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Division Theorems in Higher-codimensional Boundary Value Problems for $\mathcal{E}$-modules

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Abstract. We prove a division theorem with coefficients of relative bimicrofuntions for $\mathcal{E}$-modules. Our proof is based on the Cauchy-Kowalevski theorems for $\mathcal{E}$-modules. As one of applications, we solve Takeuchi's conjecture. We also apply his conjecture to higher-codimensional boundary value problems.

1. Introduction

Let X be a complex manifold. We denote by $\mathcal{O}_X$ the sheaf of holomorphic functions on X and by $\mathcal{D}_X$ the sheaf of differential operators with holomorphic coefficients on X. Sato-Kawai-Kashiwara [12] defined the sheaf $\mathcal{E}_X$ of microdifferential operators on the cotangent bundle T^*X. If $\pi_X : T^*X \to X$ is the projection, then $\pi_X{}^{-1}\mathcal{D}_X$ is a subring of $\mathcal{E}_X$. Kashiwara-Kawai [6] proved the division theorem with coefficients of relative microfunctions for $\mathcal{D}$-modules (also for $\mathcal{E}$-modules). In the case of $\mathcal{D}$-modules, we can prove the theorem by using the generalized Cauchy-Kowalevski theorems for $\mathcal{D}$-modules with algebraic method today (see Kashiwara [5]). In the case of $\mathcal{E}$-modules, their proof of the division theorem used the theory of quantized contact transformations. However, the author thinks that their proof is complicated, so we give another proof in this paper. Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module. Since $\mathcal{O}_X$ is not an $\mathcal{E}_X$-module, we have to find a suitable definition of the holomorphic solution complex $R\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{O}_X)$. This complex was defined through the work of Bony-Schapira [2] and Kashiwara-Schapira [9]. Ishimura [4] stated the Cauchy-Kowalevski theorem for $\mathcal{E}$-modules by using $R\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{O}_X)$. Our proof uses the Cauchy-Kowalevski theorems for $\mathcal{E}$-modules.

There are many varieties of division theorems. Tose [18] proved the division theorem with coefficients of second microfunctions in the case of 1-codimension. Kataoka-Tose [7] and Schapira-Takeuchi [13] developed the

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theory of bimicrofuntions. Takeuchi stated the division theorem with coefficients of relative bimicrofunctions for $\mathcal{D}$-modules. We prove a division theorem with coefficients of relative bimicrofunctions for $\mathcal{E}$-modules. We cannot apply the theory of quantized contact transformations, so the proof of this division theorem also uses the Cauchy-Kowalevski theorems for $\mathcal{E}$-modules.

Our aim is the study of higher-codimensional boundary value problems. Uchida [19] obtained the extension of real analytic solutions. Takeuchi [16] applied the theory of bimicrofunctions to the edge of the wedge type theorems, and he also stated a conjecture. We solve his conjecture, and we prove a local Bochner type extension theorem for hyperfunction (real analytic) solutions (*cf*. Takeuchi [17]).

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2. Review on Preliminary Notions and Results

Let us recall the definition of $R\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{O}_X)$.

Let X be an n-dimensional complex manifold. We denote by $\mathbf{D}^{\mathrm{b}}(X)$ the derived category of the complexes of sheaves on X. Take a point $p_X \in \dot{T}^*X$. Suppose that $\mathcal{M}$ is a coherent $\mathcal{E}_X$-module on an open neighborhood of p_X. Then, we can choose a finite $\mathcal{E}_X$-free resolution of $\mathcal{M}$:

$$(2.1) \qquad 0 \to \mathcal{E}_X^{N_r} \to \mathcal{E}_X^{N_{r-1}} \to \cdots \to \mathcal{E}_X^{N_1} \to \mathcal{E}_X^{N_0} \to \mathcal{M} \to 0$$

We assume that X is an open subset of $\mathbb{C}^n$. Then, there exists a natural homeomorphism $T^*X \simeq X \times \mathbb{C}^n$. We can represent p_X as $(x, \xi) \in X \times \mathbb{C}^n$. We write by $\langle\ ,\ \rangle$ the canonical pairing map $T_xX \times T_x^*X \to \mathbb{C}$. We set $\{\xi\}^{\circ a} := \{v \in T_xX : \mathrm{Re}\,\langle v, \xi\rangle \leq 0\}$. By considering $T_xX \simeq \mathbb{C}^n$, we get that $\{\xi\}^{\circ a}$ is a half space of $\mathbb{C}^n$.

Let $G \subset \mathbb{C}^n$ be a proper convex cone in $\{\xi\}^{\circ a}$ and let $D \subset X$ be an G-round open neighborhood of x. Then, Kashiwara-Schapira [8] defined the ring $E(G, D)$. By choosing G and D, the resolution (2.1) gives rise to the following complex of finite free $E(G, D)$-modules :

$$0 \to E(G,D)^{N_r} \to E(G,D)^{N_{r-1}} \to \cdots \to E(G,D)^{N_1} \to E(G,D)^{N_0} \to 0.$$

We write this complex by $M^\bullet(G, D)$.

We denote by X_G the set X endowed with the G-topology, and by $\phi_G : X \to X_G$ the natural continuous map. To construct $R\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{O}_X)$, we need the following proposition.

PROPOSITION 2.1 (Kashiwara-Schapira [8]). *There exists a G-round open neighborhood U of x such that for any pair $\Omega_0 \subset \Omega_1$ of G-open subsets of X with $\Omega_1 - \Omega_0 \subset\subset U$, we have :*

$$\phi_G{}^{-1} R\phi_{G*} \mathrm{R}\Gamma_{\Omega_1 - \Omega_0}(\mathcal{O}_X) \in \mathbf{D}^{\mathrm{b}}(E(G, D)_{\Omega_1}).$$

We set $Z := \Omega_1 - \Omega_0$. We write by $\mathcal{O}(G, Z)$ the object $\phi_G{}^{-1} R\phi_{G*} \mathrm{R}\Gamma_Z(\mathcal{O}_X)$.

We denote by $\mathbf{D}^{\mathrm{b}}(\Omega_1; p_X)$ the localization of $\mathbf{D}^{\mathrm{b}}(\Omega_1)$ at p_X. Further information of $\mathbf{D}^{\mathrm{b}}(\Omega_1; p_X)$ can be found in Kashiwara-Schapira [10]. Remark that the connection between $\mathcal{O}_X$ and $\mathcal{O}(G, Z)$ is as follows. There is a natural isomorphism

$$\mathcal{O}(G, Z) \simeq \mathcal{O}_X$$

in $\mathbf{D}^{\mathrm{b}}(\Omega_1; p_X)$. We have the natural functor $\mathbf{D}^{\mathrm{b}}(\Omega_1) \to \mathbf{D}^{\mathrm{b}}(\Omega_1; p_X)$. Following Ishimura [4], we denote by

$$R\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{O}_X)_{p_X}$$

the image of the complex $R\mathcal{H}om_{E(G,D)}(M^\bullet(G, D), \mathcal{O}(G, Z))$ in $\mathbf{D}^{\mathrm{b}}(\Omega_1)$.

For any object $F \in \mathbf{D}^{\mathrm{b}}(X)$, we write the micro-support of F by $\mathrm{SS}(F)$.

THEOREM 2.2 (Kashiwara-Schapira [9]). *Set $V := \mathrm{Int}(Z) \times \mathrm{Int}(G^{\circ a}) \subset X \times \mathbb{C}^n \simeq T^*X$. Then, we have an estimation of the micro-support :*

$$\mathrm{SS}(R\mathcal{H}om_{E(G,D)}(M^\bullet(G, D), \mathcal{O}(G, Z))) \cap V \subset \mathrm{supp}(\mathcal{M}) \cap V.$$

Let Y be a d-codimensional complex submanifold of X and let f be an embedding map from Y to X. Then we have the natural morphisms :

$$T^*Y \xleftarrow{{}^t f'} Y \times_X T^*X \xrightarrow{f_\pi} T^*X.$$

Let $p \in Y \times_X T^*X$. We set $p_X := f_\pi(p)$ and $p_Y := {}^t f'(p)$. We define the full subcategory $\mathbf{D}^{\mathrm{b}}_f(X; p_X)$ of $\mathbf{D}^{\mathrm{b}}(X; p_X)$ as :

$$\begin{aligned}&\mathbf{D}^{\mathrm{b}}_f(X; p_X)\\ &\quad := \{F \in \mathbf{D}^{\mathrm{b}}(X; p_X) \,|\, {}^t f'^{-1}(p_Y) \cap f_\pi{}^{-1}(\mathrm{SS}(F)) \subset \{p\} \text{ near } p_X\}.\end{aligned}$$

Kashiwara-Schapira [10] defined the microlocal inverse image :

$$f_p^{-1} : \mathbf{D}^{\mathrm{b}}_f(X; p_X) \longrightarrow \mathbf{D}^{\mathrm{b}}(Y; p_Y).$$

If $F \in \mathbf{D}^{\mathrm{b}}_f(X; p_X)$, then $f_p^{-1}F$ is as follows. Take the refined microlocal cut-off $F' \to F$ at p_X. Then, we may set $f_p^{-1}F := f^{-1}F'$.

We say that f is non-characteristic for $\mathcal{M}$ at p if $\mathcal{M}$ satisfies the following condition :

$$(2.2) \qquad {}^t f'^{-1}(p_Y) \cap f_\pi{}^{-1}(\mathrm{SS}(\mathcal{M})) \subset \{p\} \text{ near } p_X.$$

Suppose that f is non-characteristic for $\mathcal{M}$ at p. Then, Sato-Kawai-Kashiwara [12] defined the inverse image $\underline{f}_p^{-1}\mathcal{M}$. Note that $\underline{f}_p^{-1}\mathcal{M}$ is a coherent $\mathcal{E}_Y$-module on a neighborhood of p_Y. The Cauchy-Kowalevski theorems for $\mathcal{E}$-modules are as follows.

Theorem 2.3 (Ishimura [4]). *Suppose that f is non-characteristic for $\mathcal{M}$ at p. Then, we have the natural isomorphism :*

$$f_p^{-1} R\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{O}_X)_{p_X} \xrightarrow{\sim} R\mathcal{H}om_{\mathcal{E}_Y}(\underline{f}_p^{-1}\mathcal{M}, \mathcal{O}_Y)_{p_Y}$$

in $\mathbf{D}^{\mathrm{b}}(Y; p_Y)$.

Sugiki-Takeuchi [14] gave a more precise proof of Theorem 2.3.

Let M be an n-dimensional real analytic manifold and let N be a d-codimensional submanifold of M. We suppose that X (resp. Y) is a complexification of M (resp. N). Then we get the following diagram :

$$\begin{array}{ccccc} T^*_N Y & \xleftarrow{\ {}^t f'_N\ } & T^*_N X & & \\ \downarrow & & \downarrow & & \\ T^*Y & \xleftarrow{\ {}^t f'\ } & Y \times_X T^*X & \xrightarrow{\ f_\pi\ } & T^*X. \end{array}$$

We set :

$$\mathcal{C}_N := \mathrm{H}^{n-d}(\mu_N(\mathcal{O}_Y)) \otimes \mathrm{or}_{N|Y}$$
$$\mathcal{C}_{N|X} := \mathrm{H}^{n}(\mu_N(\mathcal{O}_X)) \otimes \mathrm{or}_{N|X},$$

where $\mu_N(\cdot)$ is the microlocalization functor and $\mathrm{or}_{N|Y}$, $\mathrm{or}_{N|X}$ are the orientation sheaves. $\mathcal{C}_N$ (resp. $\mathcal{C}_{N|X}$) is called the sheaf of microfunctions (resp. relative microfunctions).

The division theorem with coefficients of relative microfunctions for $\mathcal{E}$-modules is as follows.

Theorem 2.4 (Kashiwara-Kawai [6]). *Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module on a neighborhood of p_X. Suppose that f is non-characteristic for $\mathcal{M}$ at p. Then, we have :*

$$R\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{N|X})_{p_X}[d] \simeq R\mathcal{H}om_{\mathcal{E}_Y}(\underline{f}_p^{-1}\mathcal{M}, \mathcal{C}_N)_{p_Y}.$$

Kashiwara-Kawai [6] transformed $\mathcal{C}_{N|X}$ into the sheaf of microfuntions with holomorphic parameters $\mathcal{CO}$ by using the theory of quantized contact transformations to prove this theorem, but we give another proof here. Our proof uses the Cauchy-Kowalevski theorems for $\mathcal{E}$-modules.

Proof. Set $F := R\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{O}_X)_{p_X}$. By Theorem 2.2, we get $F \in \mathbf{D}^{\mathrm{b}}_f(X; p_X)$. Take the refined microlocal cut-off $F' \to F$ at p_X. We apply Theorem 6.7.1 of [10] to F'. Then, we obtain :

$$R^t f'_{N*}(\mu_N(F')) \xrightarrow{\sim} \mu_N(\omega_{Y|X} \otimes f^{-1}F'), \tag{2.3}$$

where $\omega_{Y|X}$ is the relative dualizing complex. We shall calculate (2.3). We get :

$$\mu_N(F')_{p_X} \simeq \mu_N(F)_{p_X} \simeq R\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mu_N(\mathcal{O}_X))_{p_X}.$$

By Theorem 2.3, we also get :

$$\mu_N(f^{-1}F')_{p_Y} \simeq \mu_N(f_p^{-1}F)_{p_Y} \simeq R\mathcal{H}om_{\mathcal{E}_Y}(\underline{f}_p^{-1}\mathcal{M}, \mu_N(\mathcal{O}_Y))_{p_Y}.$$

It completes the proof. □

3. A Division Theorem with Coefficients of Relative Bimicrofunctions for $\mathcal{E}$-modules

In this section we prove a division theorem with coefficients of relative bimicrofunctions for $\mathcal{E}$-modules. Let us recall the theory of bimicrofunctions.

Let M'' be a real analytic manifold and let X'' be a complexification of M''. Suppose that $g_M : M \to M''$ and $g : X \to X''$ are submersions. Then we get the following diagram :

$$\begin{array}{ccccc} N & \xrightarrow{f_N} & M & \xrightarrow{g_M} & M'' \\ \downarrow & & \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X & \xrightarrow{g} & X'', \end{array}$$

Moreover, we suppose that $h_N := g_M \circ f_N$ and $h := g \circ f$ are submersions.

Set $L := g^{-1}(M'')$, $H := h^{-1}(M'')$. Then, we obtain sequences of manifolds $N \subset L \subset X$, $N \subset H \subset Y$. We also get the following diagram :

$$\begin{array}{ccc} T^*_N H \times_H T^*_H Y & \xleftarrow{{}^t f'_{NL}} & T^*_N L \times_L T^*_L X \\ {\scriptstyle \pi_H}\downarrow & & {\scriptstyle \pi_L}\downarrow \\ N \times_H T^*_H Y & \xleftarrow{\sim} & N \times_L T^*_L X. \end{array}$$

Definition 3.1 (Kataoka-Tose [7] and Schapira-Takeuchi [13]). We set :

$$\mathcal{C}_{NH} := \mathrm{H}^{n-d}(\mu_{NH}(\mathcal{O}_Y)) \otimes \mathrm{or}_N$$
$$\mathcal{C}_{NL} := \mathrm{H}^{n}(\mu_{NL}(\mathcal{O}_X)) \otimes \mathrm{or}_N,$$

where $\mu_{NH}(\cdot)$ is the bimicrolocalization functor (See [13] and Takeuchi [15]). $\mathcal{C}_{NH}$ (resp.$\mathcal{C}_{NL}$) is called the sheaf of bimicrofunctions (resp. relative bimicrofunctions).

Note that we have :

$$T^*_M L \times_L T^*_L X \simeq T^*_{(M \times_L T^*_L X)}(T^*_L X) \subset T^*(T^*_L X) \simeq T_{(T^*_L X)}(T^* X),$$

where we used the Hamiltonian isomorphism $-H : T^*(T^*_L X) \simeq T_{(T^*_L X)}(T^* X)$. For any $F \in \mathbf{D}^{\mathrm{b}}(X)$, the support of the complex $\mu_{ML}(F)$ is estimated in the following way.

Proposition 3.2 (Funakoshi [3] and Koshimizu-Takeuchi [11]). *For any $F \in \mathbf{D}^{\mathrm{b}}(X)$, we have :*

$$\operatorname{supp}(\mu_{ML}(F)) \subset T^*_{(M \times_L T^*_L X)}(T^*_L X) \cap C_{T^*_L X}(\mathrm{SS}(F)).$$

We shall recall the condition of non-microcharactericity to state the division theorem. We define the relative cotangent bundle $T^*(X/X'')$ by the following exact sequence :

$$0 \longrightarrow V := X \times_{X''} T^*X'' \longrightarrow T^*X \longrightarrow T^*(X/X'') \longrightarrow 0.$$

Considering Hamiltonian isomorphism, we have :

$$V \xleftarrow{\dot{\pi}_Y} \dot{T}^*_Y X \times_X V \longrightarrow T^*(X/X'') \times_X V \simeq T_V(T^*X).$$

Note that $\dot{T}^*_Y X \times_X V \longrightarrow T^*(X/X'') \times_X V$ is a locally closed embedding.

Definition 3.3 (Bony [1]). Let $q \in V$ and let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module on a neighborhood of q. We say that $\mathcal{M}$ is non-microcharacteristic for Y along V at q if $\mathcal{M}$ satisfies $\dot{\pi}_Y^{-1}(q) \cap C_V(\operatorname{supp}(\mathcal{M})) = \varnothing$.

We state a division theorem with coefficients of relative bimicrofunctions for $\mathcal{E}$-modules.

Theorem 3.4. *Let $p \in N \times_L T^*_L X \subset V$ and let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module on a neighborhood of p_X. If $\mathcal{M}$ is non-microcharacteristic for Y along V at p, then there exists a quasi-isomorphism :*

$$\mathrm{R}^t f'_{NL!} R\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{NL})[d] \simeq R\mathcal{H}om_{\mathcal{E}_Y}(\underline{f}_p^{-1}\mathcal{M}, \mathcal{C}_{NH}) \tag{3.1}$$

on $\pi_H^{-1}(p_Y)$.

Proof. Set $F := R\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{O}_X)_{p_X}$. By the condition of non-microcharactericity, we get $F \in \mathbf{D}^{\mathrm{b}}_f(X; p_X)$. Here we use Proposition 3.4 of [17]. Take the refined microlocal cut-off $F' \to F$ at p_X. Then, there exists an open neighborhood W_0 (not necessary conic) of p_X on V and an open conic neighborhood W_1 of $\{x\}\times_X \dot{T}^*_Y X$ in T^*X such that $(W_0+W_1)\cap\mathrm{SS}(F')=\varnothing$. Hence, we obtain :

$$\mathrm{R}^t f'_{NL!}\mu_{NL}(F')\simeq\mu_{NH}(\omega_{Y|X}\otimes f^{-1}F')$$

on $\pi_H^{-1}(p_Y)$. By applying Proposition 3.2, we get :

$$\mu_{NL}(F')\simeq\mu_{NL}(F)\simeq R\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M},\mu_{NL}(\mathcal{O}_X)).$$

Moreover, by Theorem 2.3, we get

$$\begin{aligned}\mu_{NH}(f^{-1}F')\simeq\mu_{NH}(f_p^{-1}F)&\simeq\mu_{NH}(R\mathcal{H}om_{\mathcal{E}_Y}(\underline{f}_p^{-1}\mathcal{M},\mathcal{O}_Y)_{p_Y})\\&\simeq R\mathcal{H}om_{\mathcal{E}_Y}(\underline{f}_p^{-1}\mathcal{M},\mu_{NH}(\mathcal{O}_Y)).\end{aligned}$$

Therefore, it completes the proof. □

4. Applications

We solve the conjecture of Takeuchi [16].

Theorem 4.1. *Let $p\in N\times_L T^*_L X\subset V$ and let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module on a neighborhood of p_X. If $\mathcal{M}$ is non-microcharacteristic for Y along V at p, then we have :*

$$R^j\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M},\mathcal{C}_{NL})=0 \tag{4.1}$$

for any $j<d$ on $\pi_L^{-1}(p_X)$.

Proof. By the assumption of non-microcharactericity, we know the finiteness of the morphism ${}^tf'_{NL}$ on the support of the complex $R\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M},\mathcal{C}_{NL})$. Hence (4.1) follows from (3.1). It completes the proof. □

We apply this result to higher-codimensional boundary value problems. Let us recall the definition of partially ellipticity by Bony-Schapira [2]. We define the relative cotangent bundle $T^*(M/M'')$ by the following exact sequence :

$$0\longrightarrow\Lambda:=M\times_{M''}T^*_{M''}X''\longrightarrow T^*_MX\longrightarrow T^*(M/M'')\longrightarrow 0.$$

Note that $\Lambda \simeq M \times_L T^*_L X \subset V = X \times_{X''} T^* X'' \subset T^* X$. We have the injection :

$$T^*(M/M'') \times_M \Lambda \longrightarrow T^*(X/X'') \times_X V \simeq T_V(T^*X)$$

and the projection $\dot{\pi}_\Lambda : \dot{T}^*(M/M'') \times_M \Lambda \to \Lambda$.

Definition 4.2 (Bony-Schapira [2]). Let $q \in \Lambda$ and let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module on X. We say that $\mathcal{M}$ is partially elliptic along V at q if $\mathcal{M}$ satisfies $\dot{\pi}_\Lambda^{-1}(q) \cap C_V(\mathrm{char}(\mathcal{M})) = \varnothing$.

We denote by $\mathcal{B}_M$ the sheaf of hyperfuncions on M. We set the projection $\theta : T^*_N X \to T^*_N M$. Then, our main result is as follows.

Theorem 4.3. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module on X. Suppose that $\mathcal{M}$ satisfies the conditions :*

(i) *$\mathcal{M}$ is partially elliptic along V on $N \times_L \dot{T}^*_L X$,*

(ii) *$\mathcal{M}$ is non-microcharacteristic for Y along V on $N \times_L \dot{T}^*_L X$,*

(iii) *$T^*_M X \cap \mathrm{char}(\mathcal{M}) \subset T^*_L X$.*

Then we have :

$$\begin{aligned} \mathrm{H}^j(R\theta_! R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|X})) \\ \simeq \mathrm{H}^j(\mu_N R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M) \otimes \mathrm{or}_{N|M}) \end{aligned} \tag{4.2}$$

for any $j < d$.

We need some results to prove this theorem.

Definition 4.4 (Takeuchi [16]). We define the sheaf $\mathcal{C}_{NM}$ on $T^*_N M \times_M T^*_M X$ as :

$$\mathcal{C}_{NM} := \mathrm{H}^n(\mu_{NM}(\mathcal{O}_X)) \otimes \mathrm{or}_N\,.$$

We have the injection $\iota : T^*_N M \times_L T^*_L X \to T^*_N M \times_M T^*_M X$ and the projection $\rho : T^*_N L \times_L T^*_L X \to T^*_N M \times_L T^*_L X$.

Proposition 4.5 (Takeuchi [16]). *Let $q \in T_N^*M \times_L T_L^*X$ and $p \in N \times_L T_L^*X \subset \Lambda$ be its base point. Suppose that $\mathcal{M}$ is partially elliptic along V at p. Then we have a natural isomorphism at q :*

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{NM}) \xrightarrow{\sim} \iota_* R\rho_* R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{NL}).$$

We have the projection $\pi_N : T_N^*M \times_L T_L^*X \to N \times_L T_L^*X$.

Lemma 4.6. *Let $p \in N \times_L \dot{T}_L^*X \subset \Lambda \subset V$ and let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module on X. Suppose that $\mathcal{M}$ satisfies the following conditions :*

(i) *$\mathcal{M}$ is partially elliptic along V on p,*

(ii) *$\mathcal{M}$ is non-microcharacteristic for Y along V on p.*

Then, we have

$$H^j(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{NM})) = 0 \tag{4.3}$$

for $j < d$ on ${\pi_N}^{-1}(p)$.

Proof. Apply Theorem 4.1 and Proposition 4.5. □

We have the projection $\dot{\pi}_M : T_N^*M \times_M \dot{T}_M^*X \to T_N^*M$.

Proposition 4.7 (Takeuchi [17]). *There exists a distinguished triangle :*

$$\begin{aligned} R\theta_! R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|X}) \longrightarrow \mu_N R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M) \otimes \mathrm{or}_{N|M} \longrightarrow \\ R\dot{\pi}_{M*} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{NM}) \longrightarrow +1 \end{aligned} \tag{4.4}$$

in $\mathbf{D}^{\mathrm{b}}(T_N^*M)$.

We shall prove Theorem 4.3.

Proof. By applying Lemma 4.6, we get (4.3) for $j < d$ on ${\pi_N}^{-1}(N \times_L \dot{T}_L^*X)$. Hence, the isomorphism (4.2) follows from the condition of Theorem 4.3 (iii) and the distinguished triangle (4.4). □

Remark 4.8. To prove the isomorphism (4.2), Takeuchi [17] assumed that f is non-characteristic for $\mathcal{M}$ (see Definition 4.10). However, our proof did not use the assumption.

The projection $T^*_M X \to M$ induces the natural injection

$$T^*_M X \times_M T^*M \xrightarrow{\quad\kappa\quad} T^*(T^*_M X) \simeq T_{(T^*_M X)}(T^*X).$$

Definition 4.9. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module and let $p \in \dot{T}^*M$. We say that $\mathcal{M}$ is hyperbolic in the direction p if $\mathcal{M}$ satisfies

$$\kappa(T^*_M X \times_M \{q\}) \cap C_{T^*_M X}(\mathrm{char}(\mathcal{M})) = \varnothing.$$

Definition 4.10. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module.

(i) We say that $\mathcal{M}$ is elliptic if $\mathcal{M}$ satisfies $\dot{T}^*_M X \cap \mathrm{char}(\mathcal{M}) = \varnothing$.

(ii) We say that f is non-characteristic for $\mathcal{M}$ if $\mathcal{M}$ satisfies

$$\dot{T}^*_Y X \cap {f_\pi}^{-1}(\mathrm{char}(\mathcal{M})) = \varnothing.$$

We assume that $d \geq 2$ and that $p \in \dot{T}^*_N M$ is a fixed covector. Then, by some well-known results and Theorem 4.3, we obtain the following results.

Theorem 4.11. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module and let*

$$\mathrm{char}(\mathcal{M}) = \bigcup_{j=1}^{r} V_j$$

be the irreducible decomposition of the characteristic variety of $\mathcal{M}$. We assume that each V_j satisfies one of conditions below.

(i) *The hypothesis of Theorem 4.3 holds for V_j.*

(ii) *V_j is hyperbolic in a direction p and satisfies $V_j \cap N \times_L T^*_L X \subset T^*_X X$.*

(iii) *V_j is elliptic, and f is non-characteristic for V_j.*

Then, we obtain

$$\mathrm{H}^j(\mu_N R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{B}_M))_p = 0$$

for any $j < d$.

Note that (ii) is the result of Kashiwara-Schapira [8] and that (iii) follows from Kashiwara-Kawai [6].

COROLLARY 4.12 (Local Bochner type extension theorem). *Let* $\mathcal{M}$ *be as in the above theorem. Then, there exists an open convex cone* $\Omega_0 \subset M$ *with the edge* N *such that the polar set of* $C_N(\Omega_0) \subset T_N M$ *contains* p *in its interior, and every hyperfunction (real analytic) solution* $u \in \Gamma_\Omega \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{B}_M)$ *on* $\Omega := M \backslash \overline{\Omega_0}$ *automatically extends to an open neighborhood of* N *as a hyperfunction solution (real analytic) solution to* $\mathcal{M}$.

Example 4.13. Put $X := \mathbb{C}^5$ and suppose that $z = (z_1, \cdots, z_5)$ is a coordinate of X. Moreover, we set $Y := \{z_1 = z_2 = 0\} \subset X$. We write the real parts of z as $x = (x_1, \cdots, x_5)$. Then, we can represent the coordinate of M and N as $M = \{(x_1, \cdots, x_5)\} \supset N = \{x_1 = x_2 = 0\}$. We write the coordinate of T^*X by $(z; \zeta dz)$, where $\zeta = (\zeta_1, \cdots, \zeta_5) \in \mathbb{C}^5$. We define two differential operators by

$$\begin{aligned} P_1 &:= (\partial_1^2 - \partial_5^2)(\partial_1 + i\partial_3) \\ P_2 &:= (\partial_1^2 - \partial_5^2)(\partial_2 + i\partial_4). \end{aligned}$$

Let us consider the following system $\mathcal{M}$:

$$P_1 u = 0\,, \quad P_2 u = 0.$$

By easy calculation, we get the irreducible decomposition of the characteristic variety of $\mathcal{M}$:

$$\mathrm{char}(\mathcal{M}) = \{\zeta_1 + \zeta_5 = 0\} \cup \{\zeta_1 - \zeta_5 = 0\} \cup \{\zeta_1 + i\zeta_3 = \zeta_2 + i\zeta_4 = 0\}.$$

Note that f is not non-characteristic for $\mathcal{M}$. If we put $p := (0; dx_1) \in \dot{T}_N^* M$, then the system $\mathcal{M}$ satisfies the condition of Theorem 4.11. This example cannot be treated by Takeuchi [16], [17].

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