-module and $\lambda=[\mathbf{l}\,;l_0]\in\mathcal{L}_n^{+}$. Suppose the condition (7.1.3) is satisfied for them. Let $\varphi\in C^{\infty}_{\eta,\tau_\lambda}(A_n)$ and $\{f(\mathbf{m}\,;r)|\ \mathbf{m}\in\Lambda^{+}_{n-1}(\eta|\lambda)\}$ its standard coefficient. Then for $r>0,\ r\neq 1$*

$$
\begin{aligned}
(7.2.16)\quad &2\mathrm{Rad}(\Omega_{\eta,\tau})\varphi(a_r)\\
&=\sum_{\mathbf{m}\in\Lambda^{+}_{n-1}(\eta|\lambda)}\Bigg[\Big(r\frac{d}{dr}\Big)^2 f(\mathbf{m}\,;r)\\
&\quad+\Big(\frac{1}{\mathrm{th}(r)}+(2n-1)\mathrm{th}(r)\Big)r\frac{d}{dr}f(\mathbf{m}\,;r)\\
&\quad+\Bigg\{-\frac{1}{\mathrm{ch}^2(r)\mathrm{sh}^2(r)}\Big(\frac{\eta_{\mathbf{m}}-\tau_{\lambda,\mathbf{m}}}{2}\Big)^2+2\Omega_{M_n}(\mathbf{m}\,;z_0-|\mathbf{m}|)
\end{aligned}
$$

$$+ \frac{1}{\mathrm{ch}^2(r)}\Big(\tau_{\lambda,\mathbf{m}}\eta_{\mathbf{m}} - \eta_{\mathbf{m}}^2 - \tau_{\lambda,\mathbf{m}}^2 + 2\Omega_{H_n}(\eta) + 2\Omega_{K_n}(\mathbf{l}\,;l_0)$$

$$- 4\Omega_{M_n}(\mathbf{m}\,;z_0 - |\mathbf{m}|)\Big)\Big\}f(\mathbf{m}\,;r)\Big] \cdot \omega_{\eta,\lambda}(\mathbf{m})$$

$$+ \sum_{\mathbf{m}\in\Lambda_{n-1}^+(\eta|\lambda)} \frac{4\mathrm{th}(r)}{\mathrm{ch}(r)} \sum_{i=1}^{n-1}\Big(a_i(\mathbf{l}\,;\mathbf{m})A_i^\eta(\mathbf{m})f(\mathbf{m}\,;r)\cdot\omega_{\eta,\lambda}(\mathbf{m}^{+i})$$

$$- b_i(\mathbf{l}\,;\mathbf{m})B_i^\eta(\mathbf{m})f(\mathbf{m}\,;r)\cdot\omega_{\eta,\lambda}(\mathbf{m}^{-i})\Big),$$

where

$$\tau_{\lambda,\mathbf{m}} = -|\mathbf{m}| + z_0 - 2l_0, \qquad \eta_{\mathbf{m}} = |\mathbf{m}| - z_0 + 2c_0$$

with $z_0 = c_n(\tau_\lambda) = c_n(\eta)$.

7.3. A_n-radial part of Schmid operators

To begin with we introduce a notation, that will be used in the sequel: For $\lambda = [\mathbf{l}\,;l_0] \in \mathcal{L}_n^+$, $\epsilon \in \{+,-\}$ and $k \in \{1,\ldots,n\}$, put

$$\lambda^{\epsilon k} = \lambda + \epsilon\beta_k = [\mathbf{l}^{\epsilon k}\,;l_0 - \epsilon 1] \in \mathcal{L}_n.$$

Let η and $\lambda = [\mathbf{l}\,;l_0] \in \mathcal{L}_n^+$ be as in 7.1. We impose on them the condition (7.1.3). There exist the linear operator

$$\mathrm{Rad}(\nabla_{\eta,\lambda}^{\pm k}) : C_{\eta,\tau_\lambda}^\infty(A_n) \to C_{\eta,\tau_{\lambda^{\pm k}}}^\infty(A_n)$$

such that $\mathrm{Rad}(\nabla_{\eta,\lambda}^{\pm k})(F|A_n) = (\nabla_{\eta,\lambda}^{\pm k}F)|A_n$ for $F \in C_{\eta,\tau_\lambda}^\infty(H_n\backslash G_n/K_n)$, which will be called *the A_n-radial part of* $\nabla_{\eta,\lambda}^{\pm k}$. In this subsection we have an explicit form of the A_n-radial part of the Schmid operators in terms of the standard coefficients. The final result is given in Theorem 7.3.1.

First we prove lemmas for later use. Recall the map ϵ_λ in Proposition 6.1.1.

Lemma 7.3.1. *Let* $\mathbf{m} \in \Lambda_{n-1}^+(\eta|\lambda)$ *and* $\beta \in R_{\mathrm{nc}}(\lambda)$.

(1) *If* $\beta = \beta_k$, *then we have*

$$\omega_{\eta,\lambda}(\mathbf{m}) \circ \epsilon_\lambda(E_{n+1,n}) \circ I_{\beta_k}(\lambda) = \gamma^+(\mathbf{m}\,;k,\mathbf{l}) \cdot \omega_{\eta,\lambda^{+k}}(\mathbf{m}). \tag{7.3.1}$$

(2) *If $\beta = -\beta_k$, then we have*

$$\omega_{\eta,\lambda}(\mathbf{m}) \circ \epsilon_\lambda(E_{n,n+1}) \circ I_{-\beta_k}(\lambda) = \gamma^-(\mathbf{m}\ ; k, \mathbf{l}) \cdot \omega_{\eta,\lambda^{-k}}(\mathbf{m}). \tag{7.3.2}$$

Proof. We consider the case (1). Since $\mathrm{Ad}(m)(E_{n+1,n}) = E_{n+1,n}$ for $m \in M_n$, the map $\epsilon_\lambda(E_{n+1,n})$ belongs to the space

$$\mathrm{Hom}_{M_n}(\tau_\lambda \otimes \mathrm{Ad}_{\mathfrak{p}_{n,\mathbb{C}}}|M_n, \tau_\lambda|M_n).$$

Put $\mu = (\mathbf{m}\ ; c_n(\tau_\lambda) - |\mathbf{m}|)$. Then $\mathsf{p}^{\mathbf{l}}_{\mathbf{m}} \in \mathrm{Hom}_{M_n}(\tau_\lambda|M_n, \sigma_\mu)$. Hence $\mathsf{p}^{\mathbf{l}}_{\mathbf{m}} \circ \epsilon_\lambda(E_{n+1,n})$ is in

$$\mathrm{Hom}_{M_n}(\tau_\lambda \otimes \mathrm{Ad}_{\mathfrak{p}_{n,\mathbb{C}}}|M_n, \sigma_\mu). \tag{7.3.3}$$

Let $P_\beta(\lambda) : \tau_\lambda \otimes \mathrm{Ad}_{\mathfrak{p}_\mathbb{C}} \to \tau_{\lambda+\beta}$ be the K_n-projection corresponding to the decomposition (6.1.1). By (6.1.1) and Proposition 3.2.1, the linear maps $\mathsf{p}^{\mathbf{l}^{\epsilon k}}_{\mathbf{m}} \circ P_{\epsilon\beta_k}(\lambda)$ with $k \in \{1, \ldots, n\}$, $\epsilon \in \{+, -\}$ such that $\mathbf{m} \in \Delta(\mathbf{l}^{\epsilon k})$ make up a basis of the space (7.3.3). Since $\mathsf{p}^{\mathbf{l}}_{\mathbf{m}} \circ \epsilon_\lambda(E_{n+1,n})$ is zero on $W_\lambda \otimes \mathfrak{p}_n^-$, there exist constants $c_k,\ k \in \{1, \ldots, n\}$ such that

$$\mathsf{p}^{\mathbf{l}}_{\mathbf{m}} \circ \epsilon_\lambda(E_{n+1,n}) = \sum_{k=1}^n c_k \cdot \mathsf{p}^{\mathbf{l}^{+k}}_{\mathbf{m}} \circ P_{\beta_k}(\lambda).$$

Since $P_{\beta_k}(\lambda) \circ I_{\beta_k}(\lambda) = 1_{W(\mathbf{l}^{+k})}$, we have

$$\mathsf{p}^{\mathbf{l}}_{\mathbf{m}} \circ \epsilon_\lambda(E_{n+1,n}) \circ I_{\beta_k}(\lambda) = c_k \cdot \mathsf{p}^{\mathbf{l}^{+k}}_{\mathbf{m}} \tag{7.3.4}$$

for $k \in \{1, \ldots, n\}$. Let $Q = (\mathbf{l}^{+k}\ ; \mathbf{m}\ ; Q') \in GZ^{(n)}(\mathbf{l}^{+k})$ with $Q' \in GZ^{(n-2)}$. The right-hand side of (7.3.4) evaluated at $|Q\rangle$ gives $c_k \cdot \mathsf{p}^{\mathbf{l}^{+k}}_{\mathbf{m}}|Q\rangle$. Using the formula (6.1.2), we see that the left-hand side evaluated at $|Q\rangle$ becomes

$$\gamma^+(\mathbf{m}\ ; \mathbf{l}, k)\mathsf{p}^{\mathbf{l}}_{\mathbf{m}}\big|Q_n^{-k}\big\rangle.$$

Since $\mathsf{p}^{\mathbf{l}^{+k}}_{\mathbf{m}}|Q\rangle = \mathsf{p}^{\mathbf{l}}_{\mathbf{m}}\big|Q_n^{-k}\big\rangle = \big|\mathbf{m}\ ; Q'\big\rangle$, we get $c_k = \gamma^+(\mathbf{m}\ ; \mathbf{l}, k)$. This completes the proof of (7.3.1). The formula (7.3.2) can be proved in the same way. □

Lemma 7.3.2. *We have*

$$\begin{aligned}(7.3.5)\qquad &\sum_{i=1}^{n-1}\tau_\lambda(E_{i,n})\circ\epsilon_\lambda(E_{n+1,i})\\ &=\sum_{i=1}^{n-1}\epsilon_\lambda(E_{n+1,n})\circ(\tau_\lambda\otimes\mathrm{Ad}_{\mathfrak{p}_n^+})(E_{n,i}E_{i,n})\\ &\quad-\sum_{i=1}^{n-1}\tau_\lambda(E_{n,i}E_{i,n})\circ\epsilon_\lambda(E_{n+1,n})-(n-1)\epsilon_\lambda(E_{n+1,n}),\end{aligned}$$

$$\begin{aligned}(7.3.6)\qquad &-\sum_{i=1}^{n-1}\tau_\lambda(E_{n,i})\circ\epsilon_\lambda(E_{i,n+1})\\ &=\sum_{i=1}^{n-1}\epsilon_\lambda(E_{n,n+1})\circ(\tau_\lambda\otimes\mathrm{Ad}_{\mathfrak{p}_n^-})(E_{i,n}E_{n,i})\\ &\quad-\sum_{i=1}^{n-1}\tau_\lambda(E_{i,n}E_{n,i})\circ\epsilon_\lambda(E_{n,n+1})-(n-1)\epsilon_\lambda(E_{n,n+1}).\end{aligned}$$

Proof. This is a paraphrase of formulas found in [17, p.350] and easily checked by a direct computation. □

Lemma 7.3.3. *Let* $\mathbf{m}\in\Lambda_{n-1}^+(\eta|\lambda)$ *and* $\beta\in R_{\mathrm{nc}}(\lambda)$.

(1) *If* $\beta=\beta_k$, *then we have*

$$\begin{aligned}(7.3.7)\qquad &\sum_{i=1}^{n-1}\omega_{\eta,\lambda}(\mathbf{m})\circ\tau_\lambda(E_{i,n})\circ\epsilon_\lambda(E_{n+1,i})\circ I_{\beta_k}(\lambda)\\ &\qquad=-(\tau_{\lambda,\mathbf{m}}+l_0-l_k+k-1)\gamma^+(\mathbf{m}\,;\mathbf{l},k)\cdot\omega_{\eta,\lambda^{+k}}(\mathbf{m}).\end{aligned}$$

(2) *If* $\beta=-\beta_k$, *then we have*

$$\begin{aligned}(7.3.8)\qquad &\sum_{i=1}^{n-1}\omega_{\eta,\lambda}(\mathbf{m})\circ\tau_\lambda(E_{n,i})\circ\epsilon_\lambda(E_{i,n+1})\circ I_{-\beta_k}(\lambda)\\ &\qquad=-(\tau_{\lambda,\mathbf{m}}+l_0-l_k+k-n)\gamma^-(\mathbf{m}\,;\mathbf{l},k)\cdot\omega_{\eta,\lambda^{-k}}(\mathbf{m}).\end{aligned}$$

PROOF. This is essentially [17, Lemma 3.1.2]. We shall reproduce the proof in our frame work. We prove (7.3.7). The formula (7.2.10) can be rewritten as

$$\sum_{i=1}^{n-1} E_{n,i}E_{i,n} = D_M + \frac{1}{2}\Omega_{K_n} + \widetilde{H}E_{n+1,n+1} - \frac{n-1}{2}E_{n+1,n+1} - E_{n+1,n+1}^2$$

with $D_M = -\frac{1}{2}\Omega_{M_n} - \frac{1}{2}Z_0 + \frac{n}{2}\widetilde{H} - \frac{1}{4}\widetilde{H}^2$; D_M belongs to $Z(\mathfrak{m}_{n,\mathbb{C}})$. Thus if $\lambda' = [\mathbf{l}' ; l_0'] \in \mathcal{L}_n^+$ and $\psi \in \mathrm{Hom}_{M_n}(\tau_{\lambda'}|M_n, \sigma_\mu)$, then we have

$$(7.3.9) \qquad \psi \circ \tau_{\lambda'}(\sum_{i=1}^{n-1} E_{n,i}E_{i,n}) = \left(D_M(\mu) + \frac{1}{2}\Omega_{K_n}(\mathbf{l}' ; l_0') + (z_0 - |\mathbf{m}|)l_0' - \frac{n-1}{2}l_0' - {l_0'}^2\right)\psi,$$

where $\mu = (\mathbf{m} ; z_0 - |\mathbf{m}|)$ and $D_M(\mu)$ is the scalar by which the operator $\sigma_\mu(D_M)$ acts on $W(\mathbf{m})$. By (7.3.5), we have

$$(7.3.10) \qquad \sum_{i=1}^{n-1} \mathsf{p}_{\mathbf{m}}^{\mathbf{l}} \circ \tau_\lambda(E_{i,n}) \circ \epsilon_\lambda(E_{n+1,i}) \circ I_{\beta_k}(\lambda)$$
$$= (\mathsf{p}_{\mathbf{m}}^{\mathbf{l}} \circ \epsilon_\lambda(E_{n+1,n}) \circ I_{\beta_k}(\lambda)) \circ \tau_{\lambda+k}(\sum_{i=1}^{n-1} E_{n,i}E_{i,n})$$
$$- \mathsf{p}_{\mathbf{m}}^{\mathbf{l}} \circ \tau_\lambda(\sum_{i=1}^{n-1} E_{n,i}E_{i,n}) \circ \epsilon_\lambda(E_{n+1,n}) \circ I_{\beta_k}(\lambda)$$
$$- (n-1)\mathsf{p}_{\mathbf{m}}^{\mathbf{l}} \circ \epsilon_\lambda(E_{n+1,n}) \circ I_{\beta_k}(\lambda).$$

Applying (7.3.9) with taking $\mathsf{p}_{\mathbf{m}}^{\mathbf{l}} \circ \epsilon_\lambda(E_{n+1,n}) \circ I_{\beta_k}(\lambda)$ for ψ, we know that the first term of the right-hand side of (7.3.10) becomes

$$\Big(D_M(\mu) + \frac{1}{2}\Omega_{K_n}(\mathbf{l}^{+k} ; l_0 - 1) + (z_0 - |\mathbf{m}|)(l_0 - 1) - \frac{n-1}{2}(l_0 - 1) - (l_0 - 1)^2\Big) \times \mathsf{p}_{\mathbf{m}}^{\mathbf{l}} \circ \epsilon_\lambda(E_{n+1,n}) \circ I_{\beta_k}(\lambda).$$

As for the second term, we also apply (7.3.9) with taking $\mathsf{p}^{\mathbf{l}}_{\mathbf{m}}$ for ψ to get

$$-\Big(D_M(\mu)+\frac{1}{2}\Omega_{K_n}(\mathbf{l}\,;l_0)+(z_0-|\mathbf{m}|)l_0-\frac{n-1}{2}l_0-l_0^2\Big)\mathsf{p}^{\mathbf{l}}_{\mathbf{m}}\circ\epsilon_\lambda(E_{n+1,n})\circ I_{\beta_k}(\lambda).$$

Substituting these two expressions to (7.3.10), we have

$$\begin{aligned}(7.3.11)\qquad &\sum_{i=1}^{n-1}\mathsf{p}^{\mathbf{l}}_{\mathbf{m}}\circ\tau_\lambda(E_{i,n})\circ\epsilon_\lambda(E_{n+1,i})\circ I_{\beta_k}(\lambda)\\ &=\Big(\frac{1}{2}\Omega_{K_n}(\mathbf{l}^{+k}\,;l_0-1)+(z_0-|\mathbf{m}|)(l_0-1)\\ &\quad-\frac{n-1}{2}(l_0-1)-(l_0-1)^2\\ &\quad-\frac{1}{2}\Omega_{K_n}(\mathbf{l}\,;l_0)-(z_0-|\mathbf{m}|)l_0\\ &\quad+\frac{n-1}{2}l_0+l_0^2-(n-1)\Big)\mathsf{p}^{\mathbf{l}}_{\mathbf{m}}\circ\epsilon_\lambda(E_{n+1,n})\circ I_{\beta_k}(\lambda).\end{aligned}$$

After some elementary computations using the formula of $\Omega_{K_n}(\mathbf{l}^{+k}\,;l_0-1)$ and $\Omega_{K_n}(\mathbf{l}\,;l_0)$ in Proposition 6.2.2 (2), we have that the right-hand side of (7.3.11) equals

$$-(\tau_{\lambda,\mathbf{m}}+l_0-l_k+k-1)\mathsf{p}^{\mathbf{l}}_{\mathbf{m}}\circ\epsilon_\lambda(E_{n+1,n})\circ I_{\beta_k}(\lambda).$$

The formula (7.3.7) follows from this combined with Lemma 7.3.1. The formula (7.3.8) can be treated in the same way. □

LEMMA 7.3.4. *For $\mathbf{l}\in\Lambda_n^+$, $\mathbf{m}\in\Lambda_{n-1}^+$, $1\leqslant k\leqslant n$ and $1\leqslant i\leqslant n-1$, set*

$$\begin{aligned}&\gamma^+(\mathbf{m},i\,;\mathbf{l},k)\\ &=S(i,k)\left|\frac{\prod_{h\neq i,h=1}^{n-1}(m_h-l_k+k-h-1)\prod_{h\neq k,h=1}^{n}(l_h-m_i+i-h)}{\prod_{h\neq k,h=1}^{n}(l_h-l_i+i-h)\prod_{h\neq i,h=1}^{n-1}(m_h-m_i+i-h-1)}\right|^{1/2},\\ &\gamma^-(\mathbf{m},i\,;\mathbf{l},k)\end{aligned}$$

$$= S(i,k)\left|\frac{\prod_{h\neq i,h=1}^{n-1}(m_h - l_k + k - h)\prod_{h\neq k,h=1}^{n}(l_h - m_i + i - h + 1)}{\prod_{h\neq k,h=1}^{n}(l_h - l_k + k - h)\prod_{h\neq i,h=1}^{n-1}(m_h - m_i + i - h + 1)}\right|^{1/2}$$

with $S(i,k)$ expressing $+1$ or -1 according to $i \geqslant k$ or $i < k$ respectively. Let $\mathbf{m} \in \Delta(\mathbf{l})$ and $i,k \in \{1,\ldots,n\}$, $i \neq n$ satisfying $\mathbf{m}^{+i} \in \Delta(\mathbf{l}^{+k})$ resp. $\mathbf{m}^{-i} \in \Delta(\mathbf{l}^{-k})$. Then for any Gelfand-Zetlin scheme $Q = (\mathbf{q}_j)_{1 \leqslant j \leqslant n-1} \in GZ^{(n-1)}(\mathbf{m}^{+i})$ resp. $\in GZ^{(n-1)}(\mathbf{m}^{-i})$ we have

$$\begin{aligned}&\mathsf{p}^{\mathbf{l}}_{\mathbf{m}} \circ \epsilon_\lambda(E_{n+1,n-1}) \circ I_{\beta_k}(\lambda)|\mathbf{l}^{+k}, \mathbf{m}^{+i}, Q\rangle\\ &\quad = \gamma^+(\mathbf{q}_{n-2}\ ; \mathbf{m}, i)\gamma^+(\mathbf{m}, i\ ; \mathbf{l}, k)|\mathbf{m}, Q\rangle,\\ &\quad\ \text{resp.}\\ &\mathsf{p}^{\mathbf{l}}_{\mathbf{m}} \circ \epsilon_\lambda(E_{n-1,n+1}) \circ I_{-\beta_k}(\lambda)|\mathbf{l}^{-k}, \mathbf{m}^{-i}, Q\rangle\\ &\quad = \gamma^-(\mathbf{q}_{n-2}\ ; \mathbf{m}, i)\gamma^-(\mathbf{m}, i\ ; \mathbf{l}, k)|\mathbf{m}, Q\rangle.\end{aligned}$$

PROOF. This can be deduced from the formulas in [21, page 385]. □

LEMMA 7.3.5. *Let $\mathbf{m} \in \Lambda^+_{n-1}(\eta|\lambda)$ and $\beta \in R_{\mathrm{nc}}(\lambda)$.*

(1) *If $\beta = \beta_k$, then we have*

$$\begin{aligned}&\sum_{i=1}^{n-1} \eta(E_{i,n+1}) \circ \omega_{\eta,\lambda}(\mathbf{m}) \circ \epsilon_\lambda(E_{n+1,i}) \circ I_{\beta_k}(\lambda)\\ &\quad = \sum_{i=1}^{n-1} \gamma^+(\mathbf{m}, i\ ; \mathbf{l}, k) A^\eta_i(\mathbf{m}) \cdot \omega_{\eta,\lambda^{+k}}(\mathbf{m}^{+i}).\end{aligned} \tag{7.3.12}$$

(2) *If $\beta = -\beta_k$, then we have*

$$\begin{aligned}&\sum_{i=1}^{n-1} \eta(E_{n+1,i}) \circ \omega_{\eta,\lambda}(\mathbf{m}) \circ \epsilon_\lambda(E_{i,n+1}) \circ I_{-\beta_k}(\lambda)\\ &\quad = \sum_{i=1}^{n-1} \gamma^-(\mathbf{m}, i\ ; \mathbf{l}, k) B^\eta_i(\mathbf{m}) \cdot \omega_{\eta,\lambda^{-k}}(\mathbf{m}^{-i}).\end{aligned} \tag{7.3.13}$$

Proof. We prove (7.3.12). Let $\mathrm{pr}^+ : \mathfrak{p}_n^+ \to (\mathfrak{h}_n \cap \mathfrak{p}_n)^+$ denote the projection corresponding to the direct sum decomposition $\mathfrak{p}_n^+ = (\mathfrak{h}_n \cap \mathfrak{p}_n)^+ \oplus (\mathfrak{q}_n \cap \mathfrak{p}_n)^+$; pr^+ is an M_n-homomorphism and

$$\mathrm{pr}^+(Y) = \sum_{i=1}^{n-1} \langle Y, E_{n+1,i}\rangle E_{i,n+1}, \quad Y \in \mathfrak{p}_n^+.$$

Firstly we have the formula

$$\sum_{i=1}^{n-1} \eta(E_{i,n+1}) \circ \omega_{\eta,\lambda}(\mathbf{m}) \circ \epsilon_\lambda(E_{n+1,i}) = [\iota_{\mathbf{m}}^\eta]^+ \circ (\mathsf{p}_{\mathbf{m}}^{\mathbf{l}} \otimes \mathrm{pr}^+). \tag{7.3.14}$$

Indeed, for every $w \in W(\mathbf{l})$ and $Y \in (\mathfrak{h}_n \cap \mathfrak{p}_n)^+$, by the definition of $[\iota_{\mathbf{m}}^\eta]$, we have

$$\begin{aligned}
[\iota_{\mathbf{m}}^\eta]^+ \circ (\mathsf{p}_{\mathbf{m}}^{\mathbf{l}} \otimes \mathrm{pr}^+)(w \otimes Y) &= \eta(\mathrm{pr}^+(Y)) \circ \iota_{\mathbf{m}}^\eta \circ \mathsf{p}_{\mathbf{m}}^{\mathbf{l}}(w) \\
&= \eta\Big(\sum_{i=1}^{n-1} \langle Y, E_{n+1,i}\rangle E_{i,n+1}\Big) \circ \omega_{\eta,\lambda}(\mathbf{m})(w) \\
&= \sum_{i=1}^{n-1} \eta(E_{i,n+1}) \circ \omega_{\eta,\lambda}(\mathbf{m})(\langle Y, E_{n+1,i}\rangle w) \\
&= \sum_{i=1}^{n-1} \eta(E_{i,n+1}) \circ \omega_{\eta,\lambda}(\mathbf{m}) \circ \epsilon_\lambda(E_{n+1,i})(Y \otimes w).
\end{aligned}$$

Let $\Delta_{\mathbf{m}}$ denote the left hand-side of (7.3.12). Then by using (7.3.14) and (5.4.2), we have

$$\begin{aligned}
\Delta_{\mathbf{m}} &= [\iota_{\mathbf{m}}^\eta]^+ \circ (\mathsf{p}_{\mathbf{m}}^{\mathbf{l}} \otimes \mathrm{pr}^+) \circ I_{\beta_k}(\lambda) \\
&= -\sum_{i=1}^{n-1} A_i^\eta(\mathbf{m}) \iota_{\mathbf{m}^{+i}}^\eta \circ P_i^+(\mathbf{m}) \circ (\mathsf{p}_{\mathbf{m}}^{\mathbf{l}} \otimes \gamma^{-1} \circ \mathrm{pr}^+) \circ I_{\beta_k}(\lambda).
\end{aligned} \tag{7.3.15}$$

Put $\mu = (\mathbf{m}\,; z_0 - |\mathbf{m}|)$. Since $P_i^+(\mathbf{m}) \circ (\mathsf{p}_{\mathbf{m}}^{\mathbf{l}} \otimes \gamma^{-1} \circ \mathrm{pr}^+) \circ I_{\beta_k}(\lambda)$ lives in

$$\mathrm{Hom}_{M_n}(\tau_{\lambda^{+k}}|M_n, \sigma_{\mu^{+i}}) = \mathbb{C}\mathsf{p}_{\mathbf{m}^{+i}}^{\mathbf{l}^{+k}},$$

there exists a constant c_{ik} such that

$$P_i^+(\mathbf{m}) \circ (\mathsf{p}_{\mathbf{m}}^{\mathbf{l}} \otimes \gamma^{-1} \circ \mathrm{pr}^+) \circ I_{\beta_k}(\lambda) = c_{ik} \cdot \mathsf{p}_{\mathbf{m}^{+i}}^{\mathbf{l}^{+k}}. \tag{7.3.16}$$

Substituting this into the final formula of (7.3.15), we get

$$\Delta_{\mathbf{m}} = -\sum_{i=1}^{n-1} A_i^{\eta}(\mathbf{m}) c_{ik} \cdot \omega_{\eta,\lambda^{+k}}(\mathbf{m}^{+i}).$$

To conclude the proof, it suffices to determine the values of c_{ik}'s.

Noting $\sum_{i=1}^{n-1} I_i^+(\mathbf{m}) \circ P_i^+(\mathbf{m}) = 1_{W(\mathbf{m})\otimes\mathbb{C}(\mathfrak{k}_n\cap\mathfrak{q}_n)^+}$, the equations (7.3.16) for $i \in \{1, \dots, n-1\}$ gives us the identity

$$(\mathsf{p}^{\mathbf{l}}_{\mathbf{m}} \otimes \gamma^{-1} \circ \mathrm{pr}^+) \circ I_{\beta_k}(\lambda) = \sum_{i=1}^{n-1} c_{ik} \cdot I_i^+(\mathbf{m}) \circ \mathsf{p}^{\mathbf{l}^{+k}}_{\mathbf{m}^{+i}}. \tag{7.3.17}$$

Using the identity

$$-\tilde{\epsilon}_{\mu}(E_{n,n-1}) \circ (\mathsf{p}^{\mathbf{l}}_{\mathbf{m}} \otimes \gamma^{-1} \circ \mathrm{pr}^+) = \mathsf{p}^{\mathbf{l}}_{\mathbf{m}} \circ \epsilon_{\lambda}(E_{n+1,n-1}),$$

we get from (7.3.17) the formula

$$\begin{aligned} &- \mathsf{p}^{\mathbf{l}}_{\mathbf{m}} \circ \epsilon_{\lambda}(E_{n+1,n-1}) \circ I_{\beta_k}(\lambda) \\ &= \sum_{i=1}^{n-1} c_{ik} \cdot \tilde{\epsilon}_{\mu}(E_{n,n-1}) \circ I_i^+(\mathbf{m}) \circ \mathsf{p}^{\mathbf{l}^{+k}}_{\mathbf{m}^{+i}}. \end{aligned} \tag{7.3.18}$$

Let $\tilde{Q} = (\mathbf{l}^{+k}\ ; \mathbf{m}^{+i}\ ; Q') \in GZ^{(n)}(\mathbf{l}^{+k})$ with $Q' = (\mathbf{q}_j)_{1\leqslant j\leqslant n-2} \in GZ^{(n-2)}$. By Lemma 7.3.4, we have

$$\mathsf{p}^{\mathbf{l}}_{\mathbf{m}} \circ \epsilon_{\lambda}(E_{n+1,n-1}) \circ I_{\beta_k}(\lambda)|\tilde{Q}\rangle = \gamma^+(\mathbf{q}_{n-2}\ ; \mathbf{m}, i)\gamma^+(\mathbf{m}, i\ ; \mathbf{l}, k)|\mathbf{m}\ ; Q'\rangle.$$

On the other hand, the right-hand side of (7.3.18) evaluated at $\tilde{Q}$ equals

$$c_{ik} \cdot \tilde{\epsilon}_{\mu}(E_{n,n-1}) \circ I_i^+(\mathbf{m})|\mathbf{m}^{+i}\ ; Q'\rangle$$

and this becomes

$$c_{ik} \cdot \gamma^+(\mathbf{q}_{n-2}\ ; \mathbf{m}, i)|\mathbf{m}\ ; Q'\rangle$$

by (5.3.5). Hence the identity (7.3.18) evaluated at $\left|\widetilde{Q}\right\rangle$ gives us

$$-\gamma^+(\mathbf{q}_{n-2}\ ; \mathbf{m}, i)\gamma^+(\mathbf{m}, i\ ; \mathbf{l}, k)|\mathbf{m}\ ; Q'\rangle = c_{ik} \cdot \gamma^+(\mathbf{q}_{n-2}\ ; \mathbf{m}, i)|\mathbf{m}\ ; Q'\rangle.$$

Thus we get $c_{ik} = -\gamma^+(\mathbf{m}, i\ ; \mathbf{l}, k)$ to conclude the proof. □

Now we have the desired formula.

Theorem 7.3.1. *Let* $\eta = \eta_0[c_0]$ *be an irreducible* $(\mathfrak{h}_{n,\mathbb{C}}, K_n \cap H_n)$-*module and* $\lambda = [\mathbf{l}\ ; l_0] \in \mathcal{L}_n^+$. *Suppose the condition* (7.1.3) *is satisfied for them. Let* $\varphi \in C^\infty_{\eta,\tau_\lambda}(A_n)$ *with standard coefficient* $\{f(\mathbf{m}\ ; r) |\ \mathbf{m} \in \Lambda^+_{n-1}(\eta|\lambda)\}$. *For* $\beta \in R_{\mathrm{nc}}(\lambda)$, *the* A_n-*radial part of* $\nabla^\beta_{\eta,\lambda}$ *is given as follows:*

(1) *If* $\beta = \beta_k$, *then*

$$\begin{aligned}
&(7.3.19)\quad \mathrm{Rad}(\nabla^{+k}_{\eta,\lambda})\varphi(a_r)\\
&= \sum_{\mathbf{m}\in\Lambda^+_{n-1}(\eta|\lambda)} \gamma^+(\mathbf{m}\ ; \mathbf{l}, k)\Big\{\frac{1}{2} r\frac{d}{dr} f(\mathbf{m}\ ; r)\\
&+ \Big(\frac{1}{4\mathrm{sh}(r)\mathrm{ch}(r)}\eta_{\mathbf{m}} - \frac{1}{4}\big(\mathrm{th}(r) + \frac{1}{\mathrm{th}(r)}\big)\tau_{\lambda,\mathbf{m}}\\
&+ \mathrm{th}(r)(\tau_{\lambda,\mathbf{m}} + l_0 - l_k + k - 1)\Big) f(\mathbf{m}\ ; r)\Big\} \cdot \omega_{\eta,\lambda^{+k}}(\mathbf{m})\\
&+ \sum_{\mathbf{m}\in\Lambda^+_{n-1}(\eta|\lambda)} \sum_{i=1}^{n-1} \gamma^+(\mathbf{m}, i\ ; \mathbf{l}, k)\frac{A_i^\eta(\mathbf{m})}{\mathrm{ch}(r)} f(\mathbf{m}\ ; r)\\
&\cdot \omega_{\eta,\lambda^{+k}}(\mathbf{m}^{+i}).
\end{aligned}$$

(2) *If* $\beta = -\beta_k$, *then*

$$\begin{aligned}
&(7.3.20)\quad \mathrm{Rad}(\nabla^{-k}_{\eta,\lambda})\varphi(a_r)\\
&= \sum_{\mathbf{m}\in\Lambda^+_{n-1}(\eta|\lambda)} \gamma^-(\mathbf{m}\ ; \mathbf{l}, k)\Big\{\frac{1}{2} r\frac{d}{dr} f(\mathbf{m}\ ; r)\\
&+ \Big(\frac{-1}{4\mathrm{sh}(r)\mathrm{ch}(r)}\eta_{\mathbf{m}} + \frac{1}{4}\big(\mathrm{th}(r) + \frac{1}{\mathrm{th}(r)}\big)\tau_{\lambda,\mathbf{m}}\\
&- \mathrm{th}(r)(\tau_{\lambda,\mathbf{m}} + l_0 - l_k + k - n)\Big) f(\mathbf{m}\ ; r)\Big\} \cdot \omega_{\eta,\lambda^{-k}}(\mathbf{m})\\
&+ \sum_{\mathbf{m}\in\Lambda^+_{n-1}(\eta|\lambda)} \sum_{i=1}^{n-1} \gamma^-(\mathbf{m}, i\ ; \mathbf{l}, k)\frac{B_i^\eta(\mathbf{m})}{\mathrm{ch}(r)} f(\mathbf{m}\ ; r)\\
&\cdot \omega_{\eta,\lambda^{-k}}(\mathbf{m}^{-i}).
\end{aligned}$$

Here

$$\tau_{\lambda,\mathbf{m}} = -|\mathbf{m}| + z_0 - 2l_0, \qquad \eta_{\mathbf{m}} = |\mathbf{m}| - z_0 + 2c_0$$

with $z_0 = c_n(\tau_\lambda) = c_n(\eta)$. *The number* $\gamma^{\pm}(\mathbf{m}\ ;\mathbf{l}, k)$ *is given in 3.1 and* $\gamma^{\pm}(\mathbf{m}, i\ ;\mathbf{l}, k)$ *is given in Lemma 7.3.4.*

PROOF. Take an $F \in C^{\infty}_{\eta,\tau_\lambda}(H_n\backslash G_n/K_n)$ with $F|A_n = \varphi$. We give a proof of (7.3.19). By (6.1.5), we first have

$$\nabla^{+k}_{\eta,\lambda} F(a_r) = \sum_{i=1}^{n} R_{E_{i,n+1}} F(a_r) \circ \epsilon_\lambda(E_{n+1,i}) \circ I_{\beta_k}(\lambda). \tag{7.3.21}$$

Noting Lemma 7.2.1 and using the formulas (7.2.2) and (7.2.4), we have

$$\begin{aligned} &R_{E_{i,n+1}} F(a) \\ &= \frac{1}{\mathrm{ch}(r)} \eta(E_{i,n+1}) \circ F(a) - \mathrm{th}(r) F(a) \circ \tau_\lambda(E_{i,n}), \quad i \in \{1, \dots, n-1\}, \\ &R_{E_{n,n+1}} F(a) \\ &= \frac{1}{4\mathrm{sh}(r)\mathrm{ch}(r)} \eta(H') \circ F(a) + \frac{1}{2} R_{H_1} F(a) - \frac{1}{4}\big(\mathrm{th}(r) + \frac{1}{\mathrm{th}(r)}\big) F(a) \circ \tau_\lambda(H'). \end{aligned}$$

Inserting these to (7.3.21), we get

$$\begin{aligned} &\nabla^{+k}_{\eta,\lambda} F(a_r) \\ &= \bigg(\frac{1}{4\mathrm{sh}(r)\mathrm{ch}(r)} \eta(H') \circ F(a) + \frac{1}{2} R_{H_1} F(a) \\ &\qquad - \frac{1}{4}\big(\mathrm{th}(r) + \frac{1}{\mathrm{th}(r)}\big) F(a) \circ \tau_\lambda(H') \bigg) \circ \epsilon_\lambda(E_{n+1,n}) \circ I_{\beta_k}(\lambda) \\ &\quad + \sum_{i=1}^{n-1} \bigg(\frac{1}{\mathrm{ch}(r)} \eta(E_{i,n+1}) \circ F(a) \\ &\qquad - \mathrm{th}(r) F(a) \circ \tau_\lambda(E_{i,n}) \bigg) \circ \epsilon_\lambda(E_{n+1,i}) \circ I_{\beta_k}(\lambda) \end{aligned} \tag{7.3.22}$$

Now substituting the expression

$$F(a) = \sum_{\mathbf{m} \in \Lambda^{+}_{n-1}(\eta|\lambda)} f(\mathbf{m}\ ; r) \cdot \omega_{\eta,\lambda}(\mathbf{m})$$

to the right-hand side of (7.3.22), and using the formulas (7.2.12), (7.3.1), (7.3.7) and (7.3.12) we compute

$$
\begin{aligned}
&\nabla^{+k}_{\eta,\lambda}F(a_r)\\
&= \sum_{\mathbf{m}\in\Lambda^+_{n-1}(\eta|\lambda)} \Big\{\frac{1}{2}r\frac{d}{dr}f(\mathbf{m}\,;r)\\
&\quad + \Big(\frac{1}{4\mathrm{sh}(r)\mathrm{ch}(r)}\eta_{\mathbf{m}} - \frac{1}{4}\big(\mathrm{th}(r)+\frac{1}{\mathrm{th}(r)}\big)\tau_{\lambda,\mathbf{m}}\Big)f(\mathbf{m}\,;r)\Big\}\\
&\quad\quad \times \omega_{\eta,\lambda}(\mathbf{m})\circ\epsilon_\lambda(E_{n+1,n})\circ I_{\beta_k}(\lambda)\\
&\quad + \sum_{\mathbf{m}\in\Lambda^+_{n-1}(\eta|\lambda)} f(\mathbf{m}\,;r)\\
&\quad \times \Big\{\frac{1}{\mathrm{ch}(r)}\sum_{i=1}^{n-1}\eta(E_{i,n+1})\circ\omega_{\eta,\lambda}(\mathbf{m})\circ\epsilon_\lambda(E_{n+1,i})\circ I_{\beta_k}(\lambda)\\
&\quad\quad - \mathrm{th}(r)\sum_{i=1}^{n-1}\omega_{\eta,\lambda}(\mathbf{m})\circ\tau_\lambda(E_{i,n})\circ\epsilon_\lambda(E_{n+1,i})\circ I_{\beta_k}(\lambda)\Big\}\\
&= \sum_{\mathbf{m}\in\Lambda^+_{n-1}(\eta|\lambda)} \Big\{\frac{1}{2}r\frac{d}{dr}f(\mathbf{m}\,;r)\\
&\quad + \Big(\frac{1}{4\mathrm{sh}(r)\mathrm{ch}(r)}\eta_{\mathbf{m}} - \frac{1}{4}\big(\mathrm{th}(r)+\frac{1}{\mathrm{th}(r)}\big)\tau_{\lambda,\mathbf{m}}\Big)f(\mathbf{m}\,;r)\Big\}\\
&\quad\quad \times \gamma^+(\mathbf{m}\,;\mathbf{l},k)\cdot\omega_{\eta,\lambda^{+k}}(\mathbf{m})\\
&\quad + \sum_{\mathbf{m}\in\Lambda^+_{n-1}(\eta|\lambda)} f(\mathbf{m}\,;r)\Big\{\frac{1}{\mathrm{ch}(r)}\sum_{i=1}^{n-1}\gamma^+(\mathbf{m},i\,;\mathbf{l},k)A_i(\mathbf{m})\cdot\omega_{\eta,\lambda^{+k}}(\mathbf{m}^{+i})\\
&\quad\quad + \mathrm{th}(r)(\tau_{\lambda,\mathbf{m}}+l_0-l_k+k-1)\gamma^+(\mathbf{m}\,;\mathbf{l},k)\cdot\omega_{\eta,\lambda^{+k}}(\mathbf{m})\Big\}.
\end{aligned}
$$

This completes the proof. □

7.4. Difference-differential equations for standard coefficients of Shintani functions

Let $\eta = \eta_0[c_0]$ be an irreducible $(\mathfrak{h}_{n,\mathbb{C}}, K_n \cap H_n)$-module and π an irreducible $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-module. We suppose that $z_0 = c_n(\eta) = c_n(\pi)$ is satisfied. Given $\Phi \in \mathcal{I}_{\eta,\pi}$, we have defined a system of functions $\{\Phi_{\mathbf{l}} |\ \mathbf{l} \in \Lambda^+_n(\pi)\}$ in

6.3. Since $\Phi_{\mathbf{l}}$ belongs to the space $C^{\infty}_{\eta,\tau_{[\mathbf{l};z_0-|\mathbf{l}|]}}(H_n\backslash G_n/K_n)$, we can consider its standard coefficients $\{f_{\mathbf{l}}(\mathbf{m}\ ;r)|\ \mathbf{m}\in\Lambda^+_{n-1}(\eta|\lambda)\}$ as in Definition 7.1.1. In Proposition 6.3.1, we already have a system of equations for the family of the Shintani functions $\Phi_{\mathbf{l}}$'s in an abstruct form. Using the explicit formula of A_n-radial part of the Casimir operators and the Schmid operators obtained in Theorem 7.2.1 and Theorem 7.3.1, we can rewrite the equations in Proposition 6.3.1 in terms of the standard coefficients to have a system of difference-differential equations among $f_{\mathbf{l}}(\mathbf{m}\ ;r)$'s in a quite explicit form. In the sequel we write $\tau^{\mathbf{l}}_{\mathbf{m}}$ for the number $\tau_{\lambda,\mathbf{m}}$ if $\lambda=[\mathbf{l}\ ;z_0-|\mathbf{l}|]$.

THEOREM 7.4.1.

(1) *For every* $\mathbf{l}\in\Lambda^+_n(\pi)$ *and* $\mathbf{m}\in\Lambda^+_{n-1}(\eta|\lambda)$, *the functions* $f_{\mathbf{l}}(\mathbf{m}\ ;r)$, $f_{\mathbf{l}}(\mathbf{m}^{+i}\ ;r)$, $i\in\{1,\dots,n-1\}$ *and* $f_{\mathbf{l}}(\mathbf{m}^{-i}\ ;r)$, $i\in\{1,\dots,n-1\}$ *satisfy the difference-differential equation*

$$\begin{aligned}(\mathsf{C})_{\mathbf{l},\mathbf{m}}:\quad &\left(r\frac{d}{dr}\right)^2 f_{\mathbf{l}}(\mathbf{m}\ ;r)+\left(\frac{1}{\mathrm{th}(r)}+(2n-1)\mathrm{th}(r)\right)r\frac{d}{dr}f_{\mathbf{l}}(\mathbf{m}\ ;r)\\
&+\Bigg\{-\frac{1}{\mathrm{ch}^2(r)\mathrm{sh}^2(r)}\left(\frac{\eta_{\mathbf{m}}-\tau^{\mathbf{l}}_{\mathbf{m}}}{2}\right)^2\\
&\quad+2\Omega_{M_n}(\mathbf{m}\ ;z_0-|\mathbf{m}|)-2\Omega_{G_n}(\pi)\\
&\quad+\frac{1}{\mathrm{ch}^2(r)}\Big(\tau^{\mathbf{l}}_{\mathbf{m}}\eta_{\mathbf{m}}-\eta^2_{\mathbf{m}}-{\tau^{\mathbf{l}}_{\mathbf{m}}}^2+2\Omega_{H_n}(\eta)\\
&\quad+2\Omega_{K_n}(\mathbf{l}\ ;z_0-|\mathbf{l}|)-4\Omega_{M_n}(\mathbf{m}\ ;z_0-|\mathbf{m}|)\Big)\Bigg\}f_{\mathbf{l}}(\mathbf{m}\ ;r)\\
&+\frac{4\mathrm{th}(r)}{\mathrm{ch}(r)}\sum_{i=1}^{n-1}\Big(a_i(\mathbf{l}\ ;\mathbf{m}^{-i})A^{\eta}_i(\mathbf{m}^{-i})f_{\mathbf{l}}(\mathbf{m}^{-i}\ ;r)\\
&\quad-b_i(\mathbf{l}\ ;\mathbf{m}^{+i})B^{\eta}_i(\mathbf{m}^{+i})f_{\mathbf{l}}(\mathbf{m}^{+i}\ ;r)\Big)\\
&=0.\end{aligned}$$

(2) *For every* $\mathbf{l}\in\Lambda^+_n(\pi)$ *and* $\mathbf{m}\in\Lambda^+_{n-1}(\eta|\lambda)$ *such that* $m_{k-1}>l_k$, *the functions* $f_{\mathbf{l}}(\mathbf{m}\ ;r)$, $f_{\mathbf{l}}(\mathbf{m}^{-i}\ ;r)$, $i\in\{1,\dots,n-1\}$ *and* $f_{\mathbf{l}^{+k}}(\mathbf{m}\ ;r)$ *satisfies the equation*

$$(\mathsf{S}^{+k})_{\mathbf{l},\mathbf{m}}:\quad \gamma^+(\mathbf{m}\ ;\mathbf{l},k)\Bigg\{\Bigg(r\frac{d}{dr}+\frac{\eta_{\mathbf{m}}-\tau^{\mathbf{l}}_{\mathbf{m}}}{2\mathrm{sh}(r)\mathrm{ch}(r)}$$

$$+ \mathrm{th}(r)(z_0 - |\mathbf{m}| + 2k - 2l_k - 2)\Big) f_{\mathbf{l}}(\mathbf{m}\ ; r)$$

$$+ \frac{2}{\mathrm{ch}(r)} \sum_{i=1}^{n-1} \frac{a_i(\mathbf{l}\ ; \mathbf{m}^{-i}) A_i^{\eta}(\mathbf{m}^{-i})}{l_k - m_i + i - k + 1} f_{\mathbf{l}}(\mathbf{m}^{-i}\ ; r)\Big\}$$

$$= 2A_k^{\pi}(\mathbf{l}) f_{\mathbf{l}^{+k}}(\mathbf{m}\ ; r).$$

(3) *For every* $\mathbf{l} \in \Lambda_n^+(\pi)$ *and* $\mathbf{m} \in \Lambda_{n-1}^+(\eta|\lambda)$ *such that* $\mathbf{m}^{+k} \in \Lambda_{n-1}^+(\eta) - \Delta(\mathbf{l})$*, the functions* $f_{\mathbf{l}}(\mathbf{m}\ ; r)$ *and* $f_{\mathbf{l}^{+k}}(\mathbf{m}^{+k}\ ; r)$ *satisfy the equation*

$$(\mathsf{T}^{+k})_{\mathbf{l},\mathbf{m}} : \quad \frac{1}{\mathrm{ch}(r)} A_k^{\eta}(\mathbf{m}) f_{\mathbf{l}}(\mathbf{m}\ ; r) = A_k^{\pi}(\mathbf{l}) f_{\mathbf{l}^{+k}}(\mathbf{m}^{+k}\ ; r).$$

(4) *For every* $\mathbf{l} \in \Lambda_n^+(\pi)$ *and* $\mathbf{m} \in \Lambda_{n-1}^+(\eta|\lambda)$ *such that* $l_k > m_k$*, the functions* $f_{\mathbf{l}}(\mathbf{m}\ ; r)$*,* $f_{\mathbf{l}}(\mathbf{m}^{+i}\ ; r)$*,* $i \in \{1, \dots, n-1\}$ *and* $f_{\mathbf{l}^{-k}}(\mathbf{m}\ ; r)$ *satisfy the equation*

$$(\mathsf{S}^{-k})_{\mathbf{l},\mathbf{m}} : \quad \gamma^-(\mathbf{m}\ ; \mathbf{l}, k)\Big\{\Big(r\frac{d}{dr} - \frac{\eta_{\mathbf{m}} - \tau_{\mathbf{m}}^{\mathbf{l}}}{2\mathrm{sh}(r)\mathrm{ch}(r)}$$

$$- \mathrm{th}(r)(z_0 - |\mathbf{m}| + 2k - 2l_k - 2n)\Big) f_{\mathbf{l}}(\mathbf{m}\ ; r)$$

$$+ \frac{2}{\mathrm{ch}(r)} \sum_{i=1}^{n-1} \frac{b_i(\mathbf{l}\ ; \mathbf{m}^{+i}) B_i^{\eta}(\mathbf{m}^{+i})}{l_k - m_i + i - k} f_{\mathbf{l}}(\mathbf{m}^{+i}\ ; r)\Big\}$$

$$= 2B_k^{\pi}(\mathbf{l}) f_{\mathbf{l}^{-k}}(\mathbf{m}\ ; r).$$

(5) *For* $\mathbf{l} \in \Lambda_n^+(\pi)$ *and* $\mathbf{m} \in \Lambda_{n-1}^+(\eta|\lambda)$ *such that* $\mathbf{m}^{-(k-1)} \in \Lambda_{n-1}^+(\eta) - \Delta(\mathbf{l})$*, the functions* $F_{\mathbf{l}}(\mathbf{m}\ ; r)$ *and* $F_{\mathbf{l}^{-k}}(\mathbf{m}^{-(k-1)}\ ; r)$ *satisfy the equation*

$$(\mathsf{T}^{-k})_{\mathbf{l},\mathbf{m}} : \quad \frac{1}{\mathrm{ch}(r)} B_{k-1}^{\eta}(\mathbf{m}) f_{\mathbf{l}}(\mathbf{m}\ ; r) = B_k^{\pi}(\mathbf{l}) f_{\mathbf{l}^{-k}}(\mathbf{m}^{-(k-1)}\ ; r).$$

Here

$$\tau_{\mathbf{m}}^{\mathbf{l}} = 2|\mathbf{l}| - |\mathbf{m}| - z_0, \quad \eta_{\mathbf{m}} = |\mathbf{m}| - z_0 + 2c_0.$$

Proof. We can obtain the equation $(\mathsf{C})_{\mathbf{l},\mathbf{m}}$ from the equation (6.3.2) using Theorem 7.2.1 easily. Next we shall explain how to deduce the equations $(\mathsf{S}^{+k})_{\mathbf{l},\mathbf{m}}$ and $(\mathsf{T}^{+k})_{\mathbf{l},\mathbf{m}}$ from the equation (6.3.3): Take an $\mathbf{l} \in \Lambda_n^+(\pi)$

and an integer $k \in \{1, \ldots, n\}$. We start from the formula (7.3.19) with $\lambda = [\mathbf{l}\ ; z_0 - |\mathbf{l}|]$. We examine the second summation with respect to $\mathbf{m}$ in the right-hand side of (7.3.19). For simplicity, we put

$$\phi_i(\mathbf{m}) = \gamma^+(\mathbf{m}, i\ ; \mathbf{l}, k)\frac{A_i^\eta(\mathbf{m})}{\mathrm{ch}(r)} f_{\mathbf{l}}(\mathbf{m}\ ; r), \quad i \in \{1, \ldots, n-1\}$$

for $\mathbf{m} \in \Lambda_{n-1}^+(\eta|\lambda)$ and $\phi_i(\mathbf{m}) = 0$ for $\mathbf{m} \in \Lambda_{n-1} - \Lambda_{n-1}^+(\eta|\lambda)$. We have

$$\begin{aligned}
&\sum_{\mathbf{m}\in\Lambda_{n-1}^+(\eta|\lambda)} \sum_{i=1}^{n-1} \phi_i(\mathbf{m}) \cdot \omega_{\eta,\lambda^{+k}}(\mathbf{m}^{+i}) \\
&= \sum_{\mathbf{m}\in\Lambda_{n-1}^+(\eta|\lambda)} \sum_{i\neq k} \phi_i(\mathbf{m}) \cdot \omega_{\eta,\lambda^{+k}}(\mathbf{m}^{+i}) + \sum_{\mathbf{m}\in\mathcal{E}_{\eta,\lambda}^{+k}} \phi_k(\mathbf{m}) \cdot \omega_{\eta,\lambda^{+k}}(\mathbf{m}^{+k}) \\
&\quad + \sum_{\mathbf{m}\in\Lambda_{n-1}^+(\eta|\lambda)-\mathcal{E}_{\eta,\lambda}^{+k}} \phi_k(\mathbf{m}) \cdot \omega_{\eta,\lambda^{+k}}(\mathbf{m}^{+k}) \\
&= \sum_{i\neq k} \sum_{\mathbf{m}^{-i}\in\Lambda_{n-1}^+(\eta|\lambda)} \phi_i(\mathbf{m}^{-i}) \cdot \omega_{\eta,\lambda^{+k}}(\mathbf{m}) + \sum_{\mathbf{m}\in\mathcal{E}_{\eta,\lambda}^{+k}} \phi_k(\mathbf{m}) \cdot \omega_{\eta,\lambda^{+k}}(\mathbf{m}^{+k}) \\
&\quad + \sum_{\mathbf{m}^{-k}\in\Lambda_{n-1}^+(\eta|\lambda)-\mathcal{E}_{\eta,\lambda}^{+k}} \phi_k(\mathbf{m}^{-k}) \cdot \omega_{\eta,\lambda^{+k}}(\mathbf{m}).
\end{aligned}$$

Here $\mathcal{E}_{\eta,\lambda}^{+k}$ denotes the set of all $\mathbf{m} \in \Lambda_{n-1}^+(\eta|\lambda)$ such that $\mathbf{m}^{+k} \in \Lambda_{n-1}^+(\eta) - \Delta(\mathbf{l})$ when $k \in \{1, \ldots, n-1\}$, and $\mathcal{E}_{\eta,\lambda}^{+n} = \emptyset$. We show that if a given $\mathbf{m} \in \Lambda_{n-1}$ satisfies

(a) $i \neq k$, $\mathbf{m}^{-i} \in \Lambda_{n-1}^+(\eta|\lambda)$ and $\mathbf{m} \notin \Lambda_{n-1}^+(\eta|\lambda)$ or

(b) $\mathbf{m}^{-k} \in \Lambda_{n-1}^+(\eta|\lambda)$, $\mathbf{m}^{-k} \notin \mathcal{E}_{\eta,\lambda}^{+k}$ and $\mathbf{m} \notin \Lambda_{n-1}^+(\eta|\lambda)$,

then $\omega_{\eta,\lambda^{+k}}(\mathbf{m}) = 0$, i.e. $\mathbf{m} \notin \Lambda_{n-1}^+(\eta|\lambda^{+k})$.
Assume $\mathbf{m} \in \Lambda_{n-1}^+(\eta)$ satisfies the condition (a). Then we have $\mathbf{m}^{-i} \in \Delta(\mathbf{l})$ and $\mathbf{m} \notin \Delta(\mathbf{l})$, hence $m_i = l_i + 1$. Since $i \neq k$, the condition $m_i = l_i + 1$ implies $\mathbf{m} \notin \Delta(\mathbf{l}^{+k})$. Next we assume $\mathbf{m} \in \Lambda_{n-1}^+(\eta)$ satisfies the condition (b). The requirments $\mathbf{m}^{-k} \notin \mathcal{E}_{\eta,\lambda}^{+k}$ and $\mathbf{m}^{-k} \in \Lambda_{n-1}^+(\eta|\lambda)$ are equivalent to $\mathbf{m} \notin \Lambda_{n-1}^+(\eta) - \Delta(\mathbf{l})$. Since we assume $\mathbf{m} \in \Lambda_{n-1}^+(\eta)$, we have $\mathbf{m} \in \Delta(\mathbf{l})$ to get $\mathbf{m} \in \Lambda_{n-1}^+(\eta|\lambda)$. But this contradicts the assumption in (b). Thus we see that (b) implies $\mathbf{m} \notin \Lambda_{n-1}^+(\eta)$.

Consequently we have that $\mathrm{Rad}(\nabla^{+k}_{\eta,\lambda})\Phi_{\mathbf{l}}(a_r)$ is expressed as a sum of

$$(7.4.1)\qquad \sum_{\mathbf{m}\in\Lambda^+_{n-1}(\eta|\lambda)}\left(\gamma^+(\mathbf{m}\,;\mathbf{l},k)F(\mathbf{m})+\sum_{i=1}^{n-1}\phi_i(\mathbf{m}^{-i})\right)\cdot\omega_{\eta,\lambda^{+k}}(\mathbf{m})$$

with

$$F(\mathbf{m})=\frac{1}{2}r\frac{d}{dr}f_{\mathbf{l}}(\mathbf{m}\,;r)+\left\{\frac{1}{2\mathrm{sh}(r)\mathrm{ch}(r)}(\eta_{\mathbf{m}}-\tau^{\mathbf{l}}_{\mathbf{m}})\right.$$
$$\left.+\,\mathrm{th}(r)\left(\tau^{\mathbf{l}}_{\mathbf{m}}+2(l_0-l_k+k-1)\right)\right\}f_{\mathbf{l}}(\mathbf{m}\,;r)$$

and

$$(7.4.2)\qquad \sum_{\mathbf{m}\in\mathcal{E}^{+k}_{\eta,\lambda}}\phi_k(\mathbf{m})\cdot\omega_{\eta,\lambda^{+k}}(\mathbf{m}^{+k}).$$

Take an $\mathbf{m}\in\Lambda^+_{n-1}(\eta|\lambda)$ with $m_{k-1}>l_k$; then $\omega_{\eta,\lambda^{+k}}(\mathbf{m})\neq 0$. Thus comparing the coefficients of $\omega_{\eta,\lambda^{+k}}(\mathbf{m})$ in the both sides of the equation (6.3.3) evaluated at $g=a_r$ with using the formula

$$\gamma^+(\mathbf{m}^{-i},i\,;\mathbf{l},k)(l_k-m_i+i-k+1)=\gamma^+(\mathbf{m}\,;\mathbf{l},k)a_i(\mathbf{l}\,;\mathbf{m}^{-i}),$$

we get the equation $(\mathsf{S}^{+k})_{\mathbf{l},\mathbf{m}}$. Take an $\mathbf{m}\in\Lambda^+_{n-1}(\eta|\lambda)$ with $\mathbf{m}^{+k}\in\Lambda^+_{n-1}(\eta)-\Delta(\mathbf{l})$; then $\gamma^+(\mathbf{m},k\,;\mathbf{l},k)=1$. Noting this, we get the equation $(\mathsf{T}^{+k})_{\mathbf{l},\mathbf{m}}$ by comparing the coefficients of $\omega_{\eta,\lambda^{+k}}(\mathbf{m}^{+k})$ in the both sides of the equality (6.3.3). The equations $(\mathsf{S}^{-k})_{\mathbf{l},\mathbf{m}}$ and $(\mathsf{T}^{-k})_{\mathbf{l},\mathbf{m}}$ can be deduced from the equation (6.3.4) in the same way. □

8. Explicit Formula of Shintani Function and Multiplicity Free Theorem for $\mathcal{I}_{\eta,\pi}$

In this section we prove a multiplicity free theorem for the space of intertwining operators $\mathcal{I}_{\eta,\pi}$ (Theorem 8.3.1) and give an explicit formula of some of the standard coefficients of Shintani functions with a special K_n-type (Theorem 8.2.1). The proof of Theorem 8.2.1 will be given in the next section. In the first subsection we give a necessary condition for the space $\mathcal{I}_{\eta,\pi}$ to be non-zero (Theorem 8.1.1).

8.1. A neccessary condition of $\mathcal{I}_{\eta,\pi} \neq \{0\}$

Let η be an irreducible $(\mathfrak{h}_{n,\mathbb{C}}, K_n \cap H_n)$-module.
For an integer i with $1 \leqslant i \leqslant n-1$, put

$$m_i^+(\eta) = \sup\{m_i |\ \mathbf{m} = (m_j)_{1\leqslant j\leqslant n-1} \in \Lambda_{n-1}^+(\eta)\},$$
$$m_i^-(\eta) = \inf\{m_i |\ \mathbf{m} = (m_j)_{1\leqslant j\leqslant n-1} \in \Lambda_{n-1}^+(\eta)\}.$$

By the explicit form of $\Lambda_{n-1}^+(\eta)$ given by Lemma 4.2.1 and Theorem 4.3.1(1), we have $m_i^+(\eta) \geqslant m_i^-(\eta),\ 1 \leqslant i \leqslant n-1$,

$$+\infty \geqslant m_1^+(\eta) \geqslant m_2^+(\eta) \geqslant \cdots \geqslant m_{n-1}^+(\eta) > -\infty,$$
$$+\infty > m_1^-(\eta) \geqslant m_2^-(\eta) \geqslant \cdots \geqslant m_{n-1}^-(\eta) \geqslant -\infty$$

and

$$\Lambda_{n-1}^+(\eta) = \{\mathbf{m} \in \Lambda_{n-1}^+ |\ m_i^+(\eta) \geqslant m_i \geqslant m_i^-(\eta),\ 1 \leqslant i \leqslant n-1\}.$$

Put $m_n^+(\eta) = -\infty$, $m_0^-(\eta) = +\infty$. For any $\lambda = [\mathbf{l}\ ; l_0] \in \mathcal{L}_n^+$ with $\mathbf{l} = (l_k)_{1\leqslant k\leqslant n} \in \Lambda_n^+$ such that $c_n(\tau_\lambda) = c_n(\eta)$, we have

$$(8.1.1)\quad \Lambda_{n-1}^+(\eta|\lambda) = \{\mathbf{m} \in \Lambda_{n-1}^+ |\ \inf(l_i, m_i^+(\eta)) \geqslant m_i \geqslant \sup(l_{i+1}, m_i^-(\eta)),\ 1 \leqslant i \leqslant n-1\}.$$

Lemma 8.1.1. *Let η be as above and π an irreducible $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-module with $c_n(\pi) = c_n(\eta)$. Let $\Phi \in \mathcal{I}_{\eta,\pi}$ and $\{\Phi_{\mathbf{l}} |\ \mathbf{l} \in \Lambda_n^+(\pi)\}$ the corresponding system of Shintani functions. Let $\mathbf{l} = (l_k)_{1\leqslant k\leqslant n} \in \Lambda_n^+$ and k an integer with $1 \leqslant k \leqslant n$. If*

(a) *$m_k^+(\eta) > l_k$, $\mathbf{l}^{+k} \in \Lambda_n^+$ and $A_k^\pi(\mathbf{l}) = 0$, or*

(b) *$l_k > m_{k-1}^-(\eta)$, $\mathbf{l}^{-k} \in \Lambda_n^+$ and $B_k^\pi(\mathbf{l}) = 0$,*

then we have $\Phi_{\mathbf{l}}(g) = 0,\ g \in G_n$.

Proof. We assume the condition (a) and will show $\Phi_{\mathbf{l}} = 0$. Let $\{f(\mathbf{m}\ ; r) |\ \mathbf{m} \in \Lambda_{n-1}^+(\eta|\lambda)\}$ be the standard coefficients of $\Phi_{\mathbf{l}}|A_n$. Since $m_n^+(\eta) = -\infty$, the condition (a) means $1 \leqslant k < n$. The assumption $m_k^+(\eta) > l_k$ implies that the set

$$\mathcal{E}_{\eta,\lambda}^{+k} = \{\mathbf{m} \in \Lambda_{n-1}^+(\eta|\lambda) |\ \mathbf{m}^{+k} \in \Lambda_{n-1}^+(\eta) - \Delta(\mathbf{l})\}.$$

is non empty and actually coincides with the set of those $\mathbf{m} = (m_i)_{1 \leqslant i \leqslant n-1} \in \Lambda^+_{n-1}(\eta|\lambda)$ with $m_k = l_k$. (Note we always have $l_k \geqslant m_k$ for $\mathbf{m} \in \Lambda^+_{n-1}(\eta|\lambda)$.) For any $\mathbf{m} \in \Lambda^+_{n-1}(\eta|\lambda)$, set $\delta_k(\mathbf{m}) = l_k - m_k$. We shall prove $f(\mathbf{m}\,;r) = 0$ for $\mathbf{m} \in \Lambda^+_{n-1}(\eta|\lambda)$ by induction on the integer $\delta_k(\mathbf{m})$. If $\delta_k(\mathbf{m}) = 0$, then $\mathbf{m} \in \mathcal{E}^{+k}_{\eta,\lambda}$. Hence by the equation $(\mathsf{T}^{+k})_{\mathbf{l},\mathbf{m}}$ in Theorem 7.4.1, we have $A^\eta_k(\mathbf{m})f(\mathbf{m}\,;r) = 0$. Since $\mathbf{m}$, $\mathbf{m}^{+k} \in \Lambda^+_{n-1}(\eta)$, the irreducibility of η means $A^\eta_k(\mathbf{m}) \neq 0$; see [9, Corollary 7.2] and [9, Proposition 8.3]. Thus we have $f(\mathbf{m}\,;r) = 0$ in this case. Let $d > 0$ and take an index $\mathbf{m} \in \Lambda^+_{n-1}(\eta|\lambda)$ with $\delta_k(\mathbf{m}) = d$. We show $f(\mathbf{m}\,;r) = 0$, assuming that $f(\mathbf{m}'\,;r) = 0$ for all $\mathbf{m}' \in \Lambda^+_{n-1}(\eta|\lambda)$ with $\delta_k(\mathbf{m}') < d$. If $\mathbf{m}^{+k} \notin \Lambda^+_{n-1}(\eta)$, then we must have $m_k = m^+_k(\eta) > l_k$, a contradiction. Hence we have $\mathbf{m}^{+k} \in \Lambda^+_{n-1}(\eta)$ on one hand. From the assumption $\delta_k(\mathbf{m}) = d > 0$, we have $\mathbf{m}^{+k} \notin \Lambda^+_{n-1}(\eta) - \Delta(\mathbf{l})$ on the other hand. Thus $\mathbf{m}^{+k} \in \Lambda^+_{n-1}(\eta|\lambda)$. Now we can use the equation $(\mathsf{S}^{+k})_{\mathbf{l},\mathbf{m}^{+k}}$ in Theorem 7.4.1 to have

$$
\begin{aligned}
r\frac{d}{dr}&f(\mathbf{m}^{+k}\,;r) + F_k(\mathbf{m}^{+k}\,;r)f(\mathbf{m}^{+k}\,;r)\\
&+ \frac{1}{\mathrm{ch}(r)}\sum_{i=1,i\neq k}^{n-1}\frac{a_i(\mathbf{l}\,;\mathbf{m}^{-i,+k})A^\eta_i(\mathbf{m}^{-i,+k})}{l_k - m_i + i - k + 1}f(\mathbf{m}^{-i,+k}\,;r)\\
&+ \frac{1}{\mathrm{ch}(r)}\frac{a_k(\mathbf{l}\,;\mathbf{m})A^\eta_k(\mathbf{m})}{l_k - m_k}f(\mathbf{m}\,;r) = 0
\end{aligned}
$$

with a function $F_k(\mathbf{m}^{+k}\,;r)$. It is obvious that $\delta_k(\mathbf{m}^{+k}) = \delta_k(\mathbf{m}^{-i,+k}) = d - 1 < d$ for $i \neq k$. Hence by the induction-assumption, all the terms in the left-hand side except those involving $f(\mathbf{m}\,;r)$ are zero. Thus we get

$$a_k(\mathbf{l}\,;\mathbf{m})A^\eta_k(\mathbf{m})f(\mathbf{m}\,;r) = 0.$$

The number $a_k(\mathbf{l}\,;\mathbf{m})$ is not zero because $\mathbf{m}$, $\mathbf{m}^{+k} \in \Delta(\mathbf{l})$. As for the number $A^\eta_k(\mathbf{m})$, we know that it is also not zero by [9, Corollary 7.2] and [9, Proposition 8.3] since $\mathbf{m}$, $\mathbf{m}^{+k} \in \Lambda^+_{n-1}(\eta)$. Thus $f(\mathbf{m}\,;r) = 0$ as desired. □

THEOREM 8.1.1. *Let η be an irreducible $(\mathfrak{h}_{n,\mathbb{C}}, K_n \cap H_n)$-module and π an irreducible $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-module. Let $\lambda = [\mathbf{l}\,;l_0] \in \mathcal{L}^+_n(\pi)$ be a D_h-corner of π with $0 \leqslant h \leqslant n$. If $\mathcal{I}_{\eta,\pi} \neq \{0\}$, then we have*

(a) $c_n(\eta) = c_n(\pi)$, $\quad \Lambda^+_{n-1}(\eta|\lambda) \neq \emptyset$,

(b) $m^-_{k-1}(\eta) \geqslant l_k$ for $k \in \{1, \dots, h\}$ with $\mathbf{l}^{-k} \in \Lambda^+_n$,

(c) $m^+_k(\eta) \leqslant l_k$ for $k \in \{h+1, \dots, n\}$ with $\mathbf{l}^{+k} \in \Lambda^+_n$.

Here $m^+_n(\eta) = -\infty$ *and* $m^-_0(\eta) = +\infty$.

Proof. Since the $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-module π is irreducible and since $G_n = H_n A_n K_n$, $\Phi = 0$ if and only if $\Phi_{\mathbf{l}}|A_n$ is identically zero. Hence the linear map $\Phi \mapsto \Phi_{\mathbf{l}}|A_n$ is injective. In particular $\mathcal{I}_{\eta,\pi} \neq \{0\}$ implies $C^\infty_{\eta,\tau_\lambda}(A_n) \neq \{0\}$; hence $c_n(\eta) = c_n(\pi)$ and $\Lambda^+_{n-1}(\eta|\lambda) \neq \emptyset$ by Proposition 7.1.1. The remaining conditions in the theorem come from the previous lemma; indeed, that $\mathbf{l}$ is a D_h-corner of π means $B^\pi_k(\mathbf{l}) = 0$ for $k \in \{1, \dots, h\}$ and $A^\pi_k(\mathbf{l}) = 0$ for $k \in \{h+1, \dots, n\}$. $\square$

8.2. Explicit formula of Shintani functions

Let $\eta = \eta_0[c_0]$ be an irreducible $(\mathfrak{h}_{n,\mathbb{C}}, K_n \cap H_n)$-module and π an irreducible $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-module with $z_0 = c_n(\pi)$. From now on we assume the conditions (a), (b) and (c) in Theorem 8.1.1. In this subsection we give an explicit formula of A_n-radial part of Shintani functions for corner K_n-type. First of all, we introduce a notation: For an integer $h \in \{0, \dots, n\}$, put $h^+ = \sup(1, h)$ and $h^- = \inf(n-1, h)$.

We prescribe to π a triple $(\mathbf{l}_0, h, s)$ with $\mathbf{l}_0 = (l_k)_{1 \leqslant k \leqslant n} \in \Lambda^+_n(\pi)$, $h \in \{0, \dots n\}$, $s \in \mathbb{C}$ as follows:

(i) If π is elementary, then we take $s \in \mathbb{C}$ and $\mathbf{p} = (p_j)_{1 \leqslant j \leqslant n-1} \in \Lambda^+_{n-1}$ so that $\pi \cong \pi(\mathbf{p}, z_0 - |\mathbf{p}|\ ; s)$. Put $l_k = p_k,\ 1 \leqslant k \leqslant n-1,\ l_n = p_{n-1}$ and $h = n$.

(ii) If π is non elementary, then we can take $\zeta = (u, v, \mathbf{r}, z_0) \in \mathcal{S}_n$ so that π belongs to the class Π_ζ (see Theorem 4.3.1.). Here u, v and z_0 are integers such that $0 \leqslant u \leqslant v \leqslant n$, and $\mathbf{r} = (r_j)_{1 \leqslant j \leqslant n+1} \in \Lambda^+_{n+1}$ such that $|\mathbf{r}| - z_0 = u + v - n$ and either $r_u > r_{u+1}$ or $r_{v+1} > r_{v+2}$. When $r_{v+1} > r_{v+2}$ we put $\lambda_0 = \lambda^v(\zeta)$ and $h = v$. When $r_{v+1} = r_{v+2}$ we put $\lambda_0 = \lambda^u(\zeta)$ and $h = u$. Since $c_n(\pi) = z_0$, λ_0 is of the form $\lambda_0 = [\mathbf{l}_0\ ; z_0 - |\mathbf{l}_0|]$ with $\mathbf{l}_0 \in \Lambda^+_n(\pi)$. Put $s = |\mathbf{l}_0| + l_{h^+} - z_0 - 2h^+ + n$.

Note that $\lambda_0 = [\mathbf{l}_0\ ; z_0 - |\mathbf{l}_0|] \in \mathcal{L}^+_n(\pi)$ is a D_{n-1}-corner of π if π is elementary and is a fundamental D_h-corner of π and $l_h > l_{h+1}$ if π is non-elementary.

Definition 8.2.1. For an integer h with $0 \leqslant h \leqslant n$ and a subset $\mathcal{M}$ of Λ_{n-1}^{+}, we denote by $\partial^{(h)}(\mathcal{M})$ the set of all $\mathbf{m} \in \mathcal{M}$ such that $\mathbf{m}^{-i} \notin \mathcal{M}$ for $1 \leqslant i < h$ and $\mathbf{m}^{+i} \notin \mathcal{M}$ for $h < i \leqslant n-1$.

Lemma 8.2.1. *Let* $\mathbf{l} \in \Lambda_n^{+}$ *and* h *an integer with* $0 \leqslant h \leqslant n$. *Then the set* $\partial^{(h)}\Lambda_{n-1}^{+}(\eta|\lambda)$ *consists of a unique element if* $h = 0$ *or* $h = n$. *If* $0 < h < n$, *then projecting the* h*-th coordinate, we have a bijection from* $\partial^{(h)}\Lambda_{n-1}^{+}(\eta|\lambda)$ *onto the interval* $\inf(l_h, m_h^{+}(\eta)) \geqslant m_h \geqslant \sup(l_{h+1}, m_h^{-}(\eta))$ *in* $\mathbb{Z}$.

Proof. This follows readily from (8.1.1). □

For any $\mathbf{m} = (m_i)_{1 \leqslant i \leqslant n-1} \in \partial^{(h)}\Lambda_{n-1}^{+}(\eta|\lambda_0)$, put

$$\begin{aligned} \mu_{\mathbf{m}} &= m_h, \quad \text{if } 0 < h < n, \\ \mu_{\mathbf{m}} &= l_1, \quad \text{if } h = 0, \\ \mu_{\mathbf{m}} &= l_n, \quad \text{if } h = n, \end{aligned} \tag{8.2.1}$$

and define the numbers $\theta_{\mathbf{m}}$, $\beta_{\mathbf{m}}$ and $\alpha_{\mathbf{m}}$ by

$$\theta_{\mathbf{m}} = \sum_{h^{+} \leqslant k \leqslant n-1} l_{k+1} - \sum_{1 \leqslant k \leqslant h^{-}} l_k - \sum_{h^{+} \leqslant i \leqslant n-1} m_i + \sum_{1 \leqslant i \leqslant h^{-}} m_i \tag{8.2.2}$$

and

$$\beta_{\mathbf{m}} = \big||\mathbf{m}| - |\mathbf{l}_0| + c_0\big|, \tag{8.2.3}$$

$$\alpha_{\mathbf{m}} = -\beta_{\mathbf{m}} + \mu_{\mathbf{m}} - l_{h^{+}} - n + s + \theta_{\mathbf{m}}. \tag{8.2.4}$$

Theorem 8.2.1. *Let* $(\mathbf{l}_0, h, s)$ *be the triplet for* π *as above. Given* $\Phi \in \mathcal{I}_{\eta,\pi}$, *let* $\Phi_{\mathbf{l}_0}$ *be the Shintani function with* K_n*-type* $\lambda_0 = [\mathbf{l}_0\ ; z_0 - |\mathbf{l}_0|]$. *Let* $\{f(\mathbf{m}\ ; r)|\ \mathbf{m} \in \Lambda_{n-1}^{+}(\eta|\lambda_0)\}$ *be the standard coefficients for* $\Phi_{\mathbf{l}_0}|A_n$. *Then there exists a family of constants* $\{\gamma(\mathbf{m})|\ \mathbf{m} \in \partial^{(h)}\Lambda_{n-1}^{+}(\eta|\lambda_0)\}$ *such that* $f(\mathbf{m}\ ; r)$ *with* $\mathbf{m} \in \partial^{(h)}\Lambda_{n-1}^{+}(\eta|\lambda_0)$ *equals*

$$\begin{aligned} &\gamma(\mathbf{m})(\mathrm{sh}(r))^{\beta_{\mathbf{m}}}(\mathrm{ch}(r))^{\alpha_{\mathbf{m}}} \\ &\quad \times {}_2F_1\Bigg(\frac{s_0 - s + \beta_{\mathbf{m}} - \mu_{\mathbf{m}} + l_{h^{+}} + 1}{2}, \\ &\qquad\qquad \frac{-s_0 - s + \beta_{\mathbf{m}} - \mu_{\mathbf{m}} + l_{h^{+}} + 1}{2}\ ; 1 + \beta_{\mathbf{m}}\ ; \mathrm{th}^2(r)\Bigg) \end{aligned} \tag{8.2.5}$$

with s_0 *a complex number determined by*

$$s_0^2 = 2\Omega_{H_n}(\eta) - 2c_0^2 + (n-1)^2 - (|\mathbf{m}| - \mu_{\mathbf{m}} - z_0 + c_0)^2 \\ + 2\Big(-\sum_{i=1}^{n-1} m_i^2 - \sum_{i=1}^{n-1}(n-2i)m_i + \mu_{\mathbf{m}}^2 \\ + (n-2h^+)\mu_{\mathbf{m}} + |\mathbf{l}_0| - 2\sum_{h^+<k\leqslant n} l_k + \theta_{\mathbf{m}}\Big). \tag{8.2.6}$$

When $0 < h < n$, *the family* $\{\gamma(\mathbf{m})|\ \mathbf{m} \in \partial^{(h)}\Lambda^+_{n-1}(\eta|\lambda_0)\}$ *satisfies the following recurrence relations:*

$$\frac{\prod_{h<k\leqslant n}(l_k - m_h + h - k + 1)}{\prod_{h<i\leqslant n-1}(m_i - m_h + h - i)} \cdot \beta_{\mathbf{m}}\gamma(\mathbf{m}) \\ = -a_h(\mathbf{l}\ ; \mathbf{m}^{-h})A_h^\eta(\mathbf{m}^{-h})\gamma(\mathbf{m}^{-h}) \tag{8.2.7}$$

for $\mathbf{m} \in \partial^{(h)}\Lambda^+_{n-1}(\eta|\lambda_0)$ *with* $|\mathbf{m}| - |\mathbf{l}_0| + c_0 > 0$*;*

(8.2.8)

$$\frac{\prod_{1\leqslant k\leqslant h}(l_k - m_h + h - k)}{\prod_{1\leqslant i<h}(m_i - m_h + h - i)} \cdot \beta_{\mathbf{m}}\gamma(\mathbf{m}) = b_h(\mathbf{l}\ ; \mathbf{m}^{+h})B_h^\eta(\mathbf{m}^{+h})\gamma(\mathbf{m}^{+h})$$

for $\mathbf{m} \in \partial^{(h)}\Lambda^+_{n-1}(\eta|\lambda_0)$ *with* $|\mathbf{m}| - |\mathbf{l}_0| + c_0 < 0$.

Assume $0 < h < n$. Let us define $\mathbf{m}_0 = (m_i^0(\eta))_{1\leqslant i\leqslant n-1} \in \Lambda_{n-1}$ by putting

$$\begin{aligned} m_i^0(\eta) &= \sup(l_{i+1}, m_i^-(\eta)), \quad 1 \leqslant i < h, \\ m_i^0(\eta) &= \inf(l_i, m_i^+(\eta)), \quad h < i \leqslant n-1, \\ m_h^0(\eta) &= |\mathbf{l}_0| - c_0 - \sum_{i\neq h} m_i^0(\eta). \end{aligned} \tag{8.2.9}$$

Proposition 8.2.1. *Let* $\{\gamma(\mathbf{m})|\ \mathbf{m} \in \partial^{(h)}\Lambda^+_{n-1}(\eta|\lambda_0)\}$ *be a family of complex numbers satisfying* (8.2.7) *and* (8.2.8). *Then it is unique up to a multiplicative constant. Moreover* $\gamma(\mathbf{m}) = 0$ *for all* $\mathbf{m} \in \partial^{(h)}\Lambda^+_{n-1}(\eta|\lambda_0)$ *unless one of the following three conditions holds:*

(i) $m_h^0(\eta) < l_{h+1}$ *and* $m_h^-(\eta) \leqslant l_{h+1}$.

(ii) $l_h < m_h^0(\eta)$ *and* $l_h \leqslant m_h^+(\eta)$.

(iii) $\sup(l_{h+1}, m_h^-(\eta)) \leqslant m_h^0(\eta) \leqslant \inf(l_h, m_h^+(\eta))$.

Proof. We have three possibilities: (i) $m_h^0(\eta) < \sup(l_{h+1}, m_h^-(\eta))$, (ii) $\sup(l_{h+1}, m_h^-(\eta)) \leqslant m_h^0(\eta) \leqslant \inf(l_h, m_h^+(\eta))$ and (iii) $m_h^0(\eta) > \inf(l_h, m_h^+(\eta))$. If we are in the case (i), then $\beta_{\mathbf{m}} = |\mathbf{m}| - |\mathbf{l}_0| + c_0 > 0$ for all $\mathbf{m} \in \partial^{(h)}\Lambda_{n-1}^+(\eta|\lambda_0)$. Let $\mathbf{m}_1$ be the unique element of $\partial^{(h)}\Lambda_{n-1}^+(\eta|\lambda_0)$ such that $\mathbf{m}_1^{-h} \notin \partial^{(h)}\Lambda_{n-1}^+(\eta|\lambda_0)$. Apply (8.2.7) for $\mathbf{m} = \mathbf{m}_1$. Then the right-hand side of (8.2.7) is zero. In the left-hand side, the second factor $\beta_{\mathbf{m}} \neq 0$ and the first factor is zero if and only if $m_h = \sup(l_{h+1}, m_h^-(\eta))$ equals l_{h+1}. Hence $\gamma(\mathbf{m}_1) = 0$ if $l_{h+1} < m_h^-(\eta)$, which in turn gives $\gamma(\mathbf{m}) = 0$ for all $\mathbf{m} \in \partial^{(h)}\Lambda_{n-1}^+(\eta|\lambda_0)$ by (8.2.7). It is clear that (8.2.7) determines $\gamma(\mathbf{m})$ from $\gamma(\mathbf{m}_1)$ uniquely. If we are in the case (ii), $\mathbf{m}_0 \in \partial^{(h)}\Lambda_{n-1}^+(\eta|\lambda_0)$. Given the number $\gamma(\mathbf{m}_0)$, we can recursively determine $\gamma(\mathbf{m})$ with $m_h > m_h^0(\eta)$ by (8.2.7) and $\gamma(\mathbf{m})$ with $m_h < m_h^0(\eta)$ by (8.2.8). If we are in the case (iii), by a similar argument as in the case (i), using (8.2.8) in this case, we have $\gamma(\mathbf{m}) = 0$ if $l_h > m_h^-(\eta)$. □

8.3. Multiplicity free theorem

To begin with, we prepare a Lemma.

Lemma 8.3.1. *Let I be a subset of $\{1, \ldots, n-1\}$. Let $\mathbf{x} = \{x_i\}_{i\in I}$ and $\mathbf{y} = \{y_i\}_{i\in I}$ be families of indeterminates. Put $\Delta = \left(\frac{1}{y_j - x_i}\right)_{(i,j)\in I\times I}$, a matrix with coefficient in the field $\mathbb{Q}(x_i, y_i |\ i \in I)$. Then the inverse matrix of Δ is given as $\Delta^{-1} = (\Gamma_{pq}(\mathbf{x}, \mathbf{y}))_{(p,q)\in I\times I}$ with*

$$\Gamma_{pq}(\mathbf{x}, \mathbf{y}) = \frac{\prod_{i\in I}(y_i - x_q) \prod_{i\in I, i\neq q}(y_p - x_i)}{\prod_{i\in I, i\neq q}(x_i - x_q) \prod_{i\in I, i\neq p}(y_p - y_i)}, \quad p, q \in I.$$

In other words, we have

$$\sum_{q\in I} \frac{\Gamma_{pq}(\mathbf{x}, \mathbf{y})}{y_j - x_q} = \delta_{pj}, \quad p, j \in I.$$

LEMMA 8.3.2. *Let $\lambda = [\mathbf{l}\ ; z_0 - |\mathbf{l}|] \in \mathcal{L}_n^+(\pi)$ be a D_h-corner of π with $h \in \{0, \dots, n\}$. We define subsets $J^\pm$ of $\{1, \dots, n-1\}$ as*

$$J^- = \{i \in \{1, \dots, h^-\} |\ \mathbf{l}^{-i} \in \Lambda_n^+\},$$
$$J^+ = \{j \in \{h^+, \dots, n-1\} |\ \mathbf{l}^{+(j+1)} \in \Lambda_n^+\}.$$

Let $\{f(\mathbf{m}\ ; r) |\ \mathbf{m} \in \Lambda_{n-1}^+(\eta|\lambda)\}$ be the standard coefficients of $\Phi_{\mathbf{l}}$ with $\Phi \in \mathcal{I}_{\eta,\pi}$.

(1) *For every $\mathbf{m} \in \Lambda_{n-1}^+(\eta|\lambda)$ and $j \in J^+$, we have*

$$(8.3.1)\quad \frac{-2}{\mathrm{ch}(r)} a_j(\mathbf{l}\ ; \mathbf{m}^{-j}) A_j^\eta(\mathbf{m}^{-j}) f(\mathbf{m}^{-j}\ ; r)$$
$$= \sum_{k \in J^+} \Bigg\{ r\frac{d}{dr} f(\mathbf{m}\ ; r)$$
$$+ \left(\frac{\eta_{\mathbf{m}} - \tau_{\mathbf{m}}^{\mathbf{l}}}{2\mathrm{sh}(r)\mathrm{ch}(r)} + \mathrm{th}(r)\big(z_0 - |\mathbf{m}| + 2(-l_{k+1} + k)\big) \right) f(\mathbf{m}\ ; r)$$
$$+ \frac{2}{\mathrm{ch}(r)} \sum_{i \notin J^+} \frac{a_i(\mathbf{l}\ ; \mathbf{m}^{-i}) A_i^\eta(\mathbf{m}^{-i})}{l_{k+1} - m_i + i - k} f(\mathbf{m}^{-i}\ ; r) \Bigg\} c_{jk}^+(\mathbf{m}),$$

with

$$c_{jk}^+(\mathbf{m}) = \frac{\prod_{\nu \in J^+, \nu \neq j} (l_{k+1} - m_\nu + \nu - k)}{\prod_{\nu \in J^+, \nu \neq j} (m_\nu - m_j + j - \nu)} \frac{\prod_{p \in J^+} (l_{p+1} - m_j + j - p)}{\prod_{p \in J^+, p \neq k} (l_{k+1} - l_{p+1} + p - k)}, \quad k \in J^+.$$

(2) *For every $m \in \Lambda_{n-1}^+(\eta|\lambda)$ and $i \in J^-$, we have*

$$(8.3.2)\quad \frac{-2}{\mathrm{ch}(r)} b_i(\mathbf{l}\ ; \mathbf{m}^{+i}) B_i^\eta(\mathbf{m}^{+i}) f(\mathbf{m}^{+i}\ ; r)$$
$$= \sum_{k \in J^-} \Bigg\{ r\frac{d}{dr} f(\mathbf{m}\ ; r) - \bigg(\frac{\eta_{\mathbf{m}} - \tau_{\mathbf{m}}^{\mathbf{l}}}{2\mathrm{sh}(r)\mathrm{ch}(r)}$$
$$+ \mathrm{th}(r)\big(z_0 - |\mathbf{m}| + 2(-l_k + k - n)\big) \bigg) f(\mathbf{m}\ ; r)$$
$$+ \frac{2}{\mathrm{ch}(r)} \sum_{j \notin J^-} \frac{b_j(\mathbf{l}\ ; \mathbf{m}^{+j}) B_j^\eta(\mathbf{m}^{+j})}{l_k - m_j + j - k} f(\mathbf{m}^{+j}\ ; r) \Bigg\} c_{ik}^-(\mathbf{m}),$$

with

$$c_{ik}^{-}(\mathbf{m}) = \frac{\prod_{\nu\in J^-,\nu\neq i}(l_k - m_\nu + \nu - k)}{\prod_{\nu\in J^-,\nu\neq i}(m_\nu - m_i + i - \nu)} \frac{\prod_{p\in J^-}(l_p - m_i + i - p)}{\prod_{p\in J^-,p\neq k}(l_k - l_p + p - k)}, \qquad k \in J^-.$$

Proof. By the choice of $\mathbf{l}$, we have $B_k^\pi(\mathbf{l}) = 0$ for $k \in J^-$ and $A_{k+1}^\pi(\mathbf{l}) = 0$ for $k \in J^+$. We explain how to deduce (8.3.1) starting with the equations $(\mathsf{S}^{+(k+1)})_{\mathbf{l},\mathbf{m}}$ with $k \in J^+$ and $\mathbf{m} \in \Lambda_{n-1}^+(\eta|\lambda)$. Take an $\mathbf{m} \in \Lambda_{n-1}^+(\eta|\lambda)$ and put

$$X_\nu = a_\nu(\mathbf{l}_0\ ; \mathbf{m}^{-\nu}) A_\nu^\eta(\mathbf{m}^{-\nu}) \frac{f(\mathbf{m}^{-\nu}\ ; r)}{\mathrm{ch}(r)}, \quad \nu \in \{1, \ldots, n-1\}.$$

Then the equation $(\mathsf{S}^{+(k+1)})_{\mathbf{l},\mathbf{m}}$ can be written as

$$\sum_{j\in J^+} \frac{X_j}{l_{k+1} - m_j + j - k} = Y_k, \qquad k \in J^+ \tag{8.3.3}$$

with

$$Y_k = \left(r\frac{d}{dr} + \frac{\eta_{\mathbf{m}} - \tau_{\mathbf{m}}^{\mathbf{l}}}{2\mathrm{sh}(r)\mathrm{ch}(r)} + \mathrm{th}(r)(z_0 - |\mathbf{m}| + 2k - l_{k+1})\right) f(\mathbf{m}\ ; r) - \sum_{\nu\notin J^+} \frac{X_\nu}{l_{k+1} - m_\nu + \nu - k}.$$

If we put $\mathbf{x} = (m_i - i)_{i\in J^+}$ and $\mathbf{y} = (l_{h+1} - h)_{h\in J^+}$, then $c_{jk}^+(\mathbf{m}) = \Gamma_{jk}(\mathbf{x}, \mathbf{y})$; hence by Lemma 8.3.1, we have

$$\sum_{k\in J^+} \frac{c_{\nu k}^+(\mathbf{m})}{l_{k+1} - m_j + j - k} = \delta_{\nu j}, \qquad \nu,\ j \in J^+. \tag{8.3.4}$$

Since (8.3.3) is a system of linear equations with respect to X_j, $j \in J^+$, we can solve it, using (8.3.4), to get

$$X_j = \sum_{k\in J^+} Y_k c_{jk}^+(\mathbf{m}).$$

Thus we obtain (8.3.1). The formula (8.3.2) can be obtained in a similar way. □

Now we have one of the main results of this article.

Theorem 8.3.1 (Multiplicity free theorem). *Let η be an irreducible $(\mathfrak{h}_{n,\mathbb{C}}, K_n \cap H_n)$-module and π an irreducible $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-module. We then have* $\dim_{\mathbb{C}} \mathcal{I}_{\eta,\pi} \leqslant 1$. *Moreover $\mathcal{I}_{\eta,\pi} = \{0\}$ unless the conditions (a), (b) and (c) in Theorem 8.1.1 and one of the conditions (i), (ii) and (iii) in Proposition 8.2.1 are satisfied.*

Proof. Let $\lambda_0 = [\mathbf{l}_0\,; z_0 - |\mathbf{l}_0|]$ be as before; see (i) and (ii) in 8.2. By Proposition 8.2.1 the family of constants $\{\gamma(\mathbf{m})\}$ for $\Phi \in \mathcal{I}_{\eta,\pi}$ in Theorem 8.2.1 is unique up to a multiplicative constant. Moreover the vector $\{\gamma(\mathbf{m})|\ \mathbf{m} \in \partial^{(h)}\Lambda^+_{n-1}(\eta|\lambda_0)\} \in \mathbb{C}^{\partial^{(h)}\Lambda^+_{n-1}(\eta|\lambda_0)}$ depends on Φ linearly. Hence we have only to show that $\gamma(\mathbf{m}) = 0$ for $\mathbf{m} \in \partial^{(h)}\Lambda^+_{n-1}(\eta|\lambda_0)$ implies $\Phi = 0$. If $\gamma(\mathbf{m}) = 0$, then $f(\mathbf{m}\,; r) = 0$, $r > 0$ for all $\mathbf{m} \in \partial^{(h)}\Lambda^+_{n-1}(\eta|\lambda_0)$ by Theorem 8.2.1. Using the equations (8.3.1) and (8.3.2), we can show that $f(\mathbf{m}\,; r)$ is identically zero for all $\mathbf{m} \in \Lambda^+_{n-1}(\eta|\lambda_0)$ by induction on the number $\delta(\mathbf{m}) = \inf_{\mathbf{n}\in\partial^{(h)}\Lambda^+_{n-1}(\eta|\lambda_0)} |\mathbf{m} - \mathbf{n}|$. In view of the formula (7.1.2), we then have $\Phi_{\mathbf{l}_0}(a) = 0$, $a \in A_n$. As we explained in the proof of Theorem 8.1.1, the map $\Phi \mapsto \Phi_{\mathbf{l}_0}|A_n$ is a linear injection. Hence we have $\Phi = 0$ as desired. This proves the first part of the theorem. The second part is already proved in Theorem 8.1.1 and Proposition 8.2.1. □

9. Proof of Theorem 8.2.1

The aim of this section is to prove Theorem 8.2.1. Throughout this section we retain the situation of 8.2. Recall that given an irreducible $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-module π, we have attached a triple $(\mathbf{l}_0, h, s)$ to π at the beginning of 8.2. Put $\mathbf{l}_0 = (l_k)_{1\leqslant k\leqslant n} \in \Lambda^+_n$ and $z_0 = c_n(\pi)$. For simplicity we write $\tau_{\mathbf{m}}$ for the number $\tau^{\mathbf{l}_0}_{\mathbf{m}}$ (see 7.4).

9.1. Proof of Theorem 8.2.1

We first have

Lemma 9.1.1. *Let $(\mathbf{l}_0, h, s)$ be as above. If π is elementary, then we take $\mathbf{p} = (p_i)_{1\leqslant i\leqslant n-1} \in \Lambda^+_{n-1}$ so that $\pi \cong \pi(\mathbf{p}, z_0 - |\mathbf{p}|\,; s)$. Otherwise, we* *define the weight* $\mathbf{p}=(p_i)_{1\leqslant i\leqslant n-1}\in\Lambda_{n-1}^{+}$ *by putting* $p_i=l_i,\ 1\leqslant i<h^{+}$, $p_i=l_{i+1},\ h^{+}\leqslant i\leqslant n-1$. *Then* π *occurs in a* $(\mathfrak{g}_{n,\mathbb{C}},K_n)$*-subquotient of* $\pi(\mathbf{p},z_0-|\mathbf{p}|\,;s)$.

PROOF. This is a consequence of Theorem 4.3.2. □

LEMMA 9.1.2. *Let* I *be a subset of* $\{1,\ldots,n-1\}$. *Given families of indeterminates* $\{x_i\}_{i\in I}$ *and* $\{y_i\}_{i\in I}$, *we have the following formulas in the field* $\mathbb{Q}(x_i,y_i|\ i\in I)$:

$$\sum_{i\in I}\frac{\prod\limits_{h\in I}(x_h-y_i+i-h)}{\prod\limits_{j\in I,j\neq i}(y_j-y_i+i-j)}\frac{1}{x_k-y_i+i-k}=1,\quad k\in I,$$

$$\sum_{k\in I}\frac{\prod\limits_{j\in I}(x_k-y_j+j-k)}{\prod\limits_{h\in I,h\neq k}(x_k-x_h+h-k)}=\sum_{k\in I}x_k-\sum_{i\in I}y_i,$$

$$\sum_{k\in I}\frac{\prod\limits_{i\in I}(x_k-y_i+i-k)}{\prod\limits_{h\in I,h\neq k}(x_k-x_h+h-k)}(x_k-k)$$
$$=\frac{1}{2}(\sum_{k\in I}x_k-\sum_{i\in I}y_i)^2+\frac{1}{2}(\sum_{k\in I}x_k)^2-\frac{1}{2}(\sum_{i\in I}y_i)^2+\sum_{i\in I}iy_i-\sum_{k\in I}kx_k.$$

PROOF. As for the last two formula see [9, Lemma 2.1]. The first formula is obtained from the second one by differentiation with respect to x_k. □

In order to prove Theorem 8.2.1, we first show that $f(\mathbf{m}\,;r)$ with $\mathbf{m}\in\partial^{(h)}\Lambda_{n-1}^{+}(\eta|\lambda_0)$ satisfies a second order differential equation. Let $J^{\pm}$ be the set defined in Lemma 8.3.2 with $\mathbf{l}=\mathbf{l}_0$.

PROPOSITION 9.1.1. *Let* $\{f(\mathbf{m}\,;r)|\ \mathbf{m}\in\Lambda_{n-1}^{+}(\eta|\lambda_0)\}$ *be a family of* C^{∞}*-functions on* $r>0$ *which satisfies the system of difference-differential equations* $(\mathsf{C})_{\mathbf{l}_0,\mathbf{m}}$, $(\mathsf{S}^{-i})_{\mathbf{l}_0,\mathbf{m}}$ *and* $(\mathsf{S}^{+(j+1)})_{\mathbf{l}_0,\mathbf{m}}$ *for* $i\in J^{-}$, $j\in J^{+}$ *and* $\mathbf{m}\in\Lambda_{n-1}^{+}(\eta|\lambda_0)$. *For an* $\mathbf{m}\in\partial^{(h)}\Lambda_{n-1}^{+}(\eta|\lambda_0)$, *the function* $f(\mathbf{m}\,;r)$ *satisfies the* *second order differential equation*

$$(9.1.1)\quad \left(r\frac{d}{dr}\right)^2 f(\mathbf{m}\,;r) + \left(\frac{1}{\mathrm{th}(r)} + (2n + 2\theta^-_{\mathbf{m}} - 2\theta^+_{\mathbf{m}} - 1)\mathrm{th}(r)\right)$$

$$\times\, r\frac{d}{dr} f(\mathbf{m}\,;r)$$

$$+ \left\{ \frac{-1}{\mathrm{ch}^2(r)\mathrm{sh}^2(r)} \left(\frac{\eta_{\mathbf{m}} - \tau_{\mathbf{m}}}{2}\right)^2 \right.$$

$$+ \frac{1}{\mathrm{ch}^2(r)} \Big(\tau_{\mathbf{m}}\eta_{\mathbf{m}} - \eta^2_{\mathbf{m}} - \tau^2_{\mathbf{m}} + 2\Omega_{H_n}(\eta) + 2\Omega_{K_n}(\mathbf{l}\,; z_0 - |\mathbf{l}|)$$

$$- 4\Omega_{M_n}(\mathbf{m}\,; z_0 - |\mathbf{m}|) - 2((\theta^-_{\mathbf{m}})^2 + (\theta^+_{\mathbf{m}})^2)$$

$$+ 2(z_0 - |\mathbf{m}|)(\theta^-_{\mathbf{m}} + \theta^+_{\mathbf{m}}) - (\eta_{\mathbf{m}} - \tau_{\mathbf{m}})(\theta^+_{\mathbf{m}} + \theta^-_{\mathbf{m}}) - 4n\theta^-_{\mathbf{m}} + 2\Theta_{\mathbf{m}} \Big)$$

$$+ 2\Omega_{M_n}(\mathbf{m}\,; z_0 - |\mathbf{m}|) - 2\Omega_{G_n}(\pi) + 2((\theta^-_{\mathbf{m}})^2 + (\theta^+_{\mathbf{m}})^2)$$

$$\left. - 2(z_0 - |\mathbf{m}|)(\theta^-_{\mathbf{m}} + \theta^+_{\mathbf{m}}) + 4n\theta^-_{\mathbf{m}} - 2\Theta_{\mathbf{m}} \right\}$$

$$\times\, f(\mathbf{m}\,;r) = 0,$$

with

$$(9.1.2)\qquad \eta_{\mathbf{m}} = |\mathbf{m}| - z_0 + 2c_0, \qquad \tau_{\mathbf{m}} = 2|\mathbf{l}_0| - |\mathbf{m}| - z_0,$$

$$(9.1.3)\qquad \theta^-_{\mathbf{m}} = \sum_{k\in J^-} l_k - \sum_{i\in J^-} m_i, \quad \theta^+_{\mathbf{m}} = \sum_{k\in J^+} l_{k+1} - \sum_{i\in J^+} m_i,$$

$$(9.1.4)\qquad \Theta_{\mathbf{m}} = \sum_{i\in J^-} m_i^2 + \sum_{i\in J^+} m_i^2 - \sum_{k\in J^-} l_k^2 - \sum_{k\in J^+} l_{k+1}^2$$

$$- 2\Big(\sum_{i\in J^-} i m_i + \sum_{i\in J^+} i m_i\Big) + 2\Big(\sum_{k\in J^-} k l_k + \sum_{k\in J^+} k l_{k+1}\Big).$$

Proof. Since $\mathbf{m} \in \partial^{(h)}\Lambda^+_{n-1}(\eta|\lambda_0)$ we may set $f(\mathbf{m}^{-i}\,;r) = 0$ for $i \notin J^+$ and $f(\mathbf{m}^{+j}\,;r) = 0$ for $j \notin J^-$ in $(\mathsf{C})_{\mathbf{l}_0,\mathbf{m}}$, (8.3.1) and (8.3.2). By substituting (8.3.1) and (8.3.2) to the equation $(\mathsf{C})_{\mathbf{l}_0,\mathbf{m}}$, we can eliminate $f(\mathbf{m}^{-j}\,;r)$, $j \in J^+$ and $f(\mathbf{m}^{+i}\,;r)$, $i \in J^-$ to get

$$(9.1.5)\qquad \left(r\frac{d}{dr}\right)^2 f(\mathbf{m}\,;r)$$

$$
\begin{aligned}
&+\left(\frac{1}{\mathrm{th}(r)}+\big(2n+2\sum_{k\in J^-}\sum_{i\in J^-}c^-_{ik}(\mathbf{m})\right.\\
&\left.-2\sum_{k\in J^+}\sum_{j\in J^+}c^+_{jk}(\mathbf{m})-1\big)\mathrm{th}(r)\right)r\frac{d}{dr}f(\mathbf{m}\,;r)\\
&+f(\mathbf{m}\,;r)\left\{\frac{-1}{\mathrm{ch}^2(r)\mathrm{sh}^2(r)}\left(\frac{\eta_{\mathbf{m}}-\tau_{\mathbf{m}}}{2}\right)^2+2\Omega_{M_n}(\mathbf{m}\,;z_0-|\mathbf{m}|)\right.\\
&-2\Omega_{G_n}(\pi)+\frac{1}{\mathrm{ch}^2(r)}\Big(\tau_{\mathbf{m}}\eta_{\mathbf{m}}-\eta_{\mathbf{m}}^2-\tau_{\mathbf{m}}^2+2\Omega_{H_n}(\eta)\\
&+2\Omega_{K_n}(\mathbf{l}\,;z_0-|\mathbf{l}|)-4\Omega_{M_n}(\mathbf{m}\,;z_0-|\mathbf{m}|)\Big)\\
&+2\mathrm{th}(r)\sum_{k\in J^-}\sum_{i\in J^-}c^-_{ik}(\mathbf{m})F^-_k(\mathbf{m}\,;r)\\
&\left.-2\mathrm{th}(r)\sum_{k\in J^+}\sum_{j\in J^+}c^+_{jk}(\mathbf{m})F^+_k(\mathbf{m}\,;r)\right\}\\
&=0,
\end{aligned}
$$

with

$$
\begin{aligned}
F^-_k(\mathbf{m}\,;r)&=-\frac{\eta_{\mathbf{m}}-\tau_{\mathbf{m}}}{2\mathrm{sh}(r)\mathrm{ch}(r)}-\mathrm{th}(r)\Big(z_0-|\mathbf{m}|+2(-l_k+k-n)\Big),\\
F^+_k(\mathbf{m}\,;r)&=\frac{\eta_{\mathbf{m}}-\tau_{\mathbf{m}}}{2\mathrm{sh}(r)\mathrm{ch}(r)}+\mathrm{th}(r)\Big(z_0-|\mathbf{m}|+2(-l_{k+1}+k)\Big).
\end{aligned}
$$

Using Lemma 9.1.2, we can prove formulas

$$
\begin{aligned}
\sum_{k\in J^+}\sum_{j\in J^+}c^+_{jk}(\mathbf{m})&=\sum_{k\in J^+}l_{k+1}-\sum_{j\in J^+}m_j,\\
\sum_{k\in J^-}\sum_{i\in J^-}c^-_{ik}(\mathbf{m})&=\sum_{k\in J^-}l_k-\sum_{i\in J^-}m_i
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{k\in J^+}\sum_{j\in J^+}c^+_{jk}(\mathbf{m})F^+_k(\mathbf{m}\,;r)\\
&=\frac{\eta_{\mathbf{m}}-\tau_{\mathbf{m}}}{2\mathrm{sh}(r)\mathrm{ch}(r)}(\sum_{k\in J^+}l_{k+1}-\sum_{j\in J^+}m_j)
\end{aligned}
$$

$$+\,\mathrm{th}(r)\Big((\sum_{k\in J^+} l_{k+1} - \sum_{j\in J^+} m_j)(z_0 - |\mathbf{m}| - \sum_{k\in J^+} l_{k+1} + \sum_{j\in J^+} m_j)$$

$$+ \sum_{j\in J^+} m_j^2 - \sum_{k\in J^+} l_{k+1}^2 + 2\sum_{k\in J^+} k l_{k+1} - 2\sum_{j\in J^+} j m_j\Big),$$

$$\sum_{k\in J^-}\sum_{i\in J^-} c_{ik}^-(\mathbf{m}) F_k^-(\mathbf{m}\,;r)$$

$$= -\frac{\eta_{\mathbf{m}} - \tau_{\mathbf{m}}}{2\mathrm{sh}(r)\mathrm{ch}(r)}(\sum_{k\in J^-} l_k - \sum_{i\in J^-} m_i)$$

$$+\,\mathrm{th}(r)\Big((\sum_{k\in J^-} l_k - \sum_{i\in J^-} m_i)(|\mathbf{m}| - z_0 + \sum_{k\in J^-} l_k - \sum_{i\in J^-} m_i + 2n)$$

$$- \sum_{i\in J^-} m_i^2 + \sum_{k\in J^-} l_k^2 - 2\sum_{k\in J^-} k l_k + 2\sum_{i\in J^-} i m_i\Big).$$

We substitute these formulas into (9.1.5); then after some calculation, we finally get (9.1.1). □

Changing variables we see below that the differential equation (9.1.1) is transformed to the Gaussian hypergeometric differential equation to get its C^∞-solution explicitly.

Proposition 9.1.2. *The differential equation* (9.1.1) *in Proposition 9.1.1 has, up to a multiplicative constant, a unique C^∞-solution on $r > 0$ given by*

$$(9.1.6)\qquad (\mathrm{sh}(r))^\beta(\mathrm{ch}(r))^\alpha {}_2F_1(X_{\mathbf{m}}^+, X_{\mathbf{m}}^-\,;1+\beta\,;\mathrm{th}^2(r)),$$

where

$$(9.1.7)\qquad \beta = \left|\frac{\eta_{\mathbf{m}} - \tau_{\mathbf{m}}}{2}\right|,$$

α is a solution of

$$(9.1.8)\qquad (\alpha+\beta)(\alpha+\beta+2n-2\theta_{\mathbf{m}}^+ + 2\theta_{\mathbf{m}}^-) = P_{\mathbf{m}}$$

with

$$(9.1.9)\qquad P_{\mathbf{m}} = 2\Omega_{M_n}(\mathbf{m}\,;z_0 - |\mathbf{m}|) - 2\Omega_{G_n}(\pi)$$

$$+ 2({\theta^-_{\mathbf{m}}}^2 + {\theta^+_{\mathbf{m}}}^2) - 2(z_0 - |\mathbf{m}|)(\theta^-_{\mathbf{m}} + \theta^+_{\mathbf{m}}) + 4n\theta^-_{\mathbf{m}} - 2\Theta_{\mathbf{m}}$$

and $X^+_{\mathbf{m}}$, $X^-_{\mathbf{m}}$ *are two solutions of*

$$(2X + \alpha + n + \theta^-_{\mathbf{m}} - \theta^+_{\mathbf{m}} - 1)^2 = Q_{\mathbf{m}} \tag{9.1.10}$$

with

$$\begin{aligned} Q_{\mathbf{m}} = {} & (n + \theta^-_{\mathbf{m}} - \theta^+_{\mathbf{m}} - 1)^2 + \left(\frac{\eta_{\mathbf{m}} - \tau_{\mathbf{m}}}{2}\right)^2 + \tau_{\mathbf{m}}\eta_{\mathbf{m}} - \eta^2_{\mathbf{m}} - \tau^2_{\mathbf{m}} \\ & + 2\Omega_{H_n}(\eta) + 2\Omega_{K_n}(\mathbf{l}\,; z_0 - |\mathbf{l}|) - 4\Omega_{M_n}(\mathbf{m}\,; z_0 - |\mathbf{m}|) \\ & - 2({\theta^-_{\mathbf{m}}}^2 + {\theta^+_{\mathbf{m}}}^2) + 2(z_0 - |\mathbf{m}|)(\theta^-_{\mathbf{m}} + \theta^+_{\mathbf{m}}) \\ & - (\eta_{\mathbf{m}} - \tau_{\mathbf{m}})(\theta^+_{\mathbf{m}} + \theta^-_{\mathbf{m}}) - 4n\theta^-_{\mathbf{m}} + 2\Theta_{\mathbf{m}}. \end{aligned} \tag{9.1.11}$$

Proof. Setting

$$z = \mathrm{th}^2(r),$$

we get a diffeomorphism from $r > 1$ to $0 < z < 1$. Now put

$$w(z) = (\mathrm{sh}(r))^{-\beta}(\mathrm{ch}(r))^{-\alpha} f(\mathbf{m}\,; r)$$

with β and α satisfying (9.1.7) and (9.1.8). Then after some computations, the equation (9.1.1) is transformed to the hypergeometric differential equation

$$\begin{aligned} z(1-z)\frac{d^2w}{dz^2} + \left((1+\beta) - \frac{-\alpha + n + \theta^-_{\mathbf{m}} - \theta^+_{\mathbf{m}}}{2} z\right)\frac{dw}{dz} \\ - \frac{1}{4}\left((\alpha + n + \theta^-_{\mathbf{m}} - \theta^+_{\mathbf{m}} - 1)^2 - Q_{\mathbf{m}}\right) w = 0. \end{aligned}$$

Since the function $f(\mathbf{m}\,; r)$ is C^∞ around $r = 1$, it turns out that $w(z)$ is a constant multiple of the hypergeometric series ${}_2F_1(X^+_{\mathbf{m}}, X^-_{\mathbf{m}}\,; 1+\beta\,; z)$. □

Lemma 9.1.3. *Let* $\mathbf{p} \in \Lambda^+_{n-1}$ *be the weight given in Lemma 9.1.1. Then for every* $\mathbf{m} \in \partial^{(h)}\Lambda^+_{n-1}(\eta|\lambda_0)$, *we have*

$$\begin{aligned} \theta^+_{\mathbf{m}} + \theta^-_{\mathbf{m}} &= |\mathbf{l}_0| - |\mathbf{m}| - \mu_{\mathbf{m}}, \\ \theta^+_{\mathbf{m}} - \theta^-_{\mathbf{m}} &= \theta_{\mathbf{m}}, \end{aligned}$$

$$\Theta_{\mathbf{m}} = \sum_{i=1}^{n-1} m_i^2 - \sum_{k=1}^{n} l_k^2 - 2\sum_{i=1}^{n-1} i m_i + 2\sum_{k=1}^{n} k l_k$$
$$+ \mu_{\mathbf{m}}^2 - 2h^+ \mu_{\mathbf{m}} - 2 \sum_{h^+ < k \leqslant n} l_k,$$
$$|\mathbf{l}_0| = |\mathbf{p}| + l_{h^+}.$$

Proof. Let $i, j \in \{1, \ldots, n-1\}$. Then if $i \notin J^-$, $l_i = m_i$; if $j \notin J^+$, $l_{j+1} = m_j$. Form this observation, the lemma follows. □

Using Lemma 9.1.3, we can make the values of $P_{\mathbf{m}}$ and $Q_{\mathbf{m}}$ given by (9.1.9) and (9.1.11) trimmer form.

Lemma 9.1.4. *Let* $\mathbf{p} \in \Lambda_{n-1}^+$ *be the weight given in Lemma 9.1.1. Then for* $\mathbf{m} \in \partial^{(h)} \Lambda_{n-1}^+(\eta|\lambda_0)$, *we have*

$$P_{\mathbf{m}} = (\theta_{\mathbf{m}}^+ - \theta_{\mathbf{m}}^- - n + \mu_{\mathbf{m}} - l_{h^+} + s)(\theta_{\mathbf{m}}^+ - \theta_{\mathbf{m}}^- - n - \mu_{\mathbf{m}} + l_{h^+} - s),$$
$$Q_{\mathbf{m}} = 2\Omega_{H_n}(\eta) - 2c_0^2 + (n-1)^2 - (|\mathbf{m}| - \mu_{\mathbf{m}} + c_0 - z_0)^2$$
$$+ 2\Bigg(- \sum_{i=1}^{n-1} m_i^2 - \sum_{i=1}^{n-1} (n-2i) m_i + \mu_{\mathbf{m}}^2 + (n - 2h^+)\mu_{\mathbf{m}} + |\mathbf{l}_0|$$
$$- \sum_{h^+ < k \leqslant n} l_k + \theta_{\mathbf{m}}^+ - \theta_{\mathbf{m}}^- \Bigg).$$

Proof. The proof is a tedious and long calculation. By Proposition 6.2.2 and Lemma 9.1.1, we first have

$$\Omega_{M_n}(\mathbf{m}\,; z_0 - |\mathbf{m}|) - \Omega_{G_n}(\pi) = \sum_{i=1}^{n-1} m_i^2 + \sum_{i=1}^{n-1}(n-2i)m_i + \frac{1}{2}(z_0 - |\mathbf{m}|)^2$$
$$- \frac{s^2}{2} + \frac{n^2}{2} - \sum_{i=1}^{n-1} p_i^2 - \sum_{i=1}^{n-1}(n-2i)p_i$$
$$- \frac{1}{2}(z_0 - |\mathbf{p}|)^2.$$

Since $\tau_{\mathbf{m}} + 2l_0 = z_0 - |\mathbf{m}|$, we have

$$\begin{aligned}
P_{\mathbf{m}} &= 2(\sum_{i=1}^{n-1} m_i^2 - \sum_{i=1}^{n-1} p_i^2 - 2\sum_{i=1}^{n-1} im_i + 2\sum_{i=1}^{n-1} ip_i) \\
&\quad + 2n(|\mathbf{m}| - |\mathbf{p}|) + (z_0 - |\mathbf{m}|)^2 - (z_0 - |\mathbf{p}|)^2 - s^2 + n^2 \\
&\quad + (\theta_{\mathbf{m}}^+ - \theta_{\mathbf{m}}^-)^2 + (\theta_{\mathbf{m}}^+ + \theta_{\mathbf{m}}^-)^2 - 2(z_0 - |\mathbf{m}|)(\theta_{\mathbf{m}}^+ + \theta_{\mathbf{m}}^-) \\
&\quad + 4n\theta_{\mathbf{m}}^- - 2(\sum_{i=1}^{n-1} m_i^2 - \sum_{k=1}^{n} l_k^2 + 2\sum_{k=1}^{n} kl_k - 2\sum_{i=1}^{n-1} im_i \\
&\quad + \mu_{\mathbf{m}}^2 - 2h^+\mu_{\mathbf{m}} - 2\sum_{h^+<k\leqslant n} l_k) \\
&= (\theta_{\mathbf{m}}^+ - \theta_{\mathbf{m}}^- - n)^2 + 2n(\theta_{\mathbf{m}}^+ - \theta_{\mathbf{m}}^-) + 2n(|\mathbf{m}| - |\mathbf{p}|) + 4n\theta_{\mathbf{m}}^- - s^2 \\
&\quad + (z_0 - |\mathbf{m}|)^2 - (z_0 - |\mathbf{p}|)^2 + (\theta_{\mathbf{m}}^+ + \theta_{\mathbf{m}}^-)^2 - 2(z_0 - |\mathbf{m}|)(\theta_{\mathbf{m}}^+ + \theta_{\mathbf{m}}^-) \\
&\quad - 2\mu_{\mathbf{m}} + 4h^+\mu_{\mathbf{m}} + 2(-\sum_{i=1}^{n-1} p_i^2 + \sum_{k=1}^{n} l_k^2 - 2\sum_{k=1}^{n} kl_k \\
&\quad + 2\sum_{i=1}^{n-1} ip_i + 2\sum_{h^+<k\leqslant n} l_k) \\
&= (\theta_{\mathbf{m}}^+ - \theta_{\mathbf{m}}^- - n)^2 + 2n(\theta_{\mathbf{m}}^+ + \theta_{\mathbf{m}}^- - |\mathbf{p}| + |\mathbf{m}|) - s^2 \\
&\quad + (z_0 - |\mathbf{m}| - \theta_{\mathbf{m}}^+ - \theta_{\mathbf{m}}^-)^2 - (z_0 - |\mathbf{p}|)^2 - 2\mu_{\mathbf{m}}^2 + 4h^+\mu_{\mathbf{m}} \\
&\quad + 2(-\sum_{i=1}^{n-1} p_i^2 + \sum_{k=1}^{n} l_k^2 - 2\sum_{k=1}^{n} kl_k + 2\sum_{i=1}^{n-1} ip_i + 2\sum_{h^+<k\leqslant n} l_k) \\
&= (\theta_{\mathbf{m}}^+ - \theta_{\mathbf{m}}^- - n)^2 \\
&\quad + (z_0 - |\mathbf{m}| - \theta_{\mathbf{m}}^+ - \theta_{\mathbf{m}}^-)^2 + 2n(\theta_{\mathbf{m}}^+ + \theta_{\mathbf{m}}^- - |\mathbf{p}| + |\mathbf{m}|) \\
&\quad - (z_0 - |\mathbf{p}|)^2 - s^2 - 2\mu_{\mathbf{m}}^2 + 4h^+\mu_{\mathbf{m}} \\
&\quad + 2(-\sum_{i=1}^{n-1} p_i^2 + \sum_{k=1}^{n} l_k^2 - 2\sum_{k=1}^{n} kl_k + 2\sum_{i=1}^{n-1} ip_i + 2\sum_{h^+<k\leqslant n} l_k) \\
&= (\theta_{\mathbf{m}}^+ - \theta_{\mathbf{m}}^- - n)^2 \\
&\quad + (|\mathbf{p}| - |\mathbf{m}| - \theta_{\mathbf{m}}^+ - \theta_{\mathbf{m}}^-)^2 + 2(|\mathbf{p}| - z_0 + n)(\theta_{\mathbf{m}}^+ + \theta_{\mathbf{m}}^- + |\mathbf{m}| - |\mathbf{p}|) \\
&\quad - s^2 - 2\mu_{\mathbf{m}}^2 + 4h^+\mu_{\mathbf{m}} + 2l_{h^+}^2 - 4h^+ l_{h^+}
\end{aligned}$$

$$+ 2(-l_{h^+}^2 + 2h^+ l_{h^+} - \sum_{i=1}^{n-1} p_i^2 + \sum_{k=1}^{n} l_k^2 - 2\sum_{k=1}^{n} k l_k + 2\sum_{i=1}^{n-1} i p_i + 2\sum_{h^+ < k \leqslant n} l_k).$$

Now using Lemma 9.1.3, we have

$$|\mathbf{p}| - |\mathbf{m}| - \theta_{\mathbf{m}}^+ - \theta_{\mathbf{m}}^- = \mu_{\mathbf{m}} - l_{h^+}.$$

Inserting this formula to the last expression of $P_{\mathbf{m}}$, we compute

$$\begin{aligned}
P_{\mathbf{m}} &= (\theta_{\mathbf{m}}^+ - \theta_{\mathbf{m}}^- - n)^2 \\
&\quad + (\mu_{\mathbf{m}} - l_{h^+})^2 + 2(|\mathbf{p}| - z_0 + n)(l_{h^+} - \mu_{\mathbf{m}}) \\
&\quad - s^2 - 2(\mu_{\mathbf{m}} - l_{h^+})(\mu_{\mathbf{m}} + l_{h^+}) + 4h^+(\mu_{\mathbf{m}} - l_{h^+}) \\
&\quad + 2(-l_{h^+}^2 + 2h^+ l_{h^+} - \sum_{i=1}^{n-1} p_i^2 + \sum_{k=1}^{n} l_k^2 - 2\sum_{k=1}^{n} k l_k \\
&\quad + 2\sum_{i=1}^{n-1} i p_i + 2\sum_{h^+ < k \leqslant n} l_k) \\
&= (\theta_{\mathbf{m}}^+ - \theta_{\mathbf{m}}^- - n)^2 \\
&\quad - (\mu_{\mathbf{m}} - l_{h^+})^2 - s^2 \\
&\quad + 2(l_{h^+} - \mu_{\mathbf{m}})(l_{h^+} - \mu_{\mathbf{m}} + |\mathbf{p}| - z_0 + n + l_{h^+} + \mu_{\mathbf{m}} - 2h^+) \\
&\quad + 2(-l_{h^+}^2 + 2h^+ l_{h^+} - \sum_{i=1}^{n-1} p_i^2 + \sum_{k=1}^{n} l_k^2 - 2\sum_{k=1}^{n} k l_k \\
&\quad + 2\sum_{i=1}^{n-1} i p_i + 2\sum_{h^+ < k \leqslant n} l_k) \\
&= (\theta_{\mathbf{m}}^+ - \theta_{\mathbf{m}}^- - n)^2 \\
&\quad - (l_{h^+} - \mu_{\mathbf{m}} - s)^2 + 2(l_{h^+} - \mu_{\mathbf{m}})(2l_{h^+} - 2h^+ + n + |\mathbf{p}| - z_0 - s) \\
&\quad + 2(-l_{h^+}^2 + 2h^+ l_{h^+} - \sum_{i=1}^{n-1} p_i^2 + \sum_{k=1}^{n} l_k^2 - 2\sum_{k=1}^{n} k l_k \\
&\quad + 2\sum_{i=1}^{n-1} i p_i + 2\sum_{h^+ < k \leqslant n} l_k).
\end{aligned}$$

If $h = 0$ or $h = n$, then $\mu_{\mathbf{m}} = l_{h^+}$ by the definition. If $0 < h < n$, then by definition of $\mathbf{l}_0$, $\mathbf{p}$ and s we have

$$s = 2l_{h^+} - 2h^+ + n + |\mathbf{p}| - z_0.$$

Therefore the formula

$$2(l_{h^+} - \mu_{\mathbf{m}})(2l_{h^+} - 2h^+ + n + |\mathbf{p}| - z_0 - s) = 0$$

is valid. Furthermore using the explicit values of $\mathbf{l}_0$ and $\mathbf{p}$, we see that

$$-l_{h^+}^2 + 2h^+ l_{h^+} - \sum_{i=1}^{n-1} p_i^2 + \sum_{k=1}^{n} l_k^2 - 2\sum_{k=1}^{n} k l_k + 2\sum_{i=1}^{n-1} i p_i + 2\sum_{h^+ < k \leqslant n} l_k = 0.$$

Combining these observations, we finally get

$$P_{\mathbf{m}} = (\theta_{\mathbf{m}}^+ - \theta_{\mathbf{m}}^- - n)^2 - (l_{h^+} - \mu_{\mathbf{m}} - s)^2$$

to conclude the computation for $P_{\mathbf{m}}$. Next we calculate $Q_{\mathbf{m}}$. By using Proposition 6.2.2 and Lemma 9.1.3, we have

$$\begin{aligned}
&\Omega_{K_n}(\mathbf{l}_0\,; z_0 - |\mathbf{l}_0|) - \Omega_{M_n}(\mathbf{m}\,; z_0 - |\mathbf{m}|) + \Theta_{\mathbf{m}} \\
&= (z_0 - |\mathbf{l}_0|)^2 - \frac{1}{2}(z_0 - |\mathbf{m}|)^2 + n(|\mathbf{l}_0| - |\mathbf{m}|) + |\mathbf{l}_0| + \mu_{\mathbf{m}}^2 - 2h^+\mu_{\mathbf{m}} \\
&\quad - 2\sum_{h^+ < k \leqslant n} l_k.
\end{aligned}$$

By inserting this and the values of $\eta_{\mathbf{m}}$ and $\tau_{\mathbf{m}}$ given by

$$\eta_{\mathbf{m}} = |\mathbf{m}| - z_0 + 2c_0, \qquad \tau_{\mathbf{m}} = 2|\mathbf{l}_0| - |\mathbf{m}| - z_0$$

to the formula (9.1.11), we compute

$$\begin{aligned}
Q_{\mathbf{m}} &= (\theta_{\mathbf{m}}^+ - \theta_{\mathbf{m}}^- - n + 1)^2 + (|\mathbf{m}| - |\mathbf{l}_0| + c_0)^2 \\
&\quad + (|\mathbf{m}| - z_0 + 2c_0)(2|\mathbf{l}_0| - |\mathbf{m}| - z_0) - (|\mathbf{m}| - z_0 + 2c_0)^2 \\
&\quad - (2|\mathbf{l}_0| - |\mathbf{m}| - z_0)^2 - 2(|\mathbf{m}| - |\mathbf{l}_0| + c_0)(\theta_{\mathbf{m}}^+ + \theta_{\mathbf{m}}^-) \\
&\quad - 2({\theta_{\mathbf{m}}^+}^2 + {\theta_{\mathbf{m}}^-}^2) + 2(z_0 - |\mathbf{m}|)(\theta_{\mathbf{m}}^+ + \theta_{\mathbf{m}}^-) - 4n\theta_{\mathbf{m}}^- \\
&\quad + 2(z_0 - |\mathbf{l}_0|)^2 - (z_0 - |\mathbf{m}|)^2 + 2n(|\mathbf{l}_0| - |\mathbf{m}|) + 2\mu_{\mathbf{m}}^2 - 4h^+\mu_{\mathbf{m}}
\end{aligned}$$

$$
\begin{aligned}
&+ 2(|\mathbf{l}_0| - \sum_{h^+<k\leqslant n} l_k) + 2\Omega_{H_n}(\eta) - 2\Omega_{M_n}(\mathbf{m}\,; z_0 - |\mathbf{m}|) \\
= {} & (\theta^+_{\mathbf{m}} - \theta^-_{\mathbf{m}})^2 - 2(n-1)(\theta^+_{\mathbf{m}} - \theta^-_{\mathbf{m}}) + (n-1)^2 + (|\mathbf{m}| - |\mathbf{l}_0| + c_0)^2 \\
&+ (|\mathbf{m}| - z_0 + 2c_0)(2|\mathbf{l}_0| - |\mathbf{m}| - z_0 - |\mathbf{m}| + z_0 - 2c_0) \\
&- (2|\mathbf{l}_0| - |\mathbf{m}| - z_0)^2 - 2(|\mathbf{m}| - |\mathbf{l}_0| + c_0)(\theta^+_{\mathbf{m}} + \theta^-_{\mathbf{m}}) \\
&- 2(\theta^{+\,2}_{\mathbf{m}} + \theta^{-\,2}_{\mathbf{m}}) + 2(z_0 - |\mathbf{m}|)(\theta^+_{\mathbf{m}} + \theta^-_{\mathbf{m}}) - 4n\theta^-_{\mathbf{m}} \\
&+ 2(z_0 - |\mathbf{l}_0|)^2 - (z_0 - |\mathbf{m}|)^2 + 2n(|\mathbf{l}_0| - |\mathbf{m}|) + 2\mu^2_{\mathbf{m}} - 4h^+\mu_{\mathbf{m}} \\
&+ 2(|\mathbf{l}_0| - \sum_{h^+<k\leqslant n} l_k) + 2\Omega_{H_n}(\eta) - 2\Omega_{M_n}(\mathbf{m}\,; z_0 - |\mathbf{m}|) \\
= {} & -(\theta^+_{\mathbf{m}} + \theta^-_{\mathbf{m}})^2 + 2n(\theta^-_{\mathbf{m}} - \theta^+_{\mathbf{m}} + |\mathbf{l}_0| - |\mathbf{m}| - 2\theta^-_{\mathbf{m}} - \mu_{\mathbf{m}}) \\
&+ (|\mathbf{m}| - |\mathbf{l}_0| + c_0)^2 + 2(|\mathbf{m}| - z_0 + 2c_0)(|\mathbf{l}_0| - |\mathbf{m}| - c_0) \\
&- (2|\mathbf{l}_0| - |\mathbf{m}| - z_0)^2 - 2(|\mathbf{m}| - |\mathbf{l}_0| + c_0)(\theta^+_{\mathbf{m}} + \theta^-_{\mathbf{m}}) \\
&+ 2(z_0 - |\mathbf{m}|)(\theta^+_{\mathbf{m}} + \theta^-_{\mathbf{m}}) \\
&+ 2(z_0 - |\mathbf{l}_0|)^2 - 2(z_0 - |\mathbf{m}|)^2 + 2c_0^2 \\
&+ R_{\mathbf{m}},
\end{aligned}
$$

where

$$
\begin{aligned}
R_{\mathbf{m}} = {} & 2\mu^2_{\mathbf{m}} - 4h^+\mu_{\mathbf{m}} + 2n\mu_{\mathbf{m}} + 2(|\mathbf{l}_0| - \sum_{h^+<k\leqslant n} l_k) \\
&+ 2\Omega_{H_n}(\eta) - 2\Omega_{M_n}(\mathbf{m}\,; z_0 - |\mathbf{m}|) \\
&+ (z_0 - |\mathbf{m}|)^2 - 2c_0^2 + (n-1)^2 + 2(\theta^+_{\mathbf{m}} - \theta^-_{\mathbf{m}}).
\end{aligned}
$$

Since

$$
\theta^+_{\mathbf{m}} + \theta^-_{\mathbf{m}} - |\mathbf{l}_0| + |\mathbf{m}| + \mu_{\mathbf{m}} = 0, \tag{9.1.12}
$$

by Lemma 9.1.3, the second term in the last expression of $Q_{\mathbf{m}}$ is zero. Thus

$$
\begin{aligned}
Q_{\mathbf{m}} = {} & -(\theta^+_{\mathbf{m}} + \theta^-_{\mathbf{m}})^2 + (|\mathbf{m}| - |\mathbf{l}_0| + c_0)^2 \\
&+ 2(|\mathbf{m}| - z_0 + 2c_0)(|\mathbf{l}_0| - |\mathbf{m}| - c_0) \\
&- (2|\mathbf{l}_0| - |\mathbf{m}| - z_0)^2 - 2(|\mathbf{m}| - |\mathbf{l}_0| + c_0)(\theta^+_{\mathbf{m}} + \theta^-_{\mathbf{m}}) \\
&+ 2(z_0 - |\mathbf{m}|)(\theta^+_{\mathbf{m}} + \theta^-_{\mathbf{m}}) \\
&+ 2(z_0 - |\mathbf{l}_0|)^2 - 2(z_0 - |\mathbf{m}|)^2 + 2c_0^2 + R_{\mathbf{m}}
\end{aligned}
$$

$$
\begin{aligned}
&= -(\theta^+_{\mathbf{m}} + \theta^-_{\mathbf{m}})^2 + (|\mathbf{m}| - |\mathbf{l}_0| + c_0)^2 \\
&\quad + 2(|\mathbf{l}_0| - |\mathbf{m}| - c_0)(|\mathbf{m}| - z_0 + 2c_0 + \theta^+_{\mathbf{m}} + \theta^-_{\mathbf{m}}) - (2|\mathbf{l}_0| - |\mathbf{m}| - z_0)^2 \\
&\quad + 2(z_0 - |\mathbf{m}|)(\theta^+_{\mathbf{m}} + \theta^-_{\mathbf{m}}) \\
&\quad + 2(z_0 - |\mathbf{l}_0|)^2 - 2(z_0 - |\mathbf{m}|)^2 + 2c_0^2 + R_{\mathbf{m}} \\
&= -(\theta^+_{\mathbf{m}} + \theta^-_{\mathbf{m}})^2 + (|\mathbf{m}| - |\mathbf{l}_0| + c_0)^2 \\
&\quad - 2(|\mathbf{m}| - |\mathbf{l}_0| + c_0)(|\mathbf{m}| - z_0 + \theta^+_{\mathbf{m}} + \theta^-_{\mathbf{m}}) - 4c_0(|\mathbf{m}| - |\mathbf{l}_0| + c_0) + 2c_0^2 \\
&\quad - 4(z_0 - |\mathbf{l}_0|)^2 + 4(z_0 - |\mathbf{l}_0|)(z_0 - |\mathbf{m}|) - (z_0 - |\mathbf{m}|)^2 \\
&\quad + 2(z_0 - |\mathbf{m}|)(\theta^+_{\mathbf{m}} + \theta^-_{\mathbf{m}}) \\
&\quad + 2(z_0 - |\mathbf{l}_0|)^2 - 2(z_0 - |\mathbf{m}|)^2 + R_{\mathbf{m}} \\
&= -(\theta^+_{\mathbf{m}} + \theta^-_{\mathbf{m}})^2 + 2(z_0 - |\mathbf{m}|)(\theta^+_{\mathbf{m}} + \theta^-_{\mathbf{m}}) - (z_0 - |\mathbf{m}|)^2 \\
&\quad + (|\mathbf{m}| - |\mathbf{l}_0| + c_0)^2 - 2(|\mathbf{m}| - |\mathbf{l}_0| + c_0)(\theta^+_{\mathbf{m}} + \theta^-_{\mathbf{m}} + |\mathbf{m}| - z_0) \\
&\quad - 4c_0(|\mathbf{m}| - |\mathbf{l}| + c_0) + 2c_0^2 \\
&\quad - 2(z_0 - |\mathbf{m}|)^2 + 4(z_0 - |\mathbf{l}_0|)(z_0 - |\mathbf{m}|) - 2(z_0 - |\mathbf{l}_0|)^2 + R_{\mathbf{m}} \\
&= -(\theta^+_{\mathbf{m}} + \theta^-_{\mathbf{m}} + |\mathbf{m}| - z_0)^2 + (|\mathbf{m}| - |\mathbf{l}_0| + c_0)^2 \\
&\quad - 2(|\mathbf{m}| - |\mathbf{l}_0| + c_0)(\theta^+_{\mathbf{m}} + \theta^-_{\mathbf{m}} + |\mathbf{m}| - z_0) - 2(|\mathbf{m}| - |\mathbf{l}_0| + c_0)^2 + R_{\mathbf{m}} \\
&= -(\theta^+_{\mathbf{m}} + \theta^-_{\mathbf{m}} + 2|\mathbf{m}| - |\mathbf{l}_0| + c_0 - z_0)^2 + R_{\mathbf{m}} \\
&= -(|\mathbf{m}| - \mu_{\mathbf{m}} + c_0 - z_0)^2 + R_{\mathbf{m}},
\end{aligned}
$$

where to obtain the last equality we used (9.1.12). Now by Proposition 6.2.2, we have

$$\Omega_{M_n}(\mathbf{m}\,; z_0 - |\mathbf{m}|) - \frac{1}{2}(z_0 - |\mathbf{m}|)^2 = \sum_{i=1}^{n-1} m_i^2 + \sum_{i=1}^{n-1}(n-2i)m_i.$$

Hence

$$
\begin{aligned}
R_{\mathbf{m}} &= 2\Omega_{H_n}(\eta) - 2c_0^2 + (n-1)^2 \\
&\quad + 2\Big(-\sum_{i=1}^{n-1} m_i^2 - \sum_{i=1}^{n-1}(n-2i)m_i + \mu_{\mathbf{m}}^2 + 2(n-2h^+)\mu_{\mathbf{m}} + |\mathbf{l}_0| \\
&\qquad - 2\sum_{h^+ < k \leqslant n} l_k + \theta^+_{\mathbf{m}} - \theta^-_{\mathbf{m}}\Big). \ \square
\end{aligned}
$$

Proof of Theorem 8.2.1. The formula (8.2.5) of $f(\mathbf{m}\,;r)$ with $\mathbf{m} \in \partial^{(h)}\Lambda^+_{n-1}(\eta|\lambda_0)$ is no other than (9.1.6) in Proposition 9.1.2. If $h = 0$ or $h = n$, then the set $\partial^{(h)}\Lambda^+_{n-1}(\eta|\lambda_0)$ consists of a unique element (Lemma 8.2.1). Hence there is nothing to say in this case. If $0 < h < n$, then we have to deduce the recurrence relations (8.2.7) and (8.2.8) among $\gamma(\mathbf{m})$'s. We need the following formulae, whose validity is confirmed by comparing the Taylor series expansion of both side at $z = 0$:

$$(9.1.13)\quad z\frac{d}{dz}{}_2F_1(a,b\,;c\,;z) + (c-1){}_2F_1(a,b\,;c\,;z) = (c-1){}_2F_1(a,b\,;c-1\,;z),$$

$$(9.1.14)\quad z(1-z)\frac{d}{dz}{}_2F_1(a,b\,;c\,;z) + \Big((1-a-b)z + c - 1\Big){}_2F_1(a,b\,;c\,;z) = (c-1){}_2F_1(a-1,b-1\,;c-1\,;z).$$

If $|\mathbf{m}| - |\mathbf{l}_0| + c_0 > 0$, then changing variables from r to $z = \mathrm{th}^2(r)$ and using the formula (9.1.13), we can easily deduce (8.2.7) from (8.3.2). If $|\mathbf{m}|-|\mathbf{l}|+c_0 < 0$, then in the same way we obtain (8.2.8) starting with (8.3.3) with the aid of (9.1.14). This completes the proof of Theorem 8.2.1. □

9.2. The case when π is elementary

Here we take a look at a special case. Let π be an irreducible $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-module isomorphic to $\pi_n(\mathbf{p}, p_0\,; s)$ with $s \in \mathbb{C}$, $(\mathbf{p}\,; p_0) \in {}^\circ\mathcal{L}^+_n$.

Proposition 9.2.1. *Let $\eta = \pi_{n-1}(\mathbf{q}, q_0\,; t)[c_0]$ with $t \in \mathbb{C}$, $(\mathbf{q}\,; q_0) \in {}^\circ\mathcal{L}^+_{n-1}$, $c_0 \in \Lambda_1$. Put $\mathbf{p} = (p_i)_{1\leqslant i\leqslant n-1}$ and $\mathbf{q} = (q_j)_{1\leqslant j\leqslant n-2}$. Assume that η is irreducible. Then $\dim_{\mathbb{C}}\mathcal{I}_{\pi,\eta} \leqslant 1$. Moreover we have $\mathcal{I}_{\pi,\eta} = \{0\}$ unless $|\mathbf{p}| - |\mathbf{q}| = -p_0 + q_0 + c_0$ and $\mathbf{q} \subset \mathbf{p}$. Under these conditions, put*

$$\mathbf{l}_0 = (p_1, p_2 \ldots, p_{n-1}, p_{n-1}) \in \Lambda^+_n(\pi),$$
$$\mathbf{m}_0 = (q_1, q_2, \ldots, q_{n-2}, p_{n-1}) \in \Lambda^+_{n-1}(\eta|\lambda_0).$$

Let $\Phi \in \mathcal{I}_{\pi,\eta}$ and $\{f(\mathbf{m}\,;r)|\ \mathbf{m} \in \Lambda^+_{n-1}(\eta|\lambda_0)\}$ the standard coefficient of $\Phi_{\mathbf{l}_0}|A_n$. Then $f(\mathbf{m}_0\,;r)$ is, up to a constant, given by the formula

$$(\mathrm{sh}(r))^{|p_0-q_0|}(\mathrm{ch}(r))^{s-n-c_0+p_{n-1}+p_0-q_0-|p_0-q_0|} \times {}_2F_1\Big(\frac{1-s+t+|p_0-q_0|}{2}, \frac{1-s-t+|p_0-q_0|}{2}\,; 1+|p_0-q_0|\,; \mathrm{th}^2(r)\Big).$$

Remark 9.2.1. We can prove that the condition $\mathbf{q} \subset \mathbf{p}$ and $|\mathbf{p}|-|\mathbf{q}| = -p_0+q_0+c_0$ above is actually a necessary and sufficient condition for $\mathcal{I}_{\eta,\pi}$ to be non zero ([20]).

9.3. The case when π is discrete series

Recall the notations in 4.3. For each integer $0 \leqslant h \leqslant n$, put $\Xi^n_{(h)} = (\mathcal{L}_n+\rho)\cap D_h$. Then Ξ^n, the union of $\Xi^n_{(h)}$ for $h \in \{0,\dots,n\}$, is the set of Harish-Chandra parameters of discrete series representations of G_n. It can be deduced from [10, Theorem 6] that if π is a discrete series representation of G_n with Harish-Chandra parameter $\lambda = [(\lambda_j)_{1\leqslant j\leqslant n}\,;\lambda_{n+1}] \in \Xi^n_{(h)}$, then π belongs to the class Π_ζ, $\zeta = (h,h,\mathbf{r},z)$ with $z = \sum_{j=1}^{n+1}\lambda_j$, $\mathbf{r} = (r_j)_{1\leqslant j\leqslant n+1} \in \Lambda^+_{n+1}$ such that

$$\begin{aligned} r_j &= \lambda_j + \frac{2j-n}{2}, \quad j \in \{1,\dots,h\},\\ r_{j+1} &= \lambda_j + \frac{2j-n-2}{2}, \quad j \in \{h+1,\dots,n\},\\ r_{h+1} &= \lambda_{n+1} + h - \frac{n}{2}. \end{aligned}$$

Note that

$$\lambda_1 > \cdots > \lambda_h > \lambda_{n+1} > \lambda_{h+1} > \cdots > \lambda_n$$

holds. Moreover it turns out that the D_h-fundamental corner $\lambda_0 = [\mathbf{l}_0\,;z-|\mathbf{l}_0|]$ defined in 8.2 (ii) coincides with the minimal K_n-type (Blattner parameter) of π and is actually given as $\mathbf{l}_0 = (r_1,\dots,r_h,r_{h+2},\dots,r_{n+1})$.

Let $\eta = \eta_0[c_0]$ and π be discrete series representaions of H_n and G_n respectively. We shall explicitly write down the necessary conditions for $\mathcal{I}_{\eta,\pi} \neq \{0\}$ stated in Theorem 8.1.1. Here we refrain from a thorough investigation, and instead just look at a special case. Let $\mu = [(\mu_i)_{1\leqslant i\leqslant n-1}\,;\mu_n] \in \Xi^{n-1}_{(h)}$ for $h \in \{0,\dots,n-1\}$ be the Harish-Chandra parameter of η_0 and $\lambda = [(\lambda_j)_{1\leqslant j\leqslant n}\,;\lambda_{n+1}] \in \Xi^n_{(k)}$ with $k \in \{0,\dots,n\}$ that of π. We assume $k = h+1 < n$.

Proposition 9.3.1. *Let π and η be as above.*

(1) *We have $\mathcal{I}_{\eta,\pi} \leqslant 1$. Moreover $\mathcal{I}_{\eta,\pi} = \{0\}$ unless $c_0 + \sum_{i=1}^n \mu_i = \sum_{j=1}^{n+1}\lambda_j$ and one of the inequalities*

$$\lambda_1 > \mu_1 > \lambda_2 > \cdots > \lambda_h > \mu_h > \lambda_{h+1} > \lambda_{n+1}$$

$$> \mu_n > \mu_{h+1} > \lambda_{h+2} > \cdots > \mu_{n-1} > \lambda_n,$$
$$\lambda_1 > \mu_1 > \lambda_2 > \cdots > \lambda_h > \mu_h > \mu_n > \mu_{h+1}$$
$$> \lambda_{h+1} > \lambda_{n+1} > \lambda_{h+2} > \cdots > \mu_{n-1} > \lambda_n$$

holds.

(2) *Let* $[(m_i)_{1 \leqslant i \leqslant n-1}\ ; m_0] \in \mathcal{L}^+_{n-1}(\eta_0)$ *be the minimal* K_{n-1}*-type of* η_0 *and* $[(l_j)_{1 \leqslant j \leqslant n}\ ; l_0] \in \mathcal{L}^+_n(\pi)$ *the minimal* K_n*-type of* π*. Assume the condition in (1). Then the set* $\partial^{(h)}\Lambda^+_{n-1}(\eta|\lambda_0)$ *coincides with the set of*

$$\mathbf{m}_y = (m_1, \ldots, m_h, y, m_{h+2}, \ldots, m_{n-1})$$

with $\inf(l_{h+1}, m_{h+1}) \leqslant y \leqslant l_{h+2}$*. For* $\Phi \in \mathcal{I}_{\eta,\pi}$*, let* $\{f(\mathbf{m}\ ; r) |\ \mathbf{m} \in \Lambda^+_{n-1}(\eta|\lambda_0)\}$ *be the standard coefficient of* $\Phi_{\mathbf{l}_0}$*. Then the function* $f(\mathbf{m}_y\ ; r)$ *is a constant multiple of the function*

$$\begin{aligned}&(\mathrm{sh}(r))^{|\kappa|}\\ &\quad\times (\mathrm{ch}(r))^{-2\epsilon\kappa+m_0-2l_0-2h-2+\Sigma_{h+1<i\leqslant n} l_i - \Sigma_{1\leqslant i\leqslant h+1} l_i + \Sigma_{1\leqslant j\leqslant h+1} m_j - \Sigma_{h+1<j\leqslant n-1} m_j}\\ &\quad\times {}_2F_1\Big(\epsilon\kappa + 1 + l_0 - m_0, \epsilon\kappa + 2h + 2 - n + l_0 - m_{h+1}\ ; 1 + |\kappa|\ ; \mathrm{th}^2(r)\Big)\end{aligned}$$

with $\kappa = y - m_{h+1} + l_0 - m_0$ *and* ϵ *expressing* 0 *or* 1 *according as* $\kappa \geqslant 0$ *or* < 0*.*

REMARK 9.3.1. When $n = 2$ the condition in (1) is actually a necessary and sufficient condition for $\dim_{\mathbb{C}} \mathcal{I}_{\eta,\pi} = 1$ (see [24], [18]). In [20], we prove that this is true even if $n > 2$.

10. Matrix Coefficients

In this chapter we present an explicit formula of A_n-radial part of matrix coefficients of irreducible $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-modules. The computations and proofs of propositions will be omitted because they are quite similar to those for Shintani functions.

10.1. Matrix coefficients

Let $(\pi, \mathcal{H}_\pi)$ be an irreducible $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-module and $\{\iota^\pi_{\mathbf{l}} |\ \mathbf{l} \in \Lambda^+_n(\pi)\}$ the standard system for π. Since $\pi|K_n$ is a multiplicity free direct sum of images of $\iota^{\pi}_{\mathbf{l}}$, $\mathbf{l} \in \Lambda^+_n(\pi)$, we get a family of K_n-homomorphisms $\{\varpi^{\mathbf{l}}_{\pi} |\ \mathbf{l} \in \Lambda^+_n(\pi)\}$ with $\varpi^{\mathbf{l}}_{\pi} : \mathcal{H}_{\pi} \to W(\mathbf{l})$ such that $\varpi^{\mathbf{l}'}_{\pi} \circ \iota^{\pi}_{\mathbf{l}} = 0$ if $\mathbf{l}' \neq \mathbf{l}$ and $\varpi^{\mathbf{l}}_{\pi} \circ \iota^{\pi}_{\mathbf{l}} = 1_{W(\mathbf{l})}$.

Now for every $\lambda = [\mathbf{l}\ ; c_n(\pi) - |\mathbf{l}|]$, $\lambda' = [\mathbf{l}'\ ; c_n(\pi) - |\mathbf{l}'|] \in \mathcal{L}^+_n(\pi)$, put

$$F_{\lambda',\lambda}(g) = \varpi^{\mathbf{l}'}_{\pi} \circ \pi^{\infty}(g) \circ \iota^{\pi}_{\mathbf{l}}, \quad g \in G_n$$

Then $F_{\lambda',\lambda} : G_n \to \mathrm{Hom}(W(\mathbf{l}), W(\mathbf{l}'))$ is a C^{∞}-function such that

$$F_{\lambda',\lambda}(k'gk) = \tau_{\lambda'}(k') \circ F_{\lambda',\lambda}(g) \circ \tau_{\lambda}(k), \quad k',\ k \in K_n,\ g \in G_n. \tag{10.1.1}$$

The functions of the form $F_{\lambda',\lambda}$ will be called (λ',λ)-*matrix coefficient* of π. By the Cartan decomposition $G = K_nA_nK_n$ and the equivariant property (10.1.1), $F_{\lambda',\lambda}$ is completely determined by its restriction to A_n, say $\varphi_{\lambda',\lambda}$. As a consequence of (10.1.1), the image of $\varphi_{\lambda',\lambda}$ is contained in $\mathrm{Hom}_{M_n}(\tau_{\lambda}|M_n, \tau_{\lambda'}|M_n)$.

Lemma 10.1.1. *Set $\Delta(\lambda'|\lambda) = \Delta(\mathbf{l}') \cap \Delta(\mathbf{l})$ (see Lemma 3.1.1). For every $\mathbf{m} \in \Delta(\lambda'|\lambda)$, put*

$$\tilde{\omega}_{\lambda',\lambda}(\mathbf{m}) = \mathsf{j}^{\mathbf{m}}_{\mathbf{l}'} \circ \mathsf{p}^{\mathbf{l}}_{\mathbf{m}},$$

where $\mathsf{j}^{\mathbf{m}}_{\mathbf{l}'} : W(\mathbf{m}) \to W(\mathbf{l}')$ is the map defined by

$$\mathsf{j}^{\mathbf{m}}_{\mathbf{l}'} \big|Q\big\rangle = \big|\mathbf{l}'\ , Q\big\rangle$$

for $Q \in GZ^{(n-1)}(\mathbf{m})$, and $\mathsf{p}^{\mathbf{l}}_{\mathbf{m}}$ is the map in Lemma 3.1.1. Then $\{\tilde{\omega}_{\lambda',\lambda}(\mathbf{m}) |\ \mathbf{m} \in \Delta(\lambda'|\lambda)\}$ is a $\mathbb{C}$-basis of the $\mathbb{C}$-vector space $\mathrm{Hom}_{M_n}(\tau_{\lambda}|M_n, \tau_{\lambda'}|M_n)$.

By this lemma, we can express $\varphi_{\lambda',\lambda}$ of the form

$$\varphi_{\lambda',\lambda}(a_r) = \sum_{\mathbf{m} \in \Delta(\lambda'|\lambda)} f_{\lambda',\lambda}(\mathbf{m}\ ; r) \cdot \tilde{\omega}_{\lambda',\lambda}(\mathbf{m}), \quad a_r \in A_n$$

with a uniquely determined family $\{f_{\lambda',\lambda}(\mathbf{m}\ ; r) |\ \mathbf{m} \in \Delta(\lambda'|\lambda)\}$ consisting of C^{∞}-functions on $r > 0$.

Here is a necessary condition for $F_{\lambda',\lambda}$ to be non zero.

Lemma 10.1.2. *Let λ', $\lambda \in \mathcal{L}^+_n(\pi)$. Then $F_{\lambda',\lambda}$ is identically zero unless $c_n(\tau_{\lambda'}) = c_n(\tau_{\lambda})$ and $\Delta(\lambda'|\lambda) \neq \emptyset$.*

10.2. Differential equations for matrix coefficients

For the induced representation $\mathcal{M} = \mathrm{Ind}_{K_n}^{G_n}(\tau_{\lambda'})$, we have the Schmid operators as we explained in 6.1:

$$^{\rho}\nabla^{\beta}_{\lambda',\lambda} : C^{\infty}_{\tau_{\lambda'},\tau_{\lambda}}(K_n\backslash G_n/K_n) \to C^{\infty}_{\tau_{\lambda'},\tau_{\lambda+\beta}}(K_n\backslash G_n/K_n)$$

for $\lambda \in \mathcal{L}_n^+$ and $\beta \in R_{\mathrm{nc}}$. Let

$$\Omega_{\lambda',\lambda} : C^{\infty}_{\tau_{\lambda'},\tau_{\lambda}}(K_n\backslash G_n/K_n) \to C^{\infty}_{\tau_{\lambda'},\tau_{\lambda}}(K_n\backslash G_n/K_n)$$

be the Casimir operator.

Here is the system of differential equations for the matrix coefficients:

Proposition 10.2.1. *The family* $\{F_{\lambda',\lambda} |\ \lambda',\ \lambda \in \mathcal{L}_n^+(\pi)\}$ *satisfies the equations*

$$\begin{aligned}
\Omega_{\lambda',\lambda} F_{\lambda',\lambda}(g) &= \Omega_{G_n}(\pi) F_{\lambda',\lambda}(g),\\
\nabla^{+\beta_k}_{\lambda',\lambda} F_{\lambda',\lambda}(g) &= A^{\pi}_k(\mathbf{l}) F_{\lambda',\lambda^{+k}}(g),\\
\nabla^{-\beta_k}_{\lambda,\lambda} F_{\lambda',\lambda}(g) &= B^{\pi}_k(\mathbf{l}) F_{\lambda',\lambda^{+k}}(g)
\end{aligned}$$

for $k \in \{1, \ldots, n\}$. *Here* $A^{\pi}_k(\mathbf{l}),\ B^{\pi}_k(\mathbf{l})$ *are the 2n functions for* π *introduced in Proposition 4.1.1.*

10.3. Difference-differential equations

Through the same type of procedure that we explained in detail for the Shintani functions (see §7), we can write down the equations in Proposition 10.2.1 in terms of standard coefficients of $F_{\lambda',\lambda}$'s. Here is the result:

Theorem 10.3.1. *The system* $\{f_{\lambda',\lambda}(\mathbf{m}\,; r) |\ \mathbf{m} \in \Delta(\lambda'|\lambda)\}$ *satisfies the following system of differential equations:*

$$\begin{aligned}
&\left(r\frac{d}{dr}\right)^2 f_{\lambda',\lambda}(\mathbf{m}\,; r) + \left(\mathrm{th}(r) + \frac{2n-1}{\mathrm{th}(r)}\right) r\frac{d}{dr} f_{\lambda',\lambda}(\mathbf{m}\,; r)\\
&\quad + G_{\lambda',\lambda}(\mathbf{m}) f_{\lambda',\lambda}(\mathbf{m}\,; r)\\
&\quad + \frac{4\mathrm{ch}(r)}{\mathrm{sh}^2(r)} \sum_{i=1}^{n-1} \Big(a_i(\mathbf{l}'\,; \mathbf{m}) a_i(\mathbf{l}\,; \mathbf{m}) f_{\lambda',\lambda}(\mathbf{m}^{-i}\,; r)\\
&\quad + b_i(\mathbf{l}'\,; \mathbf{m}) b_i(\mathbf{l}\,; \mathbf{m}) f_{\lambda',\lambda}(\mathbf{m}^{+i}\,; r) \Big)
\end{aligned}$$

$$= 0$$

$$\begin{aligned}
&2A_k^\pi(\mathbf{l}) f_{\lambda',\lambda^{+k}}(\mathbf{m}\,;r) \\
&= \gamma^+(\mathbf{m}\,;k,\mathbf{l})\Big\{ r\frac{d}{dr} f_{\lambda',\lambda}(\mathbf{m}\,;r) \\
&\quad + \left(\frac{\tau_{\lambda',\mathbf{m}} + \tau_{\lambda,\mathbf{m}}}{2\mathrm{sh}(r)\mathrm{ch}(r)} + \frac{1}{\mathrm{th}(r)}(\tau_{\lambda,\mathbf{m}} + 2(z_0 - |\mathbf{l}| - l_k + k - 1)) \right) \\
&\quad \times f_{\lambda',\lambda}(\mathbf{m}\,;r) + \frac{2}{\mathrm{sh}(r)} \sum_{i=1}^{n-1} \frac{a_i(\mathbf{l}'\,;\mathbf{m}) a_i(\mathbf{l}\,;\mathbf{m})}{l_k - m_i + i - k + 1} f_{\lambda',\lambda}(\mathbf{m}^{-i}\,;r) \Big\}
\end{aligned}$$

$$\begin{aligned}
&2B_k^\pi(\mathbf{l}) f_{\lambda',\lambda^{-k}}(\mathbf{m}\,;r) \\
&= \gamma^-(\mathbf{m}\,;k,\mathbf{l})\Big\{ r\frac{d}{dr} f_{\lambda',\lambda}(\mathbf{m}\,;r) \\
&\quad - \left(\frac{\tau_{\lambda',\mathbf{m}} + \tau_{\lambda,\mathbf{m}}}{2\mathrm{sh}(r)\mathrm{ch}(r)} + \frac{1}{\mathrm{th}(r)}(\tau_{\lambda,\mathbf{m}} + 2(z_0 - |\mathbf{l}| - l_k + k - n)) \right) \\
&\quad \times f_{\lambda',\lambda}(\mathbf{m}\,;r) - \frac{2}{\mathrm{sh}(r)} \sum_{i=1}^{n-1} \frac{b_i(\mathbf{l}'\,;\mathbf{m}) b_i(\mathbf{l}\,;\mathbf{m})}{l_k - m_i + i - k} f_{\lambda',\lambda}(\mathbf{m}^{+i}\,;r) \Big\}
\end{aligned}$$

with

$$\begin{aligned}
&G_{\lambda',\lambda}(\mathbf{m}) \\
&= -\frac{1}{\mathrm{ch}^2(r)\mathrm{sh}^2(r)} \left(\frac{\tau_{\lambda',\mathbf{m}} + \tau_{\lambda,\mathbf{m}}}{2} \right)^2 \\
&\quad + \frac{1}{\mathrm{sh}^2(r)} \Big((\tau_{\lambda',\mathbf{m}})^2 + \tau_{\lambda',\mathbf{m}}\tau_{\lambda,\mathbf{m}} + (\tau_{\lambda,\mathbf{m}})^2 - 2\Omega_{K_n}(\mathbf{l}'\,;z_0 - |\mathbf{l}'|) \\
&\qquad - 2\Omega_{K_n}(\mathbf{l}\,;z_0 - |\mathbf{l}|) + 4\Omega_{M_n}(\mathbf{m}\,;z_0 - |\mathbf{m}|) \Big) \\
&\quad + 2\Omega_{M_n}(\mathbf{m}\,;z_0 - |\mathbf{m}|) - 2\Omega_{G_n}(\pi).
\end{aligned}$$

10.4. Explicit formula for matrix coefficients with corner K_n-type

For an irreducible $(\mathfrak{g}_{n,\mathbb{C}}, K_n)$-module π, we have attached a triple $(\mathbf{l}_0, h, s)$ at the beginning of 8.2. Put $z_0 = c_n(\pi)$ and $\lambda_0 = [\mathbf{l}_0\,;z_0 - |\mathbf{l}|] \in \mathcal{L}_n^+$.

Then, by definition, λ_0 is D_{n-1}-corner or D_h-corner according as π is elementary or not. Now we give an explicit formula of (λ', λ_0)-matrix coefficients of π. Put $\mathbf{l}' = (l'_k)_{1 \leqslant k \leqslant n}$ and $\mathbf{l}_0 = (l_k)_{1 \leqslant k \leqslant n}$. Then

$$\Delta(\lambda'|\lambda_0) = \{\mathbf{m} \in \Lambda^+_{n-1} | \ \inf(l'_i, l_i) \geqslant m_i \geqslant \sup(l'_{i+1}, l_{i+1}), \ i = 1, \ldots, n-1\}.$$

Thus the set $\Delta(\lambda'|\lambda_0)$ is non-empty if and only if

$$\inf(l'_i, l_i) \geqslant \sup(l'_{i+1}, l_{i+1}), \quad i \in \{1, \ldots, n-1\}.$$

We assume that $\mathbf{l}'$ satisfies this inequality as well as $c_n(\tau_{\lambda'}) = c_n(\tau_{\lambda_0})$. Let $\partial^{(h)}\Delta(\lambda'|\lambda_0)$ be the subset of Λ^+_{n-1} defined as in Definition 8.2.1.
For $\mathbf{m} \in \Delta(\lambda'|\lambda_0)$, let $\mu_{\mathbf{m}}$ and $\theta_{\mathbf{m}}$ be the numbers defined by (8.2.1) and (8.2.2) respectively.

Theorem 10.4.1. *Let* $\mathbf{m} \in \partial^{(h)}\Delta(\lambda'|\lambda_0)$. *Then the function* $f_{\lambda',\lambda_0}(\mathbf{m} \,; r)$ *is, up to a constant, given by the formula*

$$(\mathrm{sh}(r))^{\theta_{\mathbf{m}}-n+1+Z_{\mathbf{m}}}(\mathrm{ch}(r))^{s+\mu_{\mathbf{m}}-l_{h^+}-1-Z_{\mathbf{m}}}{}_2F_1(Y^+_{\mathbf{m}}, Y^-_{\mathbf{m}} \,; 1+Z_{\mathbf{m}} \,; \mathrm{th}^2(r))$$

with $Z_{\mathbf{m}} \geqslant 0$,

$$\begin{aligned} Z^2_{\mathbf{m}} = {} & 2\Big(\sum_{k=1}^{n} {l'_k}^2 + \sum_{k=1}^{n}(n-2k+1)l'_k\Big) - (|\mathbf{m}| - |\mathbf{l}_0| - \mu_{\mathbf{m}})^2 + (n-1)^2 \\ & + 2\Big(-\sum_{i=1}^{n-1} m_i^2 - \sum_{i=1}^{n-1}(n-2i)m_i + \mu^2_{\mathbf{m}} + (n-2h^+)\mu_{\mathbf{m}} + |\mathbf{l}_0| \\ & \quad - 2\sum_{h^+ < k \leqslant n} l_k + \theta_{\mathbf{m}}\Big) \end{aligned}$$

and

$$Y^+_{\mathbf{m}} = \frac{1 - \mu_{\mathbf{m}} + l_{h^+} - s + |\mathbf{l}'| + |\mathbf{l}_0| - |\mathbf{m}| - z_0 + Z_{\mathbf{m}}}{2},$$
$$Y^-_{\mathbf{m}} = \frac{1 - \mu_{\mathbf{m}} + l_{h^+} - s - |\mathbf{l}'| - |\mathbf{l}_0| + |\mathbf{m}| + z_0 + Z_{\mathbf{m}}}{2}.$$

10.5. A few examples

We consider the case when the representation π is an elementary representation of G_n.

Proposition 10.5.1. *Let* $\pi = \pi_n(s\ ; \mathbf{p}, p_0)$ *with* $s \in \mathbb{C}$, $(\mathbf{p}\ ; p_0) \in {}^{\circ}\mathcal{L}^+_{n-1}$ *be an irreducible elementary representation of* G_n. *Let* $\mathbf{l}_0 \in \Lambda_n^+(\pi)$ *be the as in 8.2 (i) and put* $\lambda_0 = [\mathbf{l}_0\ ; z_0 - |\mathbf{l}_0|]$ *with* $z_0 = |\mathbf{p}_0| + p_0$. *Then for* $\mathbf{m}_0 = (p_2, \dots, p_{n-1}, p_{n-1}) \in \Delta(\lambda_0|\lambda_0)$, *the function* $f_{\lambda_0,\lambda_0}(\mathbf{m}_0\ ; r)$ *is given by the formula*

$$(\mathrm{ch}(r))^{s-n+p_{n-1}-p_1}{}_2F_1\left(p_1 - \frac{s-n+p_0}{2}, -p_{n-1} - \frac{s-n-p_0}{2}\ ;\right.$$
$$\left. n + p_1 - p_{n-1}\ ; \mathrm{th}^2(r)\right).$$

Next we consider the non-elementary representations.

Proposition 10.5.2. *Let* π *be a non-elementary representation with the triplet* $(\mathbf{l}_0, h, s)$ *as in 8.2 (ii). Assume* $0 < h < n$. *Then for* $\mathbf{m} = (m_i)_{1 \leqslant i \leqslant n-1} \in \partial^{(h)}\Delta(\lambda_0|\lambda_0)$, *the function* $f_{\lambda_0,\lambda_0}(\mathbf{m}\ ; r)$ *is given by*

$$(\mathrm{ch}(r))^{s+m_h-l_h-l_1+l_n-n}{}_2F_1(h-m_h+l_1, z_0-|\mathbf{l}_0|-l_n+h\ ; l_1-l_n+n\ ; \mathrm{th}^2(r)).$$

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