J. Math. Sci. Univ. Tokyo
8 (2001), 541–558.

The Generating Function for Certain Cohomology Intersection Pairings of the Moduli Space of Flat Connections

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Abstract. We consider the cohomology ring of the moduli space of flat connections on a closed oriented surface with n marked points. We give the generating function for certain cohomology intersection pairings, which Weitsman considered in [11].

1. Introduction

Let Σ^g be a closed oriented surface of genus g with n distinct points $p_1, ..., p_n$. For $t_1, ..., t_n \in [0,1]$, we consider the moduli space $\mathcal{M}_g(t_1, ..., t_n)$ of flat $SU(2)$ connections on $\Sigma^g \backslash \{p_1, ..., p_n\}$, whose holonomies around p_m are conjugate to $\begin{pmatrix} e^{i\pi t_m} & 0 \\ 0 & e^{-i\pi t_m} \end{pmatrix}$. If t_m's satisfy certain conditions, this space is a smooth symplectic manifold with symplectic form $\omega^g_{t_1,...,t_n}$.

On $\mathcal{M}_g(t_1, ..., t_n)$, there are n circle bundles $V_1^g(t_1, ..., t_n), ..., V_n^g(t_1, ..., t_n)$ corresponding to each of the marked points $p_1, ..., p_n$. We denote by $r_1^g, ..., r_n^g$ the Chern classes of $V_1^g(t_1, ..., t_n), ..., V_n^g(t_1, ..., t_n)$, respectively. In [11], Weitsman found the Poincaré duals to r_m^g for $m = 1, ..., n$, and use these duals to obtain the recursion relations according to the intersection pairings for $r_1^g, ..., r_n^g$, and the cohomology class $[\omega^g_{t_1,...,t_n}]$. This is an analog of the Witten conjecture in [13] for the recursion relations among certain cohomology intersection pairings of the moduli space of curves.

In this paper, we study the relation with these intersection pairings and the symplectic volume of $\mathcal{M}_g(t_1, ..., t_n)$, and give the way for calculating the intersection pairings. The important fact in [11], is that the circle bundles $V_m^g(t_1, ..., t_n)$ (for $m = 1, \ldots n$) are obtained from symplectic quotients for the Hamiltonian torus action, which is considered by Goldman [4], and Jeffrey-Weitsman [7, 8]. This description for $V_m^g(t_1, ..., t_n)$ also plays an

1991 *Mathematics Subject Classification.* 57R57; 58D29, 58F05.

important role in this paper. In fact, this description allows us to use Duistermaat-Heckman's theorem [3], which shows that r_m^g is given by the variation of the symplectic form $\omega^g_{t_1,...,t_n}$ as t_m varies ([11], and see also Section 3). Using this result, we can calculate the intersection pairings from the symplectic volume of $\mathcal{M}_g(t_1,...,t_n)$. The symplectic volume of $\mathcal{M}_g(t_1,...,t_n)$ is calculated by Donaldson [2], Jeffrey-Weitsman [8], and Witten [12]. The following is the main result.

THEOREM 1.1. *For $x_1,...,x_n \in \mathbb{R}$, we have*

$$\sum_{\substack{k_1,...,k_n\geq 0\\ k\leq 3g-3+n}} \frac{x_1^{k_1}}{k_1!}\cdots\frac{x_n^{k_n}}{k_n!}\int_{\mathcal{M}_g(t_1,...,t_n)}(r_1^g)^{k_1}\cdots(r_n^g)^{k_n}\frac{(\omega^g_{t_1,...,t_n})^{3g+n-3-k}}{(3g-3+n-k)!}$$
$$= \frac{1}{2^{g-2}\pi^{2g-2+n}}\sum_{m=1}^{\infty}\frac{\prod_{j=1}^{n}\sin(\pi m(t_j+x_j))}{m^{2g-2+n}},$$

where $k=\sum_{j=1}^n k_j$. If $x_1,...,x_n$ are sufficiently close to 0, this formula also equals to the volume of $\mathcal{M}_g(t_1+x_1,...,t_n+x_n)$.

For $n=1$, the right hand side of the above equation is rewritten by using the Bernoulli polynomials [2], [9]. Then we have the explicit formula for intersection pairings.

THEOREM 1.2. *For n=1, putting $r^g=r_1^g$, we have*

$$\int_{\mathcal{M}_g(t)}(r^g)^k\frac{(\omega_t^g)^{3g-2-k}}{(3g-2-k)!} = (-1)^g 2^{g-k}\mathrm{P}_{2g-1-k}(\frac{t}{2}),$$

where $\mathrm{P}_\lambda(x)$ is the λth Bernoulli polynomial.

A similar formula for $\mathcal{M}_g(1)$ is given in [2], [10]. But we do not treat this space in this paper.

In [11], Weitsman also used the Poincaré duals to r_m^g to get the analog of Newstead conjecture, which tells, for the case $n=1$, that the equality $(r^g)^k=0$ holds for $k\geq 2g$. Using Theorem 1.2, it is easy to see that $k=2g$ is the infimum.

Corollary 1.3. *For n=1, $(r^g)^{2g-1}$ does not vanish.*

This paper is organized as follows. In section 2, we define the moduli spaces $\mathcal{M}_g(t_1, ..., t_n)$, $\overline{\mathcal{M}}_g$, and also construct circle bundles $V_1^g(t_1, ..., t_n)$, ..., $V_n^g(t_1, ..., t_n)$ on $\mathcal{M}_g(t_1, ..., t_n)$. In section 3, we recall the torus action on the open dense set of the moduli space, and we see how $\mathcal{M}_g(t_1, ..., t_n)$ arises as a symplectic quotient of this torus action. Section 4 is devoted to prove Theorem 1.1. We also prove some theorems for intersection pairings in Section 4.

Acknowledgment. The author would like to thank Professor Toshitake Kohno for kindly encouragements. The author also would like to thank the referee for useful comments.

2. Moduli Spaces of Flat Connections on Surfaces

In this section, we recall the moduli spaces and circle bundles which will be the basic objects of study in this paper.

Let G denote $SU(2)$ and $\mathfrak{g}$ its Lie algebra. We denote by T the subgroup of G consisting of diagonal matrices.

2.1. The moduli space of flat connections on a closed surface

In this subsection, we recall well known facts about the moduli space of flat connections on a closed surface. For more details, see [1], [4].

Let Σ^g be a closed oriented surface of genus $g(\geq 2)$, and $E \to \Sigma^g$ be a principal G bundle on Σ^g. Since G is simply connected, all G bundles on Σ^g are topologically trivial. Fixing a trivialization, the space $\mathcal{A}$ of connections on E is identified with the space $\Omega^1(\Sigma^g, \mathfrak{g})$ of $\mathfrak{g}$-valued 1-forms on Σ^g, and the gauge group $\mathcal{G}$ is identified with the space $\mathrm{Map}(\Sigma^g, G)$ of maps from Σ^g to G. The gauge group $\mathcal{G}$ acts on $\mathcal{A}$, with $g \in \mathcal{G}$ taking $A \in \mathcal{A}$ to $\psi_g(A) = g^{-1}Ag + g^{-1}dg$. A connection A is called flat if the curvature $F_A = dA + A \wedge A$ of A equals to 0. We denote by $\mathcal{A}_F$ the space of flat connections. Since $\mathcal{G}$-action on $\mathcal{A}$ preserves $\mathcal{A}_F$, we define the moduli space $\overline{\mathcal{M}}_g$ of flat connections on Σ^g by $\mathcal{A}_F/\mathcal{G}$.

Associating the conjugacy class of the holonomy representation to the class of flat connections on $\overline{\mathcal{M}}_g$, we may identify $\overline{\mathcal{M}}_g$ with Hom $(\pi_1(\Sigma^g), G)$ $/G$, the space of conjugacy classes of representations of $\pi_1(\Sigma^g)$ into G. In

$\overline{\mathcal{M}}_g$, there is the open dense set $\mathcal{M}_g$, which consists of conjugacy classes of irreducible representations. In [6], it is well known that $\mathcal{M}_g$ is a $(6g-6)$-dimensional smooth manifold.

On $\mathcal{M}_g$, there is a natural symplectic form ω. As in [4], we can identify the tangent space $T_{[\rho]}\mathcal{M}_g$ of $\mathcal{M}_g$ at $[\rho]$ with the $\mathfrak{g}_{\mathrm{Ad}\rho}$-valued 1st cohomology $H^1(\pi_1(\Sigma^g),\ \mathfrak{g}_{\mathrm{Ad}\rho})$ of the group $\pi_1(\Sigma^g)$. We define the symplectic form ω on $\mathcal{M}_g$ by the Poincaré duality

$$\omega : H^1(\pi_1(\Sigma^g), \mathfrak{g}_{\mathrm{Ad}\rho}) \times H^1(\pi_1(\Sigma^g), \mathfrak{g}_{\mathrm{Ad}\rho}) \to H^2(\pi_1(\Sigma^g), \mathbb{R}) \cong \mathbb{R}$$

which is defined by the cup product and the adjoint invariant bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{g}$.

2.2. The moduli space of flat connections on a surface with n marked points

In this subsection, we define the moduli space of flat connections on a closed oriented surface with n marked points, and n circle bundles on this moduli space. For more details, see [8], [11].

Let Σ^g be a closed oriented surface of genus g as before, and $p_1, ..., p_n \in \Sigma^g$ be n distinct points in Σ^g. Assume that $n \geq 1$ and $3g-3+n > 0$. Then $\pi_1(\Sigma^g \backslash \{p_1, ..., p_n\})$ can be described as the quotient of the free group with the $2g+n$ standard generators $A_1, ..., A_g$, $B_1, ..., B_g$, $C_1, ..., C_n$ by the single relation $\prod_{i=1}^{g}[A_i, B_i]\prod_{j=1}^{n}C_j = 1$, where each of the generator C_i can be chosen to correspond to the point p_i.

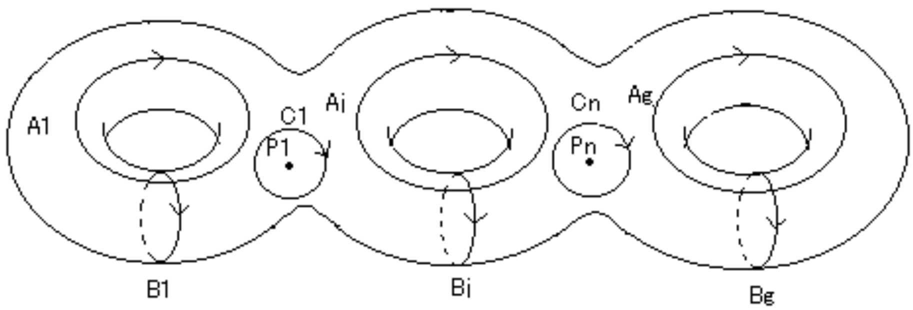

Fig. 1. generators of $\pi_1(\Sigma^g \backslash \{p_1, ..., p_n\})$.

Given $t_1, ..., t_n \in [0,1]$, we may define the moduli space of flat connections on $\Sigma^g \backslash \{p_1, ..., p_n\}$ by

$$\begin{aligned}&\mathcal{M}_g(t_1, ..., t_n)\\ &= \{\rho \in \mathrm{Hom}(\pi_1(\Sigma^g \backslash \{p_1, ..., p_n\}), G) |\ \mathrm{Tr}\rho(C_j) = 2\mathrm{cos}\pi t_j,\ j = 1, ..., n\}/G,\end{aligned}$$

where G acts by conjugation.

For $m = 1, ..., n$, we also define the space $V_m^g(t_1, ..., t_n)$ by

$$\begin{aligned}V_m^g&(t_1, ..., t_n)\\ &= \{\rho \in \mathrm{Hom}(\pi_1(\Sigma^g \backslash \{p_1, ..., p_n\}), G) |\ \rho(C_m) = e^{i\pi t_m},\\ &\qquad\qquad \mathrm{Tr}\rho(C_j) = 2\mathrm{cos}\pi t_j \text{ for } j\},\end{aligned}$$

where $e^{i\pi t_m} = \begin{pmatrix} e^{i\pi t_m} & 0 \\ 0 & e^{-i\pi t_m} \end{pmatrix}$.

We study the smoothness of $\mathcal{M}_g(t_1, ..., t_n)$, $V_1^g(t_1, ..., t_n)$, ..., and $V_n^g(t_1, ..., t_n)$. Define the map $f : G^{2g+n} \to G$ by

$$f(A_1, B_1, ..., A_g, B_g, C_1, ..., C_n) = \prod_{j=1}^{g} [A_j, B_j] \prod_{l=1}^{n} {C_l}^{-1} e^{-i\pi t_l} C_l$$

for $(A_1, B_1, ..., A_g, B_g, C_1, ..., C_n) \in G^{2g+n}$, and put $U_g(t_1, ..., t_n) = f^{-1}(I)$.

Proposition 2.1. *If f has $I \in G$ as a critical value, there exists $J \subset \{1, ..., n\}$ such that $\sum_{j \in J} t_j - \sum_{j \notin J} t_j \in \mathbb{Z}$.*

Proof. We differentiate $f(A_j, B_j, C_l) = \prod_{j=1}^{g} [A_j, B_j] \cdot \prod_{l=1}^{n} {C_l}^{-1} e^{-i\pi t_l} C_l$ at $(A_j, B_j, C_l) \in f^{-1}(I)$,

$$\begin{aligned}df = &\sum_{m=1}^{g} \mathrm{Ad}(\prod_{j=1}^{m-1} [A_j, B_j])\\ &\times (\mathrm{Ad}(A_m B_m)((B_m^{-1} \delta A_m B_m - \delta A_m) + (\delta B_m - A_m^{-1} \delta B_m A_m)))\\ &\qquad - \sum_{k=1}^{n} \mathrm{Ad}((\prod_{l>k} {C_l}^{-1} e^{-i\pi t_l} C_l)^{-1})((\mathrm{Ad}({C_k}^{-1} e^{-i\pi t_k} C_k) - 1)\delta C_k)\end{aligned}$$

where $\delta A_m := {A_m}^{-1} dA_m$ etc.

We first consider the case $g = 0$. In this case,

$$
\begin{aligned}
df &= -\sum_{k=1}^{n} \mathrm{Ad}((\prod_{l\geq k} {C_l}^{-1} e^{-i\pi t_l} C_l)^{-1})\delta C_k + \sum_{k=1}^{n} \mathrm{Ad}((\prod_{l>k} {C_l}^{-1} e^{-i\pi t_l} C_l)^{-1})\delta C_k \\
&= -\sum_{k=1}^{n-1} \mathrm{Ad}((\prod_{l>k} {C_l}^{-1} e^{-i\pi t_l} C_l)^{-1})(\delta C_{k+1} - \delta C_k) + \delta C_n - \delta C_1 \\
&= \sum_{k=1}^{n-1} (1 - \mathrm{Ad}((\prod_{l>k} {C_l}^{-1} e^{-i\pi t_l} C_l)^{-1}))(\delta C_{k+1} - \delta C_k).
\end{aligned}
$$

We put $X_l := {C_l}^{-1} e^{-i\pi t_l} C_l$. The argument in [6] tells that if df is not surjective, X_n, $X_n X_{n-1}$, ... , and $X_1 \cdots X_n$ commute each other. Then X_1, ..., X_n are simultaneously diagonalized. So there exists $g \in G$ such that

$$
\begin{aligned}
I &= g X_1 \cdots X_n g^{-1} \\
&= g X_1 g^{-1} \cdots g X_n g^{-1} \\
&= e^{\pm i\pi t_1} \cdots e^{\pm i\pi t_n}.
\end{aligned}
$$

Then we put $J = \{j \in \{1, ..., n\} |\ g X_j g^{-1} = e^{+i\pi t_j}\}$.

When $g > 1$, from the existence of the term $({B_m}^{-1}\delta A_m B_m - \delta A_m)$ $+(\delta B_m - {A_m}^{-1}\delta B_m A_m)$ in df, if df is not surjective, by the same argument as above, A_m and B_m commute each other. This brings us back to the case $g = 0$. □

DEFINITION 2.2. We call $(t_1, ..., t_n)$ *admissible* if $\sum_{j=1}^{n} \epsilon_j t_j \notin \mathbb{Z}$ for any $\epsilon_j \in \{-1, 0, 1\}$ except for $(\epsilon_1, \cdots, \epsilon_n) = (0, \cdots, 0)$.

The space $U_g(t_1, ..., t_n)$ is equipped with a natural action of $G \times T_1 \times \cdots \times T_n$, where $T_j \cong T$. Namely $G \times T_1 \times \cdots \times T_n$ acts on $U_g(t_1, ..., t_n)$ by

$$
(g, \xi_1, ..., \xi_n) \cdot (A_j, B_j, C_l) = (g A_j g^{-1}, g B_j g^{-1}, \xi_l C_l g^{-1})
$$

for $(g, \xi_1, ..., \xi_n) \in G \times T_1 \times \cdots \times T_n$ and $(A_j, B_j, C_l) \in U_g(t_1, \cdots, t_n)$. When $(t_1, \cdots, t_n)$ is admissible, this action has the global stabilizer $\mathbb{Z}_2$. In this case, we have

$$
\begin{aligned}
V_m^g(t_1, ..., t_n) &= U_g(t_1, ..., t_n)/(G \times T_1 \times \cdots \times \widehat{T_m} \times \cdots \times T_n), \\
\mathcal{M}_g(t_1, ..., t_n) &= U_g(t_1, ..., t_n)/(G \times T_1 \times \cdots \times T_n),
\end{aligned}
$$

and $T = T_m$ acts on $V_m^g(t_1, ..., t_n)$ by conjugation.

Corollary 2.3. *If* $(t_1, ..., t_n)$ *is admissible,* $\pi_m : V_m^g(t_1, ..., t_n) \to \mathcal{M}_g(t_1, ..., t_n)$ *is a smooth circle bundle.*

From now on, we assume $(t_1, ..., t_n)$ is admissible, and we denote the Chern class of $V_m^g(t_1, ..., t_n)$ by $r_m^g(t_1, ..., t_n) \in \mathrm{H}^2(\mathcal{M}_g(t_1, ..., t_n))$ or simply r_m^g, where there is no confusion.

Then $\mathcal{M}_g(t_1, ..., t_n)$ is a $(6g - 6 + 2n)$-dimensional smooth manifold. Jeffrey-Weitsman proved in [8] that $\mathcal{M}_g(t_1, ..., t_n)$ has a symplectic structure which we denote by $\omega^g_{t_1,...,t_n}$.

3. Torus Actions on Moduli Spaces

In [5], Goldman finds a Hamiltonian torus action on the moduli space of flat connections on a surface. We recall this torus action in Section 3.1, and how $\mathcal{M}_g(t_1, ..., t_n)$ appears as the symplectic quotient of this action in Section 3.2. For more details, see [5], [7], and [8].

3.1. Hamiltonian circle actions associated with simple closed curves

Let C be an oriented simple closed curve in a closed oriented surface Σ^g. For C, let us define a function $f_C : \overline{\mathcal{M}}_g \to [0, 1]$ as follows. We mark a point in C. By choosing an arc joining the marked point in C and a base point of $\pi_1(\Sigma^g)$, we obtain the element $[C] \in \pi_1(\Sigma^g)$. Then we define a function $\widetilde{f}_C : \mathrm{Hom}(\pi_1(\Sigma^g), G) \longrightarrow [0, 1]$ by

$$\widetilde{f}_C(\rho) = \frac{1}{\pi}\cos^{-1}\frac{1}{2}\mathrm{Tr}\rho([C])$$

for each $\rho \in \mathrm{Hom}(\pi_1(\Sigma^g), G)$. Since $\widetilde{f}_C$ is invariant under the conjugation action, $\widetilde{f}_C$ descends to give the function f_C on $\overline{\mathcal{M}}_g$. Note that f_C is smooth only on $U_C = f_C^{-1}((0\ 1))$. In [5], Goldman showed

Proposition 3.1 ([5]). *If* C *and* C' *are disjoint oriented simple closed curves in* Σ^g*, the Poisson bracket* $\{f_C\ f_{C'}\}$ *vanishes.*

We consider the Hamiltonian flow of f_C on U_C. To define the flow, we need the following preliminaries. For $t \in \mathbb{R}$ and $\rho \in \widetilde{U}_C = \widetilde{f}_C^{-1}((0\ 1))$, we define $\zeta_t^C(\rho)$ by the element of the centralizer $Z(\rho([C]))$ of $\rho([C])$, which is conjugate to $\begin{pmatrix} e^{-2\pi i t} & 0 \\ 0 & e^{2\pi i t} \end{pmatrix}$. Note that $Z(\rho([C])) \cong T$, since $\rho([C]) \neq \pm I$.

Now we define the Hamiltonian flow. When C is a nonseparating curve in Σ^g i.e $\Sigma^g \backslash C$ is connected, there exists another oriented simple closed curve $B \subset \Sigma^g$ which intersects once transverse to C. the fundamental group $\pi_1(\Sigma^g)$ is generated by the two subgroups $\pi_1(\Sigma^g \backslash C)$ and $\langle [B] \rangle$ with relation $[B]A_+[B]^{-1}A_- = 1$, where A_+ and A_- are the elements of $\pi_1(\Sigma^g \backslash C)$ whose image in $\pi_1(\Sigma^g)$ are $[C]$ and $[B][C]^{-1}[B]^{-1}$ respectively. For $t \in \mathbb{R}$ and $\rho \in \widetilde{U}_C$, we define a flow $\widetilde{\Phi}_t^C : \widetilde{U}_C \to \widetilde{U}_C$ by

$$\widetilde{\Phi}_t^C(\rho)(\alpha) = \begin{cases} \rho(\alpha) & \alpha \in \pi_1(\Sigma^g \backslash C) \\ \rho(\alpha)\zeta_t^C(\rho) & \alpha \in \langle [B] \rangle \ . \end{cases}$$

If C is a separating curve; let Σ_1, Σ_2 be the two components of $\Sigma^g \backslash C$. The fundamental group $\pi_1(\Sigma^g)$ is then generated by $\pi_1(\Sigma_1)$ and $\pi_1(\Sigma_2)$, amalgamated over the subgroup generated by $[C]$. For $t \in \mathbb{R}$ and $\rho \in \widetilde{U}_C$, we define a flow $\widetilde{\Phi}_t^C : \widetilde{U}_C \to \widetilde{U}_C$ by

$$\widetilde{\Phi}_t^C(\rho)(\alpha) = \begin{cases} \rho(\alpha) & \alpha \in \pi_1(\Sigma_1) \\ \zeta_t^C(\rho)\rho(\alpha)\zeta_t^C(\rho)^{-1} & \alpha \in \pi_1(\Sigma_2) \ . \end{cases}$$

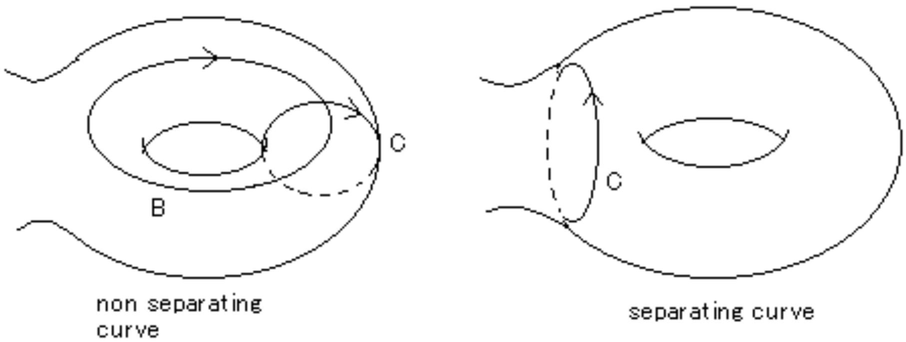

Fig. 2. separating curve and nonseparating curve.

THEOREM 3.2 ([5]). *The flow $\widetilde{\Phi}_t^C$ on $\widetilde{U}_C$ covers the Hamiltonian flow Φ_t^C on U_C associated with the function $f_C : \overline{\mathcal{M}}_g \to [0,1]$.*

REMARK 3.3. It is apparent from the definition of $\zeta_t^C(\rho)$ that the Hamiltonian flow associated with f_C has the period 1 if C is a nonseparating curve, and the period $\frac{1}{2}$ if C is a separating curve. Moreover from Proposition 3.1, if $\{C_j\}$ $j = 1, ..., l$ are disjoint oriented simple closed curves in Σ^g, the Hamiltonian flows of f_{C_j} $j = 1, ..., l$ commute each other. Then these flows define the Hamiltonian l-dimensional torus action.

3.2. Symplectic quotient description of $\mathcal{M}_g(t_1, ..., t_n)$

In this subsection, we see that $\mathcal{M}_g(t_1, ..., t_n)$ appears as a symplectic quotient of the torus action defined in the previous section, as Jeffrey-Weitsman pointed out in [8].

Let Σ^{g+n} be a closed surface which is formed by attaching one-holed tori N_j, $j = 1, ..., n$ to the boundary circles C_j of $\Sigma_n^g = \Sigma^g \backslash \coprod_{j=1}^n D_j$, where D_j is the small disc centered at p_j. Let us also assume that N_j is equipped with a distinguished nonseparating oriented simple closed curve, which we denote by C_{n+j}, so that $N_j \backslash C_{n+j}$ is a trinion or a sphere S^2 with three holes.

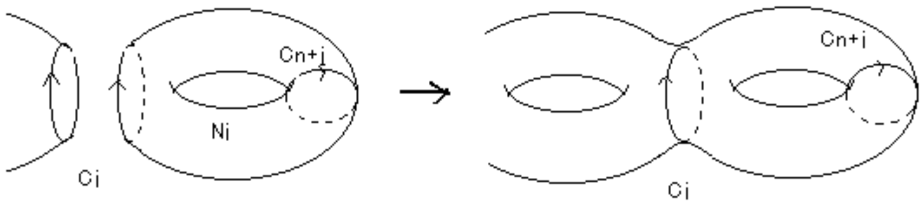

Fig. 3. attaching one holed tori to $\Sigma^g \backslash \{p_1, ..., p_n\}$.

From the Remark 3.3, the Hamiltonian flows of the functions f_{C_j}, $j = 1, ..., 2n$ are defined on $U' = \bigcap_{j=1}^{2n} f_{C_j}^{-1}((0,1)) \subset \overline{\mathcal{M}}_{g+n}$. As in Remark 3.3, these flows define a torus $T^{2n} = T_1 \times \cdots \times T_{2n}$ action with the moment map $\mu = (f_{C_1}, ..., f_{C_{2n}})$, where T_j is the circle corresponding to the flow of f_{C_j}. As Jeffrey-Weitsman pointed out, we have the following.

LEMMA 3.4 ([8], [11]). *If* $\mathbf{x} = (t_1, ..., t_n, x_{n+1}, ..., x_{2n})$ *satisfies the following inequalities*

$$0 < t_j < min\{2x_{n+j},\ 2 - 2x_{n+j}\}$$

for any j, the circle bundle

$$\mu^{-1}(\mathbf{x})/T_1 \times \cdots \widehat{T_m} \times \cdots \times T_{2n} \longrightarrow \mu^{-1}(\mathbf{x})/T^{2n}$$

is equal to

$$V_m^g(t_1, ..., t_n) \longrightarrow \mathcal{M}_g(t_1, ..., t_n)$$

for $m = 1, ..., n$, and is equal to

$$\mathcal{M}_g(t_1, ..., t_n) \times S^1 \longrightarrow \mathcal{M}_g(t_1, ..., t_n)$$

for $m = n + 1, ..., 2n$.

Since there is no proof in [8], [11], we give a proof of this lemma. In order to this, we give another description of the moment map μ. Restricting each representation $\rho \in \mathrm{Hom}(\pi_1(\Sigma^{g+n}), G)$ to the fundamental groups $\pi_1(\Sigma_n^g)$, $\pi_1(N_1)$, ..., and $\pi_1(N_n)$ of Σ_n^g and one-hole tori $N_1, ..., N_n$ respectively, we obtain the map

$$\alpha_1 : U' \to \bigcup\nolimits_{t_1,...,t_n \in (0\ 1)} \mathcal{M}_g(t_1, ..., t_n) \times \mathcal{M}_1(t_1) \times \cdots \times \mathcal{M}_1(t_n)$$

where $\mathcal{M}_1(t_j)$ is the moduli space of flat connections on N_j. Each one-holed torus N_j is obtained by gluing two boundary components D_{j_1} and D_{j_2} of the trinion $P_j = N_j \backslash C_{n+j}$. We also denote the other boundary component of the trinion P_j by D_{j_3}.

Then $\pi_1(P_j)$ is generated by three generators induced from the three boundary components, which we denote by D_{j_1}, D_{j_2} and D_{j_3} again, with the relation $D_{j_1} D_{j_2} D_{j_3} = 1$. The natural map $\imath : P_j \hookrightarrow N_j$ induces the homomorphism $\imath_* : \pi_1(P_j) \to \pi_1(N_j)$ which associates $[C_{n+j}], [C_j]$ and $[B][C_{n+j}]^{-1}[B]^{-1}$ to D_{j_1}, D_{j_3} and D_{j_2} respectively, where B is the oriented simple closed curve which intersects C_{n+j} once transversely and $[B]$ is the element of $\pi_1(N_j)$ represented by B. Then $\pi_1(N_j)$ is generated by the image $\imath_* \pi_1(P_j)$ and $\langle [B] \rangle$ with relation $[B] \imath_* D_{j_1} [B]^{-1} \imath_* D_{j_2} = 1$. Then the map $\imath_*$ induces the map β_j from $\mathcal{M}_1(t_j)$ to the moduli space $\mathcal{M}(P_j)$ of flat connections on the trinion ; $\beta_j : \mathcal{M}_1(t_j) \to \mathcal{M}(P_j)$. The moduli space $\mathcal{M}(P_j)$ is studied by Jeffrey-Weitsman [7]. They investigated the function $f_{D_{j_k}}$ for $k = 1, ..., 3$ on $\mathcal{M}(P_j)$ defined before and obtained the following result.

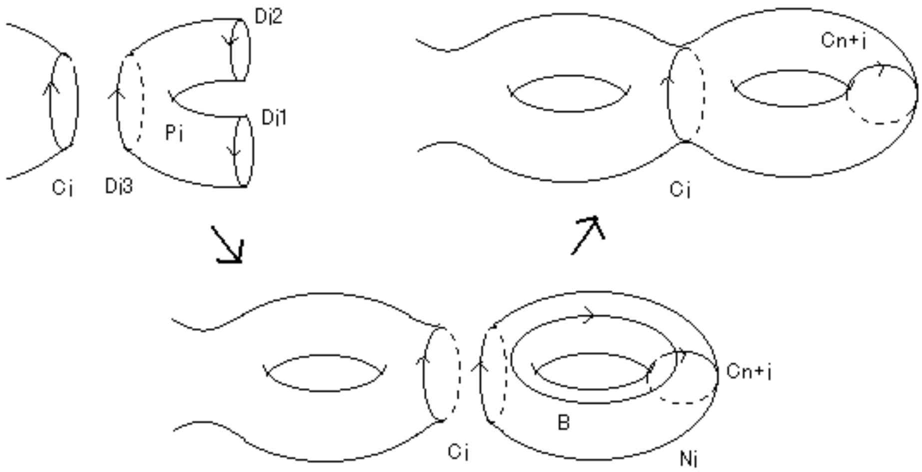

Fig. 4.

PROPOSITION 3.5 ([7]). *The map* $f := (f_{D_{j_1}}, f_{D_{j_2}}, f_{D_{J_3}}) : \mathcal{M}(P_j) \to [0,1]^3$ *sends* $\mathcal{M}(P_j)$ *bijectively to the set of* $(t_1, t_2, t_3) \in [0,1]^3$ *satisfying the inequalities*

$$|t_2 - t_3| \leq t_1 \leq min\{t_2 + t_3,\ 2 - (t_2 + t_3)\}.$$

Now the following is clear.

PROPOSITION 3.6. *The map* μ *is decomposed into the maps* α_1, α_2 *and* α_3

$$\begin{aligned}\mu : U' &\xrightarrow{\alpha_1} \bigcup\nolimits_{t_1,\ldots,t_n \in (0,1)} \mathcal{M}_g(t_1, ..., t_n) \times \mathcal{M}_1(t_1) \times \cdots \times \mathcal{M}_1(t_n) \\ &\xrightarrow{\alpha_2} \bigcup\nolimits_{t_1,\ldots,t_n \in (0,1)} \mathcal{M}_g(t_1, ..., t_n) \times \mathcal{M}(P_1) \times \cdots \times \mathcal{M}(P_n) \\ &\xrightarrow{\alpha_3} [0,1]^{2n},\end{aligned}$$

where the maps α_2 *and* α_3 *are defined as follows*

$\alpha_2 = id \times \beta_1 \times \cdots \times \beta_n$

$$
\begin{aligned}
&: \bigcup_{t_1,...,t_n \in (0,1)} \mathcal{M}_g(t_1,...,t_n) \times \mathcal{M}_1(t_1) \times \cdots \times \mathcal{M}_1(t_n) \\
&\qquad \longrightarrow \bigcup_{t_1,...,t_n \in (0,1)} \mathcal{M}_g(t_1,...,t_n) \times \mathcal{M}(P_1) \times \cdots \times \mathcal{M}(P_n) \\
\alpha_3 &= (f_{D_{1_3}}, ..., f_{D_{n_3}}, f_{D_{1_1}}, ..., f_{D_{n_1}}) \\
&: \bigcup_{t_1,...,t_n \in (0,1)} \mathcal{M}_g(t_1,...,t_n) \times \mathcal{M}(P_1) \times \cdots \times \mathcal{M}(P_n) \\
&\qquad \longrightarrow [0,1]^{2n}.
\end{aligned}
$$

We use this description of μ to prove Lemma 3.4.

Proof of Lemma 3.4. For $m = n+1, ..., 2n$, it is easy calculation from the definition of the action. For $m = 1, ..., n$, we consider the case $n = 1$ first. For simplicity, we omit the index j and denote the one-holed torus by N, the trinion by P etc. For $(t_1, t_2) \in \mathrm{Im}\mu$, which is included in the interior of the set in Proposition 3.5, we identify the level set $\mu^{-1}(t_1, t_2) = \alpha_1^{-1}(\alpha_2^{-1}(\alpha_3^{-1}(t_1,\ t_2)))$. From proposition 3.5, for $(t_1,\ t_2) \in \mathrm{Im}\mu$, we can easily see that there only exists $\rho \in \mathrm{Hom}(\pi_1(P), G)$ such that

$$
\mathrm{Im}\alpha_2 \cap \alpha_3^{-1}(t_1,\ t_2) = \mathcal{M}_g(t_1) \times \{[\rho]\}.
$$

If necessary, replacing the representative, we can take the representative $\rho \in \mathrm{Hom}(\pi_1(P), G)$ which satisfies $\mathrm{Tr}\rho(D_k) = 2\cos\pi t_2$ for $k = 1,\ 2$ and $\rho(D_3) = e^{i\pi t_1}$. The condition of $(t_1,\ t_2)$ implies that the representation ρ is irreducible.

Next we identify $\alpha_2^{-1}(\alpha_3^{-1}(t_1,\ t_2))$. Since it is easy to see $\alpha_2^{-1}(\alpha_3^{-1}(t_1,\ t_2)) = \mathcal{M}_g(t_1) \times \beta^{-1}([\rho])$, we see the inverse image of $\beta : \mathcal{M}_1(t) \to \mathcal{M}(P)$. First we choose an arbitrary element $[\rho_1] \in \beta^{-1}([\rho])$ such that the representative of this class satisfies $\rho_1 \circ \imath_* = \rho$ i.e.

$$
\begin{aligned}
&\rho_1([C_2]) = \rho(D_1),\ \rho_1([C_1]) = \rho(D_3) = e^{i\pi t_1} \text{ and} \\
&\rho_1([B][C_2]^{-1}[B]^{-1}) = \rho(D_2), \qquad (1)
\end{aligned}
$$

and we fix this class. If the class $[\rho_2]$ is in $\beta^{-1}([\rho])$, if necessary, replacing the representative, we can assume ρ_2 satisfies the condition (1). By the first two equations, there exists an element H in the centralizer $Z(\rho_1([C_2]))$ of $\rho_1([C_2])$ such that $\rho_2([B]) = \rho_1([B])H$. Conversely, it is easy to see that the conjugacy classes of representations which satisfy the condition (1) are included in $\beta^{-1}([\rho])$. Moreover if two representatives ρ' and ρ'' satisfy the condition (1), then because of the irreducibility of ρ, the class $[\rho']$ and $[\rho'']$ in $\mathcal{M}_1(t_1)$ are equal if only if ρ' and ρ'' are equal. Then we identify

$$\beta^{-1}([\rho]) = \{\phi \in \mathrm{Hom}(\pi_1(N), G)| \text{ there exists } H \in Z(\rho([C_2])) \\ \text{such that } \phi(\gamma) = \begin{cases} \rho_1(\gamma) & \gamma \in \pi_1(P) \\ \rho_1(\gamma)H & \gamma \in \langle [B] \rangle \end{cases} \}$$

Then it is clear that $\beta^{-1}([\rho])$ is the orbit of the Hamiltonian flow of the function f_{C_2} trough ρ_1.

Finally we identify $\mu^{-1}(t_1, t_2) = {\alpha_1}^{-1}({\alpha_2}^{-1}({\alpha_3}^{-1}(t_1, t_2)))$. From the above argument, we could identify ${\alpha_2}^{-1}({\alpha_3}^{-1}(t_1, t_2)) = \mathcal{M}_g(t_1) \times \beta^{-1}([\rho])$. Then it is clear that

$$\begin{aligned} &{\alpha_1}^{-1}({\alpha_2}^{-1}({\alpha_3}^{-1}(t_1, t_2))) \\ &\qquad = \{[\varphi] \in U' | \ \varphi|_{\pi_1(\Sigma_1^g)} \in V^g(t_1) \text{ and } \varphi|_{\pi_1(N)} \in \beta^{-1}([\rho])\}. \end{aligned}$$

If $[\rho]$ and $[\rho'] \in {\alpha_1}^{-1}({\alpha_2}^{-1}({\alpha_3}^{-1}(t_1, t_2)))$ are equal, where the representatives ρ and ρ' satisfy the condition $\rho|_{\pi_1(\Sigma_1^g)}$, $\rho'|_{\pi_1(\Sigma_1^g)} \in V^g(t_1)$ and $\rho|_{\pi_1(N)}$, $\rho|_{\pi_1(N)} \in \beta^{-1}([\rho])$. Then there exists $g \in G$ such that $g\rho g^{-1} = \rho'$. So $g\rho|_{\pi_1(N)}g^{-1} = \rho'|_{\pi_1(N)}$. But the above argument implies $g = \pm I$. Then we identify

$$\mu^{-1}(t_1, t_2) = {\alpha_1}^{-1}({\alpha_2}^{-1}({\alpha_3}^{-1}(t_1, t_2))) = V^g(t_1) \times \beta^{-1}([\rho]).$$

As we describe above, $\beta^{-1}([\rho])$ is the orbit of the Hamiltonian flow of the function f_{C_2}, and by the definition of a Hamiltonian flow, the Hamiltonian flow of the function f_{C_1} is defined for each $t \in \mathbb{R}$ and $\tau \in \mu^{-1}(t_1, t_2)$ by

$$\Phi_t^{C_1}(\tau)(\gamma) = \begin{cases} \tau(\gamma) & \gamma \in \pi_1(N) \\ e^{-2i\pi t}\tau(\gamma)e^{2i\pi t} & \gamma \in \pi_1(\Sigma_1^g). \end{cases}$$

Then

$$\mu^{-1}(t_1, t_2)/T_2 = V^g(t_1) \text{ and } {\mu'}^{-1}(t_1, t_2)/T_1 \times T_2 = \mathcal{M}_g(t_1),$$

where T_1 and T_2 actions are induced from the Hamiltonian flows of f_{C_1} and f_{C_2}, respectively.

For the case $n > 1$, we first perform the above reduction for the curves C_1, C_{n+1} in N_1. Since each T_j actions commute each other, remaining actions descend to the quotient $\mu^{-1}(t_1, t_{n+1})/T_1 \times T_{n+1} = \mathcal{M}_{g+n-1}(t_1)$. Next we perform the reduction for the curves C_2, C_{n+2} in N_2 on $\mathcal{M}_{g+n-1}(t_1)$, then we can obtain $\mathcal{M}_{g+n-2}(t_1, t_2)$. Repeating the above operation, we can obtain the above result. □

We can apply the Duistermaat-Heckman theorem to this situation.

COROLLARY 3.7 ([11]). *Suppose* $(t_1, ..., t_n)$ *and* $(t'_1, ..., t'_n)$ *are sufficiently close. Then* $\mathcal{M}_g(t_1, ..., t_n)$ *and* $\mathcal{M}_g(t'_1, ..., t'_n)$ *are diffeomorphic;*

$$q : \mathcal{M}_g(t_1, ..., t_n) \cong \mathcal{M}_g(t'_1, ..., t'_n)$$

and for symplectic forms $\omega^g_{t_1,...,t_n}$ *and* $\omega^g_{t'_1,...,t'_n}$,

$$q^*\omega^g_{t'_1,...,t'_n} = \omega^g_{t_1,...,t_n} + \sum_{j=1}^{n}(t'_j - t_j)r^g_j(t_1, ..., t_n) \in \mathrm{H}^2(\mathcal{M}_g(t_1, ..., t_n)).$$

4. The Generating Function for Intersection Pairings

In this section, we give the way for calculating the intersection pairings $\int_{\mathcal{M}_g(t_1,...,t_n)} (r^g_1)^{k_1} \cdots (r^g_n)^{k_n} (\omega^g_{t_1,...,t_n})^{3g-3+n-k}$. On $\mathcal{M}_g(t_1, ..., t_n)$, there is a natural orientation induced by the symplectic volume $(\omega^g_{t_1,...,t_n})^{3g-3+n}/(3g-3+n)!$ such that $\int_{\mathcal{M}_g(t_1,...,t_n)} (\omega^g_{t_1,...,t_n})^{3g-3+n}/(3g-3+n)! > 0$.

THEOREM 4.1. *For* $x_1, ..., x_n \in \mathbb{R}$, *we have*

$$\sum_{\substack{k_1,...,k_n \geq 0 \\ k \leq 3g-3+n}} \frac{x_1^{k_1}}{k_1!} \cdots \frac{x_n^{k_n}}{k_n!} \int_{\mathcal{M}_g(t_1,...,t_n)} (r^g_1)^{k_1} \cdots (r^g_n)^{k_n} \frac{(\omega^g_{t_1,...,t_n})^{3g+n-3-k}}{(3g-3+n-k)!}$$

$$= \frac{1}{2^{g-2}\pi^{2g-2+n}} \sum_{m=1}^{\infty} \frac{\prod_{j=1}^{n} \sin(\pi m(t_j + x_j))}{m^{2g-2+n}},$$

where $k = \sum_{j=1}^{n} k_j$. *If* $x_1, ..., x_n$ *are sufficiently close to* 0, *this formula also equals to the volume of* $\mathcal{M}_g(t_1 + x_1, ..., t_n + x_n)$.

PROOF. By integrating n-th wedge product of the equation in Corollary 3.7, we see that the polynomial in Theorem 4.1 equals to the volume of $\mathcal{M}_g(t_1+x_1,...,t_n+x_n)$. The symplectic volume of $\mathcal{M}_g(t_1,...,t_n)$ is first calculated by Witten [12], and Jeffrey-Weitsman take in [8], an another approach by using the torus action in the previous section. They give the following volume formula

$$\mathrm{Vol}(\mathcal{M}_g(t_1,...,t_n)) = \frac{1}{2^{g-2}\pi^{2g-2+n}} \sum_{m=1}^{\infty} \frac{\prod_{j=1}^{n} \sin(\pi m t_j)}{m^{2g-2+n}}.$$

The equality in this theorem is obtained from this formula. □

In the case $n=1$, we have the explicit formula for intersection pairings.

COROLLARY 4.2. *For $n=1$, putting $r^g = r_1^g$, we have*

$$\frac{1}{(3g-2-k)!}\int_{\mathcal{M}_g(t)} (r^g)^k (\omega_t^g)^{3g-2-k} = (-1)^g 2^{g-k} P_{2g-1-k}(\frac{t}{2}),$$

where $P_\lambda(x)$ is the λth Bernoulli polynomial, a polynomial of degree λ characterized the following properties

$$1)\ \frac{d}{dx}P_\lambda(x) = P_{\lambda-1}(x),$$
$$2)\ \int_0^1 P_\lambda(x)dx = 0$$

for each $\lambda \geq 1$, and

$$3)\ P_0(x) = 1.$$

PROOF. For $n=1$, by Donaldson [2], and Jeffrey-Weitsman [9], the above volume formula is rewritten by using the Bernoulli polynomials as follows. For $t \neq 0,\ 1$,

$$\mathrm{Vol}(\mathcal{M}_g(t)) = \frac{1}{2^{g-2}\pi^{2g-1}} \sum_{m=1}^{\infty} \frac{\sin(\pi m t)}{m^{2g-1}}$$
$$= (-1)^g 2^g P_{2g-1}(\frac{t}{2}).$$

From the volume formula and the properties of the Beroulli polynomial, we can easily prove the above equality. □

REMARK 4.3. Donaldson and Thaddeus find similar formulae for intersection pairings of $\mathcal{M}_g(1)$ (for more details, see [2], [10]). But this space is not in our case, because $t = 1$ is not admissible.

Using Theorem 4.1, we can give another, directly proof of Weitsman's recursion relation.

THEOREM 4.4 (Weitsman's recursion relation [11]). *Suppose* $k_1, \ldots,$ $k_n \in \mathbb{Z}_{\geq 0}$ *and* $k_n = 2r < 2g$ *is even. Then,*

$$\begin{aligned}
&\int_{\mathcal{M}_g(t_1,\ldots,t_n)} (r_1^g)^{k_1} \cdots (r_n^g)^{k_n} (\omega_{t_1,\ldots,t_n}^g)^{3g+n-3-k} \\
&= (-1)^r 2^{-r} r! \begin{pmatrix} 3g+n-k-3 \\ r \end{pmatrix} \\
&\qquad \times \int_{\mathcal{M}_{g-r}(t_1,\ldots,t_n)} (r_1^{g-r})^{k_1} \cdots (r_{n-1}^{g-r})^{k_{n-1}} (\omega_{t_1,\ldots,t_n}^{g-r})^{3(g-r)+n-3-(k-2r)}.
\end{aligned}$$

PROOF. From Theorem 4.1, it is clear that

$$\begin{aligned}
&\frac{1}{(3g-3+n-k)!} \int_{\mathcal{M}_g(t_1,\ldots,t_n)} (r_1^g)^{k_1} \cdots (r_n^g)^{k_n} (\omega_{t_1,\ldots,t_n}^g)^{3g-3+n-k} \\
&\qquad = \left.\frac{\partial^{k_1}}{\partial {t'_1}^{k_1}}\right|_{t'_1=t_1} \cdots \left.\frac{\partial^{k_n}}{\partial {t'_n}^{k_n}}\right|_{t'_n=t_n} \mathrm{Vol}(\mathcal{M}_g(t'_1, \ldots, t'_n)).
\end{aligned}$$

Then calculating and comparing both side of the equation of the Theorem, we obtain the above fact. □

In [11], Weitsman obtain the following result as an analog of Newstead conjecture.

THEOREM 4.5 (The analog of Newstead conjecture[11]). *Let* r_i^g *be the Chern classes of the circle bundle* V_i^g, *and* $k_1, \ldots, k_n \in \mathbb{Z}_{\geq 0}$. *Then,*

$$(r_1^g)^{k_1} \cdots (r_n^g)^{k_n} = 0 \qquad \text{if } \; k \geq 2g+n-1.$$

From Corollary 4.5 and the properties of Bernoulli polynomials, it is easy to see that $k = 2g$ is the infimum.

COROLLARY 4.6. *For $n = 1$, $(r^g)^{2g-1}$ does not vanish.*

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(Received December 24, 1999)
(Revised May 15, 2001)

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