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Self-Similar Solutions of a Nonlinear Heat Equation

By Thierry Cazenave, Flávio Dickstein, Miguel Escobedo
and Fred B. Weissler

Abstract. In this paper, we study the asymptotic behavior of certain solutions of the nonlinear heat equation $u_t - \Delta u + |u|^\alpha u = 0$ in $(0, \infty) \times \mathbb{R}^N$, where $\alpha > 0$. We focus especially on solutions that may change sign and that do not necessarily have a radial behavior as $|x| \to \infty$.

1. Introduction

Given $\alpha > 0$, consider the nonlinear heat equation

$$u_t - \Delta u + |u|^\alpha u = 0 \quad t > 0,\ x \in \mathbb{R}^N. \tag{1.1}$$

It is well known that if $u_0 \in C_0(\mathbb{R}^N)$, there exists a unique, global solution u of (1.1), $u \in C([0, \infty), C_0(\mathbb{R}^N))$, satisfying the initial condition $u(0) = u_0$. For $t > 0$, u is as smooth as the regularity of the mapping $u \mapsto |u|^\alpha u$ allows, and at least u is C^1 in t and C^2 in x.

The asymptotic behavior of these solutions as $t \to \infty$ has been studied in particular by Gmira and Véron [13]; Kamin and Peletier [16]; Brezis, Peletier and Terman [3]; Escobedo and Kavian [7, 8]; Escobedo, Kavian and Matano [9], Mizoguchi and Yanagida [18], Herraiz [15], Kwak [17]. It is determined by the decay of $u_0(x)$ as $|x| \to \infty$, as well as the oscillatory properties of u_0, and its description often involves self-similar solutions of (1.1). (We recall that a solution u of (1.1) is self-similar if it can be expressed in terms of its profile $f(x)$ as $u(t, x) = t^{-\frac{1}{\alpha}} f(x/\sqrt{t})$.)

In this paper, we are interested in the case where the initial value $u_0(x)$ is not necessarily asymptotically radial as $|x| \to \infty$ and may change sign. Nonnegative initial data of such type have been studied in [16] using a scaling argument. By this, one constructs self-similar solutions, thus solutions with homogeneous initial values. One also obtains, at the same time, the

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asymptotically self-similar behavior for a class of general solutions. Here, we consider separately the questions of the existence of self-similar solutions and of the convergence of general solutions. This may, in some instances, give more flexibility in the methods one can apply, see the comments below. Our main results are the following.

Theorem 1.1. *Suppose $\alpha > 0$ and let $\omega \in C(\mathbb{R}^N \setminus \{0\})$ be a homogeneous function of degree 0. It follows that there exists a self-similar solution u_ω of (1.1) with profile $f_\omega \in C_0(\mathbb{R}^N)$ such that $|x|^{\frac{2}{\alpha}} f_\omega(x) - \omega(x) \to 0$ as $|x| \to \infty$.*

Theorem 1.2. *Suppose $\alpha > 0$, let $\omega \in C(\mathbb{R}^N \setminus \{0\})$ be a homogeneous function of degree 0, and let u_ω be a self-similar solution of (1.1) with profile $f_\omega \in C_0(\mathbb{R}^N)$ such that $|x|^{\frac{2}{\alpha}} f_\omega(x) - \omega(x) \to 0$ as $|x| \to \infty$. Given $u_0 \in C_0(\mathbb{R}^N)$, let u be the solution of (1.1) with the initial condition $u(0) = u_0$. If $\alpha < 2/N$ suppose, in addition, that $\omega \geq 0$, $\omega \not\equiv 0$, that $f_\omega \geq 0$ and that $u_0 \geq 0$. If $|x|^{\frac{2}{\alpha}} u_0(x) - \omega(x) \to 0$ as $|x| \to \infty$, then*

$$\sup_{x \in \mathbb{R}^N} (|x|^2 + t)^{\frac{1}{\alpha}} |u(t,x) - u_\omega(t,x)| \to 0, \tag{1.2}$$

as $t \to \infty$. In particular, $t^{\frac{1}{\alpha}} \|u(t)\|_{L^\infty} \to \|f_\omega\|_{L^\infty}$ as $t \to \infty$.

Theorem 1.3. *Suppose $\alpha > 2/N$ and let $\omega \in C(\mathbb{R}^N \setminus \{0\})$ be a homogeneous function of degree 0. Let $u_0 \in C_0(\mathbb{R}^N)$ and let u be the solution of (1.1) with the initial condition $u(0) = u_0$.*

(i) *If $|x|^\sigma u_0(x) - \omega(x) \to 0$ as $|x| \to \infty$, for some $2/\alpha < \sigma < N$, then*

$$\sup_{x \in \mathbb{R}^N} (|x|^2 + t)^{\frac{\sigma}{2}} |u(t,x) - e^{t\Delta} v_0(x)| \to 0, \tag{1.3}$$

as $t \to \infty$, where $v_0(x) = \omega(x)|x|^{-\sigma}$. In particular, $t^{\frac{\sigma}{2}} \|u(t)\|_{L^\infty} \to C > 0$ as $t \to \infty$ if $\omega \not\equiv 0$.

(ii) *If $|x|^N u_0(x) - \omega(x) \to 0$ as $|x| \to \infty$, then*

$$\sup_{x \in \mathbb{R}^N} \frac{(t + |x|^2)^{\frac{N}{2}}}{\log t} |u(t,x) - \ell(\omega) G_t(x) \log t| \underset{t \to \infty}{\longrightarrow} 0, \tag{1.4}$$

where $G_t(x) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}}$ *is the heat kernel and* $\ell(\omega) = \frac{1}{2}\int_{\{|\xi|=1\}} \omega(\xi)\, d\xi$. *It follows in particular that* $\frac{(4\pi t)^{\frac{N}{2}}}{\log t}\|u(t)\|_{L^\infty} \to |\ell(\omega)|$ *as* $t \to \infty$.

In the case $\alpha < 2/N$, there are the restrictions $\omega \geq 0$ and $u_0 \geq 0$ in Theorem 1.2. Even if $\omega > 0$, the restriction $u_0 \geq 0$ is essential, as shown in Kwak [17]. Indeed, there may exist many different self-similar solutions of (1.1) with profile $f \in C_0(\mathbb{R}^N)$ such that $|y|^{\frac{2}{\alpha}} f(y) - \omega(y) \to 0$ as $|y| \to \infty$. The following two results concern the cases $u_0 \not\geq 0$ and $\omega \not\geq 0$.

THEOREM 1.4. *If* $\alpha < 2/N$, *then there exists a constant* $\gamma > 0$ *with the following property. Let* $\omega \in C(\mathbb{R}^N \setminus \{0\})$ *be a homogeneous function of degree* 0 *with* $\omega \geq \gamma$, *and let* u_ω *be a self-similar solution of* (1.1) *with profile* $f_\omega \in C_0(\mathbb{R}^N)$ *such that* $|x|^{\frac{2}{\alpha}} f_\omega(x) - \omega(x) \to 0$ *as* $|x| \to \infty$. *Let* $u_0 \in C_0(\mathbb{R}^N)$ *and let* u *be the solution of* (1.1) *with the initial condition* $u(0) = u_0$. *If* $|x|^{\frac{2}{\alpha}} u_0(x) - \omega(x) \to 0$ *as* $|x| \to \infty$, *then* (1.2) *holds.*

THEOREM 1.5. *Suppose* $\alpha < 2/N$ *and let* $\omega \in C(\mathbb{R}^N \setminus \{0\})$ *be a homogeneous function of degree* 0. *Let* $\mathcal{F}_\omega$ *be the set of the profiles* $f \in C_0(\mathbb{R}^N)$ *of self-similar solutions of* (1.1) *such that* $|y|^{\frac{2}{\alpha}} f(y) - \omega(y) \to 0$ *as* $|y| \to \infty$. *Let* $u_0 \in C_0(\mathbb{R}^N)$ *satisfy* $|x|^{\frac{2}{\alpha}} u_0(x) - \omega(x) \to 0$ *as* $|x| \to \infty$ *and let* u *be the solution of* (1.1) *with the initial condition* $u(0) = u_0$. *The following properties hold.*

(i) $\mathcal{F}_\omega$ *has a minimal element* f_ω^- *and a maximal element* f_ω^+, *corresponding to the self-similar solutions* $u_\omega^\pm(t,x) = t^{-\frac{1}{\alpha}} f_\omega^\pm(x/\sqrt{t})$.

(ii) $\limsup_{t\to\infty} (|x|^2 + t)^{\frac{1}{\alpha}}(u(t,x) - u_\omega^+(t,x)) \leq 0$ *and* $\liminf_{t\to\infty} (|x|^2 + t)^{\frac{1}{\alpha}}(u(t,x) - u_\omega^-(t,x)) \geq 0$, *uniformly in* $x \in \mathbb{R}^N$.

(iii) *If* $u_0(x) \leq u_\omega^-(\tau, x)$ *for some* $\tau > 0$, *then* (1.2) *holds with* $u_\omega(t,x)$ *replaced by* $u_\omega^-(t,x)$.

(iv) *If* $u_0(x) \geq u_\omega^+(\tau, x)$ *for some* $\tau > 0$, *then* (1.2) *holds with* $u_\omega(t,x)$ *replaced by* $u_\omega^+(t,x)$.

It follows from Theorem 1.4 that if $\alpha < 2/N$ and if $u_0 \in C_0(\mathbb{R}^N)$ is such that $\liminf_{|x|\to\infty} |x|^{\frac{2}{\alpha}} u_0(x) \geq \gamma$, then the corresponding solution u of (1.1) is positive for t large. This property raises the more general question. If $\liminf_{|x|\to\infty} |x|^\sigma u_0(x) = c > 0$ for some $\sigma > 0$, does $u(t)$ become positive for t large? Here is an answer to this question.

Theorem 1.6. *Let $\alpha, \sigma, c > 0$, let $u_0 \in C_0(\mathbb{R}^N)$ and let u be the solution of* (1.1) *with the initial condition $u(0) = u_0$. Assume further that $\liminf_{|x|\to\infty} |x|^\sigma u_0(x) = c$.*

(i) *Suppose $\alpha \ge 2/N$. If $\sigma \le N$, then $u(t) > 0$ for t large. If $\sigma > N$, then u_0 can be chosen so that $u(t)$ takes both positive and negative values for all $t > 0$.*
(ii) *If $\alpha < 2/N$, then there exists a $\overline{c} > 0$ satisfying the following. If $\sigma < 2/\alpha$, or if $\sigma = 2/\alpha$ and $c > \overline{c}$ then $u(t) > 0$ for t large. If $\sigma > 2/\alpha$ or if $\sigma = 2/\alpha$ and $c \le \overline{c}$ then u_0 can be chosen so that $u(t)$ takes both positive and negative values for all $t > 0$.*

Remark 1.7. Here are some comments on the above results.

(i) The case $\alpha > 2/N$ of Theorem 1.1 was already established in [16, Theorem 2]. (Theorem 2 of [16] is stated for nonnegative ω, but the proof clearly applies to the general case.)
(ii) The case $\alpha > 2/N$ of Theorem 1.2 was essentially obtained in [16, Theorem 2] (see (i) above for the positivity assumption), the main difference being the optimal convergence rate (1.2). The same observation applies to part (i) of Theorem 1.3.
(iii) In some of the cases not covered by Theorems 1.2—1.5, the asymptotic behavior of solutions that change sign and that have non-asymptotically radial initial values is already known. (See the papers cited above.) In some other cases, it can be deduced from previous results and Theorem 1.6. For example, let $u_0 \in C_0(\mathbb{R}^N)$ and let u be the corresponding solution of (1.1). If $|x|^{\frac{2}{\alpha}} u_0(x) \to \infty$ as $|x| \to \infty$, then it follows from Theorem 1.6 that $u(t) > 0$ for t large. On the other hand, $|x|^{\frac{2}{\alpha}} u(t,x) \to \infty$ as $|x| \to \infty$ by Proposition 5.5. Therefore, the asymptotic behavior of $u(t)$ is described by Theorem 2.1 in Gmira and Véron [13].
(iv) It seems that the case of initial values u_0 satisfying $|x|^{\frac{2}{\alpha}} u_0(x) \approx \omega(x)$ as $|x| \to \infty$ with $|\omega(x)| < \infty$ in certain directions and $|\omega(x)| = \infty$

in other directions is open. For example, in the case $N = 1$, one can construct self-similar solutions whose profile f satisfies $f(x) \to \alpha^{-\frac{1}{\alpha}}$ as $x \to +\infty$ and $|x|^{\frac{2}{\alpha}} f(x) \to \ell$ as $x \to -\infty$ for some finite ℓ. Does it describe the behavior of the solutions of (1.1) whose initial value satisfies $|x|^{\frac{2}{\alpha}} u_0(x) \to +\infty$ as $x \to +\infty$ and $|x|^{\frac{2}{\alpha}} u_0(x) \to \ell$ as $x \to -\infty$?

(v) It follows from Theorem 1.2 that the self-similar solution u_ω with profile $f_\omega \in C_0(\mathbb{R}^N)$ such that $|x|^{\frac{2}{\alpha}} f_\omega(x) - \omega(x) \to 0$ as $|x| \to \infty$ is unique if $\alpha \geq 2/N$. If $\alpha < 2/N$, it follows from Theorem 1.2 that it is unique if $\omega \geq 0$, $\omega \not\equiv 0$ and $u_\omega \geq 0$; and it follows from Theorem 1.4 that it is unique if $\omega \geq \gamma$ (without requiring $u_\omega \geq 0$).

Our proof of Theorem 1.1 is based on the following elementary observation. If $u(t, x)$ is a self-similar solution of (1.1) and has an initial value φ at $t = 0$, then φ is clearly homogeneous of degree $-2/\alpha$. Conversely, suppose one can construct a solution u of (1.1) with some initial value φ which is homogeneous of degree $-2/\alpha$. Clearly, $\lambda^{\frac{2}{\alpha}} u(\lambda^2 t, \lambda x)$ is also a solution with the same initial value φ. If, for example, u is the unique solution corresponding to φ, then $u(t, x) \equiv \lambda^{\frac{2}{\alpha}} u(\lambda^2 t, \lambda x)$, i.e. u is self-similar. The problem of existence of self-similar solutions is then essentially reduced to the solvability of the Cauchy problem for homogeneous initial values of degree $-2/\alpha$. Note that if $\alpha > 2/N$, then $\omega(x)|x|^{-\frac{2}{\alpha}}$ belongs to $L^1(\mathbb{R}^N) + L^p(\mathbb{R}^N)$ for some $p < \infty$, so it is easy to solve the Cauchy problem for such initial values (see Theorem 8.8). However, if $\alpha \leq 2/N$, then $|x|^{\frac{2}{\alpha}} \notin L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Our proof (which works in both the cases $\alpha > 2/N$ and $\alpha \leq 2/N$) is based on an explicit supersolution and the maximum principle. As far as we are aware, the idea of constructing self-similar solutions by solving the initial value problem for homogeneous initial data was first used by Giga and Miyakawa [12], for the Navier-Stokes equation in vorticity form. The idea of [12] was used later by Cannone and Planchon [4], Planchon [20] (for the Navier-Stokes equation); Angenent and Aronson [1], Ribaud [21, 22], Kwak [17], Snoussi *et al.* [25, 26] (for nonlinear parabolic problems); Cazenave and Weissler [5, 6], Ribaud and Youssfi [23], Furioli [11] (for the nonlinear Schrödinger equation); Ribaud and Youssfi [24] and Pecher [19] (for the nonlinear wave equation).

Our proof of Theorem 1.2 in the case $\alpha > 2/N$ is based on simple estimates for the heat semigroup. When $\alpha = 2/N$, we also use an explicit supersolution of (1.1) and a uniqueness result of Brezis and Friedman [2]. For $\alpha < 2/N$, we essentially use the techniques of Escobedo, Kavian and Matano [9]. Theorem 1.3 relies only on estimates of the heat semigroup. Theorems 1.4 and 1.5 use a uniqueness property for radially symmetric self-similar solutions, and Theorem 1.6 relies mostly on Theorems 1.2–1.4.

The remaining of the paper is organized as follows. Section 2—7 are devoted to the proofs of Theorems 1.1—1.6. For completeness, we collect in the appendix (Section 8) some estimates for the linear heat equation, and a well-posedness result for the equation (1.1) with initial conditions in $L^1(\mathbb{R}^N) + L^p(\mathbb{R}^N)$.

Notation. If $\alpha < 2/N$, we denote by Γ_0 the profile of the positive, radially symmetric, self-similar solution of (1.1) with exponential decay. (See [3, 7, 8, 9, 27].) Moreover, $C_0(\mathbb{R}^N)$ is the space of continuous functions on $\mathbb{R}^N$ converging to 0 as $|x| \to \infty$ and $\mathcal{S}'(\mathbb{R}^N)$ is the space of tempered distributions on $\mathbb{R}^N$. Finally, $L^1(\mathbb{R}^N) + L^p(\mathbb{R}^N)$ is the subset of $\mathcal{S}'(\mathbb{R}^N)$ whose elements can be expressed as the sum of a function in $L^1(\mathbb{R}^N)$ and a function in $L^p(\mathbb{R}^N)$.

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2. Construction of Self-Similar Solutions

In this section, we prove Theorem 1.1. We will use the following two lemmas.

Lemma 2.1. *Let $u_0 \in C_0(\mathbb{R}^N)$ and let u be the corresponding solution of* (1.1). *If $|u_0(x)| \le C|x|^{-\frac{2}{\alpha}}$, then $|u(t,x)| \le \max\{C, (4(\alpha+1)/\alpha^2)^{\frac{1}{\alpha}}\} \cdot (|x|^2+t)^{-\frac{1}{\alpha}}$.*

Proof. Set $w(t,x) = k(|x|^2+t)^{-\frac{1}{\alpha}}$. We have

$$\begin{aligned} w_t - \Delta w + w^{\alpha+1} &= k(|x|^2+t)^{-\frac{\alpha+1}{\alpha}}\Big(-\frac{1}{\alpha} + \frac{2N}{\alpha} - \frac{4(\alpha+1)|x|^2}{\alpha^2(|x|^2+t)} + k^\alpha\Big) \\ &\ge k(|x|^2+t)^{-\frac{\alpha+1}{\alpha}}\Big(-\frac{4(\alpha+1)}{\alpha^2} + k^\alpha\Big). \end{aligned}$$

Therefore, if $k \geq (4(\alpha+1)/\alpha^2)^{\frac{1}{\alpha}}$, then w is a supersolution of (1.1). Fix $k > \max\{C, (4(\alpha+1)/\alpha^2)^{\frac{1}{\alpha}}\}$. For ε sufficiently small, $|u_0(x)| \leq w(\varepsilon, x)$. By the maximum principle, $|u(t,x)| \leq w(t+\varepsilon, x)$, and the result follows by letting $\varepsilon \downarrow 0$. $\square$

LEMMA 2.2. *Let Ω be a smooth, bounded domain of $\mathbb{R}^N$, let $\Omega' \subset\subset \Omega$, let $\varphi \in C(\overline{\Omega})$, let $C, T > 0$ and set $Q = (0,T)\times\Omega$. It follows that there exists $\gamma \in C([0,T])$ with $\gamma(0) = 0$ such that if the function $u \in C([0,T], L^1(\Omega))$ is C^1 in $t \in (0,T)$ and C^2 in $x \in \overline{\Omega}$ and satisfies $|u_t - \Delta u| + |u| \leq C$ in Q and $u(0,x) = \varphi(x)$ in Ω, then $\|u(t) - \varphi\|_{L^\infty(\Omega')} \leq \gamma(t)$ for all $t \in [0,T]$.*

PROOF. By the maximum principle, $v_- \leq u \leq v_+$, where $v_\pm$ is the solution of

$$\begin{cases} \partial_t v_\pm - \Delta v_\pm = \pm C \quad \text{in} \quad Q, \\ v_\pm = \pm C \quad \text{in} \quad (0,T)\times\partial\Omega, \\ v_\pm(0,x) = \varphi(x) \quad \text{in} \quad \Omega. \end{cases}$$

In particular, $v_- - \varphi \leq u - \varphi \leq v_+ - \varphi$, so that we need only show that $\|v_\pm - \varphi\|_{L^\infty(\Omega')} \to 0$ as $t \downarrow 0$. If we denote by $\mathfrak{T}(t)$ the heat semigroup with Dirichlet boundary condition in Ω, then

$$v_\pm(t) - \varphi = (\pm C - \varphi) - \mathfrak{T}(t)(\pm C - \varphi) \pm C \int_0^t \mathfrak{T}(s) 1 \, ds.$$

Since the integral clearly converges to 0 in $L^\infty(\Omega)$ as $t \downarrow 0$, we need only show that if $\psi \in C(\overline{\Omega})$, then $\|\psi - \mathfrak{T}(t)\psi\|_{L^\infty(\Omega')} \to 0$ as $t \downarrow 0$. To see this, we fix a function $\xi \in C_c^\infty(\Omega)$ such that $\xi = 1$ in Ω', and we set $w(t) = \mathfrak{T}(t)\psi$ and $z(t) = \xi w(t)$. It follows that

$$\begin{cases} z_t - \Delta z = -w\Delta\xi - 2\nabla w \cdot \nabla\xi \quad \text{in} \quad Q, \\ z = 0 \quad \text{in} \quad (0,T)\times\partial\Omega, \\ z(0,x) = \xi(x)\psi(x) \quad \text{in} \quad \Omega, \end{cases}$$

So that

$$z(t) - \mathfrak{T}(t)(\xi\psi) = -\int_0^t \mathfrak{T}(t-s)(w(s)\Delta\xi + 2\nabla w(s)\cdot\nabla\xi)\, ds.$$

Therefore,

$$\begin{aligned}\|z(t)-\mathcal{T}(t)(\xi\psi)\|_{L^\infty(\Omega)} &\le C\int_0^t (t-s)^{-\frac{1}{4}}(\|w(s)\|_{L^{2N}(\Omega)}+\|\nabla w(s)\|_{L^{2N}(\Omega)})\\ &\le C\int_0^t (t-s)^{-\frac{1}{4}}(1+s^{-\frac{1}{2}}) \underset{t\downarrow 0}{\longrightarrow} 0,\end{aligned}$$

where the second inequality follows from the analyticity of $\mathcal{T}(t)$ in $L^{2N}(\Omega)$. On the other hand, $\xi\psi \in C_0(\Omega)$, so that $\|\xi\psi - \mathcal{T}(t)(\xi\psi)\|_{L^\infty(\Omega)} \to 0$ as $t \downarrow 0$; and so,

$$\begin{aligned}\|\psi-\mathcal{T}(t)\psi\|_{L^\infty(\Omega')} \le \|\xi\psi - z(t)\|_{L^\infty(\Omega)} &\le \|\xi\psi-\mathcal{T}(t)(\xi\psi)\|_{L^\infty(\Omega)}\\ &+\|\mathcal{T}(t)(\xi\psi)-z(t)\|_{L^\infty(\Omega)} \underset{t\downarrow 0}{\longrightarrow} 0,\end{aligned}$$

which completes the proof. □

PROOF OF THEOREM 1.1. Set $\psi(x)=\omega(x)|x|^{-\frac{2}{\alpha}}$ and, given $\mu,\nu>0$, set

$$\psi_\nu^\mu(x)=\min\{\mu,\max\{-\nu,\psi(x)\}\},\quad \psi^\mu=\min\{\mu,\psi(x)\}.$$

Let u_ν^μ be the solution of (1.1) with the initial value ψ_ν^μ. It follows that u_ν^μ is nondecreasing in μ and nonincreasing in ν. By Lemma 2.1, there exists C_ω, depending on $\|\omega\|_{L^\infty}$, such that

$$|u_\nu^\mu(t,x)| \le C_\omega(|x|+\sqrt{t})^{-\frac{2}{\alpha}}, \tag{2.1}$$

for all $t\ge 0$, $x\in\mathbb{R}^N$. Therefore, there exists a function $u^\mu(t,x)$ such that $u_\nu^\mu(t,x)\downarrow u^\mu(t,x)$ as $\nu\to\infty$. It follows from (2.1) that

$$|u^\mu(t,x)| \le C_\omega(|x|+\sqrt{t})^{-\frac{2}{\alpha}}, \tag{2.3}$$

for all $t\ge 0$, $x\in\mathbb{R}^N$, and that

$$u_t^\mu - \Delta u^\mu + |u^\mu|^\alpha u^\mu = 0, \tag{2.4}$$

in $(0, \infty) \times \mathbb{R}^N$. In addition, if $\Omega \subset\subset \mathbb{R}^N \setminus \{0\}$, then it follows from (2.3) that there exists C such that $|\partial_t u^\mu - \Delta u^\mu| + |u^\mu| \le C$ on $(0, \infty) \times \Omega$. Since $\psi^\mu(x) = \psi^\mu_\nu(x)$ in Ω for ν sufficiently large, we deduce from Lemma 2.2 that

$$(2.5) \qquad u^\mu(t,x) \underset{t\downarrow 0}{\longrightarrow} \psi^\mu(x) \quad \text{uniformly on compact subsets of } \mathbb{R}^N \setminus \{0\}.$$

It is clear that u^μ is nondecreasing in μ. Therefore, we let $\mu \to \infty$, and we see as above that $u^\mu \uparrow u$ as $\mu \to \infty$, where u satisfies

$$(2.6) \qquad \begin{cases} u_t - \Delta u + |u|^\alpha u = 0 & t > 0, x \in \mathbb{R}^N, \\ |u| \le C_\omega(|x| + \sqrt{t})^{-\frac{2}{\alpha}} & t \ge 0, x \in \mathbb{R}^N, \\ u(t,x) \underset{t\downarrow 0}{\longrightarrow} \psi(x) & \text{uniformly on compact subsets of } \mathbb{R}^N \setminus \{0\}. \end{cases}$$

Given $\lambda > 0$, we define, for $w = w(t,x)$, $w_\lambda(t,x) = \lambda^{\frac{2}{\alpha}} w(\lambda^2 t, \lambda x)$. Of course, if w satisfies (1.1), then so does w_λ. It follows that $(u^\mu_\nu)_\lambda$ is the solution of (1.1) with the initial value

$$(u^\mu_\nu)_\lambda(0,x) = \lambda^{\frac{2}{\alpha}} \psi^\mu_\nu(\lambda x) = \psi^{\lambda^{\frac{2}{\alpha}}\mu}_{\lambda^{\frac{2}{\alpha}}\nu}(x).$$

Therefore, $(u^\mu_\nu)_\lambda = u^{\lambda^{\frac{2}{\alpha}}\mu}_{\lambda^{\frac{2}{\alpha}}\nu} \downarrow u^{\lambda^{\frac{2}{\alpha}}\mu}$ as $\nu \uparrow \infty$. On the other hand, it is clear that $(u^\mu_\nu)_\lambda \downarrow (u^\mu)_\lambda$, and we conclude that $(u^\mu)_\lambda = u^{\lambda^{\frac{2}{\alpha}}\mu}$ for all $\lambda > 0$. Letting $\mu \to \infty$, we deduce that $(u^\mu)_\lambda \uparrow u$. Moreover, it is also clear that $(u^\mu)_\lambda \uparrow u_\lambda$ as $\mu \to \infty$, and we conclude that $u_\lambda = u$ for all $\lambda > 0$, i.e. u is self-similar. If f is the profile of u, i.e. $u(t,x) = t^{-\frac{1}{\alpha}} f(x/\sqrt{t})$, then we deduce from the last condition in (2.6) that $|x|^{\frac{2}{\alpha}} f(x) - \omega(x) \to 0$ as $|x| \to \infty$. $\square$

REMARK 2.3. We note the following two properties of the self-similar solutions constructed in the proof of Theorem 1.1.

(i) If $\omega \ge \omega'$, then $u_\omega \ge u_{\omega'}$.
(ii) If $\omega(x) \equiv c \in \mathbb{R}$, then u_ω is radially symmetric.

3. Proof of Theorem 1.2

We note first that it suffices to prove Theorem 1.2 where u_ω is the self-similar solution constructed in the proof of Theorem 1.1. To see this, we apply (1.2) where $u(t,x)$ is the solution of (1.1) whose initial value is the profile of a self-similar solution with the stated properties. We begin with the following lemma, which is an application of Kato's inequality.

LEMMA 3.1. *Let u, v be two solutions of (1.1) with the initial conditions $u(0) = u_0$ and $v(0) = v_0$, $u_0, v_0 \in C_0(\mathbb{R}^N)$. It follows that $|u - v| \le 2w$, where w is the solution of (1.1) with the initial condition $w(0) = |u_0 - v_0|/2$. In particular, $|u(t) - v(t)| \le e^{t\Delta}|u_0 - v_0|$.*

PROOF. Set $z = |u - v|$, so that $z_t - \Delta z + \big|\, |u|^\alpha u - |v|^\alpha v\big| \le 0$, by Kato's parabolic inequality. Since $\big|\, |u|^\alpha u - |v|^\alpha v\big| \ge 2^{-\alpha}|u - v|^{\alpha+1}$, we deduce that $z_t - \Delta z + 2^{-\alpha} z^{\alpha+1} \le 0$. Setting $\widetilde{z} = z/2$, we have $\widetilde{z}_t - \Delta \widetilde{z} + \widetilde{z}^{\alpha+1} \le 0$ and $\widetilde{z}(0) = w(0)$, so that $\widetilde{z} \le w$. Hence the result. □

We now consider separately the cases $\alpha > 2/N$, $\alpha = 2/N$ and $\alpha < 2/N$.

THE CASE $\alpha > 2/N$. Let $w_0 = |u_0 - f_\omega|$, so that $|x|^{\frac{2}{\alpha}} w_0(x) \to 0$ as $|x| \to \infty$. It follows from Lemma 3.1 that $|u(t) - u_\omega(t+1)| \le e^{t\Delta} w_0$, so that by Corollary 8.4,

$$\sup_{x \in \mathbb{R}^N} (|x|^2 + t)^{\frac{1}{\alpha}} |u(t,x) - u_\omega(t+1,x)| \to 0, \tag{3.1}$$

as $t \to \infty$. Note that we may apply the estimate (3.1) with $u_0 = u_\omega(2)$, i.e. $u(t) = u_\omega(t+2)$. Changing t to $t-1$, we deduce that

$$\sup_{x \in \mathbb{R}^N} (|x|^2 + t)^{\frac{1}{\alpha}} |u_\omega(t,x) - u_\omega(t+1,x)| \to 0, \tag{3.2}$$

(1.2) now follows from (3.1) and (3.2).

THE CASE $\alpha = 2/N$. We will use the following two lemmas. The first one is elementary and the second one is an application of Brezis and Friedman [2].

LEMMA 3.2. *Let K be a compact metric space and let $(\varphi_n)_{n\geq 0} \subset C(K,\mathbb{R})$. If $\varphi_n(x) \downarrow 0$ as $n \to \infty$ for all $x \in K$, then $\sup\limits_{x\in K} \varphi_n(x) \downarrow 0$ as $n \to \infty$.*

PROOF. Supppose by contradiction that there exist $\delta > 0$ and $(x_n)_{n\geq 0} \subset K$ such that $\varphi_n(x_n) \geq \delta$. Without loss of generality, we may assume that $x_n \to x \in K$ as $n \to \infty$. Given $n \geq m \geq 0$, we have $\varphi_n(x_n) \leq \varphi_m(x_n) \to \varphi_m(x)$ as $n \to \infty$. Therefore, $\limsup\limits_{n\to\infty} \varphi_n(x_n) \leq \varphi_m(x)$. We obtain a contradiction by letting $m \to \infty$. □

LEMMA 3.3. *Suppose $\alpha = 2/N$. Given $\varepsilon \geq 0$, let u_ε be the self-similar solution of* (1.1) *constructed in the proof of Theorem* 1.1 *with $\omega(x) \equiv \varepsilon$, and let f_ε be its profile. It follows that $\sup\limits_{x\in\mathbb{R}^N} (1+|x|^2)^{\frac{N}{2}} f_\varepsilon(x) \to 0$ as $\varepsilon \downarrow 0$.*

PROOF. It follows from Remark 2.3 that u_ε is nondecreasing in ε. In particular, $u_\varepsilon \geq u_0 = 0$. We first claim that $u_\varepsilon(t,x) \downarrow 0$ as $\varepsilon \downarrow 0$, for all $t > 0, x \in \mathbb{R}^N$. Indeed, let $u \geq 0$ be the limit of u_ε as $\varepsilon \downarrow 0$. We see that u satisfies (1.1) in $(0,\infty)\times\mathbb{R}^N$. Also, if $\Omega \subset\subset \mathbb{R}^N \setminus \{0\}$ and $\varepsilon > 0$,

$$\limsup_{t\downarrow 0} \|u(t)\|_{L^\infty(\Omega)} \leq \limsup_{t\downarrow 0} \|u_\varepsilon(t)\|_{L^\infty(\Omega)} = \varepsilon \sup_{x\in\Omega} |x|^{-N}.$$

Letting $\varepsilon \downarrow 0$, we see that $\|u(t)\|_{L^\infty(\Omega)} \to 0$ as $t \downarrow 0$. By Theorem 2 of [2], we conclude that $u = 0$, which proves the claim. In particular, we see that $f_\varepsilon(x) \downarrow 0$ as $\varepsilon \downarrow 0$. Applying Lemma 3.2, we deduce that $f_\varepsilon(x) \downarrow 0$ uniformly on compact sets of $\mathbb{R}^N$. Fix now $\delta > 0$. Since $(1+|x|^2)^{\frac{N}{2}} f_\varepsilon(x) \to \varepsilon$ as $|x| \to \infty$, we see that if K is sufficiently large and if $0 < \varepsilon \leq \delta/2$, then

$$\sup_{|x|>K} (1+|x|^2)^{\frac{N}{2}} f_\varepsilon(x) \leq \sup_{|x|>K} (1+|x|^2)^{\frac{N}{2}} f_{\frac{\delta}{2}}(x) \leq \delta.$$

On the other hand, it follows from what precedes that if ε is sufficiently small, then

$$\sup_{|x|<K} (1+|x|^2)^{\frac{N}{2}} f_\varepsilon(x) \leq \delta.$$

Since $\delta > 0$ is arbitrary, this completes the proof. □

Proposition 3.4. *Suppose* $\alpha = 2/N$. *Let* $u_0, v_0 \in C_0(\mathbb{R}^N)$ *and let* u, v *be the corresponding solutions of* (1.1). *If* $|x|^N |u_0(x) - v_0(x)| \to 0$ *as* $|x| \to \infty$, *then*

$$\sup_{x\in\mathbb{R}^N} (|x|^2 + t)^{\frac{N}{2}} |u(t,x) - v(t,x)| \underset{t\to\infty}{\longrightarrow} 0.$$

Proof. Let $w_0 = |u_0 - v_0|/2$, so that

$$|x|^N w_0(x) \underset{|x|\to\infty}{\longrightarrow} 0. \tag{3.3}$$

If w is the corresponding solution of (1.1), then it follows from Lemma 3.1 that $|u - v| \le 2w$, so that we need only show that $\sup\limits_{x\in\mathbb{R}^N} (|x|^2 + t)^{\frac{N}{2}} w(t,x) \underset{t\to\infty}{\longrightarrow} 0$.

Fix $\varepsilon > 0$. Given $R > 0$, let $z_0 = w_0 1_{\{|x|<R\}}$ and let z be the corresponding solution of (1.1). It follows from (3.3) that if R is sufficiently large, then $0 \le w_0 - z_0 \le f_\varepsilon/2$, where f_ε is as in Lemma 3.3. We deduce from Lemma 3.1 that

$$\begin{aligned}\limsup_{t\to\infty} \sup_{x\in\mathbb{R}^N} & (|x|^2 + t)^{\frac{N}{2}} |w(t,x) - z(t,x)| \\ &\le 2 \limsup_{t\to\infty} \sup_{x\in\mathbb{R}^N} (|x|^2 + t)^{\frac{N}{2}} u_\varepsilon(t,x) = 2 \sup_{x\in\mathbb{R}^N} (|x|^2 + 1)^{\frac{N}{2}} f_\varepsilon(x).\end{aligned} \tag{3.4}$$

Given $t_0 \ge e^4$ and $A \ge (Ne^4)^{\frac{N}{2}}$, set

$$Z(t,x) = \frac{A}{[(t_0 + t)\log(t_0 + t)]^{\frac{N}{2}}} e^{-\frac{|x|^2}{4(t_0+t)}\left(1 - \frac{1}{\log(t_0+t)}\right)}.$$

It follows from a straightforward calculation that, setting $\tau = t + t_0$ and $\rho = |x|^2/\tau$,

$$\begin{aligned} Z_t - \Delta Z + Z^{\alpha+1} &= \frac{Z}{\tau\log\tau}\Big[-N + \frac{\rho}{8}\Big(2 - \frac{4}{\log\tau}\Big) + A^{\frac{2}{N}} e^{-\frac{\rho}{2N}\left(1-\frac{1}{\log\tau}\right)}\Big] \\ &\ge \frac{Z}{\tau\log\tau}\Big[-N + \frac{\rho}{8}\Big(2 - \frac{4}{\log t_0}\Big) + A^{\frac{2}{N}} e^{-\frac{\rho}{2N}}\Big] \end{aligned}$$

$$\geq \frac{Z}{\tau \log \tau} \Big[-N + \frac{\rho}{8} + A^{\frac{2}{N}} e^{-\frac{\rho}{2N}} \Big] \geq 0.$$

Therefore, choosing A large enough so that $Z(0) \geq z_0$, we deduce that $z(t) \leq Z(t)$. Therefore, $(|x|^2 + t)^{\frac{N}{2}} z(t,x) \leq C[\log(t+t_0)]^{-\frac{N}{2}}$. Applying now (3.4), we conclude that

$$\limsup_{t\to\infty} \sup_{x\in\mathbb{R}^N} (|x|^2 + t)^{\frac{N}{2}} w(t,x) \leq 2 \sup_{x\in\mathbb{R}^N} (|x|^2 + 1)^{\frac{N}{2}} f_\varepsilon(x).$$

The result follows by letting $\varepsilon \downarrow 0$ and applying Lemma 3.3. □

We now can prove (1.2) in the case $\alpha = 2/N$. It follows from Proposition 3.4 that

$$\sup_{x\in\mathbb{R}^N} (|x|^2 + t)^{\frac{N}{2}} |u(t,x) - u_\omega(t+1,x)| \underset{t\to\infty}{\longrightarrow} 0,$$

and one concludes as in the case $\alpha > 2/N$.

The case $\alpha < 2/N$. It is convenient to introduce the self-similar variables. We recall that $u(t,x)$ satisfies (1.1) if and only if $v(s,y) = e^{\frac{s}{\alpha}} u(t,x)$ with $s = \log(1+t)$ and $y = x/\sqrt{1+t}$ satisfies

$$v_s - \Delta v - \frac{1}{2} y \cdot \nabla v - \frac{1}{\alpha} v + |v|^\alpha v = 0. \tag{3.5}$$

Also, $f \in C_0(\mathbb{R}^N)$ is the profile of a self-similar solution of (1.1) iff

$$-\Delta f - \frac{1}{2} y \cdot \nabla f - \frac{1}{\alpha} f + |f|^\alpha f = 0. \tag{3.6}$$

The following lemma is an essential tool in our proof. (See the proof of Lemma 2.6 in [9].)

Lemma 3.5. *Let $v_{1,0}, v_{2,0}, v_{3,0} \in C_0(\mathbb{R}^N)$ such that*

$$0 \leq v_{1,0}, v_{2,0} \leq v_{3,0} \leq v_{1,0} + v_{2,0},$$

and let v_1, v_2, v_3 *be the solutions of* (3.5) *with the initial values* $v_{1,0}, v_{2,0}$ *and* $v_{3,0}$*, respectively. It follows that*

$$v_1(s) + v_2(s) - v_3(s) \geq \xi(s)$$

for all $s > 0$*, where* ξ *it the solution of* (3.5) *with the initial value* $\xi(0) = v_{1,0} + v_{2,0} - v_{3,0}$.

Proof. By the maximum principle, $0 \leq v_1(s), v_2(s) \leq v_3(s)$ for all $s > 0$. We now claim that $v_3(s) \leq v_1(s) + v_2(s)$. Indeed, $z = v_1 + v_2$ satisfies $z(0) \geq v_{3,0}$ and

$$z_s - \Delta z - \frac{1}{2} y \cdot \nabla z - \frac{1}{\alpha} z + |z|^\alpha z = (v_1 + v_2)^{\alpha+1} - v_1^{\alpha+1} - v_2^{\alpha+1} \geq 0,$$

so that by the maximum principle $z \geq v_3$, which proves the claim. We next observe that, since $0 \leq v_1, v_2 \leq v_3 \leq v_1 + v_2$,

$$(v_1 + v_2 - v_3)^{\alpha+1} \geq v_1^{\alpha+1} + v_2^{\alpha+1} - v_3^{\alpha+1}.$$

Therefore, $z = v_1 + v_2 - v_3$ satisfies $z(0) = v_{1,0} + v_{2,0} - v_{3,0}$ and

$$z_t - \Delta z - \frac{1}{2} y \cdot \nabla z - \frac{1}{\alpha} z + |z|^\alpha z = (v_1 + v_2 - v_3)^{\alpha+1} - v_1^{\alpha+1} - v_2^{\alpha+1} + v_3^{\alpha+1} \geq 0,$$

so that by the maximum principle $z \geq \xi$. $\square$

Lemma 3.6. *Let* $w_0, z_0 \in C_0(\mathbb{R}^N)$, $w_0, z_0 \geq 0$, $w_0 \not\equiv 0$, $z_0 \not\equiv 0$ *satisfy* $|y|^{\frac{2}{\alpha}}(w_0(y) + z_0(y)) \to 0$ *as* $|y| \to \infty$*, and let* w *and* z *be the corresponding solutions of* (3.5)*. It follows that* $\sup_{y \in \mathbb{R}^N} (1 + |y|^{\frac{2}{\alpha}})|w(s,y) - z(s,y)| \to 0$ *as* $s \to \infty$.

Proof. Fix $\varepsilon > 0$. Let f_ε be as in the proof of Theorem 1.1 with $\omega(x) \equiv \varepsilon$. For $M > 0$ sufficiently large, we have $w_0 1_{\{|y|>M\}} \leq f_\varepsilon$. If $w_1(s)$ is the solution of (3.5) with the initial value $w_0 1_{\{|y|>M\}}$, we deduce that $w_1(s) \leq f_\varepsilon$. On the other hand, there exists $c \geq 1$ such that $w_0 1_{\{|y|<M\}} \leq c\Gamma_0$. Since $c\Gamma_0$ is a supersolution of (3.5) for all $c \geq 1$, we deduce that $w_2(s) \leq c\Gamma_0$, where w_2 is the solution of (3.5) with the initial value

$w_0 1_{\{|y|<M\}}$. Since $|w(s) - w_1(s)| \le 2w_2(s)$ by Lemma 3.1, we obtain that $w(s) \le f_\varepsilon + 2c\Gamma_0$. As well, $z(s) \le f_\varepsilon + 2c\Gamma_0$, by possibly choosing c larger. It follows that for $R > 0$ sufficiently large, $(1 + |y|^{\frac{2}{\alpha}})|w(s,y) - z(s,y)| \le 2\varepsilon$ for $|y| \ge R$. Next, $w(s) \to \Gamma_0$ and $z(s) \to \Gamma_0$ uniformly as $s \to \infty$ by Theorem 1.2 of [9]; and so, $(1 + |y|^{\frac{2}{\alpha}})|w(s,y) - z(s,y)| \le \varepsilon$ for s sufficiently large and all $|y| \le R$. It follows that

$$\sup_{y \in \mathbb{R}^N} (1 + |y|^{\frac{2}{\alpha}})|w(s,y) - z(s,y)| \le 2\varepsilon,$$

for s sufficiently large. Hence the result, since ε is arbitrary. □

COROLLARY 3.7. *Let $\widetilde{w}_0, \widetilde{z}_0 \in C_0(\mathbb{R}^N)$ and let $\widetilde{w}(s)$ and $\widetilde{z}(s)$ be the corresponding solutions of* (3.5). *If $\widetilde{w}_0 \ge 0$, $\widetilde{w}_0 \not\equiv 0$ and*

$$\limsup_{|y|\to\infty} |y|^{\frac{2}{\alpha}} (\widetilde{z}_0(y) - \widetilde{w}_0(y)) \le 0, \tag{3.7}$$

then

$$\limsup_{s\to\infty} \sup_{y\in\mathbb{R}^N} (1 + |y|^{\frac{2}{\alpha}})(\widetilde{z}(s,y) - \widetilde{w}(s,y)) \le 0. \tag{3.8}$$

In particular, if $\widetilde{w}_0, \widetilde{z}_0 \ge 0$, $\widetilde{w}_0, \widetilde{z}_0 \not\equiv 0$ and $|y|^{\frac{2}{\alpha}}|\widetilde{w}_0(y) - \widetilde{z}_0(y)| \underset{|y|\to\infty}{\longrightarrow} 0$, then $\sup_{y\in\mathbb{R}^N} (1 + |y|^{\frac{2}{\alpha}})|\widetilde{w}(s,y) - \widetilde{z}(s,y)| \underset{s\to\infty}{\longrightarrow} 0.$

PROOF. Let $\overline{z}_0 = \max\{\widetilde{w}_0, \widetilde{z}_0\}$ and let $\overline{z}(s)$ be the corresponding solution of (3.5). Since $\overline{z}(s) \ge \widetilde{z}(s)$, we need only show that

$$\limsup_{s\to\infty} \sup_{y\in\mathbb{R}^N} (1 + |y|^{\frac{2}{\alpha}})(\overline{z}(s,y) - \widetilde{w}(s,y)) \le 0. \tag{3.9}$$

We claim that

$$\limsup_{|y|\to\infty} |y|^{\frac{2}{\alpha}} (\overline{z}_0(y) - \widetilde{w}_0(y)) \le 0. \tag{3.10}$$

Indeed, consider $|y_n| \to \infty$ such that $|y_n|^{\frac{2}{\alpha}}(\overline{z}_0(y_n) - \widetilde{w}_0(y_n)) = \sigma$. If $\overline{z}_0(y_n) = \widetilde{z}_0(y_n)$ for n large then $\sigma \le 0$ by (3.7). Otherwise, $\sigma = 0$. This shows the claim. Let now $\varphi_0 \in C_c^\infty(\mathbb{R}^N)$, $\varphi_0 \ge 0$, $\varphi_0 \not\equiv 0$, $\varphi_0 \le \widetilde{w}_0$, and let $\varphi(s)$ be the corresponding solution of (3.5). Finally, let

$$z_0 = \overline{z}_0 - \widetilde{w}_0 + \varphi_0 \le \overline{z}_0, \tag{3.11}$$

and let $z(s)$ be the corresponding solution of (3.5). Note that $z_0 \ge \varphi_0 \not\equiv 0$ and that $|y|^{\frac{2}{\alpha}} z_0(y) \to 0$ as $|y| \to \infty$ by (3.10). Since $0 \le \widetilde{w}_0, z_0 \le \overline{z}_0 \le \widetilde{w}_0 + z_0$ by (3.11), we deduce from Lemma 3.5 that $\widetilde{w}(s) + z(s) - \overline{z}(s) \ge \varphi(s)$. Therefore, $\overline{z}(s) - \widetilde{w}(s) \le z(s) - \varphi(s)$ and (3.9) follows by applying Lemma 3.6 to the right-hand side. □

We now can prove (1.2) in the case $\alpha < 2/N$. We need to show that if $v_0 \ge 0$ satisfies $|y|^{\frac{2}{\alpha}} v_0(y) - \omega(y) \to 0$ as $|y| \to \infty$ and if v is the corresponding solution of (3.5), then

$$\sup_{y \in \mathbb{R}^N} (1 + |y|^{\frac{2}{\alpha}}) |v(s,y) - f_\omega(y)| \underset{s \to \infty}{\longrightarrow} 0.$$

This follows from Corollary 3.7, by letting $\widetilde{w}_0 = v_0$ and $\widetilde{z}_0 = f_\omega$.

4. Proof of Theorem 1.3

PROOF OF PROPERTY (i). We have $|u(t)| \le e^{t\Delta}|u_0|$, thus $|u(t,x)| \le C(1 + t + |x|^2)^{-\frac{\sigma}{2}}$ by Corollary 8.3. Fix

$$0 < \varepsilon < \min\{N - \sigma, \alpha\sigma - 2\}.$$

We deduce that

$$||u|^\alpha u|(s,x) \le C(1 + s + |x|^2)^{-\frac{\sigma(\alpha+1)}{2}} \le C(1+s)^{-\frac{\alpha\sigma - \varepsilon}{2}} (1 + s + |x|^2)^{-\frac{\sigma+\varepsilon}{2}};$$

and so, by using again Corollary 8.3,

$$\begin{aligned} |e^{(t-s)\Delta}|u(s)|^\alpha u(s)| &\le C(1+s)^{-\frac{\alpha\sigma-\varepsilon}{2}} (1 + s + (t-s) + |x|^2)^{-\frac{\sigma+\varepsilon}{2}} \\ &= C(1+s)^{-\frac{\alpha\sigma-\varepsilon}{2}} (1 + t + |x|^2)^{-\frac{\sigma+\varepsilon}{2}}. \end{aligned}$$

Therefore,

$$\Big|\int_0^t e^{(t-s)\Delta}|u(s)|^\alpha u(s)\,ds\Big| \le C(t+|x|^2)^{-\frac{\sigma+\varepsilon}{2}}\int_0^t (1+s)^{-\frac{\alpha\sigma-\varepsilon}{2}}\,ds \le C(t+|x|^2)^{-\frac{\sigma+\varepsilon}{2}}. \tag{4.1}$$

We deduce that

$$\begin{aligned}&(t+|x|^2)^{\frac{\sigma}{2}}|u(t,x)-e^{t\Delta}v_0(x)|\\ &\quad\le (t+|x|^2)^{\frac{\sigma}{2}}(|u(t,x)-e^{t\Delta}u_0(x)|+|e^{t\Delta}(u_0-v_0)(x)|)\\ &\quad\le C(t+|x|^2)^{-\frac{\varepsilon}{2}}+(t+|x|^2)^{\frac{\sigma}{2}}|e^{t\Delta}(u_0-v_0)(x)| \underset{t\to\infty}{\longrightarrow} 0,\end{aligned}$$

uniformly in $x\in\mathbb{R}^N$, by using (4.1) and Corollary 8.4. □

Proof of property (ii). We have

$$u(t)-e^{t\Delta}u_0 = -\int_0^t e^{(t-s)\Delta}|u(s)|^\alpha u(s)\,ds.$$

Since $|u(s)|\le e^{s\Delta}|u_0|$, it follows from Lemma 8.5 that $|u(s)|\le C(1+s+|x|^2)^{-\frac{N}{2}}\log(2+s)$. Applying Lemma 8.7, we deduce that $e^{(t-s)\Delta}|u(s)|^{\alpha+1}\le C(1+t+|x|^2)^{-\frac{N}{2}}(1+s)^{-\frac{N\alpha}{2}}(\log(2+s))^{\alpha+1}$. Therefore,

$$\begin{aligned}&\sup_{x\in\mathbb{R}^N}\frac{(t+|x|^2)^{\frac{N}{2}}}{\log t}|u(t,x)-e^{t\Delta}u_0(x)|\\ &\quad\le C\frac{1}{\log t}\int_0^t (1+s)^{-\frac{N\alpha}{2}}(\log(2+s))^{\alpha+1}\,ds \underset{t\to\infty}{\longrightarrow} 0,\end{aligned}$$

and the result follows from Lemma 8.6. □

5. Proof of Theorem 1.4

Throughout this section, we assume that $\alpha<2/N$. Given $\omega\in C(\mathbb{R}^N\setminus\{0\})$ a homogeneous function of degree 0, we denote by $\mathcal{F}_\omega$ the set of the profiles $f\in C_0(\mathbb{R}^N)$ of self-similar solutions of (1.1) such that $|y|^{\frac{2}{\alpha}}f(y)-$

$\omega(y) \to 0$ as $|y| \to \infty$. We begin with the following result, inspired by Lemma 2.5 of [9].

LEMMA 5.1. *Let $f \in \mathcal{F}_\omega$ and $k \geq 2$. Let $v_0 = f + k\Gamma_0$ and $w_0 = f - k\Gamma_0$, and let v and w be the corresponding solutions of* (3.5). *It follows that $v(s)$ is nonincreasing in s and $w(s)$ is nondecreasing in s.*

PROOF. Note that

$$|x + y|^\alpha (x + y) - |x|^\alpha x \geq 2^{-\alpha} y^{\alpha+1}, \tag{5.1}$$

for all $x \in \mathbb{R}$ and $y \geq 0$. We have

$$\begin{aligned} -\Delta v_0 - \frac{1}{2} y \cdot \nabla v_0 - \frac{1}{\alpha} v_0 + |v_0|^\alpha v_0 &= |f + k\Gamma_0|^\alpha (f + k\Gamma_0) - |f|^\alpha f - k\Gamma_0^{\alpha+1} \\ &\geq (k^\alpha 2^{-\alpha} - 1) k \Gamma_0^{\alpha+1} \geq 0, \end{aligned}$$

by (5.1). Therefore, v_0 is a supersolution of (3.6); and so, $v(s)$ is nonincreasing. The other statement is proved in the same way. □

The next ingredient we need concerns the solutions of the problem

$$\begin{cases} u'' + \Big(\dfrac{N-1}{r} + \dfrac{r}{2}\Big) u' + \dfrac{1}{\alpha} u - |u|^\alpha u = 0, & r > 0, \\ u(0) = \theta, \quad u'(0) = 0, \end{cases} \tag{5.2}$$

where $\theta > 0$. If $\theta = \pm\alpha^{-\frac{1}{\alpha}}$, then u is constant. If $|\theta| > \alpha^{-\frac{1}{\alpha}}$, then it is not difficult to see that u blows up in finite time. If $|\theta| < \alpha^{-\frac{1}{\alpha}}$, then one can easily adapt the techniques of [3] to show there exists $c = c(\theta)$ such that $r^{\frac{2}{\alpha}} u(r) \to c$ as $r \to \infty$. It is clear that c is odd, and it is well-known that there exists $0 < \theta^* < \alpha^{-\frac{1}{\alpha}}$ such that $c(\theta^*) = 0$ and $c(\theta) > 0$ for $\theta^* < \theta < \alpha^{-\frac{1}{\alpha}}$. In addition, $u > 0$ if $\theta^* \leq \theta < \alpha^{-\frac{1}{\alpha}}$ while u changes sign if $0 < \theta < \theta^*$. Moreover, c is increasing and convex on $[\theta^*, \alpha^{-\frac{1}{\alpha}})$ and $c(\theta) \to \infty$ as $\theta \to \alpha^{-\frac{1}{\alpha}}$. (For all these properties, see [9]). In particular, c is continuous on $[\theta^*, \alpha^{-\frac{1}{\alpha}})$. The same property holds on $[0, \theta^*]$, as shows the following result. (See [14] for the analogous result for the equation with the other sign.)

LEMMA 5.2. *The mapping $\theta \mapsto c(\theta)$ is continuous $(-\alpha^{-\frac{1}{\alpha}}, \alpha^{-\frac{1}{\alpha}}) \to \mathbb{R}$.*

PROOF. Fix $\theta_0 \in (-\alpha^{-\frac{1}{\alpha}}, \alpha^{-\frac{1}{\alpha}})$ and let u_0 be the corresponding solution of (5.2). Let $\theta \in (-\alpha^{-\frac{1}{\alpha}}, \alpha^{-\frac{1}{\alpha}})$ and let u be the corresponding solution of (5.2). Set $c_0 = c(\theta_0)$ and $c = c(\theta)$. Next, let $v_0(r) = r^{\frac{1}{\alpha}} u_0(r)$ and $v(r) = r^{\frac{1}{\alpha}} u(r)$. It follows that $v_0(r) \to c_0$ and $v(r) \to c$ as $r \to \infty$, and that v_0 and v are solutions of the equation

$$v'' + \Big(\frac{r}{2} + \frac{N-1}{r} - \frac{4}{\alpha r}\Big)v' + \Big(\frac{4}{\alpha^2} - \frac{2(N-2)}{\alpha} - |v|^\alpha\Big)\frac{v}{r^2} = 0. \tag{5.3}$$

By continuous dependence, $v \to v_0$ in $C^1([0,R])$ as $\theta \to \theta_0$ for every $R > 0$. Finally, let

$$\phi(r) = \frac{r}{2} + \frac{N-1}{r} - \frac{4}{\alpha r}, \quad \Phi(r) = \int_0^r \phi(s)\, ds.$$

It follows from (5.3) that

$$(e^{\Phi(r)} v'(r))' = -e^{\Phi(r)}\Big(\frac{4}{\alpha^2} - \frac{2(N-2)}{\alpha} - |v|^\alpha\Big)\frac{v}{r^2}.$$

Integrating between $\underline{r} > 0$ and $r > \underline{r}$, we deduce that

$$|v'(r)| \le e^{-\Phi(r)+\Phi(\underline{r})}|v'(\underline{r})| + \int_{\underline{r}}^r \frac{e^{-\Phi(r)+\Phi(s)}}{s^2}\Big|\frac{4}{\alpha^2} - \frac{2(N-2)}{\alpha} - |v|^\alpha\Big|\,|v|\, ds.$$

Integrating again,

$$\begin{aligned}|v(r)| \le |v(\underline{r})| + |v'(\underline{r})| \int_{\underline{r}}^r e^{-\Phi(s)+\Phi(\underline{r})}\, ds \\ + \Big[\sup_{[\underline{r},r]}\Big|\frac{4}{\alpha^2} - \frac{2(N-2)}{\alpha} - |v|^\alpha\Big|\,|v|\Big] \int_{\underline{r}}^r \int_{\underline{r}}^s \frac{e^{-\Phi(s)+\Phi(\sigma)}}{\sigma^2}\, d\sigma\, ds.\end{aligned}$$

Note that

$$\int_{\underline{r}}^r e^{-\Phi(s)+\Phi(\underline{r})}\, ds \le \int_{\underline{r}}^\infty e^{-\Phi(s)+\Phi(\underline{r})}\, ds = \int_0^\infty e^{-\int_{\underline{r}}^{\underline{r}+s} \phi(\sigma)\, d\sigma}\, ds \underset{\underline{r}\to\infty}{\longrightarrow} 0,$$

by dominated convergence. Also,

$$\int_{\underline{r}}^{r}\int_{\underline{r}}^{s}\frac{e^{-\Phi(s)+\Phi(\sigma)}}{\sigma^2}\,d\sigma\,ds=\int_{\underline{r}}^{r}\int_{\underline{r}}^{s}\frac{e^{-\int_\sigma^s\phi(\tau)\,d\tau}}{\sigma^2}\,d\sigma\,ds.$$

By L'Hôpital's rule,

$$\int_1^s\frac{e^{-\int_\sigma^s\phi(\tau)\,d\tau}}{\sigma^2}\,d\sigma=O(s^{-3}),$$

as $s\to\infty$, so that

$$\int_1^\infty\int_1^s\frac{e^{-\int_\sigma^s\phi(\tau)\,d\tau}}{\sigma^2}\,d\sigma\,ds<\infty.$$

Therefore,

$$\int_{\underline{r}}^{r}\int_{\underline{r}}^{s}\frac{e^{-\Phi(s)+\Phi(\sigma)}}{\sigma^2}\,d\sigma\,ds\underset{\underline{r}\to\infty}{\longrightarrow}0,$$

uniformly in $r\ge\underline{r}$. It follows that

$$|v(r)|\le|v(\underline{r})|+|v'(\underline{r})|\varepsilon(\underline{r})+\Big[\sup_{[\underline{r},r]}\Big|\frac{4}{\alpha^2}-\frac{2(N-2)}{\alpha}-|v|^\alpha\Big|\,|v|\Big]\varepsilon(\underline{r}),\tag{5.4}$$

for $r\ge\underline{r}$, with $\varepsilon(\underline{r})\to0$ as $\underline{r}\to\infty$. We now fix $\underline{r}$ such that

$$\begin{gathered}2\varepsilon(\underline{r})\le c_0,\quad 6c_0\Big(\frac{4}{\alpha^2}+\frac{2|N-2|}{\alpha}+(6c_0)^\alpha\Big)\varepsilon(\underline{r})\le c_0,\\ |v_0(\underline{r})|\le2c_0,\quad |v_0'(\underline{r})|\le1.\end{gathered}\tag{5.5}$$

For $|\theta-\theta_0|$ sufficiently small, we have in particular

$$|v(\underline{r})|\le3c_0,\quad |v'(\underline{r})|\le2.\tag{5.6}$$

We claim that $|v(r)|<6c_0$ for all $r\ge\underline{r}$. Indeed, assume by contradiction that there exists $R>\underline{r}$ such that $|v(r)|\le6c_0$ for all $\underline{r}\le r\le R$ and $|v(R)|=6c_0$. It follows from (5.4), (5.5) and (5.6) that $|v(R)|\le5c_0$, which is absurd. Consider now $\overline{r}>\underline{r}$. We obtain as above that

$$|v(r)-v_0(r)|\le|v(\overline{r})-v_0(\overline{r})|+|v'(\overline{r})-v_0'(\overline{r})|\varepsilon(\overline{r})+C\varepsilon(\overline{r}),\tag{5.7}$$

for all $r \geq \overline{r}$. Given $\delta > 0$, fix $\overline{r}$ large enough so that $(C+1+|v_0'(\overline{r})|)\varepsilon(\overline{r}) \leq \delta/2$ and suppose $|\theta - \theta_0|$ is small enough so that $|v(\overline{r}) - v_0(\overline{r})| \leq \delta/2$ and $|v'(\overline{r}) - v_0'(\overline{r})| \leq 1 + |v_0'(\overline{r})|$. It then follows from (5.7) that $|v(r) - v_0(r)| \leq \delta$ for all $r \geq \overline{r}$. Letting $r \to \infty$, we obtain $|c - c_0| \leq \delta$. Hence the result. □

The following result is then an immediate consequence of Lemma 5.2.

Corollary 5.3. *Let $\overline{c} = \sup\{|c(\theta)|;\, 0 \leq \theta < \theta^*\}$. If $K > \overline{c}$, then the radially symmetric self-similar solution of* (1.1) *with profile $f \in C_0(\mathbb{R}^N)$ such that $r^{\frac{2}{\alpha}} f(r) \to K$ as $r \to \infty$ is unique and positive. If $0 < K \leq \overline{c}$, then there exists a radially symmetric self-similar solution of* (1.1) *with profile $f \in C_0(\mathbb{R}^N)$ such that $r^{\frac{2}{\alpha}} f(r) \to K$ as $r \to \infty$ and f changes sign.*

Proposition 5.4. *Suppose $\alpha < 2/N$ and let $\overline{c} = \sup\{|c(\theta)|;\, 0 \leq \theta < \theta^*\}$. Let $u_0 \in C_0(\mathbb{R}^N)$ and let u be the corresponding solution of* (1.1). *If $\liminf_{|x|\to\infty} |x|^{\frac{2}{\alpha}} u_0(x) > \overline{c}$, then $u(t) > 0$ for t sufficiently large.*

Proof. Let v be the solution of (3.5) with initial value u_0. We need to show that $v(s) > 0$ for s large. Let $K > K' > \overline{c}$. Note that by Corollary 5.3 and Remark 2.3 (ii), the profile of the (unique) radially symmetric self-similar solution of (1.1) which behaves like $K' r^{-\frac{2}{\alpha}}$ as $r \to \infty$ is $f_{K'}$. Next, we observe that there exists $k \geq 2$ such that $u_0 \geq w_0$ with $w_0 = f_{K'} - k\Gamma_0$. If we denote by w the solution of (3.5) with initial value w_0, it follows from Lemma 5.1 that $w(s)$ is nondecreasing. Since $w(s) \leq f_{K'}$ by the maximum principle, we see that $w(s)$ has a limit w_∞ as $s \to \infty$. Since w_0 is radially, symmetric, so is w_∞. Also, since $|w_0 - f_{K'}| \leq k\Gamma_0$ and $k\Gamma_0$ is a supersolution of (3.5), we deduce from Lemma 3.1 that $|w(s) - f_{K'}| \leq 2k\Gamma_0$, so that

$$f_{K'} - 2k\Gamma_0 \leq w(s) \leq f_{K'}. \tag{5.8}$$

Clearly w_∞ is a solution of (3.6), so it follows from (5.8) that $w_\infty \in \mathcal{F}_{K'}$. Since w_∞ is radially symmetric, we have $w_\infty = f_{K'}$ by Corollary 5.3. Applying (5.8), we see that there exists R such that $w(s,y) \geq f_{K'}(y)/2$ for $|y| \geq R$ and all $s > 0$. We now deduce from Lemma 3.2 that $w(s,y) \geq f_{K'}(y)/2$ for $|y| \leq R$ and s sufficiently large. It follows that $w(s,y) \geq f_{K'}(y)/2 > 0$ for s sufficiently large and all $y \in \mathbb{R}^N$. The result follows, since $v(s) \geq w(s)$. □

Theorem 1.4 now follows from Proposition 5.4, Theorem 1.2 and the following result.

Proposition 5.5. *Let* $\sigma > 0$ *and let* $\omega \in C(\mathbb{R}^N \setminus \{0\})$ *be homogeneous of degree* 0. *Let* $u_0 \in C_0(\mathbb{R}^N)$ *and let* u *be the corresponding solution of* (1.1). *If* $|x|^\sigma u_0(x) - \omega(x) \to 0$ *as* $|x| \to \infty$, *then for every* $T > 0$, $|x|^\sigma u(t,x) - \omega(x) \to 0$ *as* $|x| \to \infty$, *uniformly in* $t \in [0,T]$.

Proof. Fix $T > 0$. We observe that $|u(t)| \le e^{t\Delta}|u_0|$, and we deduce from Lemma 8.1 that there exists M such that $u(t,x) \le M(1+|x|)^{-\sigma}$ for all $0 \le t \le T$. Applying again Lemma 8.1, we obtain that, by possibly choosing M larger,

$$|e^{(t-s)\Delta}|u(s)|^\alpha u(s)| \le M(1+|x|)^{-(\alpha+1)\sigma}, \tag{5.9}$$

for all $0 \le s \le t \le T$. Next, since

$$u(t) - e^{t\Delta}u_0 = -\int_0^t e^{(t-s)\Delta}|u(s)|^\alpha u(s)\, ds,$$

we deduce from (5.9) that

$$|u(t) - e^{t\Delta}u_0| \le C(1+|x|)^{-(\alpha+1)\sigma}, \tag{5.10}$$

for all $0 \le t \le T$. Finally,

$$|\,|x|^\sigma u(t) - \omega(x)| \le |\,|x|^\sigma e^{t\Delta}u_0 - \omega(x)| + |x|^\sigma |u(t) - e^{t\Delta}u_0|.$$

The result follows by applying Lemma 8.1 to the first term on the right-hand side and (5.10) to the second term. □

6. Proof of Theorem 1.5

Throughout this section, we assume that $\alpha < 2/N$. We begin with the following lemma.

Lemma 6.1. *Let* $\omega_1, \omega_2 \in C(\mathbb{R}^N \setminus \{0\})$ *be homogeneous of degree* 0, *and let* $f \in \mathcal{F}_{\omega_1}$ $g \in \mathcal{F}_{\omega_2}$. *If* $\omega_1 \ge \omega_2$, *then there exist* $h \in \mathcal{F}_{\omega_1}$ *and* $k \in \mathcal{F}_{\omega_2}$ *such that* $k \le f, g \le h$.

Proof. Let $v_0 = \max\{f, g\}$ and let v be the corresponding solution of (3.5). Since v_0 is a subsolution of (3.6), we see that $v(s)$ is nondecreasing and we denote by v_∞ its limit. Let $w_0 = |f - v_0|/2$ and let w be the solution of (3.5) with the initial value w_0. It follows from Lemma 3.1 that $|v(s) - f| \leq 2w(s)$, so that

$$\max\{f, g\} \leq v(s) \leq f + 2w(s).$$

Note that $|y|^{\frac{2}{\alpha}} w_0(y) \to 0$ as $|y| \to \infty$, so that $w(s) \to \Gamma_0$ as $s \to \infty$ by Theorem 1.2 of [9]; and so,

$$\max\{f, g\} \leq v_\infty \leq f + 2\Gamma_0. \tag{6.1}$$

Since v_∞ is clearly a solution of (3.6), we deduce from (6.1) that $v_\infty \in \mathcal{F}_{\omega_1}$. The first statement follows with $h = v_\infty$ and the second statement is proved in the same way. □

Corollary 6.2. *$\mathcal{F}_\omega$ has a minimal element f_ω^- and a maximal element f_ω^+, i.e. $f_\omega^- \leq f \leq f_\omega^+$ for every $f \in \mathcal{F}_\omega$.*

Proof. We first claim that if $f, g \in \mathcal{F}_\omega$, then

$$|f - g| \leq 2\Gamma_0. \tag{6.2}$$

Indeed, $|y|^{\frac{2}{\alpha}} |f - g| \to 0$ as $|y| \to \infty$, so that by Lemma 3.1 and Theorem 1.2 of [9], $|f - g| \leq 2v(s)$ with $v(s) \to \Gamma_0$ as $s \to \infty$. Hence (6.2). Let now

$$m = \inf_{f \in \mathcal{F}_\omega} f(0),$$

so that here exists $(f_n)_{n \geq 0} \subset \mathcal{F}_\omega$ such that $f_n(0) \to m$. It follows easily from (6.2) and a local compactness argument that there exist a subsequence n_k and a solution f of (3.6) such that $f_{n_k} \to f$ uniformly on bounded subsets of $\mathbb{R}^N$. Applying again (6.2), we see that $f \in \mathcal{F}_\omega$. We claim that f is minimal. Otherwise, there exist $g \in \mathcal{F}_\omega$ and $x_0 \in \mathbb{R}^N$ such that $g(x_0) < f(x_0)$. By Lemma 6.1, there exists $h \in \mathcal{F}_\omega$ such that $h \leq f$ and $h \not\equiv f$. By the strong maximum principle, $h(0) < f(0) = m$, which is absurd. The maximal element is constructed in the same way. □

COROLLARY 6.3. *With the notation of Corollary 6.2, the following properties hold.*

(i) *If $\omega_1, \omega_2 \in C(\mathbb{R}^N \setminus \{0\})$ are homogeneous of degree 0 and $\omega_1 \geq \omega_2$, then $f^+_{\omega_1} \geq f^+_{\omega_2}$ and $f^-_{\omega_1} \geq f^-_{\omega_2}$.*

(ii) *If $\omega \in C(\mathbb{R}^N \setminus \{0\})$ is homogeneous of degree 0, then $\sup_{y\in\mathbb{R}^N} (1 + |y|^2)^{\frac{1}{\alpha}} |f^+_{\omega+\varepsilon} - f^+_\omega| \underset{\varepsilon\downarrow 0}{\longrightarrow} 0$ and $\sup_{y\in\mathbb{R}^N} (1 + |y|^2)^{\frac{1}{\alpha}} |f^-_\omega - f^-_{\omega-\varepsilon}| \underset{\varepsilon\downarrow 0}{\longrightarrow} 0$.*

PROOF. (i) By Lemma 6.1, there exists $g \in \mathcal{F}_{\omega_1}$ such that $g \geq f^+_{\omega_1}$ and $g \geq f^+_{\omega_2}$. By maximality, $g = f^+_{\omega_1}$, so that $f^+_{\omega_1} \geq f^+_{\omega_2}$. The second statement is proved in the same way.

(ii) Here also, we only prove the first statement. By (i), $f^+_{\omega+\varepsilon}$ is nondecreasing in ε. Therefore, $f^+_{\omega+\varepsilon}$ has a limit f as $\varepsilon \downarrow 0$, and

$$f^+_{\omega+\varepsilon} \geq f \geq f^+_\omega. \tag{6.3}$$

It is clear that f is a solution of (3.6), and it follows from (6.3) that $f \in \mathcal{F}_\omega$. Using again (6.3), we deduce by maximality that $f = f^+_\omega$. Therefore, $(1 + |y|^2)^{\frac{1}{\alpha}} f^+_{\omega+\varepsilon} \downarrow (1 + |y|^2)^{\frac{1}{\alpha}} f^+_\omega$ as $\varepsilon \downarrow 0$. Using Lemma 3.2, we conclude that the convergence is uniform. □

We are now in a position to prove Theorem 1.5. Property (i) is simply Corollary 6.2. To prove property (ii), we need only show (using the self-similar variables) that if $v_0 \in C_0(\mathbb{R}^N)$ satisfies $|y|^{\frac{2}{\alpha}} v_0(y) - \omega(y) \to 0$ as $|y| \to \infty$, then

$$\begin{aligned}(1 + |y|^2)^{\frac{1}{\alpha}} f^-_\omega(y) &\leq \liminf_{s\to\infty} (1 + |y|^2)^{\frac{1}{\alpha}} v(s, y) \\ &\leq \limsup_{s\to\infty} (1 + |y|^2)^{\frac{1}{\alpha}} v(s, y) \leq (1 + |y|^2)^{\frac{1}{\alpha}} f^+_\omega(y).\end{aligned} \tag{6.4}$$

Given any $\varepsilon > 0$, there exists $k \geq 2$ such that $v_0 \leq w_0$ with $w_0 = f^+_{\omega+\varepsilon} + k\Gamma_0$. If w denotes the solution of (3.5) with the initial value w_0, then on one hand $v(s) \leq w(s)$, and on the other hand, it follows from Lemma 5.1 that $w(s)$ is nonincreasing. Therefore, $w(s) \downarrow w_\infty$ as $s \to \infty$, and it follows easily that w_∞ is a solution of (3.6). Since $|w_0 - f^+_{\omega+\varepsilon}| \leq k\Gamma_0$ and $k\Gamma_0$ is a supersolution of (3.5), we deduce from Lemma 3.1 that $|w(s) - f^+_{\omega+\varepsilon}| \leq 2k\Gamma_0$.

Therefore, $|w_\infty - f^+_{\omega+\varepsilon}| \le 2k\Gamma_0$ and it follows that $w_\infty \in \mathcal{F}_{\omega+\varepsilon}$; and so, since $w(s) \ge f^+_{\omega+\varepsilon}$, $w_\infty = f^+_{\omega+\varepsilon}$. Therefore, $(1+|y|^2)^{\frac{1}{\alpha}} w(s) \downarrow (1+|y|^2)^{\frac{1}{\alpha}} f^+_{\omega+\varepsilon}$ and, by Lemma 3.2, the convergence is uniform. We deduce that

$$\limsup_{s\to\infty}(1+|y|^2)^{\frac{1}{\alpha}} v(s,y) \le \limsup_{s\to\infty}(1+|y|^2)^{\frac{1}{\alpha}} w(s,y) \le (1+|y|^2)^{\frac{1}{\alpha}} f^+_{\omega+\varepsilon}(y),$$

uniformly in $y \in \mathbb{R}^N$. By letting $\varepsilon \downarrow 0$ and applying Corollary 6.3, we deduce the right-hand side estimate of (6.4). The left-hand side estimate is proved similarly. Finally, properties (iii) and (iv) follow easily from (6.4). Indeed, if $v_0 \ge f^+_\omega$ (respectively, $v_0 \le f^-_\omega$), then $v(s) \ge f^+_\omega$ (respectively, $v(s) \le f^-_\omega$).

7. Proof of Theorem 1.6

We first prove the positive part of properties (i) and (ii), i.e. that $u(t) > 0$ for t large. We consider separately the cases $\alpha < 2/N$, $\alpha = 2/N$ and $\alpha \ge 2/N$.

The case $\alpha < 2/N$. The property follows from Proposition 5.4.

The case $\alpha = 2/N$. Without loss of generality, we may assume that $|x|^N u_0(x) \to c$ as $|x| \to \infty$. Letting $\omega(x) \equiv c$, we deduce from Theorem 1.2 that (1.2) holds. Note that

$$\begin{aligned}\inf_{x\in\mathbb{R}^N}(|x|^2+t)^{\frac{1}{\alpha}} u(t,x) \ge \inf_{x\in\mathbb{R}^N}&(|x|^2+t)^{\frac{1}{\alpha}} u_\omega(t,x)\\ &- \sup_{x\in\mathbb{R}^N}(|x|^2+t)^{\frac{1}{\alpha}} |u(t,x)-u_\omega(t,x)|.\end{aligned}$$

Using (1.2), we deduce that

$$\inf_{x\in\mathbb{R}^N}(|x|^2+t)^{\frac{1}{\alpha}} u(t,x) \ge \inf_{y\in\mathbb{R}^N}(|y|^2+1)^{\frac{1}{\alpha}} f_\omega(y) - \varepsilon(t),$$

with $\varepsilon(t) \to 0$ as $t\to\infty$. Since $\inf_{y\in\mathbb{R}^N}(|y|^2+1)^{\frac{1}{\alpha}} f_\omega(y) > 0$, we see that $u(t) > 0$ for t large. This proves the part of (i) corresponding to the case $\alpha = 2/N$.

THE CASE $\alpha > 2/N$. We need only prove that if $\liminf_{|x|\to\infty} |x|^N u_0(x) > 0$, then $u(t) > 0$ for t large. Without loss of generality, we may assume that $|x|^N u_0(x) \to c > 0$ as $|x| \to \infty$, that $u_0(x)$ is radially symmetric and that there exists $\overline{r} > 0$ such that $u_0(r) < 0$ for $0 \le r < \overline{r}$ and $u_0(r) > 0$ for $r > \overline{r}$. We then apply Theorem 1.3, and we deduce from (1.4) that $u(t,0) > 0$ for t large, say for $t \ge t_0$. On the other hand, it follows from Proposition 5.5 that there exists $R > 0$ such that $u(t,x) > 0$ for $0 \le t \le t_0$ and $|x| \ge R$. Since $u_0(r)$ has only one zero on the interval $0 \le r \le R$, we deduce that $u(t_0, r)$ has at most one zero on the interval $0 \le r \le R$ (apply e.g. Lemma 2.2 of [10]). Since $u(t_0,0), u(t_0,R) > 0$, it follows that $u(t_0) > 0$ on $\{|x| \le R\}$. Thus $u(t_0) > 0$, so $u(t) > 0$ for $t \ge t_0$.

We now prove the negative part of properties (i) and (ii). We consider $u_0 \in C_0(\mathbb{R}^N)$ such that $|x|^\sigma u_0(x) \to c > 0$ as $|x| \to \infty$. It follows from Proposition 5.5 that, given any $t > 0$, $u(t,x) > 0$ for $|x|$ large. Therefore, we need only construct u_0 as above such that $u(t)$ takes negative values for all $t > 0$. We consider separately the cases $\alpha < 2/N$ and $\alpha \ge 2/N$.

THE CASE $\alpha < 2/N$. It follows from Corollary 5.3 that given $0 < c \le \overline{c}$ there exists a changing sign self-similar solution u_c of (1.1) satisfying $|x|^{2/\alpha} u_c(t,x) \to c$ as $|x| \to \infty$ for all $t > 0$. Suppose now that $\sigma > 2/\alpha$, $d > 0$ and let $u_0(x) = \min\{u_c(0,x), d|x|^{-\sigma}\}$. We have $|x|^\sigma u_0(x) \to d$ as $|x| \to \infty$. On the other hand, $u(t) \le u_c(t)$, so that $u(t)$ takes negative values for every $t > 0$.

THE CASE $\alpha \ge 2/N$. We will use the following lemma.

LEMMA 7.1. *Suppose $\alpha \ge 2/N$. Given $\sigma' > N$, there exist $\widetilde{u}_0 \in C_0(\mathbb{R}^N)$ and $c' > 0$ such that $|x|^{\sigma'} \widetilde{u}_0(x) \to c'$ as $|x| \to \infty$ and such that the corresponding solution $\widetilde{u}$ of* (1.1) *satisfies $\widetilde{u}(t,0) < 0$ for all $t \ge 0$.*

PROOF. We need only show that the solution v of (3.5) with the initial condition $v(0) = \widetilde{u}_0$ satisfies $v(s,0) < 0$ for all $s \ge 0$. Let $h \in C^\infty(\mathbb{R}^N)$,

$h \geq 0$ satisfy $h(y) = |y|^{-\sigma'}$ for $|y| > 1$. Set

$$(7.1)\quad k = \frac{N+\sigma'}{4} - \frac{1}{\alpha} > 0$$

$$(7.2)\quad \rho = \max\Big\{1, \frac{4\sigma'(\sigma' - N + 2)}{\sigma' - N}\Big\}$$

$$(7.3)\quad a = \max\Big\{2h(0), \frac{8}{\sigma' - N} e^{\frac{\rho}{4}} \Big\| \Delta h + \frac{1}{2} y \cdot \nabla h + \frac{1+k\alpha}{\alpha} h \Big\|_{L^\infty(\{|y|^2 < \rho\})}\Big\}$$

$$(7.4)\quad c' = \Big(\frac{\sigma' - N}{8}\Big)^{\frac{1}{\alpha}} (\|h\|_{L^\infty} + a)^{-1}$$

Let $w(s,y) = c' e^{-ks}(h(y) - a e^{-\frac{|y|^2}{4}})$, so that by (7.1),

$$w_s - \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{1}{\alpha} w = c' e^{-ks}\Big[-\Delta h - \frac{1}{2} y \cdot \nabla h - \frac{1+k\alpha}{\alpha} h + a \frac{\sigma' - N}{4} e^{-\frac{|y|^2}{4}}\Big].$$

Note that $|w|^\alpha \leq c'^\alpha(\|h\|_{L^\infty} + a)^\alpha \leq (\sigma' - N)/8$ by (7.4); and so, $|w|^\alpha w \geq -c' e^{-ks} a e^{-\frac{|y|^2}{4}} (\sigma' - N)/8$. It follows that

$$(7.5)\quad \begin{aligned} & w_s - \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{1}{\alpha} w + |w|^\alpha w \\ & \geq c' e^{-ks}\Big[-\Delta h - \frac{1}{2} y \cdot \nabla h - \frac{1+k\alpha}{\alpha} h + a \frac{\sigma' - N}{8} e^{-\frac{|y|^2}{4}}\Big]. \end{aligned}$$

For $|y|^2 \geq \rho$, we have

$$\begin{aligned} |y|^{\sigma'}\Big(-\Delta h - \frac{1}{2} y \cdot \nabla h - \frac{1+k\alpha}{\alpha} h\Big) &= \frac{\sigma' - N}{4} - \sigma'(\sigma' - N + 2)|y|^{-2} \\ &\geq \Big(\frac{\sigma' - N}{4} - \frac{\sigma'(\sigma' - N + 2)}{\rho}\Big) \geq 0, \end{aligned}$$

by (7.1) and (7.2). For $|y|^2 < \rho$,

$$-\Delta h - \frac{1}{2} y \cdot \nabla h - \frac{1+k\alpha}{\alpha} h + a \frac{\sigma' - N}{8} e^{-\frac{|y|^2}{4}} \geq 0,$$

by (7.3). Therefore, we deduce from (7.5) that w is a supersolution of (3.5). Note that $|y|^{\sigma'} v(0,y) \to c' > 0$ and that $w(s,0) = c' e^{-ks}(h(0) - a) < 0$

by (7.3). Consider now $\widetilde{u}_0(x) = w(0,x)$. We have $v(s,0) \le w(s,0) < 0$, which completes the proof. □

Let now $\sigma > N$ and $c > 0$. Fix $N < \sigma' < \sigma$ and let $\widetilde{u}_0$ be given by Lemma 7.1. Set $u_0(x) = \min\{\widetilde{u}_0(x), c|x|^{-\sigma}\}$. We have $|x|^\sigma u_0(x) \to c$ as $|x| \to \infty$. On the other hand, $u(t) \le \widetilde{u}(t)$, so that $u(t)$ takes negative values for every $t > 0$.

8. Appendix

We study here the asymptotic behavior as $|x| \to \infty$ and as $t \to \infty$ of solutions of the linear heat equation. We also present an existence and uniqueness result for the nonlinear heat equation (1.1) with initial data $u_0 \in L^1(\mathbb{R}^N) + L^p(\mathbb{R}^N)$, $p < \infty$.

Lemma 8.1. *Let $\omega \in C(\mathbb{R}^N \setminus \{0\})$ be homogeneous of degree 0 and $\sigma > 0$. If $u_0 \in C_0(\mathbb{R}^N)$ satisfies $|x|^\sigma u_0(x) - \omega(x) \to 0$ as $|x| \to \infty$, then*

$$|x|^\sigma e^{t\Delta} u_0(x) - \omega(x) \underset{|x|\to\infty}{\longrightarrow} 0,$$

uniformly in $t \in [0,T]$, for every $T < \infty$.

Proof. We have

$$(4\pi t)^{\frac{N}{2}} (|x|^\sigma e^{t\Delta} u_0(x) - \omega(x)) = \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} (|x|^\sigma u_0(y) - \omega(x))\, dy. \tag{8.1}$$

We first consider the case $u_0(x) = \omega(x)|x|^{-\sigma} 1_{\{|x|>R\}}$, where $R > 0$. We deduce from (8.1) that

$$(4\pi t)^{\frac{N}{2}} \big|\, |x|^\sigma e^{t\Delta} u_0(x) - \omega(x) \big| \le \int_{\{|y|>R\}} e^{-\frac{|x-y|^2}{4t}} \big|\, |x|^\sigma |y|^{-\sigma} \omega(y) - \omega(x) \big|\, dy.$$

Let $B > 0$. Given $|x| > 2\min\{B, R\}$, we break the integral I on the right-hand side of the above inequality in three parts,

$$I \le I_1 + I_2 + I_3 = \int_{\{R<|y|<|x|/2\}} + \int_{\{|y|>|x|/2\}\cap\{|x-y|>B\}}$$

$$+ \int_{\{|y|>|x|/2\}\cap\{|x-y|<B\}} .$$

Since $|x-y| \geq |x|/2$ on $\{|y| < |x|/2\}$,

$$I_1 \leq e^{-\frac{|x|^2}{16t}} \int_{\{R<|y|<|x|/2\}} \big|\, |x|^\sigma |y|^{-\sigma} \omega(y) - \omega(x) \big|\, dy \leq C|x|^{N+\sigma} e^{-\frac{|x|^2}{16t}};$$

and since $\big|\, |x|^\sigma |y|^{-\sigma} \omega(y) - \omega(x)\big| \leq K < \infty$ when $|y| > |x|/2$, we see that

$$I_2 \leq K e^{-\frac{B^2}{8t}} \int_{\mathbb{R}^N} e^{-\frac{|z|^2}{8t}}\, dz \leq C t^{\frac{N}{2}} e^{-\frac{B^2}{8t}}.$$

Finally,

$$\begin{aligned} I_3 &\leq \Big(\int_{\mathbb{R}^N} e^{-\frac{|z|^2}{4t}}\, dz \Big) \sup_{\{|x-y|<B\}} \big|\, |x|^\sigma |y|^{-\sigma} \omega(y) - \omega(x) \big| \\ &\leq C t^{\frac{N}{2}} \sup_{\{|x-y|<B\}} \big|\, |x|^\sigma |y|^{-\sigma} \omega(y) - \omega(x) \big|. \end{aligned}$$

Set $\rho(a) = \sup\limits_{|z|=1} \sup\limits_{|w|\leq a} |\omega(z+w) - \omega(z)|$, so that $\rho(a) \to 0$ as $a \downarrow 0$. Since

$$\begin{aligned} \big|\, |x|^\sigma |y|^{-\sigma} \omega(y) - \omega(x) \big| &= |(|y|/|x|)^{-\sigma} \omega(y/|x|) - \omega(x/|x|)| \\ &\leq |(|y|/|x|)^{-\sigma} - 1|\, \|\omega\|_{L^\infty} + |\omega(y/|x|) - \omega(x/|x|)|, \end{aligned}$$

we deduce that for $|x| > 2B$ and $|x-y| < B$,

$$\big|\, |x|^\sigma |y|^{-\sigma} \omega(y) - \omega(x) \big| \leq C \frac{B}{|x|} + \rho\Big(\frac{B}{|x|}\Big).$$

Therefore,

$$I_3 \leq C t^{\frac{N}{2}} \Big(\frac{B}{|x|} + \rho\Big(\frac{B}{|x|}\Big)\Big);$$

and so,

$$\big|\, |x|^\sigma e^{t\Delta} u_0(x) - \omega(x) \big| \leq C|x|^{N+\sigma} t^{-\frac{N}{2}} e^{-\frac{|x|^2}{16t}} + C e^{-\frac{B^2}{8t}} + C \frac{B}{|x|} + C\rho\Big(\frac{B}{|x|}\Big),$$

where C is independent of B. Setting $B = \sqrt{x}$, we then see that, given any $T > 0$,

$$||x|^\sigma e^{t\Delta} u_0(x) - \omega(x)| \le \varepsilon_\omega(t, |x|), \tag{8.2}$$

where

$$\varepsilon_\omega(t, |x|) \to 0,$$

as $|x| \to \infty$, uniformly in $t \in [0, T]$, for every $T < \infty$.

We now consider the general case. Given $R > 0$, we write $u_0 = u_R + v_R + w_R$, where $u_R = 1_{\{|x|<R\}} u_0$, $v_R = 1_{\{|x|\ge R\}} |x|^{-\sigma} \omega$, and $w_R = 1_{\{|x|\ge R\}}(u_0 - |x|^{-\sigma}\omega)$. By assumption, $|w_R| \le \delta(R) 1_{\{|x|\ge R\}} |x|^{-\sigma}$ with $\delta(R) \to 0$ as $R \to \infty$. Therefore,

$$\begin{aligned} ||x|^\sigma e^{t\Delta} u_0 - \omega| \le |x|^\sigma |e^{t\Delta} u_R| + ||x|^\sigma e^{t\Delta} v_R - \omega| \\ + \delta(R) ||x|^\sigma e^{t\Delta}(1_{\{|x|\ge R\}} |x|^{-\sigma}) - 1| + \delta(R). \end{aligned}$$

Since u_R has compact support, it is clear that

$$|e^{t\Delta} u_R| \le C t^{-\frac{N}{2}} e^{-\frac{|x|^2}{8t}}.$$

Using (8.2) twice, we deduce that

$$||x|^\sigma e^{t\Delta} u_0(x) - \omega(x)| \le C|x|^\sigma t^{-\frac{N}{2}} e^{-\frac{|x|^2}{8t}} + \varepsilon_\omega(t, |x|) + \varepsilon_1(t, |x|) + \delta(R),$$

and the result follows. □

LEMMA 8.2. *Let $\omega \in C(\mathbb{R}^N \setminus \{0\})$ be homogeneous of degree 0, let $0 < \sigma < N$ and set $\psi(x) = \omega(x)|x|^{-\sigma}$. It follows that*

$$e^{t\Delta} \psi(x) = t^{-\frac{\sigma}{2}} g\Big(\frac{x}{\sqrt{t}}\Big), \tag{8.3}$$

where $g \in C^\infty(\mathbb{R}^N)$ and $|x|^\sigma g(x) - \omega(x) \to 0$ as $|x| \to \infty$.

PROOF. Given $\lambda > 0$, we define the dilation operators d_λ and D_λ by $d_\lambda \varphi(x) \equiv \lambda^\sigma \varphi(\lambda x)$ and $D_\lambda u(t, x) \equiv \lambda^\sigma u(\lambda^2 t, \lambda x)$ for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$ and all $u \in C([0, \infty), \mathcal{S}(\mathbb{R}^N))$. These operators are extended by duality to $\mathcal{S}'(\mathbb{R}^N)$

and $C([0,\infty),\mathcal{S}'(\mathbb{R}^N))$, respectively. Since clearly $e^{t\Delta}(d_\lambda\varphi) = D_\lambda(e^{t\Delta}\varphi)$ for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$, the same relation holds by duality for all $\varphi \in \mathcal{S}'(\mathbb{R}^N)$. Since $\psi \in \mathcal{S}'(\mathbb{R}^N)$ satisfies $d_\lambda\psi = \psi$, it follows that $v(t) = e^{t\Delta}\psi$ satisfies $D_\lambda v = v$ for all $\lambda > 0$. Moreover, $\psi \in L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ for $1 \le p < N/\sigma < q \le \infty$, so that $v \in C^\infty((0,\infty)\times\mathbb{R}^N)$ by the smoothing effect of the heat equation. It follows that $\lambda^\sigma v(\lambda^2 t, \lambda x) = v(t,x)$ for all $\lambda > 0$, i.e. v is a self-similar solution of the heat equation. In particular, setting $g = v(1) \in C^\infty(\mathbb{R}^N)$ and letting $\lambda = t^{-\frac{1}{2}}$, we obtain the formula (8.3). In addition, one verifies easily that $v(t,x) \to \psi(x)$ as $t \to 0$ uniformly on $\{|x| > \varepsilon\}$, for every $\varepsilon > 0$. This implies that

$$t^{-\frac{\sigma}{2}} g\Big(\frac{x}{\sqrt{t}}\Big) \underset{t\downarrow 0}{\longrightarrow} \omega(x),$$

for $|x| = 1$; and so, $|x|^\sigma g(x) - \omega(x) \to 0$ as $|x| \to \infty$. □

Corollary 8.3. *Given $0 < \sigma < N$ and $A \ge 0$, there exists C such that if $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ satisfies $|u_0(x)| \le A(\tau + |x|^2)^{-\frac{\sigma}{2}}$ for some $\tau \ge 0$, then*

$$|e^{t\Delta}u_0| \le C(\tau + t + |x|^2)^{-\frac{\sigma}{2}},$$

for all $t > 0$ and all $x \in \mathbb{R}^N$.

Proof. Let $\psi(x) = |x|^{-\sigma}$ and $g = e^{\Delta}\psi$. It follows from Lemma 8.2 that there exists C such that $g(x) \le C(1 + |x|^2)^{-\frac{\sigma}{2}}$, so that by (8.3), $[e^{t\Delta}\psi](x) \le C(t + |x|^2)^{-\frac{\sigma}{2}}$. Therefore,

$$e^{t\Delta}[(1 + |x|^2)^{-\frac{\sigma}{2}}] \le C(1 + t + |x|^2)^{-\frac{\sigma}{2}}.$$

By scaling, $e^{t\Delta}[(\tau + |x|^2)^{-\frac{\sigma}{2}}] \le C(\tau + t + |x|^2)^{-\frac{\sigma}{2}}$ and the result follows. □

Corollary 8.4. *If $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ satisfies $|x|^\sigma u_0(x) \to 0$ as $|x| \to \infty$ for some $0 < \sigma < N$, then*

$$\sup_{x\in\mathbb{R}^N} (t + |x|^2)^{\frac{\sigma}{2}} |e^{t\Delta}u_0(x)| \underset{t\to\infty}{\longrightarrow} 0.$$

Proof. Given $R > 0$, let $\varphi_R = |u_0| 1_{\{|x|<R\}}$ and $\varphi^R = |u_0| 1_{\{|x|>R\}}$. It is clear that $|e^{t\Delta}u_0| \le e^{t\Delta}\varphi_R + e^{t\Delta}\varphi^R$. Furthermore, $\varphi_R \in L^1(\mathbb{R}^N)$ and

$\varphi^R \le \varepsilon(R)|x|^{-\sigma}$ with $\varepsilon(R) \to 0$ as $R \to \infty$. It follows from Corollary 8.3 that

$$e^{t\Delta}\varphi^R(x) \le \varepsilon(R)C(t+|x|^2)^{-\frac{\sigma}{2}}, \tag{8.4}$$

for all $t > 0$ and all $x \in \mathbb{R}^N$. We now estimate $e^{t\Delta}\varphi_R$. We have

$$\begin{aligned}(4\pi t)^{\frac{N}{2}} e^{t\Delta}\varphi_R(x) &= \int_{\{|y|<R\}} e^{-\frac{|x-y|^2}{4t}} |u_0(y)|\,dy \\ &\le e^{-\frac{|x|^2}{4t}} \int_{\{|y|<R\}} e^{\frac{x\cdot y}{2t}} |u_0(y)|\,dy \\ &\le e^{-\frac{|x|^2}{8t}} \int_{\{|y|<R\}} e^{\frac{|y|^2}{2t}} |u_0(y)|\,dy \\ &\le e^{-\frac{|x|^2}{8t}} e^{\frac{R^2}{2t}} \|u_0\|_{L^1(\{|x|<R\})};\end{aligned}$$

and so,

$$(t+|x|^2)^{\frac{\sigma}{2}} e^{t\Delta}\varphi_R(x) \le C(R)t^{-\frac{N-\sigma}{2}} e^{-\frac{|x|^2}{8t}} \Big(1+\frac{|x|^2}{t}\Big)^{\frac{\sigma}{2}}.$$

Since the function $s \mapsto e^{-\frac{s^2}{8}}(1+s^2)^{\frac{\sigma}{2}}$ is bounded, we deduce that

$$(t+|x|^2)^{\frac{\sigma}{2}} e^{t\Delta}\varphi_R(x) \le C(R)t^{-\frac{N-\sigma}{2}}. \tag{8.5}$$

The result follows from (8.4) and (8.5) by letting $t \to \infty$ then $R \to \infty$. □

In the case of initial values that behave like $|x|^{-N}$ as $|x| \to \infty$, there are the following results.

LEMMA 8.5. *There exists C such that if $\varphi(x) = (1+|x|^2)^{-\frac{N}{2}}$, then*

$$e^{t\Delta}\varphi(x) \le C\log(2+t)(1+t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}} + C(1+t+|x|^2)^{-\frac{N}{2}},$$

for all $t > 0$, $x \in \mathbb{R}^N$.

PROOF. Setting $u(t) = e^{t\Delta}\varphi$ and $v(s,y) = e^{\frac{Ns}{2}}u(t,x)$ with $s = \log(1+t)$ and $y = x/\sqrt{1+t}$, we need only show that $v(s,y) \le C(se^{-\frac{|y|^2}{4}} + (1+|y|^2)^{-\frac{N}{2}})$. Note that

$$v_s - \Delta v - \frac{1}{2}y\cdot\nabla v - \frac{N}{2}v = 0, \tag{8.6}$$

and, setting $f(s,y) = se^{-\frac{|y|^2}{4}}$, $f_s - \Delta f - \frac{1}{2}y\cdot\nabla f - \frac{N}{2}f = e^{-\frac{|y|^2}{4}}$. Consider now $\phi \in C^\infty(\mathbb{R}^N)$ such that $\phi(y) = |y|^{-N} + |y|^{-N-1}$ for $|y| > 1$. It follows that

$$-\Delta\phi - \frac{1}{2}y\cdot\nabla\phi - \frac{N}{2}\phi = -\frac{1}{2}|y|^{-N-1} - 2N|y|^{-N-2} - 3(N+1)|y|^{-N-3} \ge 0,$$

for $|y| > r_0$ sufficiently large. On the other hand, for $|y| < r_0$, $-\Delta\phi - \frac{1}{2}y\cdot\nabla\phi - \frac{N}{2}\phi \ge -Me^{-\frac{|y|^2}{4}}$ if M is large enough; and so, $Mf(s,y) + \phi(y)$ is a supersolution of (8.6). The result follows easily. □

LEMMA 8.6. *Let $\omega \in C(\mathbb{R}^N \setminus \{0\})$ be homogeneous of degree 0. If $u_0 \in C_0(\mathbb{R}^N)$ satisfies $|x|^N u_0(x) - \omega(x) \to 0$ as $|x| \to \infty$, then*

$$\sup_{x\in\mathbb{R}^N} \frac{(t+|x|^2)^{\frac{N}{2}}}{\log t}|e^{t\Delta}u_0(x) - \ell(\omega)G_t(x)\log t| \underset{t\to\infty}{\longrightarrow} 0,$$

where $G_t(x) = (4\pi t)^{-\frac{N}{2}}e^{-\frac{|x|^2}{4t}}$ is the heat kernel and $\ell(\omega) = \frac{1}{2}\int_{\{|\xi|=1\}}\omega(\xi)\,d\xi$.

PROOF. Set $A(t,x) = |e^{t\Delta}u_0(x) - \ell(\omega)G_t(x)\log t|$. It follows from Lemma 8.5 that, given $K > 0$,

$$\sup_{\{|x|^2>Kt\}} \frac{(t+|x|^2)^{\frac{N}{2}}}{\log t}A(t,x) \le \frac{C}{\log t} + C\sup_{\{|y|^2>K\}}(1+|y|^2)e^{-\frac{|y|^2}{4}};$$

and so,

$$\limsup_{t\to\infty}\sup_{\{|x|^2>Kt\}} \frac{(t+|x|^2)^{\frac{N}{2}}}{\log t}A(t,x) \le C\sup_{\{|y|>K\}}(1+|y|^2)e^{-\frac{|y|^2}{4}} \underset{K\to\infty}{\longrightarrow} 0.$$

Therefore, we need only consider the set $\{|x|^2 < Kt\}$; and so, it suffices to show that

$$\sup_{x\in\mathbb{R}^N} \frac{t^{\frac{N}{2}}}{\log t} A(t,x) \underset{t\to\infty}{\longrightarrow} 0. \tag{8.7}$$

We show (8.7) in two steps.

Step 1. Consider first $M > 0$ and $u_0(x) = |x|^{-N}\omega(x)1_{\{|x|>M\}}$. By scaling, we need only consider the case $M = 1$. Setting $v(t) = e^{t\Delta}u_0$, we have

$$(4\pi t)^{\frac{N}{2}} v(t,\sqrt{t}x) = \int_{\{|y|>t^{-\frac{1}{2}}\}} e^{-\frac{|x-y|^2}{4}} |y|^{-N}\omega(y)\, dy = I_1 + I_2,$$

where $I_1 = \int_{\{|y|>\delta\}}$ and $I_2 = \int_{\{\delta>|y|>t^{-\frac{1}{2}}\}}$. First,

$$I_1 \le \|\omega\|_{L^\infty}\delta^{-N}\int_{\mathbb{R}^N} e^{-\frac{|y|^2}{4}}\, dy = (4\pi)^{\frac{N}{2}}\|\omega\|_{L^\infty}\delta^{-N}.$$

Next, we observe that

$$\int_{\{\delta>|y|>t^{-\frac{1}{2}}\}} |y|^{-N}\omega(y)\, dy = \int_{t^{-\frac{1}{2}}}^{\delta} \frac{dr}{r}\Big(\int_{\{|\xi|=1\}} \omega(\xi)\, d\xi\Big) = 2\ell(\omega)\log(\delta\sqrt{t}).$$

Therefore,

$$\begin{aligned}
|I_2 - \ell(\omega)e^{-\frac{|x|^2}{4}}\log t| &\le 2e^{-\frac{|x|^2}{4}}|\ell(\omega)\log\delta| \\
&\quad + \|\omega\|_{L^\infty}\int_{\{\delta>|y|>t^{-\frac{1}{2}}\}} |e^{-\frac{|x-y|^2}{4}} - e^{-\frac{|x|^2}{4}}|\,|y|^{-N}\, dy \\
&\le 2|\ell(\omega)\log\delta| \\
&\quad + C\log(\delta\sqrt{t})\|\omega\|_{L^\infty}\sup_{\{\delta>|y|>t^{-\frac{1}{2}}\}} |e^{-\frac{|x-y|^2}{4}} - e^{-\frac{|x|^2}{4}}|.
\end{aligned}$$

It follows from the above estimates that

$$\Big|\frac{(4\pi t)^{\frac{N}{2}}}{\log t} v(t,\sqrt{t}x) - \ell(\omega)e^{-\frac{|x|^2}{4}}\Big|$$

$$\leq \frac{(4\pi)^{\frac{N}{2}}\|\omega\|_{L^\infty}\delta^{-N} + 2|\ell(\omega)\log\delta|}{\log t}$$
$$+ C\Big(\frac{1}{2} + \frac{|\log\delta|}{\log t}\Big)\|\omega\|_{L^\infty} \sup_{\{\delta>|y|>t^{-\frac{1}{2}}\}} |e^{-\frac{|x-y|^2}{4}} - e^{-\frac{|x|^2}{4}}|.$$

Given $\varepsilon > 0$, we first fix $\delta > 0$ small enough so that $C\|\omega\|_{L^\infty} \sup\limits_{x\in\mathbb{R}^N} \sup\limits_{\{\delta>|y|\}} \cdot$ $|e^{-\frac{|x-y|^2}{4}} - e^{-\frac{|x|^2}{4}}| \leq \varepsilon$. We then see that

$$\limsup_{t\to\infty} \sup_{x\in\mathbb{R}^N} \Big|\frac{(4\pi t)^{\frac{N}{2}}}{\log t} v(t, \sqrt{t}x) - \ell(\omega)e^{-\frac{|x|^2}{4}}\Big| \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, (8.7) follows.

Step 2. Conclusion. Given $M \geq 1$, let $u_0 = u_1 + u_2$ with $u_1 = u_0 1_{\{|x|<M\}}$. Setting $\varphi_{\omega,M} = |x|^{-N}\omega(x)1_{\{|x|>M\}}$, we have $|u_2 - \varphi_{\omega,M}| \leq \delta(M)\varphi_{1,1}$ with $\delta(M) \to 0$ as $M \to \infty$ and $u_1 \in L^1(\mathbb{R}^N)$. Set

$$B = \frac{t^{\frac{N}{2}}}{\log t}\|e^{t\Delta}u_0 - \ell(\omega)e^{-\frac{|x|^2}{4}}\log t\|_{L^\infty}.$$

Since $u_0 = u_1 + \varphi_{\omega,M} + (u_2 - \varphi_{\omega,M})$, we have

$$B \leq \frac{1}{\log t}\|u_1\|_{L^1} + \frac{t^{\frac{N}{2}}}{\log t}\|e^{t\Delta}\varphi_{\omega,M} - \ell(\omega)e^{-\frac{|x|^2}{4}}\log t\|_{L^\infty}$$
$$+ \delta(M)\frac{t^{\frac{N}{2}}}{\log t}\|e^{t\Delta}\varphi_{1,1}\|_{L^\infty} = B_1 + B_2 + B_3.$$

Given $\varepsilon > 0$, we first fix M large enough so that $B_3 \leq \varepsilon/3$ for t large, which is possible by Lemma 8.5. Next, it follows from Step 1 that for t large enough $B_2 \leq \varepsilon/3$. Thus $B \leq \varepsilon$ for t sufficiently large. Hence (8.7), since $\varepsilon > 0$ is arbitrary. $\square$

We finally consider initial values that behave like $|x|^{-\sigma}$ as $|x| \to \infty$, for $\sigma > N$.

Lemma 8.7. *Suppose $\sigma > N$. There exists $C > 0$ such that if $a > 0$ and $u_0(x) = (a + |x|^2)^{-\frac{\sigma}{2}}$, then*

$$(a + t + |x|^2)^{\frac{N}{2}} e^{t\Delta} u_0(x) \le C a^{-\frac{\sigma - N}{2}}, \tag{8.8}$$

for all $x \in \mathbb{R}^N$, $t \ge 0$.

Proof. By scaling, we need only consider the case $a = 1$. Since $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, we have the elementary estimate $e^{t\Delta} u_0(x) \le C(1+t)^{-\frac{N}{2}}$. This proves (8.8) for $|x|^2 \le 1 + t$. Next, we have that

$$\begin{aligned} &(1 + t + |x|^2)^{\frac{N}{2}} e^{t\Delta} u_0(x) \\ &\quad = \Big(\frac{1 + t + |x|^2}{4\pi t} \Big)^{\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} (1 + |y|^2)^{-\frac{\sigma}{2}} \, dy. \end{aligned} \tag{8.9}$$

We estimate

$$\begin{aligned} \int_{\{|y| < |x|/2\}} e^{-\frac{|x-y|^2}{4t}} (1 + |y|^2)^{-\frac{\sigma}{2}} \, dy &\le e^{-\frac{|x|^2}{16t}} \int_{\mathbb{R}^N} (1 + |y|^2)^{-\frac{\sigma}{2}} \, dy \\ &\le C t^{\frac{N}{2}} |x|^{-N}, \end{aligned} \tag{8.10}$$

and,

$$\begin{aligned} \int_{\{|y| > |x|/2\}} e^{-\frac{|x-y|^2}{4t}} (1 + |y|^2)^{-\frac{\sigma}{2}} \, dy &\le C(1 + |x|^2)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} \, dy \\ &\le C t^{\frac{N}{2}} |x|^{-N}. \end{aligned} \tag{8.11}$$

We deduce from (8.9), (8.10) and (8.11) that

$$(1 + t + |x|^2)^{\frac{N}{2}} e^{t\Delta} u_0(x) \le C(1 + t + |x|^2)^{\frac{N}{2}} |x|^{-N} \le C,$$

for $|x|^2 \ge 1 + t$. This completes the proof. □

Our last result concerns the initial value problem for the equation (1.1) for initial values $u_0 \in L^1(\mathbb{R}^N) + L^p(\mathbb{R}^N)$, $p < \infty$.

Theorem 8.8. *Suppose $u_0 \in L^1(\mathbb{R}^N) + L^p(\mathbb{R}^N)$ for some $p < \infty$. It follows that there exists a unique solution $u \in C((0, \infty), C_0(\mathbb{R}^N))$ of* (1.1) *such that* $|u(t) - u_0| \to 0$ *in* $\mathcal{S}'(\mathbb{R}^N)$ *as* $t \downarrow 0$. *Moreover,* $u = v + w$ *with* $v \in C([0,\infty), L^1(\mathbb{R}^N))$ *and* $w \in C([0,\infty), L^p(\mathbb{R}^N))$.

PROOF. Given $n \in \mathbb{N}$, let $u_{0,n} = \min\{n, \max\{u_0(x), -n\}\}$. It follows that $u_{0,n} \in L^p(\mathbb{R}^N)$, so that there exists a unique solution $u_n \in C([0,\infty), L^p(\mathbb{R}^N)) \cap C((0,\infty), C_0(\mathbb{R}^N))$ of (1.1) with the initial condition $u_n(0) = u_{0,n}$. We deduce from Lemma 3.1 that, given $n, \ell \in \mathbb{N}$,

$$|u_n(t) - u_\ell(t)| \le e^{t\Delta}|u_{0,n} - u_{0,\ell}|. \tag{8.12}$$

Now, we observe that (even though possibly $u_0 \notin L^1(\mathbb{R}^N)$) $u_{0,n} - u_{0,\ell} \in L^1(\mathbb{R}^N)$ and $\|u_{0,n} - u_{0,\ell}\|_{L^1} \to 0$ as $n, \ell \to \infty$. Indeed, if $u_0 = \varphi + \psi$ with $\varphi \in L^1(\mathbb{R}^N)$, $\psi \in L^p(\mathbb{R}^N)$ and $\varphi\psi = 0$ then, assuming $n < \ell$,

$$\int_{\mathbb{R}^N} |u_{0,n} - u_{0,\ell}| \le \int_{\{|u_0|>n\}} |u_0| \le \int_{\{|\varphi|>n\}} |\varphi| + \int_{\{|\psi|>n\}} |\psi| \underset{n\to\infty}{\longrightarrow} 0.$$

Applying (8.12), we deduce in particular that $u_n(t) - u_\ell(t) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ for all $t > 0$ and that

$$\|u_n(t) - u_\ell(t)\|_{L^1} + t^{\frac{N}{2}}\|u_n(t) - u_\ell(t)\|_{L^\infty} \le \|u_{0,n} - u_{0,\ell}\|_{L^1} \to 0, \tag{8.13}$$

as $n, \ell \to \infty$. It follows from the second inequality in (8.13) that there exists a function $u \in C((0,\infty), C_0(\mathbb{R}^N))$ such that $u_n \to u$ in $L^\infty((\varepsilon,\infty) \times \mathbb{R}^N)$ for every $\varepsilon > 0$. One sees easily that u satisfies the equation (1.1). Next, using the first inequality in (8.13), we deduce that $u_n - u_1$ is a Cauchy sequence in $C([0,\infty), L^1(\mathbb{R}^N))$. Therefore, $u - u_1 \in C([0,\infty), L^1(\mathbb{R}^N))$. The existence part follows, with $v = u - u_1$ and $w = u_1$.

We now prove uniqueness. Let u and v be two solutions as stated. It follows from Lemma 3.1 that

$$|u(t) - v(t)| \le e^{(t-s)\Delta}|u(s) - v(s)| \le e^{(t-s)\Delta}|u(s) - u_0| + e^{(t-s)\Delta}|v(s) - u_0|,$$

for all $t > s > 0$. Letting $s \downarrow 0$, we deduce that $u(t) = v(t)$. □

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Thierry Cazenave
Analyse Numérique, UMR CNRS 7598
B.C. 187, Université Pierre et Marie Curie
4, place Jussieu
75252 Paris Cedex 05, France
E-mail: `cazenave@ccr.jussieu.fr`

Flávio Dickstein
Instituto de Matemática
Universidade Federal do Rio de Janeiro
Caixa Postal 68530
21944–970, Rio de Janeiro, R.J., Brazil
E-mail: `flavio@labma.ufrj.br`

Miguel Escobedo
Departamento de Matemáticas
Universidad del País Vasco
Apartado 644
48080 Bilbao, Spain
E-mail: mtpesmam@lg.ehu.es

Fred B. Weissler
LAGA, UMR CNRS 7539
Institut Galilée–Université Paris XIII
99, Avenue J.-B. Clément
93430 Villetaneuse, France
E-mail: weissler@math.univ-paris13.fr