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Diffeomorphism Types of Good Torus Fibrations with Twin Singular Fibers

By Masatomo Toda

Abstract. The diffeomorphism type of a good torus fibration with twin singular fibers whose 1-st Betti number is odd is determined by some topological invariants.

1. Introduction

In this paper, we will study some remaining problems about the diffeomorphism types of good torus fibrations (GTF) with twin singular fibers.

The types of singular fibers of good torus fibrations were classified. Among the singular fibers which are not multiple, the simplest one seems to be I_1^+ or I_1^-. A singular fiber of type I_1^+ (resp. I_1^-) consists of an immersed 2-sphere which intersects itself transversely at one point with intersection number $+1$ (resp. -1). In this paper, all the diffeomorphisms will be assumed to be orientation-preserving. The following theorem is known.

Theorem 1.1 (Matsumoto. [5], [4]). *Let $f_i : M_i \to B_i$ $(i = 1, 2)$ be GTF's over a closed surface with at least one singular fiber. Suppose that each singular fiber is of type I_1^+ or I_1^- and that $\sigma(M_1) \neq 0$. Then M_1 is diffeomorphic to M_2 if and only if $g(B_1) = g(B_2)$, $e(M_1) = e(M_2)$ and $\sigma(M_1) = \sigma(M_2)$. The symbols g, e and σ represent the genus, the Euler number and the signature, respectively.*

Remark 1. Let $f : M \to B$ be a GTF satisfying the condition of Theorem 1.1. Let k_+, k_- be the numbers of the singular fibers of type I_1^+, I_1^- of M, respectively. Then $\sigma(M) = -(2/3)(k_+ - k_-)$ and $e(M) = k_+ + k_-$ ([6], [7]). Therefore, the diffeomorphism type of the total space is determined by k_+, k_- and $g(B)$ if $k_+ \neq k_-$.

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What happens if $\sigma(M) = 0$, that is, $k_+ = k_-$? In this case, if the base space is 2-sphere, it is known that we can deform the projection map slightly so that all the singular fibers are simple twin singular fibers ([4]). The twin singular fiber consists of two smoothly embedded 2-spheres intersecting each other at two points whose signs of intersections are $+1$ and -1.

The following Theorem is known.

THEOREM 1.2 (Iwase. [1]). *Let $f : M \to S^2$ be a GTF over a sphere. Suppose that each singular fiber is non-multiple twin singular fiber. Then the diffeomorphism type of M is determined by (i) $e(M)$, (ii) $\pi_1(M)$, (iii) the information about whether M is spin or not, (iv) the information about whether M is of typeI or typeII. Here $\pi_1(M)$ is the fundamental group of M.*

However, the case when $\sigma(M) = 0$ and the base space is an arbitrary oriented surface has not been solved. In this paper, we consider this case. Unfortunately the author could not get a complete solution in this paper, but we could have the following Theorem. Hereafter, all base spaces which are treated in this paper are closed, and $H_1(M) = H_1(M; \boldsymbol{Z})$. We call a twin singular fiber even if $m + n \equiv 0 \mod 2$, where $mR + nS$ is the divisor of this singular fiber. (See §2.)

THEOREM 1.3. *Let $f : M \to B$ be a GTF with at least one even twin singular fiber. Suppose that each singular fiber is non-multiple twin singular fiber, and that $\operatorname{rank} H_1(M)$ is odd. Then the diffeomorphism type of M is determined by (i) $e(M)$, (ii) $\operatorname{rank} H_1(M)$ or $g(B)$, (iii) the information about whether M is of typeI or typeII.*

REMARK 2. If $\operatorname{rank} H_1(M)$ is odd, then $\operatorname{rank} H_1(M) = 2g(B) + 1$. In the case when each singular fiber is of type I_1^+, I_1^- or Tw, it is known that if $\sigma(M) \neq 0$, then $\pi_1(M)$ is isomorphic to $\pi_1(B)$ ([5]). Therefore, if $\operatorname{rank} H_1(M)$ is odd, then $\sigma(M) = 0$.

If each type of singular fibers is of type I_1^+ or I_1^-, then Theorem 1.3 can be rewritten as follows.

THEOREM 1.4. *Let $f : M \to B$ be a GTF. Suppose that each singular fiber is of type I_1^+ or I_1^-, and that $e(M) \neq 0$, ($\sigma(M) = 0$) and* $\operatorname{rank} H_1(M)$

is odd. Then the diffeomorphism type of M *is determined by (i)* $e(M)$*, (ii)* $\operatorname{rank} H_1(M)$ *or* $g(B)$*.*

As a corollary, we have a similar result about the case when each singular fiber is of type I_1^+, I_1^- or twin.

What happens if $\operatorname{rank} H_1(M)$ is even? We consider the case when $e(M) = 2$ and $g(B) = 1$.

THEOREM 1.5. *Let* $f : M \to B$ *be a GTF. Suppose that each singular fiber is non-multiple even twin singular fiber, and that* $e(M) = 2$ *and* $g(B) = 1$*. Then if* $\pi_1(M)$ *is isomorphic to the following one, then the diffeomorphism type of* M *is determined.*

Class 1. $< \alpha, \beta, \tau \mid \tau^c = 1, [\tau, \alpha] = [\tau, \beta] = [\alpha, \beta] = 1 >$ $(c \geq 0)$.

Class 1'. $< \alpha, \beta, \tau \mid \tau^c = 1, [\tau, \alpha] = [\tau, \beta] = 1, [\alpha, \beta] = \tau^{-n} >$ $(c = 2n, 3n, 4n, 6n, n \geq 1)$.

Class 2. $< \alpha, \beta, \tau \mid \tau^c = 1, \alpha^{-1}\tau\alpha = \tau^d, [\tau, \beta] = [\alpha, \beta] = 1 >$ $(1 < d, d+1 < c, c|d^2 - 1)$.

Class 2'. $< \alpha, \beta, \tau \mid \tau^c = 1, \alpha^{-1}\tau\alpha = \tau^d, [\tau, \beta] = 1, [\alpha, \beta] = \tau^{-n} >$ $(1 < d, d+1 < c, c|d^2 - 1, \gcd(c, d-1) = 2n, 3n, 4n, 6n, n \geq 1)$.

Class 3. $< \alpha, \beta, \tau \mid [\tau, \alpha] = [\tau, \beta] = 1, [\alpha, \beta] = \tau^{-n} >$ $(n \geq 1)$.

Class 4. $< \alpha, \beta, \tau \mid [\tau, \alpha] = 1, \beta^{-1}\tau\beta = \tau^{-1}, [\alpha, \beta] = \tau^{-n} >$ $(n = 0, 1)$.

As an extension of Theorem 1.5, we have the following Theorem.

THEOREM 1.6. *The same result of Theorem 1.5 holds if we change the clause "Suppose that each singular fiber is non-multiple even twin singular fiber, and that* $e(M) = 2$ *and* $g(B) = 1$*" in Theorem 1.5, to "Suppose that each singular fiber is neither multiple nor odd twin singular fiber, and that* $e(M) = 2$*,* $g(B) = 1$ *and* $\sigma(M) = 0$*."*

2. Definitions

In this section, we give the definitions of good torus fibrations, twin singular fibers and singular fibers of type I_1^+ and I_1^-.

First we give a precise definition of good torus fibrations (GTF) ([4]). A *proper* map $f : M \to B$ between manifolds is a map such that the preimage of each compact subset of B is compact and $f^{-1}(\partial B) = \partial M$.

DEFINITION 2.1 ([3], [4]). Let M and B be oriented 4 and 2-dimensional smooth manifolds, respectively. Let $f : M \to B$ be a proper, surjective and smooth map. We call $f : M \to B$ a *good torus fibration* (GTF) if it satisfies the following conditions:

(i) at each point $p \in IntM$ (resp. $f(p) \in IntB$), there exist local complex coordinates z_1, z_2 with $z_1(p) = z_2(p) = 0$ (resp. local complex coordinate ξ with $\xi(f(p)) = 0$), so that f is locally written as $\xi = f(z_1, z_2) = z_1^m z_2^n$ or $(\bar{z}_1)^m z_2^n$, where m, n are non-negative integers with $m+n \geq 1$, and $\bar{z}_1$ is the complex conjugate of z_1;

(ii) there exists a set Γ of isolated points of $IntB$ so that

$$f|f^{-1}(B-\Gamma) : f^{-1}(B-\Gamma) \to B-\Gamma$$

is a smooth T^2-bundle over $B-\Gamma$.

We call f, M and B the *projection*, the *total space* and the *base space*, respectively. Given a good torus fibration $f : M \to B$, those points p of $IntM$ at which $m+n \geq 2$ make a nowhere dense subset Σ. We may assume that $f(\Sigma) = \Gamma$. We call Γ the *set of singular values*. The fiber $F_x = f^{-1}(x)$ is a *general* or *singular* fiber according as $x \in B-\Gamma$ or $x \in \Gamma$.

A singular fiber has a finite number of normal crossings. The complement $F_x - \{\text{normal crossings}\}$ is divided into a finite number of connected components. The closure of each component is called an *irreducible component* of F_x. Irreducible components are smoothly immersed surfaces, and F_x is the union of them: $F_x = \Theta_1 \cup \cdots \cup \Theta_s$. Each irreducible component is naturally oriented. Thus it represents a homology class $[\Theta_i]$ in $H_2(f^{-1}(D_x); \boldsymbol{Z})$, where D_x ($\subset IntB$) denotes a small 2-disk centered at x such that $D_x \cap \Gamma = x$. $H_2(f^{-1}(D_x); \boldsymbol{Z})$ is a free abelian group with basis $[\Theta_1], \cdots, [\Theta_s]$, with which the homology class $[F_y]$ of a nearby general fiber F_y ($y \in D_x - x$) is written as $[F_y] = m_1[\Theta_1] + \cdots + m_s[\Theta_s]$, $m_i \geq 1$. The formal sum $\Sigma m_i \Theta_i$ is called the *divisor* of the singular fiber F_x. F_x is said to be *simple* or *multiple* according as $gcd(m_1, \cdots, m_s) = 1$ or > 1.

Let F_0 be a general fiber over a base point $x_0 \in B - \Gamma$. Let $l : [0,1] \to B - \Gamma$ be a loop based at x_0. As is easily seen, there exists a map $h : F_0 \times [0,1] \to M - f^{-1}(\Gamma)$ such that (i)$f(h(p,t)) = l(t)$ for all $(p,t) \in F_0 \times [0,1]$; (ii)the map $h_t : F_0 \to F_t$ defined by $h_t(p) = h(p,t)$ is a homeomorphism, where $F_t = f^{-1}(l(t))$; (iii)h_0 =identity of F_0. The isotopy class of h_1 :

$F_0 \to F_1 = F_0$ is determined by x_0 together with the homotopy class $[l]$. h_1 induces an automorphism $(h_1)_* : H_1(F_0; \boldsymbol{Z}) \to H_1(F_0; \boldsymbol{Z})$. Fix an ordered basis $< \mu, \lambda >$ of $H_1(F_0; \boldsymbol{Z})$ so that it is compatible with the orientation of F_0. Then $(h_1)_*$ is represented by a matrix A called the *monodromy matrix*. This gives a map $\rho : \pi_1(B - \Gamma, x_0) \to SL(2, \boldsymbol{Z})$. Recalling that the product $l \cdot l'$ of loops is the loop which goes first round l and then l', we easily see that to make ρ an anti-homomorphism we must adopt the following rule assigning $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to $(h_1)_*$: $(h_1)_*(\mu) = a\mu + c\lambda$, $(h_1)_*(\lambda) = b\mu + d\lambda$. This rule is written as $(\ (h_1)_*(\mu),\ (h_1)_*(\lambda)\ ) = (\ \mu,\ \lambda\ )A$. This convention coincides with the one in [3] but is different from the one in [4]. A different basis $(\ \mu',\ \lambda'\ )$ gives a different anti-homomorphism $\rho' : \pi_1(B - \Gamma, x_0) \to SL(2, \boldsymbol{Z})$. ρ' is related to ρ by $\rho' = C^{-1} \cdot \rho \cdot C$, C being a matrix in $SL(2, \boldsymbol{Z})$. The conjugacy class of matrix $\rho([l])$ is called the *monodromy* associated with $[l]$.

Let x be a point of Γ, D_x a small disk in $IntB$ such that $D_x \cap \Gamma = x$. Let x' be a point on ∂D_x. Then ∂D_x is considered as a loop based at x'. (The direction of ∂D_x is determined by the orientation of D_x.) The monodromy associated with the loop ∂D_x is called the *local monodromy* of the singular fiber F_x.

To this paper only three types of singular fibers are relevant. They are I_1^+, I_1^- and Tw.

Definition 2.2. A singular fiber is of *type* I_1^+ (resp. *type* I_1^-) if it is a simple singular fiber consisting of a smooth immersed 2-sphere (in the total space) which intersects itself transversely at one point, where the sign of intersection is $+1$ (resp. -1). The local monodromy of a singular fiber of type I_1^+ (resp. I_1^-) is represented by $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $(resp. \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix})$.

Definition 2.3. A singular fiber is of type Tw if it consists of two smoothly embedded 2-spheres R, S intersecting each other transversely at two points p_+, p_-. The sign of intersection at p_+ (resp. p_-) is $+1$ (resp. -1).

The divisor is $mR+nS$. We call this singular fiber $(m,n)-$*twin singular fiber*. When $m + n \equiv 0 \mod 2$ (resp. $m + n \equiv 1 \mod 2$), this is said to be *even* (resp. *odd*).

If F_x is a twin singular fiber, the intersection numbers $R \cdot R$, $R \cdot S$, $S \cdot S$ are zero. Therefore, the neighborhood $f^{-1}(D_x)$ is obtained by plumbing $D^2 \times S^2$ and $S^2 \times D^2$. This plumbing manifold is called a *twin*. We denote it by the symbol Tw. The boundary $\partial(Tw) = \partial(f^{-1}(D_x))$ is diffeomorphic to $T^3 = S^1 \times S^1 \times S^1$, and the local monodromy is trivial $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

3. Properties of a Twin

First we recall some properties of a twin. A twin is a manifold which consists of two $S^2 \times D^2$'s plumbed at two points with opposite signs. Let R, S be the core of two $S^2 \times D^2$'s. They generate $H_2(Tw; \boldsymbol{Z}) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}$. Let $D(r), D(s)$ be 2-disks properly embedded in Tw such that $R \cdot D(r) = S \cdot D(s) = 1$ and $R \cdot D(s) = S \cdot D(r) = 0$. $\partial D(r)$ and $\partial D(s)$ are circles in $\partial(Tw) = T^3$. We call them r and s, respectively. Choose a circle l in $\partial(Tw)$ such that $< l,\ r,\ s >$ is an oriented basis of $H_1(\partial(Tw); \boldsymbol{Z})$. The ambiguity of the choice of l is not essential, because if l_1 and l_2 are two choices of l, then there exists a diffeomorphism $\bar{h} : Tw \to Tw$ such that ($\bar{h}_*(l_1)$, $\bar{h}_*(r)$, $\bar{h}_*(s)$) = (l_2, r, s).

PROPOSITION 3.1 ([9]). *For any diffeomorphism $h : \partial(Tw) \to \partial(Tw)$, define $A^h \in GL(3, \boldsymbol{Z})$ by ($h_*(l)$, $h_*(r)$, $h_*(s)$)=(l, r, s)A^h in $H_1(\partial(Tw); \boldsymbol{Z})$. Then h can be extended to a diffeomorphism $\bar{h} : Tw \to Tw$ if and only if $A^h \in H_1$, where*

$$H_1 = \{ \begin{bmatrix} \pm 1 & 0 & 0 \\ * & a & b \\ * & c & d \end{bmatrix} \in GL(3, \boldsymbol{Z}) | a + b + c + d \equiv 0 \mod 2 \}.$$

Let $\bar{l}$, $\bar{r}$, $\bar{s}$ be the circles $\partial D^2 \times \{*\} \times \{*\}$, $\{*\} \times S^1 \times \{*\}$, $\{*\} \times \{*\} \times S^1$ in $\partial(D^2 \times T^2)$, respectively.

PROPOSITION 3.2 ([1]). *For any diffeomorphism $h : \partial(D^2 \times T^2) \to \partial(D^2 \times T^2)$, define $A^h \in GL(3, \boldsymbol{Z})$ by ($h_*(\bar{l})$, $h_*(\bar{r})$, $h_*(\bar{s})$) = ($\bar{l}$, $\bar{r}$, $\bar{s}$)A^h in $H_1(\partial(D^2 \times T^2); \boldsymbol{Z})$. Then h can be extended to a diffeomorphism $\bar{h} : D^2 \times T^2 \to D^2 \times T^2$ if and only if $A^h \in H_2$, where*

$$H_2 = \{ \begin{bmatrix} \pm 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \in GL(3, \boldsymbol{Z}) \}.$$

For any integers $m \geq 3$, let S_m be a compact, connected, planar surface whose boundary has m components $S^1_1, \cdots, S^1_m$. The boundary of the manifold $S_m \times T^2 = S_m \times S^1 \times S^1$ has m copies of T^3. Let $\bar{l}_i$, $\bar{r}_i$, $\bar{s}_i$ be the circles $S^1_i \times \{*\} \times \{*\}$, $\{*_i\} \times S^1 \times \{*\}$, $\{*_i\} \times \{*\} \times S^1$ $(*_i \in S^1_i)$ in $\partial(S_m \times T^2)$, respectively.

Proposition 3.3 ([1]). *For any diffeomorphism $h : \partial(S_m \times T^2) \to \partial(S_m \times T^2)$ which satisfies $h(S^1_i \times S^1 \times S^1) = S^1_i \times S^1 \times S^1$ $(i = 1, \cdots, m)$, define $A^h = A^h_1 \oplus \cdots \oplus A^h_m \in \oplus^m_{i=1} GL(3, \boldsymbol{Z})$ by $(\, h_*(\bar{l}_i), h_*(\bar{r}_i), h_*(\bar{s}_i)\,) = (\, \bar{l}_i, \bar{r}_i, \bar{s}_i) A^h_i$ in $H_1(\partial(S_m \times T^2); \boldsymbol{Z})$. Then h can be extended to a diffeomorphism $\bar{h} : S_m \times T^2 \to S_m \times T^2$ if and only if $A^h \in H_3$, where*

$$H_3 = \{\oplus^m_{i=1} \begin{bmatrix} \epsilon & 0 & 0 \\ p_i & a & b \\ q_i & c & d \end{bmatrix} \in \oplus^m_{i=1} GL(3, \boldsymbol{Z}) | \sum_{i=1}^m p_i = 0, \sum_{i=1}^m q_i = 0\}.$$

Remark 3 ([1]). Let $f_i : T^3 \to T^3$ $(i = 1, 2)$ be a diffeomorphism. Then, by Waldhausen's theorem, f_1 is isotopic to f_2 if and only if $(f_1)_* = (f_2)_*$, where $(f_i)_* : H_1(T^3; \boldsymbol{Z}) \to H_1(T^3; \boldsymbol{Z})$.

Proposition 3.4 ([1]). *For $i = 1, 2$ let $f_i : \partial(Tw) \to \partial D^2$ be a map which is decomposed as $f_i = p \circ \phi_i$, where $p : \partial D^2 \times S^1 \times S^1 \to \partial D^2$ is the projection map to the first factor and $\phi_i : \partial(Tw) \to \partial D^2 \times S^1 \times S^1$ is a diffeomorphism. Assume that $(f_i)* : H_1(\partial(Tw); \boldsymbol{Z}) \to H_1(\partial D^2; \boldsymbol{Z})$ maps l, r, s to $a\gamma$, $b\gamma$, $c\gamma$, respectively, where a, b, c are integers and γ is a generator of $H_1(\partial D^2; \boldsymbol{Z})$. Let f be either f_1 or f_2.*

(i) There exists a diffeomorphism $\phi : \partial(Tw) \to \partial(Tw)$ such that ϕ is isotopic to identity and that $f_1 = f_2 \circ \phi$.

(ii) The map f can be extended to a projection map of a GTF $\bar{f} : Tw \to D^2$ which has only one twin singular fiber as a singular fiber if and only if $b \neq 0$ or $c \neq 0$.

(iii) The twin singular fiber of $\bar{f}$ is simple if and only if $gcd(b, c) = 1$.

Corollary 3.1 ([9], [3], [1]). *Let f be as in Proposition 3.4 and assume that $gcd(b, c) = 1$. Let m, n be non-negative integers such that*

$gcd(m,n)=1$. Then there exists an $\bar{f}$, an extension of f such that $\bar{f}$ is fiber preserving, orientation preserving diffeomorphic to a (m,n)-twin singular fiber if and only if $b+c \equiv m+n \mod 2$.

It is well-known that an even twin singular fiber is not fiber preserving, orientation preserving diffeomorphic to an odd twin singular fiber.

PROPOSITION 3.5 ([1]). *(i) Let λ be a simple loop in $D^2 \times T^2$ such that $[\lambda] \in \pi_1(D^2 \times T^2)$ is a primitive element. Then $\chi(D^2 \times T^2; \lambda)$ is diffeomorphic to Tw for any framing of λ.*
(ii) $\chi(Tw; S)$ is diffeomorphic to $D^2 \times T^2$.

Let M be a manifold and Σ be a sphere embedded in $Int(M)$ which has trivial normal bundle. Then $\chi(M;\Sigma)$ means the manifold obtained from M by the Milnor surgery on Σ.

PROPOSITION 3.6 ([1]). *Let C_1, C_2 be two curves in $\partial(Tw)$ such that there exists a diffeomorphism $h : \partial(Tw) \to S^1 \times S^1 \times S^1$ which maps C_1, C_2 to the curves $\{*\} \times \{*\} \times S^1$ and $\{*'\} \times \{*\} \times S^1$ $(* \neq *')$. Assume that $[C_1] = [C_2] = 2al + br + cs \in H_1(\partial(Tw); \boldsymbol{Z})$ for some integers a, b, c. Let D_i be a 2-chain in Tw such that*
(i) $\partial D_i(mod\ 2) = C_i$ $(i = 1, 2)$,
(ii) $[D_1] = [D_2]$ in $H_2(Tw, \partial(Tw); \boldsymbol{Z})$,
(iii) D_1 and D_2 meet transversely.
Then $D_1 \cdot D_2 \equiv bc \mod 2$, where $D_1 \cdot D_2$ is the intersection number.

4. Main Results

First we determine the symbol

$$M(A_1, B_1, \cdots, A_g, B_g; C_1, \cdots, C_k; D_1, D_2)$$

for any A_j, $B_j \in SL(2,\boldsymbol{Z})$ s.t. $A_1^{-1}B_1^{-1}A_1B_1 \cdots A_g^{-1}B_g^{-1}A_gB_g = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and C_i, D_1, $D_2 \in SL(3,\boldsymbol{Z})$. Let B be an oriented closed surface whose genus is g. Cutting B along a set of loops based at $v_1' \in B$ gives a $4g$-gon P with sides $a_1^{-1}b_1^{-1}a_1b_1 \cdots a_g^{-1}b_g^{-1}a_gb_g$ to be identified in pairs. Removing an open disk D_0 around v_1' gives a polygon $\bar{P}$ which is P with a sector of a disk removed from each vertex. $\bar{P} \times T^2$ is a 4-manifold. We represent each straight edge of $\bar{P}$ which is obtained from a_j (resp. b_j) by the same symbol a_j (resp. b_j). Let x_{a_j} (resp. x_{b_j}) be a point on the straight edge a_j (resp. b_j) and let x'_{a_j} (resp. x'_{b_j}) be a point on the straight edge a_j^{-1} (resp. b_j^{-1}) which will be identified with x_{a_j} (resp. x_{b_j}) in B and let α'_j (resp. β'_j) be an element of $\pi_1(B, v'_0)$ which is represented by the path formed by the straight lines v'_0 to $x'_{b_{g-j}}$ (resp. $x_{a_{g-j}}$) and $x_{b_{g-j}}$ (resp. $x'_{a_{g-j}}$) to v'_0 which is the loop in B based at v'_0, where v'_0 is the center of P. Let $\rho : \pi_1(B, v_0) \to SL(2, \boldsymbol{Z})$ (Automorphism of $\pi_1(T^2)$) be an anti-homomorphism which satisfies $\rho(\alpha'_j) = A_j$ and $\rho(\beta'_j) = B_j$. Identifying pairs of fibers over the straight edges of $\bar{P}$ by the homeomorphisms which is induced by this anti-homomorphism gives a 4-manifold M_0 with a boundary $\partial D_0 \times T^2 = T^3$.

Let l_0, r_0, s_0 be the circles $\partial D_0 \times \{*\} \times \{*\}$, $\{*\} \times S^1 \times \{*\}$, $\{*\} \times \{*\} \times S^1$, respectively. We denote $(S_{k+2} \times T^2) \cup_{\phi_{C_i}} \cup_{i=1}^{k} Tw_i \cup_{\phi_{D_1}} (D^2 \times T^2) \cup_{\phi_{D_2}} M_0$ by

$M(A_1, B_1, \cdots, A_g, B_g; C_1, \cdots, C_k; D_1; D_2)$, where ϕ_{C_i}, ϕ_{D_1} and ϕ_{D_2} are diffeomorphisms such that

$$\begin{aligned}
&\phi_{C_i} : \partial(Tw_i) \to S_i^1 \times T^2 \ (i = 1, \cdots, k),\\
&\phi_{D_1} : \partial(D^2 \times T^2) \to S_{k+1}^1 \times T^2,\\
&\phi_{D_2} : \partial M_0 \to S_{k+2}^1 \times T^2,
\end{aligned}$$

which are defined by the elements of $SL(3, \boldsymbol{Z})$, respectively as follows, by Remark 3.

Choose a basis (l_i, r_i, s_i) of $H_1(\partial(Tw_i); \boldsymbol{Z})$ as in §3. The diffeomorphism ϕ_{C_i} is determined by $C_i \in SL(3, \boldsymbol{Z})$ so that ($\phi_{C_i *}(l_i)$, $\phi_{C_i *}(r_i)$, $\phi_{C_i *}(s_i)$) $=$ ($\bar{l}_i$, $\bar{r}_i$, $\bar{s}_i$)C_i. Likewise choose a canonical basis ($\bar{l}$, $\bar{r}$, $\bar{s}$) of $H_1(\partial(D^2 \times T^2); \boldsymbol{Z})$ as in §3. The diffeomorphism ϕ_{D_1} is determined by $D_1 \in SL(3, \boldsymbol{Z})$ so that ($\phi_{D_1 *}(\bar{l})$, $\phi_{D_1 *}(\bar{r})$, $\phi_{D_1 *}(\bar{s})$) $=$ ($\bar{l}_{k+1}$, $\bar{r}_{k+1}$, $\bar{s}_{k+1}$)D_1. The diffeomorphism $\phi_{D_2 *}$ is determined by ($\phi_{D_2 *}(l_0)$, $\phi_{D_2 *}(r_0)$, $\phi_{D_2 *}(s_0)$) $=$ ($\bar{l}_{k+2}$, $\bar{r}_{k+2}$, $\bar{s}_{k+2}$)D_2. Here $D_2 = \begin{bmatrix} 1 & 0 & 0 \\ m & 1 & 0 \\ n & 0 & 1 \end{bmatrix}$.

At present, we can not see whether this differential manifold

$$M(A_1, B_1, \cdots, A_g, B_g; C_1, \cdots, C_k; D_1; D_2)$$

has GTF structures or not, but we will see later any manifold which we deal with in this paper has at least one GTF structure.

Let $f : M \to B$ be a good torus fibration which has k non-multiple twin singular fibers as singular fibers, where B is an oriented closed surface whose genus is g. Evidently there exists A_1, B_1,$\cdots$,A_g, $B_g \in SL(2, \boldsymbol{Z})$ and $C_1, \cdots, C_k$, D_1, $D_2 \in SL(3, \boldsymbol{Z})$ so that

$$M(A_1, B_1, \cdots, A_g, B_g; C_1, \cdots, C_k; D_1; D_2)$$

is diffeomorphic to M. Here

$$D_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } D_2 = \begin{bmatrix} 1 & 0 & 0 \\ m & 1 & 0 \\ n & 0 & 1 \end{bmatrix}.$$

LEMMA 4.1.

$$M = M(A_1, B_1, \cdots, A_g, B_g; C_1, \cdots, C_k; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \begin{bmatrix} 1 & 0 & 0 \\ m & 1 & 0 \\ n & 0 & 1 \end{bmatrix})$$

is diffeomorphic to

$$\begin{aligned} M' = M(&A_1, B_1, \cdots, A_g, B_g; \\ &\begin{bmatrix} 0 & 1 & \epsilon_1 \\ * & 0 & * \\ * & 0 & * \end{bmatrix}, \cdots, \begin{bmatrix} 0 & 1 & \epsilon_k \\ * & 0 & * \\ * & 0 & * \end{bmatrix}; \begin{bmatrix} 1 & 0 & 0 \\ m' & 1 & 0 \\ n' & 0 & 1 \end{bmatrix}; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}), \end{aligned}$$

where ϵ_i $(i = 1, \cdots, k)$ *is* 0 *or* 1. m', n' *are integers which are determined by* $C_1, \cdots, C_k$, *and* m, n.

PROOF. Define $C_i = \begin{bmatrix} a_i & b_i & c_i \\ * & * & * \\ * & * & * \end{bmatrix}$ $(i = 1, \cdots, k)$, where $C_i \in SL(3, \boldsymbol{Z})$ are determined by $(\ \phi_{C_i*}(l_i),\ \phi_{C_i*}(r_i),\ \phi_{C_i*}(s_i)\ ) = (\ \bar{l}_i,\ \bar{r}_i,\ \bar{s}_i\ )C_i$. We consider $\bar{f}_i : Tw_i \to D^2$ to be the projection map which is obtained by restricting the projection map of the GTF $f : M \to B$. We consider

$f_i : \partial(Tw_i) \to \partial D^2$ to be the restriction of $\bar{f}_i : Tw_i \to D_i$. $\bar{l}_i$ can be considered as a generator of $H_1(\partial D^2; \boldsymbol{Z})$, and $(\phi_{C_i *}(l_i), \phi_{C_i *}(r_i), \phi_{C_i *}(s_i)) = (\bar{l}_i, \bar{r}_i, \bar{s}_i)C_i$, so $(f_i)_* : H_1(\partial(Tw); \boldsymbol{Z}) \to H_1(\partial D^2; \boldsymbol{Z})$ maps r_i, s_i to $b_i\bar{l}_i$, $c_i\bar{l}_i$, respectively. Recall that all twin singular fibers are assumed not to be multiple. Thus by Proposition 3.4 (iii), $gcd(b_i, c_i) = 1$, and there exists integers x_i, y_i such that $x_i b_i + y_i c_i = 1$ because of the Euclidean algorithm. By Proposition 3.1, the diffeomorphism type of the total space does not change if we change C_i to C_iA_i, where $A_i \in H_1$ ([1]). Therefore, we can change C_i to $C'_i = \begin{bmatrix} 0 & 1 & \epsilon_i \\ * & d_i & * \\ * & e_i & * \end{bmatrix}$, where ϵ_i $(i = 1, \cdots, k)$ is 0 or 1. For example, if $x_i - c_i + y_i + b_i \equiv 0 \mod 2$, we define $A_i = \begin{bmatrix} 1 & 0 & 0 \\ -x_i a_i & x_i & -c_i \\ -y_i a_i & y_i & b_i \end{bmatrix}$.
Likewise, by Proposition 3.3, for

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -d_1 & 1 & 0 \\ -e_1 & 0 & 1 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 1 & 0 & 0 \\ -d_k & 1 & 0 \\ -e_k & 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 0 & 0 \\ \Sigma_{i=1}^k d_i + m & 1 & 0 \\ \Sigma_{i=1}^k e_i + n & 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 0 & 0 \\ -m & 1 & 0 \\ -n & 0 & 1 \end{bmatrix} \in H_3,$$

we can see that M is diffeomorphic to M'. $\square$

Hereafter we will denote

$$M(A_1, B_1, \cdots, A_g, B_g; \begin{bmatrix} 0 & 1 & \epsilon_1 \\ * & 0 & * \\ * & 0 & * \end{bmatrix}, \cdots, \begin{bmatrix} 0 & 1 & \epsilon_k \\ * & 0 & * \\ * & 0 & * \end{bmatrix}; \begin{bmatrix} 1 & 0 & 0 \\ m' & 1 & 0 \\ n' & 0 & 1 \end{bmatrix}; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix})$$

by $M(A_1, B_1, \cdots, A_g, B_g; \begin{bmatrix} 0 & 1 & \epsilon_1 \\ * & 0 & * \\ * & 0 & * \end{bmatrix}, \cdots, \begin{bmatrix} 0 & 1 & \epsilon_k \\ * & 0 & * \\ * & 0 & * \end{bmatrix}; \begin{bmatrix} m' \\ n' \end{bmatrix})$.

Let α_1, $\beta_1, \cdots, \alpha_g$, β_g be homotopy classes which are represented by loops in M whose base point is v_0, and which determine monodromy matrices A_1, $B_1, \cdots, A_g$, B_g, respectively. Let σ, τ be generators of the fundamental group $\pi_1(f^{-1}(v_0)) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}$ corresponding to $\bar{r}_i$, $\bar{s}_i$ which are the elements of the canonical basis (l_i, r_i, s_i) of $H_1(\partial(S_{k+2}\times T^2);\boldsymbol{Z})$, respectively. Note that r_i (resp. s_i) is homologous to r_j (resp. s_j) for $i\neq j$.

We can calculate the fundamental group of M' by making use of van Kampen's theorem as follows.

PROPOSITION 4.1. *Let $\pi_1(M')$ be the fundamental group of*

$$M'=M(A_1,B_1,\cdots,A_g,B_g;\begin{bmatrix}0&1&\epsilon_1\\x_1&0&y_1\\z_1&0&t_1\end{bmatrix},\cdots,\begin{bmatrix}0&1&\epsilon_k\\x_k&0&y_k\\z_k&0&t_k\end{bmatrix};\begin{bmatrix}m\\n\end{bmatrix}),$$

where $A_j=\begin{bmatrix}a_j&b_j\\c_j&d_j\end{bmatrix}$, $B_j=\begin{bmatrix}p_j&q_j\\r_j&s_j\end{bmatrix}$ $(j=1,\cdots,g)$.

Then $\pi_1(M')=<\alpha_1,\ \beta_1,\cdots,\alpha_g,\ \beta_g,\ \sigma,\ \tau\mid\sigma\tau=\tau\sigma\ ,\alpha_j^{-1}\sigma\alpha_j=\sigma^{a_j}\tau^{c_j}$, $\alpha_j^{-1}\tau\alpha_j=\sigma^{b_j}\tau^{d_j}$, $\beta_j^{-1}\sigma\beta_j=\sigma^{p_j}\tau^{r_j}$, $\beta_j^{-1}\tau\beta_j=\sigma^{q_j}\tau^{s_j}$ $(j=1,\cdots,g)$, $1=\sigma^{y_i}\tau^{t_i}$ $(i=1,\cdots,k)$, $\alpha_1^{-1}\beta_1^{-1}\alpha_1\beta_1\cdots\alpha_g^{-1}\beta_g^{-1}\alpha_g\beta_g=\sigma^{-m}\tau^{-n}>$.

PROPOSITION 4.2. *$H_1(M')$ is isomorphic to $\boldsymbol{Z}^{2g}\oplus(\boldsymbol{Z}^2/K)$, where K is the subgroup of $\boldsymbol{Z}^2$ generated by $\begin{bmatrix}-m\\-n\end{bmatrix}$, $\begin{bmatrix}y_i\\t_i\end{bmatrix}$ $(i=1,\cdots,k)$ and the column vectors of A_j-E and B_j-E $(j=1,\cdots,g)$ (E stands for $\begin{bmatrix}1&0\\0&1\end{bmatrix}$).*

LEMMA 4.2. *For any element P of $SL(2,\boldsymbol{Z})$,*

$$M'=M(A_1,B_1,\cdots,A_g,B_g;\begin{bmatrix}0&1&\epsilon_1\\x_1&0&y_1\\z_1&0&t_1\end{bmatrix},\cdots,\begin{bmatrix}0&1&\epsilon_k\\x_k&0&y_k\\z_k&0&t_k\end{bmatrix};\begin{bmatrix}m\\n\end{bmatrix})$$

is diffeomorphic to

$$\begin{aligned}\tilde{M}'=&M(P^{-1}A_1P,P^{-1}B_1P,\cdots,P^{-1}A_gP,P^{-1}B_gP;\\&\begin{bmatrix}0&1&\epsilon_1\\\check{x}_1&0&\check{y}_1\\\check{z}_1&0&\check{t}_1\end{bmatrix},\cdots,\begin{bmatrix}0&1&\epsilon_k\\\check{x}_k&0&\check{y}_k\\\check{z}_k&0&\check{t}_k\end{bmatrix};P^{-1}\begin{bmatrix}m\\n\end{bmatrix}),\end{aligned}$$

where $\begin{bmatrix}\check{x}_i&\check{y}_i\\\check{z}_i&\check{t}_i\end{bmatrix}=P^{-1}\begin{bmatrix}x_i&y_i\\z_i&t_i\end{bmatrix}$ $(i=1,\cdots,k)$.

PROOF. We can easily prove by using the diffeomorphism which is determined by transformation of basis of fibers. □

In this section, we treat only the case when $\operatorname{rank} H_1(M) = odd$. Of course, $\operatorname{rank} H_1(M')$ is odd, too. In this case, a diffeomorphism type of M (i.e. M') is very simple.

LEMMA 4.3.

$$M' = M(A_1, B_1, \cdots, A_g, B_g; \begin{bmatrix} 0 & 1 & \epsilon_1 \\ x_1 & 0 & y_1 \\ z_1 & 0 & t_1 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & 1 & \epsilon_k \\ x_k & 0 & y_k \\ z_k & 0 & t_k \end{bmatrix}; \begin{bmatrix} m \\ n \end{bmatrix})$$

is diffeomorphic to $M'' = M(\begin{bmatrix} 1 & e_1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & f_1 \\ 0 & 1 \end{bmatrix}, \cdots, \begin{bmatrix} 1 & e_g \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & f_g \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 0 & 1 & \epsilon_1 \\ g_1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & 1 & \epsilon_k \\ g_k & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}; \begin{bmatrix} m' \\ 0 \end{bmatrix})$, *where* $\operatorname{rank} H_1(M')$ *is odd and* e_i, f_i $(i = 1, \cdots, g)$, $g_i (i = 1, \cdots, k)$ *are integers which are determined by* M'.

PROOF. By Proposition 4.2, $\operatorname{rank} H_1(M) = 2g + 1$, and $\operatorname{rank}(A_1 - E, B_1 - E, \cdots, A_g - E, B_g - E, \begin{bmatrix} y_1 \\ t_1 \end{bmatrix}, \cdots, \begin{bmatrix} y_k \\ t_k \end{bmatrix}, \begin{bmatrix} m \\ n \end{bmatrix}) = 1$ (E stands for $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$). Since $\operatorname{rank}(A_1 - E) \leq 1$, $det(A_1 - E) = 0$ holds. That is, for $A_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $(ad - bc = 1)$, $(a - 1)(d - 1) - bc = 0$ holds. Thus $traceA_1 = a + d = 2$. By Lemma 4.2, we can assume that $A_1 = \begin{bmatrix} 1 & e_1 \\ 0 & 1 \end{bmatrix}$ $(e_1 \in \boldsymbol{Z})$. See [10]. By $\operatorname{rank}(\begin{bmatrix} 0 & e_1 \\ 0 & 0 \end{bmatrix}, B_1 - E, A_2 - E, \cdots, A_g - E, B_g - E, \begin{bmatrix} y_1 \\ t_1 \end{bmatrix}, \cdots, \begin{bmatrix} y_k \\ t_k \end{bmatrix}, \begin{bmatrix} m \\ n \end{bmatrix}) = 1$, in the case when $e_1 \neq 0$, all elements of the second row of each matrix are zero. Thus in this case, M' is diffeomorphic to M''. Likewise in the case when $e_1 = 0$, we can see that M' is diffeomorphic to M''. □

REMARK 4. We will see that $M'' = M(\begin{bmatrix}1 & e_1\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & f_1\\0 & 1\end{bmatrix}, \cdots, \begin{bmatrix}1 & e_g\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & f_g\\0 & 1\end{bmatrix}; \begin{bmatrix}0 & 1 & \epsilon_1\\g_1 & 0 & 1\\1 & 0 & 0\end{bmatrix}, \cdots, \begin{bmatrix}0 & 1 & \epsilon_k\\g_k & 0 & 1\\1 & 0 & 0\end{bmatrix}; \begin{bmatrix}m'\\0\end{bmatrix})$ has a GTF structure as follows. M can be decomposed so that $M'' = (\overline{M'' - \cup_{i=1}^k Tw_i}) \cup_{\phi_{C_i}} Tw_i$, and $\overline{M'' - \cup_{i=1}^k Tw_i}$ has a structure of a torus fibration $f' : \overline{M'' - \cup_{i=1}^k Tw_i} \to B - \cup_{i=1}^k Int D_i^2$. Since

$$(\phi_{C_i*}(l_i), \phi_{C_i*}(r_i), \phi_{C_i*}(s_i)) = (\bar{l}_i, \bar{r}_i, \bar{s}_i)\begin{bmatrix}0 & 1 & \epsilon_i\\g_i & 0 & 1\\1 & 0 & 0\end{bmatrix} (i = 1, \cdots, k),$$

$(f' \circ \phi_{C_i})_*; H_1(\partial(Tw_i); \boldsymbol{Z}) \to H_1(\partial D_i^2; \boldsymbol{Z})$ maps l_i, r_i, s_i to 0, $\bar{l}_i$, $\epsilon_i \bar{l}_i$, respectively. Thus by Proposition 3.4 (ii), we can extend a GTF structure on $\overline{M'' - \cup_{i=1}^k Tw_i}$ to a GTF structure on Tw_i $(i = 1, \cdots, k)$. By Proposition 3.4 (iii), all singular fibers are simple. By Corollary 3.1, the i-th twin singular fiber can be regarded as (m,n)-twin singular fiber, where m, n are non-negative integers so that $m + n \equiv 1 + \epsilon_i \mod 2$. In particular it is even (resp. odd) if $\epsilon_i = 1$ (resp. 0).

Comparing the first GTF structure of M with the one of M'' which is induced as above, we can see that the i-th twin singular fiber of M'' is even (resp. odd) if and only if the corresponding twin in M is even (resp. odd).

Hereafter we will assume that M has at least one even twin singular fiber. Obviously we can assume that the first twin singular fiber is even without loss of generality.

LEMMA 4.4. *$M'' = M(\begin{bmatrix}1 & e_1\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & f_1\\0 & 1\end{bmatrix}, \cdots, \begin{bmatrix}1 & e_g\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & f_g\\0 & 1\end{bmatrix}; \begin{bmatrix}0 & 1 & \epsilon_1\\g_1 & 0 & 1\\1 & 0 & 0\end{bmatrix}, \cdots, \begin{bmatrix}0 & 1 & \epsilon_k\\g_k & 0 & 1\\1 & 0 & 0\end{bmatrix}; \begin{bmatrix}m'\\0\end{bmatrix})$ is diffeomorphic to $M''' = M(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \cdots, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}; \begin{bmatrix}0 & 1 & \epsilon_1\\0 & 0 & 1\\1 & 0 & 0\end{bmatrix}, \cdots, \begin{bmatrix}0 & 1 & \epsilon_k\\0 & 0 & 1\\1 & 0 & 0\end{bmatrix}; \begin{bmatrix}0\\0\end{bmatrix})$.*

PROOF. By Corollary 3.1, we can assume that the first twin singular fiber is (1,1)-twin singular fiber. We can deform this singular fiber to two singular fibers which are of I_1^+ and I_1^- ([4]). By Picard-Lefschetz formula ([8]), the monodromy matrices of singular fibers of type I_1^+ (resp. I_1^-) are $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, respectively.

$\overline{M'' - \cup_{i=2}^k Tw_i - D^2 \times T^2}$ is the total space of a GTF which has two singular fibers of type I_1^+ andI_1^- and has a boundary. We define homology classes γ_1^+ and γ_1^- which are associated with I_1^+ and I_1^-, respectively. a_+ and a_- are critical values.

In what follows, we want to prove that

$$N_1 = \overline{M'' - \cup_{i=2}^k Tw_i - (D^2 \times T^2)}$$

is isomorphic to

$$N_2 = \overline{M''' - \cup_{i=2}^k Tw_i - (D^2 \times T^2)}$$

whose monodromy matrices are trivial. Here we say that two GTF $f_1 : N_1 \to B_1$ and $f_2 : N_2 \to B_2$ are isomorphic to each other if and only if there exists an orientation preserving diffeomorphisms $\Psi : N_1 \to N_2$ and $\Phi : B_1 \to B_2$, where $\Phi \circ f_1 = f_2 \circ \Psi$. By seeing the proof of the statement of [8] (p. 169), it is simply extended to the case of containing singular fibers of type I_1^- and having an arbitrary oriented base space. Note that this statement does not assume that the monodromies are epimorphisms. Thus we can use the following assertion.

Assertion. *Let $f_1 : M \to B$ and $f_2 : M \to B$ be two GTF's each of whose singular fibers is of type I_1^+ or I_1^- and whose base space B satisfies that $\partial B \neq \phi$, with the same set of critical values, say $a_1, \cdots, a_\mu$. Suppose that $f_1^{-1}(v_0) = f_2^{-1}(v_0)$ and the corresponding canonical anti-homomorphisms $\pi_1(B - \cup_{i=1}^\mu a_i, v_0) \to Aut_+(H_1(f_1^{-1}(v_0), \boldsymbol{Z}))$ and $\pi_1(B - \cup_{i=1}^\mu a_i, v_0) \to Aut_+(H_1(f_2^{-1}(v_0), \boldsymbol{Z})$ coincide, that is, the two sets of monodromy matrices are completely coincide. Then there exists an isomorphism $\Phi : f_1 \to f_2$, $\Phi = \{\Phi : B \to B, \Psi : M_1 \to M_2\}$ such that $\Phi(v_0) = v_0$, Φ induces identity on $\pi_1(B - \cup_{i=1}^\mu a_i)$ and $\Psi | f^{-1}(v_0) = identity$.*

We assume that by choosing an appropriate configuration

$$C = (\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g; \gamma_1^+, \gamma_1^-, \gamma_2, \cdots, \gamma_k, \gamma_{k+1})$$

which consists of homotopy classes which are represented by loops on B_1, $\rho_1(\alpha_j) = \rho_1(\beta_j) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $(j = 1, \cdots, g)$, $\rho_1(\gamma_1^+) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\rho_1(\gamma_1^-) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, $\rho_1(\gamma_i) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $(i = 2, \cdots, k+1)$ hold, where ρ_1 is the monodromy representation associated with f_1. Obviously there is a configuration $C' = (\alpha_1', \beta_1', \cdots, \alpha_g', \beta_g'; \gamma_1^{+\prime}, \gamma_1^{-\prime}, \gamma_2', \cdots, \gamma_{k+1}')$ on B_2 such that $\rho_2(\alpha_j') = \rho_2(\beta_j') = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $(i = 1, \cdots, g)$, $\rho_2(\gamma_1^{+\prime}) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\rho_2(\gamma_1^{-\prime}) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, $\rho_2(\gamma_i') = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $(i = 2, \cdots, k+1)$, where ρ_2 is the monodromy representation associated with f_2. Clearly there exists an orientation preserving homeomorphism $\Phi : B_1 \to B_2$ such that $\Phi(a_+) = a_+'$, $\Phi(a_-) = a_-'$, $\Phi_\#(C) = C'$. By deforming it if necessary, we may assume Φ is a diffeomorphism. Then N_2 is considered a GTF over B_1 with the projection $\Phi^{-1} \circ f_2 : N_2 \to B_1$. Thus from the beginning we can assume that $B_1 = B_2 = B$ and $C = C'$. Obviously $f_1 : N_1 \to B$ and $f_2 : N_2 \to B$ satisfy the assumption of the Assertion. Thus we have only to prove the existence of such a configuration C in order to prove that f_1 is isomorphic to f_2. A trivial configuration $C_0 = (\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g; \gamma_1^+, \gamma_1^-, \gamma_2, \cdots, \gamma_{k+1})$ on B_1 satisfies $\rho_1(\alpha_j) = \begin{bmatrix} 1 & e_1 \\ 0 & 1 \end{bmatrix}$, $\rho_1(\beta_j) = \begin{bmatrix} 1 & f_1 \\ 0 & 1 \end{bmatrix}$, $(j = 1, \cdots, g)$, $\rho_1(\gamma_1^+) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\rho_1(\gamma_1^-) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, $\rho_1(\gamma_i) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $(i = 2, \cdots, k+1)$, where ρ_1 is the monodromy representation associated with f_1. There exist homeomorphisms $\mathcal{L}_j^+$, $\mathcal{L}_j^-$, $\mathcal{M}_j^+$ and $\mathcal{M}_j^- : \{B_1 - \{a_+, a_-\}, v_0\} \to \{B_1 - \{a_+, a_-\}, v_0\}$ $(1 \le j \le g)$ so that $C = (\mathcal{M}_g^{-f_g} \circ \cdots \circ \mathcal{M}_1^{-f_1} \circ \mathcal{L}_g^{-e_g} \circ \cdots \circ \mathcal{L}_1^{-e_1})_\#(C_0)$, where $\mathcal{L}_j^n = \begin{cases} (\mathcal{L}_j^+)^n & n > 0, \\ \textit{identity} & n = 0, \\ (\mathcal{L}_j^-)^n & n < 0, \end{cases}$ $\mathcal{M}_j^n = \begin{cases} (\mathcal{M}_j^+)^n & n > 0, \\ \textit{identity} & n = 0, \\ (\mathcal{M}_j^-)^n & n < 0 \end{cases}$ $(j = 1, \cdots, g)$ is the configuration required.

For example, the homeomorphism $\mathcal{L}_j^+$ is characterized by

$$\begin{aligned} &\mathcal{L}_{j\#}^+(\alpha_1, \beta_1, \cdots, \alpha_j, \beta_j, \cdots, \alpha_g, \beta_g; \gamma_1^+, \gamma_1^-, \gamma_2, \cdots, \gamma_{k+1}) \\ &= (\alpha_1, \beta_1, \cdots, \bar{\alpha}_j, \beta_j, \cdots, \alpha_g, \beta_g; \bar{\gamma}_1^+, \gamma_1^-, \gamma_2, \cdots, \gamma_{k+2}), \end{aligned}$$

where $\bar{\alpha}_j = \alpha_j[\beta_{j-1}, \alpha_{j-1}] \cdots [\beta_1, \alpha_1]\gamma_1^+[\alpha_1, \beta_1] \cdots [\alpha_{j-1}, \beta_{j-1}]$ and $\bar{\gamma}_1^+$ is a certain conjugate of γ_1^+. See [5].

Therefore, $f_1 : N_1 \to B_1$ is isomorphic to $f_2 : N_2 \to B_2$. The diffeomorphism $\Psi : N_1 \to N_2$ induces the diffeomorphism $\Psi \cup id : N_1 \cup_{\phi_{C_i}} \cup_{i=2}^k Tw_i \cup_{\phi_{D_1}} (D^2 \times T^2) \to N_2 \cup_{\Psi\circ\phi_{C_i}} \cup_{i=2}^k Tw_i \cup_{\Psi\circ\phi_{D_1}} (D^2 \times T^2)$. Hence $M'' = M(\begin{bmatrix} 1 & e_1 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 1 & f_1 \\ 0 & 1 \end{bmatrix}, \cdots, \begin{bmatrix} 1 & e_g \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & f_g \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 0 & 1 & \epsilon_1 \\ g_1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & 1 & \epsilon_k \\ g_k & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}; \begin{bmatrix} m' \\ 0 \end{bmatrix})$ is diffeomorphic to $\bar{M}'' = M(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \cdots, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & \epsilon_2 \\ g_2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & 1 & \epsilon_k \\ g_k & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}; \begin{bmatrix} \bar{m}' \\ 0 \end{bmatrix})$. By seeing the assertion of [8] (p. 169), all twists of the structure of the torus fibration of the boundary of the diffeomorphism $\Psi : N_1 \to N_2$ can be gathered into $S^1_{k+1} \times T^2$. Hence we can assume that ϕ is identity on $S^1_i \times T^2$ $(i = 2, \cdots, k)$, that is $\Psi \circ \phi_{C_i} = \phi_{C_i}$ $(i = 2, \cdots, k)$.

By Proposition 3.1, the diffeomorphism type of $\bar{M}''$ does not change if we change $\bar{C}_i'' = \begin{bmatrix} 0 & 1 & \epsilon_i \\ g_i & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $(i = 2, \cdots, k)$ of $\bar{M}''$ to $\bar{C}_i'' A_i = \begin{bmatrix} 0 & 1 & \epsilon_i \\ g_i & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ g_i\epsilon_i & 1 & 0 \\ -g_i & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & \epsilon_i \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, where $A_i = \begin{bmatrix} 1 & 0 & 0 \\ g_i\epsilon_i & 1 & 0 \\ -g_i & 0 & 1 \end{bmatrix} \in H_1$. Thus $\bar{M}''$ is diffeomorphic to $\tilde{M}'' = M(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \cdots, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & \epsilon_2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & 1 & \epsilon_k \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}; \begin{bmatrix} \bar{m}' \\ 0 \end{bmatrix})$.

A similar argument of the proof of Lemma 4.1 shows that $\tilde{M}''$ is diffeomorphic to $\tilde{M}_1'' = M(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \cdots, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & \epsilon_2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & 1 & \epsilon_k \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ \bar{m}' & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix})$. Finally we will see that $\tilde{M}_1''$ ($\bar{m}' \neq 0$) is diffeomorphic to M''' by using an isotopy which does not change the neighborhood of singular fibers. For a GTF $\tilde{f} : \tilde{M}_1'' \to \tilde{B}$, we construct an isotopy $\phi_t^n : \tilde{B} \to \tilde{B}$, $t \in [0,1]$, $n \in \boldsymbol{Z} - 0$. Let $\Gamma = \{a_+, a_-, a_2, \cdots, a_k\}$ be the set of singular values. Denote a disk which is surrounded by a circle S_{k+1}^1 in $\tilde{B}$ by D_a, and choose a base point a in D_a. For $n \in \boldsymbol{Z} - \{0\}$, let $\gamma_n : [0,1] \to \tilde{B} - \Gamma$ be a loop which satisfies $\gamma_n(0) = \gamma_n(1) = a$ and goes round the singular value of the singular fiber of type I_1^+ if $n > 0$ or of type I_1^- if $n < 0$ $|n|$ times toward the positive direction. There exists an isotopy $\phi_t^n : \tilde{B} \to \tilde{B}$, $t \in [0,1]$, $n \in \boldsymbol{Z} - \{0\}$, $\phi_0 = identity$ such that for some open $U \subset \tilde{B} - D_a$, $\phi_t^n|_U = identity$, $\phi_t^n(a) = \gamma_n(t)$, $\phi_1^n(D_a) = D_a$. A diffeomorphism $\phi_1^{-\bar{m}'} : \tilde{B} \to \tilde{B}$ induces a diffeomorphism $\psi_1^{-\bar{m}'} : \tilde{M}_1'' \to M_1'''$, where $M_1''' = M(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \cdots, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & \epsilon_2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & 1 & \epsilon_k \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix})$. Actually an isotopy $\phi_1^{-\bar{m}'}$ induces a diffeomorphism which influence only a gluing map ϕ_{D_1} which represents an obstruction for existing a cross-section. See [8]. Therefore, $\tilde{M}_1''$ is diffeomorphic to M_1'''. A similar argument shows that M_1''' is diffeomorphic to M'''. Hence M'' is diffeomorphic to M'''. □

Hereafter we will denote $M(E, \cdots, E; C_1, \cdots, C_k; \begin{bmatrix} m \\ n \end{bmatrix})$ by $M(E^{2g}; C_1, \cdots, C_k; \begin{bmatrix} m \\ n \end{bmatrix})$. A closed 4-manifold M is said to be *spin* if and only if for any $x \in H_2(M; \boldsymbol{Z}/2)$, $x \circ x = 0 \in \boldsymbol{Z}/2$, where $x \circ x$ is the self-intersection number. A closed oriented 4-manifold M is said to be of *typeII* if and only if for any $x \in H_2(M; \boldsymbol{Z})$, $x \circ x = 0 \mod 2$, where $x \circ x$ is a self-intersection number. Otherwise, it is said to be of *typeI*. The following Lemma means that there exists both GTF of *typeI* and one of *typeII*. Hereafter we will denote $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ by C.

LEMMA 4.5.

$$M''' = M(E^{2g}; \begin{bmatrix} 0 & 1 & \epsilon_1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & \epsilon_2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & 1 & \epsilon_k \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}; \begin{bmatrix} 0 \\ 0 \end{bmatrix})$$

$(\epsilon_1 = 1, k \geq 1)$ *is of typeI if* $\epsilon_1\epsilon_2\cdots\epsilon_k = 0$, *and* M''' *is diffeomorphic to* $M(E^{2g}; C\ ; \begin{bmatrix} 0 \\ 0 \end{bmatrix})\ \#(k-2)(S^2\times S^2)\#(S^2\tilde{\times}S^2)$. *If* $\epsilon_1\epsilon_2\cdots\epsilon_k = 1$, *then* M''' *is of typeII and* M''' *is diffeomorphic to* $M(E^{2g}; C\ ; \begin{bmatrix} 0 \\ 0 \end{bmatrix})\#(k-1)(S^2\times S^2)$.

PROOF. First we show that $\underline{M} = M(E^{2g}; C\ ; \begin{bmatrix} 0 \\ 0 \end{bmatrix})$ is of *typeII*. Let F_g be an oriented, closed surface whose genus is g. Then $\underline{M}$ is decomposed as $\underline{M} = ((F_g - IntD^2) \times T^2) \cup_{\phi_{C_1}} Tw_1$. We consider the Meyer-Vietoris sequence

$$\begin{gathered} H_2((F_g - IntD^2) \times T^2; \boldsymbol{Z}/2) \oplus H_2(Tw_1; \boldsymbol{Z}/2) \xrightarrow{j_*} H_2(\underline{M}; \boldsymbol{Z}/2) \\ \xrightarrow{\Delta} H_1(\partial((F_g - IntD^2) \times T^2); \boldsymbol{Z}/2) \xrightarrow{i_*} \\ H_1((F_g - IntD^2) \times T^2; \boldsymbol{Z}/2) \oplus H_1(Tw_1; \boldsymbol{Z}/2), \end{gathered}$$

where Δ is the connecting homomorphism and j_*, i_* are maps which are induced by inclusions. Since any homology group is a vector space, there exists a subspace V so that $H_2(\underline{M}; \boldsymbol{Z}/2) = Im(j_*)\oplus V$. The self-intersection number of any element of $H_2((F_g - IntD^2)\times T^2; \boldsymbol{Z}/2)$, $H_2(Tw_1; \boldsymbol{Z}/2)$ is zero, thus the self-intersection number of any element of $Im(j_*)$ is zero, too. The map which gives a self-intersection number for each element of $H_2(X; \boldsymbol{Z}/2)$ is linear, thus we have only to examine self-intersection numbers on V to see whether $\underline{M}$ is spin or not. An element $v \in V$ which is not zero is distinguished by $\Delta(v) \neq 0$ and $i_*(\Delta(v)) = 0$. Choose a basis $< l_1 >$ of $H_1(Tw_1; \boldsymbol{Z}/2)$ and a basis $< \tilde{\alpha_1}, \tilde{\beta}_1, \cdots, \tilde{\alpha_g}, \tilde{\beta}_g, \tilde{r}, \tilde{s} >$ of $H_1((F_g - IntD^2)\times T^2; \boldsymbol{Z}/2) = \oplus^{2g+2}\boldsymbol{Z}/2$. $\tilde{\alpha}_1$, $\tilde{\beta}_1, \cdots, \tilde{\alpha}_g$, $\tilde{\beta}_g$ and $\tilde{r}$, $\tilde{s}$ are homology classes which correspond to α_1, β_1, $\cdots$, α_g, β_g and $\bar{r}_1$, $\bar{s}_1$, respectively.

Since $(\ \phi_{C_1*}(l_1), \phi_{C_1*}(r_1), \phi_{C_1*}(s_1)\ ) = (\ \bar{l}_1, \bar{r}_1, \bar{s}_1\ )\ C$,

$$\begin{aligned} \bar{l}_1 &= \phi_{C_1*}(r_1), \\ \bar{r}_1 &= -\phi_{C_1*}(r_1) + \phi_{C_1*}(s_1), \\ \bar{s}_1 &= \phi_{C_1*}(l_1). \end{aligned}$$

Since 1-chain $\partial(F_g - IntD^2) \times \{*\} \times \{*\}$ is homologous to 0 in $(F_g - IntD^2) \times T^2$,

$$\begin{aligned} i_*(\bar{l}_1) &= (0\tilde{r} \oplus 0\tilde{s}) \oplus 0l_1, \\ i_*(\bar{r}_1) &= (1\tilde{r} \oplus 0\tilde{s}) \oplus 0l_1, \\ i_*(\bar{s}_1) &= (0\tilde{r} \oplus 1\tilde{s}) \oplus (-1)l_1. \end{aligned}$$

Hence $Ker(i_*)$ is generated by $\bar{l}_1$. Let C be a 2-chain $(F_g - IntD^2) \times \{*\} \times \{*\}$ in $(F_g - IntD^2) \times T^2$, and C_1 be the 2-chain in Tw so that $\partial C_1 \mod 2$ is equal to r_1. Then $\Delta([C + C_1]) = \bar{l}_1$. By Proposition 3.6, $[C + C_1]^2 (mod\ 2) = 0$ holds. Therefore, $\underline{M} = M(E^{2g};\ C\ ;\ \begin{bmatrix} 0 \\ 0 \end{bmatrix}\ )$ is *spin*. Thus $\underline{M}$ is of *typeII*.

Define $D_i^2 \times T^2 = \chi(Tw_i; S_i)$ $(i = 2, \cdots, k)$, where χ is the Milnor surgery and Tw_i is the i-th twin in M''', see [4]. Since l_i, r_i and s_i in $H_1(\partial Tw_i; \boldsymbol{Z})$ correspond to $-\tilde{r}_i = -(\{*\} \times S^1 \times \{*\})$, $\tilde{l}_i = \partial D_i^2 \times \{*\} \times \{*\}$, $\tilde{s}_i = \{*\} \times \{*\} \times S^1$ in $H_1(\partial(D_i^2 \times T^2); \boldsymbol{Z})$, respectively, gluing maps of Tw_i $\begin{bmatrix} 0 & 1 & \epsilon_i \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $(i = 2, \cdots, k)$ correspond to gluing maps of $D_i^2 \times T^2$ $\begin{bmatrix} 1 & 0 & \epsilon_i \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$ $(i = 2, \cdots, k)$, respectively. These matrices are elements of H_2 in Proposition 3.2, thus we can change these matrices to unit matrices without changing the diffeomorphism type of M'''. Therefore, by Proposition 3.5, we can change M''' to $\underline{M}$ by performing the Milnor surgery on all twins except the first twin.

Conversely, s_i is isotopic to s_1, and s_1 is isotopic to 0 in Tw_1, thus M''' is obtained from $\underline{M}$, by performing the Milnor surgery on $k-1$ disjoint loops $\{s'_2, \cdots, s'_k\}$ in $\underline{M}$ which is isotopic to s_1. Since $(S^2 \tilde{\times} S^2) \# (S^2 \tilde{\times} S^2) = (S^2 \times S^2) \# (S^2 \tilde{\times} S^2)$ and $\underline{M}$ is of *typeII*, M''' is diffeomorphic to $\underline{M} \# (k-2)(S^2 \times S^2) \# (S^2 \tilde{\times} S^2)$ if M''' is of *typeI*, and M''' is diffeomorphic to $\underline{M} \# (k-1)(S^2 \times S^2)$ if M''' is of *typeII* ([11]).

In what follows, we will see that $X_1 = M(E^{2g};\ C\ ,\ C\ ;\ \begin{bmatrix} 0 \\ 0 \end{bmatrix}\ )$ is of *typeII*, and $X_0 = M(E^{2g};\ C\ ;\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\ ;\ \begin{bmatrix} 0 \\ 0 \end{bmatrix}\ )$ is of *typeI*. We consider the Meyer-Vietoris sequence for the decomposition $X_{\epsilon_2} = ((F_g - (IntD_1^2 \cup IntD_2^2)) \times T^2) \cup_{\phi_{C_1}} Tw_1 \cup_{\phi_{C_2}} Tw_2$ ($\epsilon_2 = 0,\ 1$) as above.

$$\begin{aligned} &H_2((F_g - (IntD_1^2 \cup IntD_2^2)) \times T^2; \boldsymbol{Z}/2) \oplus H_2(Tw_1 \cup Tw_2; \boldsymbol{Z}/2) \\ &\xrightarrow{j_*} H_2(X_{\epsilon_2}; \boldsymbol{Z}/2) \xrightarrow{\Delta} H_1(\partial((F_g - (IntD_1^2 \cup IntD_2^2)) \times T^2); \boldsymbol{Z}/2) \\ &\xrightarrow{i_*} H_1((F_g - (IntD_1^2 \cup IntD_2^2)) \times T^2; \boldsymbol{Z}/2) \oplus H_1(Tw_1 \cup Tw_2; \boldsymbol{Z}/2). \end{aligned}$$

We have

$$\begin{aligned} &\bar{l}_i = \phi_{C_i *}(r_i), \\ &\bar{r}_i = -\epsilon_i \phi_{C_i *}(r_i) + \phi_{C_i *}(s_i), \\ &\bar{s}_i = \phi_{C_i *}(l_i)\ (i = 1,\ 2), \end{aligned}$$

where $\epsilon_2 = 0$ or 1. For their images,

$$\begin{aligned} &i_*(\bar{l}_1) = (0\bar{r} \oplus 0\bar{s} \oplus 1\bar{l}_1) \oplus (0l_1 \oplus 0l_2), \\ &i_*(\bar{r}_1) = (1\bar{r} \oplus 0\bar{s} \oplus 0\bar{l}_1) \oplus (0l_1 \oplus 0l_2), \\ &i_*(\bar{s}_1) = (0\bar{r} \oplus 1\bar{s} \oplus 0\bar{l}_1) \oplus ((-1)l_1 \oplus 0l_2, \\ &i_*(\bar{l}_2) = (0\bar{r} \oplus 0\bar{s} \oplus (-1)\bar{l}_1) \oplus (0l_1 \oplus 0l_2), \\ &i_*(\bar{r}_2) = (1\bar{r} \oplus 0\bar{s} \oplus 0\bar{l}_1) \oplus (0l_1 \oplus 0l_2), \\ &i_*(\bar{s}_2) = (0\bar{r} \oplus 1\bar{s} \oplus \bar{l}_1) \oplus (0l_1 \oplus (-1)l_2) \end{aligned}$$

hold.

Therefore, $Ker(i_*)$ is generated by $\bar{l}_1 + \bar{l}_2$ and $\bar{r}_1 - \bar{r}_2$. Let C be the 2-chain $(F_g - (IntD_1^2 \cup IntD_2^2)) \times \{*\} \times \{*\}$ in $(F_g - (IntD_1^2 \cup IntD_2^2)) \times T^2$, and let C_i ($i = 1,\ 2$) be 2-chains in Tw_i such that $\partial C_i (mod\ 2)$ is equal to r_i. $\Delta([C + C_1 + C_2]) = \bar{l}_1 + \bar{l}_2$ holds. Let C' be the 2-chain $I \times S^1 \times \{*\}$ in $(F_g - (IntD_1^2 \cup IntD_2^2)) \times T^2$, I being an arc in $F_g - (IntD_1^2 \cup D_2^2)$ connecting ∂D_1^2 and ∂D_2^2, and let C_i' be a 2-cochain in T_{w_i} such that $\partial C_i' (mod\ 2)$ is equal to $-\epsilon_i r_i + s_i$. $\Delta([C' + C_1' + C_2']) = \bar{r}_1 + \bar{r}_2$ holds. $H_2(X_{\epsilon_2}; \boldsymbol{Z}/2) = Im(j_*) \oplus < [C + C_1 + C_2], [C' + C_1' + C_2'] >$ and any self-intersection number on $Im(j_*)$ is zero, thus we have only to examine $[C + C_1 + C_2]^2 (mod\ 2)$ and $[C' + C_1' + C_2']^2 (mod\ 2)$. By Proposition 3.6, $[C + C_1 + C_2]^2 (mod\ 2) = 0$, $[C' + C_1' + C_2']^2 (mod\ 2) = C_1' \cdot C_1' + C_2' \cdot C_2' (mod\ 2) = -1 - \epsilon_2$ hold.

Therefore, $X_1(\epsilon_2 = 1)$ is spin. Thus X_1 is of *typeII*. We change X_1 to X_0 by performing Gluck-surgery on 2-sphere S^2 in Tw_2. Thus $X_1 = \underline{M}\#(S^2\times S^2)$ and $X_0 = \tau(X_1; S^2) = \tau(\underline{M}\#(S^2\times S^2); S^2) = \underline{M}\#\tau(S^2\times S^2; \{*\}\times S^2) = \underline{M}\#(S^2\tilde{\times}S^2)$. See [1]. Therefore, X_0 is of *typeI*.

$\epsilon_1\epsilon_2\cdots\epsilon_k = 0$ means that some of $\epsilon_2,\cdots,\epsilon_k$ are zero. Then M''' is constructed by a connected sum of X_0 and some $S^2\times S^2$ or $S^2\tilde{\times}S^2$. Since X_0 is of *typeI*, M''' is of *typeI*, too. Hence $M''' = M(E^{2g};\, C\ ;\ \begin{bmatrix}0\\0\end{bmatrix}\ )$ $\#(k-2)(S^2\times S^2)\#(S^2\tilde{\times}S^2)$.

$\epsilon_1\epsilon_2\cdots\epsilon_k = 1$ means that all $\epsilon_2,\cdots,\epsilon_k$ are equal to 1. M''' is constructed by a connected sum of X_0 and some $S^2\times S^2$ or $S^2\tilde{\times}S^2$. If $S^2\tilde{\times}S^2$ is contained in them, M''' is constructed by $\underline{M}\#(S^2\tilde{\times}S^2)$ and some $S^2\times S^2$ or $S^2\tilde{\times}S^2$. $\underline{M}\#(S^2\tilde{\times}S^2)$ is diffeomorphic to X_1 because all $\epsilon_2,\cdots,\epsilon_{k-1}$ and ϵ_k are equal to 1. On the other hand, $\underline{M}\#(S^2\tilde{\times}S^2)$ is of *typeI*, thus this is X_0. But X_0 and X_1 are of *typeI* and of *typeII*, respectively. Thus it is not diffeomorphic to each other. This is a contradiction. Therefore, $M''' = M(E^{2g};\, C\ ;\ \begin{bmatrix}0\\0\end{bmatrix}\ )\#(k-1)(S^2\times S^2)$. Of course, M''' is of *typeII*. $\square$

THEOREM 4.1. *Let $f : M \to B$ be a good torus fibration with at least one even twin singular fiber. Suppose that each singular fiber is non-multiple twin singular fiber, and that* $\operatorname{rank} H_1(M)$ *is odd. Then the diffeomorphism type of M is determined by (i) $e(M)$, (ii)* $\operatorname{rank} H_1(M)$ *or $g(B)$, (iii) the information about whether M is typeI or typeII, as follows.*

If M is of typeI, then

$$M = M(E^{2g(B)};\, C\ ;\ \begin{bmatrix}0\\0\end{bmatrix}\ )\#(\tfrac{1}{2}e(M)-2)(S^2\times S^2)\#(S^2\tilde{\times}S^2).$$

If M is of typeII, then

$$M = M(E^{2g(B)};\, C\ ;\ \begin{bmatrix}0\\0\end{bmatrix}\ )\#(\tfrac{1}{2}e(M)-1)(S^2\times S^2).$$

Here $e(M)$, $H_1(M)$ and $g(B)$ are the Euler number of M, the 1-dimensional homology group of M and the genus of B, respectively.

Conversely, these manifolds have such GTF structures.

PROOF. $\frac{1}{2}e(M)$ is a number of twin singular fibers. By Lemmas 4.1, 4.3, 4.4 and 4.5, the first part of this Theorem is obvious. By Remark 4, we can see that each manifold has such a GTF structure. $\square$

REMARK 5. If $\operatorname{rank} H_1(M)$ is odd, then $\operatorname{rank} H_1(M) = 2g(B) + 1$. In the case when each singular fiber is of type I_1^+, I_1^- or Tw, it is known that if $\sigma(M) \neq 0$, then $\pi_1(M)$ is isomorphic to $\pi_1(B)$ ([5]). Therefore, if $\operatorname{rank} H_1(M)$ is odd, then $\sigma(M) = 0$.

If each type of singular fibers is of type I_1^+ or I_1^-, then Theorem 4.1 can be rewritten as follows.

THEOREM 4.2. *Let $f : M \to B$ be a good torus fibration. Suppose that each singular fiber is of type I_1^+ or I_1^-, and that $e(M) \neq 0$, and that $\operatorname{rank} H_1(M)$ is odd, ($\sigma(M) = 0$). Then the diffeomorphism type of M is determined by (i) $e(M)$, (ii)$\operatorname{rank} H_1(M)$ or $g(B)$, as follows.*

$$M = M(E^{2g(B)}; C ; \begin{bmatrix} 0 \\ 0 \end{bmatrix})\#(\tfrac{1}{2}e(M) - 1)(S^2 \times S^2).$$

Here $e(M)$, $H_1(M)$ and $g(B)$ are the Euler number of M, the 1-dimensional homology group of M and the genus of B, respectively.

Conversely, these manifolds have such GTF structures.

As the Corollary, we have a similar result about the case when each singular fiber is of type I_1^+, I_1^- or Tw.

COROLLARY 4.1. *Let $f_i : M \to B$ be a good torus fibrations. Suppose that each singular fiber is of type I_1^+ or I_1^-, or non-multiple twin singular fiber, and that $e(M) \neq 0$ and $\operatorname{rank} H_1(M)$ is odd ($\sigma(M) = 0$). Then the same result of Theorem 4.1 holds.*

5. Some Special Cases

In this section, we assume that each singular fiber is simple and each twin singular fiber is even. In this section, we will consider the special case when $e(M) = 2$, $g(B) = 1$ and $\sigma(M) = 0$ ($\operatorname{rank} H_1(M) = even$). It means that a singular fiber of GTF $f : M \to B$ is an even twin singular fiber or a pair of singular fibers whose type are I_1^+ and I_1^-, respectively.

LEMMA 5.1. *(i) For any $\begin{bmatrix} p & q \\ r & s \end{bmatrix} \in SL(2, \boldsymbol{Z})$, $M = M(A, B; C ; \begin{bmatrix} 0 \\ n \end{bmatrix})$ is diffeomorphic to $M(A^pB^r, A^qB^s; C ; \begin{bmatrix} 0 \\ n \end{bmatrix})$.*

(ii) For any $P \in SL(2, \boldsymbol{Z})$, $M = M(A, B;\ C\ ;\ \begin{bmatrix} 0 \\ n \end{bmatrix})$ *is diffeomorphic to* $M(P^{-1}AP, P^{-1}BP;\ \begin{bmatrix} 0 & 1 & 1 \\ \check{x}_1 & 0 & \check{y}_1 \\ \check{z}_1 & 0 & \check{t}_1 \end{bmatrix}\ ;\ P^{-1}\begin{bmatrix} 0 \\ n \end{bmatrix})$, *where* $\begin{bmatrix} \check{x}_1 & \check{y}_1 \\ \check{z}_1 & \check{t}_1 \end{bmatrix} = P^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(iii) Let $< A-E, B-E >$ *be the subgroup of* $\boldsymbol{Z}^2$ *generated by the column vectors of* $A - E$ *and* $B - E$. *(*E *stands for* $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$*.) Assume* $\begin{bmatrix} m' \\ n' \end{bmatrix} - \begin{bmatrix} 0 \\ n \end{bmatrix} \in < A-E, B-E >$. *Then* $M = M(A, B;\ C\ ;\ \begin{bmatrix} 0 \\ n \end{bmatrix})$ *is diffeomorphic to* $M(A, B;\ C\ ;\ \begin{bmatrix} m' \\ n' \end{bmatrix}) = M(A, B;\ C\ ;\ \begin{bmatrix} 0 \\ n' \end{bmatrix})$.

We omit the proof of Lemma 5.1, because it is easily proved by seeing Proposition 2 of [10].

LEMMA 5.2. *For* $M = M(A, B;\ C\ ;\ \begin{bmatrix} 0 \\ n \end{bmatrix})$, *there exists a certain* $A' \in SL(2, \boldsymbol{Z})$, *so that it is diffeomorphic to either* $M(A', E;\ C\ ;\ \begin{bmatrix} 0 \\ n \end{bmatrix})$ *or* $M(A', -E;\ C\ ;\ \begin{bmatrix} 0 \\ n \end{bmatrix})$. *(*$E$ *stands for* $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$*.)*

PROOF. By [10], we see that there exists $C \in GL(2, \boldsymbol{Z})$ so that $A = \pm C^s$ and $B = \pm C^{-q}$ for some integers s and q, because $AB = BA$. By replacing C by C^g, where $g = gcd(s, -q)$, we may assume that s and $-q$ are relatively prime, i.e., $ps - qr = 1$ for some integers p and r. For A, $B \in SL(2, \boldsymbol{Z})$, $A^pB^r = \pm C^{ps-qr} = \pm C$ and $A^qB^s = \pm C^{sq}C^{-sq} = \pm E$. We note that $A^pB^r = \pm C$ is an element of $SL(2, \boldsymbol{Z})$. By applying Lemma 5.1 (i), this completes the proof. □

By Lemma 5.2, hereafter we will be able to assume that $B = \pm E$.

LEMMA 5.3. *(i)* $M = M(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};\ C\ ;\ \begin{bmatrix} 0 \\ n \end{bmatrix})$ *is diffeomorphic to* $M(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & e \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};\ C\ ;\ \begin{bmatrix} 0 \\ \bar{n} \end{bmatrix}) = M(\begin{bmatrix} a & ea+b \\ c & ec+d \end{bmatrix},$

$\pm\begin{bmatrix}1&0\\0&1\end{bmatrix}$; C ; $\begin{bmatrix}0\\\bar{n}\end{bmatrix}$). *e is any integer, and $\bar{n}$ is an integer which is determined by M.*

(ii) $M(\begin{bmatrix}a&b\\c&d\end{bmatrix}, \pm\begin{bmatrix}1&0\\0&1\end{bmatrix}; C; \begin{bmatrix}0\\n\end{bmatrix})$ is diffeomorphic to $M(\begin{bmatrix}1&e\\0&1\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}, \pm\begin{bmatrix}1&0\\0&1\end{bmatrix}; C; \begin{bmatrix}0\\\bar{n}\end{bmatrix}) = M(\begin{bmatrix}a+ec&b+ed\\c&d\end{bmatrix}, \pm\begin{bmatrix}1&0\\0&1\end{bmatrix}; C; \begin{bmatrix}0\\\bar{n}\end{bmatrix})$. e is any integer, and $\bar{n}$ is an integer which is determined by M.

Proof. (i) This is easily proved by a similar argument of the proof of Lemma 4.4. Note that we use only two homeomorphisms $\mathcal{L}_1^+$ and $\mathcal{L}_1^-$, and that $\rho_1(\beta_1) = \pm\begin{bmatrix}1&0\\0&1\end{bmatrix}$ is commutative with any element of $SL(2,\boldsymbol{Z})$.

(ii) By Lemma 5.3 (i), M is diffeomorphic to $M(\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}1&e\\0&1\end{bmatrix}, \pm\begin{bmatrix}1&0\\0&1\end{bmatrix}; C; \begin{bmatrix}0\\\bar{n}\end{bmatrix})$. By applying Lemma 5.1 (ii) for defining $P = \begin{bmatrix}1&-e\\0&1\end{bmatrix}$, we see that M is diffeomorphic to $M(\begin{bmatrix}1&e\\0&1\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}, \pm\begin{bmatrix}1&0\\0&1\end{bmatrix}; \begin{bmatrix}0&1&1\\e&0&1\\1&0&0\end{bmatrix}; \begin{bmatrix}e\bar{n}\\\bar{n}\end{bmatrix}) = M(\begin{bmatrix}1&e\\0&1\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}, \pm\begin{bmatrix}1&0\\0&1\end{bmatrix}; C; \begin{bmatrix}0\\\bar{n}\end{bmatrix})$. □

Let us consider a necessary condition under which the fundamental group of $M = M(A,B;C;\begin{bmatrix}0\\n\end{bmatrix})$ ($A = \begin{bmatrix}a&b\\c&d\end{bmatrix}$, $B = \begin{bmatrix}p&q\\r&s\end{bmatrix} = \pm\begin{bmatrix}1&0\\0&1\end{bmatrix}$) is isomorphic to that of $M' = M(A',B';C;\begin{bmatrix}0\\n'\end{bmatrix})$ ($A' = \begin{bmatrix}a'&b'\\c'&d'\end{bmatrix}$, $B' = \begin{bmatrix}p'&q'\\r'&s'\end{bmatrix} = \pm\begin{bmatrix}1&0\\0&1\end{bmatrix}$). By Proposition 4.1, the fundamental groups of M and M' have the presentation $\pi_1(M) =< \alpha,\beta,\tau|\tau^c = 1, \alpha^{-1}\tau\alpha = \tau^d, \beta^{-1}\tau\beta = \tau^s, \alpha^{-1}\beta^{-1}\alpha\beta = \tau^{-n} >$ and $\pi_1(M') =< \alpha',\beta',\tau'|\tau'^{c'} = 1, \alpha'^{-1}\tau'\alpha' = \tau'^{d'}, \beta'^{-1}\tau'\beta' = \tau'^{s'}, \alpha'^{-1}\beta'^{-1}\alpha'\beta' = \tau'^{-n'} >$, respectively. We can easily prove the following Proposition.

Proposition 5.1. *Any element of $\pi_1(M)$ (resp. $\pi_1(M')$) can be* *uniquely represented by* $\alpha^u\beta^v\tau^w$ *(resp.* $\alpha'^u\beta'^v\tau'^w$*)* $0 \le w < o(\tau)$ *(resp.* $0 \le w' < o(\tau')$*), where* $o(\tau)$ *(resp.* $o(\tau')$*) is the order of* τ *(resp.* τ'*).*

In this section, we are assuming that $\operatorname{rank} H_1(M) = 2$ (i.e. $\operatorname{rank} H_1(M) = even$). In this case, the following Proposition holds.

PROPOSITION 5.2. *Let* $\phi : \pi_1(M') \to \pi_1(M)$ *be an isomorphism. (*$\operatorname{rank} H_1(M) = 2$*.) Then there exists a certain integer* t *so that* $\phi(\tau') = \tau^t$*.*

PROOF. By Proposition 5.1, we can put $\phi(\tau') = \alpha^u\beta^v\tau^t$ $(0 \le t < o(\tau))$. Let $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\tau}$ (resp. $\bar{\tau}'$) be the elements of $H_1(M)$ (resp. $H_1(M')$) which are images of α, β and $\tau \in \pi_1(M)$ (resp. $\tau' \in \pi_1(M')$) by the Hurwitz homomorphism. Let $\bar{\phi} : H_1(M') \to H_1(M)$ be the isomorphism which is induced by ϕ. Since $\operatorname{rank} H_1(M) = 2$ and $H_1(M') =< \bar{\alpha}', \bar{\beta}', \bar{\tau}' | [\bar{\alpha}', \bar{\beta}'] = 1, \bar{\tau}'^{c'} = \bar{\tau}'^{d'-1} = \bar{\tau}'^{-n'} = 1 >$, the order of $\bar{\tau}'$ is finite. Thus there exists a certain integer $\tilde{n} \neq 0$ so that $\bar{\tau}'^{\tilde{n}} = 1$. Hence $\bar{\phi}(\bar{\tau}'^{\tilde{n}}) = 1 \Rightarrow (\bar{\phi}(\bar{\tau}'))^{\tilde{n}} = 1 \Rightarrow (\bar{\alpha}^u\bar{\beta}^v\bar{\tau}^t)^{\tilde{n}} = 1 \Rightarrow \bar{\alpha}^{\tilde{n}u}\bar{\beta}^{\tilde{n}v}\bar{\tau}^{\tilde{n}t} = 1 \Rightarrow \tilde{n}u = \tilde{n}v = 0 \Rightarrow u = v = 0$. $\square$

Let H be the cyclic subgroup of $\pi_1(M)$ which is generated by τ. Obviously H is a normal subgroup of $\pi_1(M)$. Then the order of H (i.e. $\#H$) is $|c|$. By using the Reidemeister Schreier rewriting process, we can prove the following Lemma.

LEMMA 5.4. *Define* $G = \pi_1(M) =< \alpha, \beta, \tau | \tau^c = 1, \alpha^{-1}\tau\alpha = \tau^d, \beta^{-1}\tau\beta = \tau^s, \alpha^{-1}\beta^{-1}\alpha\beta = \tau^{-n} >$ $(s = \pm 1)$ *and let* H *be a normal subgroup of* G *which is generated by* τ*. Then the order of the cyclic group* H *(i.e.* $\#H$*) is* $|c|$*. That is,* $H =< \tau | \tau^c = 1 >$*.*

In what follows, let us consider an invariant.

LEMMA 5.5. *Define* $\pi_1(M) =< \alpha, \beta, \tau | \tau^c = 1, \alpha^{-1}\tau\alpha = \tau^d, \beta^{-1}\tau\beta = \tau^s, \alpha^{-1}\beta^{-1}\alpha\beta = \tau^{-n} >$ *and* $\pi_1(M') =< \alpha', \beta', \tau' | \tau'^{c'} = 1, \alpha'^{-1}\tau'\alpha' = \tau'^{d'}, \beta'^{-1}\tau'\beta' = \tau'^{s'}, \alpha'^{-1}\beta'^{-1}\alpha'\beta' = \tau'^{-n'} >$*. If* $\pi_1(M)$ *is isomorphic to* $\pi_1(M')$*, then* $|c| = |c'|$*. That is, this number is an invariant of the group which has a presentation as above.*

PROOF. Let H (resp. H') be the subgroup of $\pi_1(M)$ (resp. $\pi_1(M')$) which is generated by τ (resp. τ'). By Lemma 5.4, $H =< \tau | \tau^c = 1 >$

and $H' =< \tau' | \tau'^{c'} = 1 >$ hold. On the other hand, let $Tor(\pi_1(M))$ (resp. $Tor(\pi_1(M')))$ be the subset of $\pi_1(M)$ (resp. $\pi_1(M')$) which consists of elements of $\pi_1(M)$ (resp. $\pi_1(M')$) whose order is finite. By Proposition 5.1, $Tor(\pi_1(M)) \subset H =< \tau | \tau^c = 1 >$ (resp. $Tor(\pi_1(M')) \subset H' =< \tau | \tau'^{c'} = 1 >$) holds. Obviously,

$$Tor(\pi_1(M)) \cong \begin{cases} \boldsymbol{Z}/|c|\boldsymbol{Z} & |c| \neq 0, \\ < 1 > & |c| = 0, \end{cases}$$

and

$$Tor(\pi_1(M')) \cong \begin{cases} \boldsymbol{Z}/|c'|\boldsymbol{Z} & |c'| \neq 0, \\ < 1 > & |c'| = 0 \end{cases}$$

hold. ($Tor(\pi_1(M))$ (resp. $Tor(\pi_1(M')))$ is the subgroup of $\pi_1(M)$ (resp. $\pi_1(M')$).)

Since $\pi_1(M)$ is isomorphic to $\pi_1(M')$, $Tor(\pi_1(M))$ is isomorphic to $Tor(\pi_1(M'))$. Thus

$$\begin{cases} |c'| = |c| & |c| \geq 2, \\ Tor(\pi_1(M)) = Tor(\pi(M')) =< 1 > & |c| = 0, 1. \end{cases}$$

Thus we have only to examine the case when $|c| = 0$ or 1. That is, we have only to show that $G_0 =< \alpha, \beta, \tau | \tau^0 = 1, \alpha^{-1}\tau\alpha = \tau^d, \beta^{-1}\tau\beta = \tau^s, \alpha^{-1}\beta^{-1}\alpha\beta = \tau^{-n} >$ ($c = 0$) is not isomorphic to $G_1 =< \alpha', \beta', \tau' | \tau'^1 = 1, \alpha'^{-1}\tau'\alpha' = \tau'^{d'}, \beta'^{-1}\tau'\beta' = \tau'^{s'}, \alpha'^{-1}\beta'^{-1}\alpha'\beta' = \tau'^{-n'} > \cong \boldsymbol{Z}^2$ ($c' = 1$). If these two groups were isomorphic to each other, then G_0 would have to be abelian, too. Thus $\tau^{d-1} = \tau^{s-1} = \tau^{-n} = 1$. By Lemma 5.4, $H_0 =< \tau | \tau^0 = 1 >$ (H_0 is the subgroup of G_0 which is generated by τ), that is, the order of τ is infinite. Thus $d = 1$, $s = 1$ and $n = 0$ hold. Then $G_0 \cong \boldsymbol{Z}^3$. This is the contradiction. This completes the proof. (Concretely, the invariant $|c|$ is represented by $|c| = \#Tor(\pi_1(M)) - \operatorname{rank} H_1(M) + 2$). $\square$

Remark 6. By seeing the proof of Lemma 5.5, we have that

$Tor(\pi_1(M)) = H$ if $|c| \neq 0$. ($Tor(\pi_1(M)) =< 1 > \neq H =< \tau | \tau^0 = 1 > \cong \boldsymbol{Z}$ if $|c| = 0$.)

Therefore, if $|c| \neq 0$, then the isomorphism $\phi : \pi_1(M') \to \pi_1(M)$ induces the isomorphism between H' and H, that is $\phi(H') = H$. Thus $\phi(\tau') = \tau^t$ is the generator of H, that is, $gcd(t, c) = 1$ holds. (See Proposition 5.2.)

If $|c| = 0$ and $\operatorname{rank} H_1(M) = 2$, then there exist t, $t' \in \boldsymbol{Z}$ such that $\phi \circ \phi^{-1}(\tau) = (\tau^t)^{t'}$, where $\phi : \pi_1(M') \to \pi_1(M)$ is the isomorphism. (See Proposition 5.2.) $t = 1$ holds. That is, $t = 1$ or -1. Therefore, $gcd(t, c) = 1$ holds.

LEMMA 5.6. *Let* $\phi : \pi_1(M') \to \pi_1(M)$ *be an isomorphism.* *(*$\operatorname{rank} H_1(M) = 2$.*) Then there exists some* $\begin{bmatrix} u & v \\ u' & v' \end{bmatrix} \in GL(2, \boldsymbol{Z})$ *so that*

$$d^u s^v \equiv d' \mod |c|,$$

$$d^{u'} s^{v'} \equiv s' \mod |c|$$

hold.

REMARK 7. Since $ad - bc = 1$, $ad \equiv 1 \mod |c|$ holds. That is, $d^{-1} \equiv a \mod |c|$ holds. Since, $s = 1$ or -1, $s^{-1} = s (= p)$ holds.

PROOF OF LEMMA 5.6. By Proposition 5.1 and Proposition 5.2, we can write $\phi(\alpha') = \alpha^u \beta^v \tau^w$, $\phi(\beta') = \alpha^{u'} \beta^{v'} \tau^{w'}$, $\phi(\tau') = \tau^t$. Since the images of relators are 1, $\phi(\alpha'^{-1} \tau' \alpha') = \phi(\tau'^{d'})$ holds.

Hence

$$\begin{cases} (\tau^t)^{d^u s^v} = (\tau^t)^{d'} & u \geq 0, \\ (\tau^t)^{a^{-u} s^v} = (\tau^t)^{d'} & u \leq 0 \end{cases}$$

hold. (We note that $s^{-1} = p$ holds by Remark 7.)

By Remark 6, $gcd(t, c) = 1$ holds. By Lemma 5.4, the order of τ is $|c|$. Therefore,

$$\begin{cases} d^u s^v \equiv d' \mod |c| & u \geq 0, \\ a^{-u} s^v \equiv d' \mod |c| & u \leq 0 \end{cases}$$

hold.

By Remark 7, $a^{-1} \equiv d \mod |c|$. Thus regardless of the sign of u, $d^u s^v \equiv d' \mod |c|$ holds. Likewise we have $d^{u'} s^{v'} \equiv s' \mod |c|$ from the relations $\beta'^{-1} \tau' \beta' = \tau'^{s'}$.

We will show $\begin{bmatrix} u & v \\ u' & v' \end{bmatrix} \in GL(2, \boldsymbol{Z})$ (i.e. $det \begin{bmatrix} u & v \\ u' & v' \end{bmatrix} = \pm 1$). By Proposition 5.1 and Proposition 5.2, we can write $\phi^{-1}(\alpha) = \alpha'^{\bar{u}} \beta'^{\bar{v}} \tau'^{\bar{u}}$, $\phi^{-1}(\beta) = \alpha'^{\bar{u'}} \beta'^{\bar{v'}} \tau'^{\bar{w'}}$, $\phi^{-1}(\tau) = \tau'^{\bar{t}}$.

Since $\phi \circ \phi^{-1}(\alpha) = \alpha$, $(\alpha^u\beta^v\tau^w)^{\bar{u}}(\alpha^{u'}\beta^{v'}\tau^{w'})^{\bar{v}}(\tau^t)^{\bar{w}} = \alpha$, then $\alpha^{u\bar{u}+u'\bar{v}}\beta^{v\bar{u}+v'\bar{v}}\tau^* = \alpha$. Thus $u\bar{u}+u'\bar{v} = 1$, $v\bar{u}+v'\bar{v} = 0$ hold. Likewise we have $u\bar{u}'+u'\bar{v}' = 0$, $v\bar{u}'+v'\bar{v}' = 0$ from $\phi\circ\phi^{-1}(\beta) = \beta$. Therefore, $\begin{bmatrix} \bar{u} & \bar{v} \\ \bar{u}' & \bar{v}' \end{bmatrix} \begin{bmatrix} u & v \\ u' & v' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Hence $det \begin{bmatrix} u & v \\ u' & v' \end{bmatrix} = \pm 1$ holds. □

Lemma 5.7. *Let $\phi : \pi_1(M') \to \pi_1(M)$ be an isomorphism. (*$\operatorname{rank} H_1(M) = 2$.*) Then if $n = 0$,*

$$n' \equiv 0 \mod gcd(c', d'-1, s'-1)$$

holds.

Proof. By Proposition 5.1 and Proposition 5.2, we can write $\phi(\alpha') = \alpha^u\beta^v\tau^w$, $\phi(\beta') = \alpha^{u'}\beta^{v'}\tau^{w'}$, $\phi(\tau') = \tau^t$. Since $\alpha'^{-1}\beta'^{-1}\alpha'\beta' = \tau'^{-n'}$, $\alpha'\beta' = \beta'\alpha'\tau'^{-n'}$. Thus $\phi(\alpha')\phi(\beta') = \phi(\beta')\phi(\alpha')\phi(\tau')^{-n'}$. That is, $(\alpha^u\beta^v\tau^w)(\alpha^{u'}\beta^{v'}\tau^{w'}) = (\alpha^{u'}\beta^{v'}\tau^{w'})(\alpha^u\beta^v\tau^w)(\tau^t)^{-n'}$.

The left-hand side $= \begin{cases} \alpha^u\beta^v\alpha^{u'}\beta^{v'}\tau^{wd^{u'}s^{v'}+w'} & u' \geq 0, \\ \alpha^u\beta^v\alpha^{u'}\beta^{v'}\tau^{wa^{-u'}s^{v'}+w'} & u' \leq 0. \end{cases}$

Similarly,

the right-hand side$= \begin{cases} \alpha^{u'}\beta^{v'}\alpha^u\beta^v\tau^{w'd^us^v+w-t} & u \geq 0, \\ \alpha^{u'}\beta^{v'}\alpha^u\beta^v\tau^{w'a^{-u}s^v+w-t} & u \leq 0. \end{cases}$

Since $n = 0$ and $\alpha^{-1}\beta^{-1}\alpha\beta = 1$, by Lemma 5.4,

$wd^{u'}s^{v'} + w' \equiv w'd^us^v + w - t \mod |c|$

i.e. $w(d^{u'}s^{v'} - 1) - w'(d^us^v - 1) + t \equiv 0 \mod |c|$

holds regardless of the signs of u and u'.

By Lemma 5.6, $w(s'-1) - w'(d'-1) + t^{n'} \equiv 0 \mod |c|$ holds.

Hence $t \equiv 0 \mod gcd(c, d'-1, s'-1)$ holds.

By Remark 6, $n' \equiv 0 \mod gcd(c, d'-1, s'-1)$ holds.

By Lemma 5.5, $n' \equiv 0 \mod gcd(c', d'-1, s'-1)$ holds. □

Remark 8. It does not go well that we try to remove the assumption $n = 0$ from Lemma 5.7. For example, we have the following result by practicing the same method of Lemma 5.6 and Lemma 5.7 for the relation $\alpha'^{-1}\beta'^{-1}\alpha'\beta' = \tau^{-n'}$.

RESULT. *Let* $\phi : \pi_1(M') \to \pi_1(M)$ *be an isomorphism (*$\operatorname{rank} H_1(M) = 2$*). Then*

$$\begin{cases} (1-d)n' \equiv 0 \mod gcd(c', d'-1, s'-1) & s = 1, \\ (1-d)n' \equiv 0 \mod gcd(2, c', d'-1, s'-1) & s = -1 \end{cases}$$

hold.

However, these are identical equations if $d = d'$. Thus these do not match the methods of this paper.

THEOREM 5.1. *Let* $f : M \to B$ *be a good torus fibration. Suppose that each singular fiber is non-multiple even twin singular fiber, and that* $e(M) = 2$ *and* $g(B) = 1$*. Then if* $\pi_1(M)$ *is isomorphic to the fundamental group of one of the following manifolds, then* M *is diffeomorphic to that selected manifold.*

Class 1. $M_{c,1,0} = M(\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; C; \begin{bmatrix} 0 \\ 0 \end{bmatrix})$

$(c \geq 0)$

$\pi_1(M_{c,1,0}) = < \alpha, \beta, \tau | \tau^c = 1, [\tau, \alpha] = [\tau, \beta] = [\alpha, \beta] = 1 >$.

Class 1'. $M_{c,1,n} = M(\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; C; \begin{bmatrix} 0 \\ n \end{bmatrix})$

$(c = 2n,\ 3n,\ 4n,\ 6n,\ n \geq 1)$

$\pi_1(M_{c,1,n}) = < \alpha, \beta, \tau | \tau^c = 1, [\tau, \alpha] = [\tau, \beta] = 1, [\alpha, \beta] = \tau^{-n} >$.

Class 2. $M_{c,d,0} = M(\begin{bmatrix} a_0 & b_0 \\ c & d \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; C; \begin{bmatrix} 0 \\ 0 \end{bmatrix})$

$(1 < d,\ d+1 < c,\ c | d^2 - 1$ *(*c *is the natural number which is larger than* $d+1$ *and is divisor of* $d^2 - 1$*.),* (a_0, b_0) *is the pair* (a, b) *whose* a_0 *is the smallest natural number* a *which satisfies* $ad - bc = 1$*.)*

$\pi_1(M_{c,d,0}) = < \alpha, \beta, \tau | \tau^c = 1, \alpha^{-1}\tau\alpha = \tau^d, [\tau, \beta] = [\alpha, \beta] = 1 >$.

Class 2'. $M_{c,d,n} = M(\begin{bmatrix} a_0 & b_0 \\ c & d \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; C; \begin{bmatrix} 0 \\ n \end{bmatrix})$

$(1 < d,\ d+1 < c,\ c | d^2 - 1,\ gcd(c, d-1) = 2n,\ 3n,\ 4n,\ 6n,\ n \geq 1$ *,* (a_0, b_0) *is the pair* (a, b) *whose* a_0 *is the smallest natural number* a *which satisfies* $ad - bc = 1$*.)*

$\pi_1(M_{c,d,n}) = < \alpha, \beta, \tau | \tau^c = 1, \alpha^{-1}\tau\alpha = \tau^d, [\tau, \beta] = 1, [\alpha, \beta] = \tau^{-n} >$.

Class 3. $M_{0,1,n} = M(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; C; \begin{bmatrix} 0 \\ n \end{bmatrix})$

$(n \geq 1)$
$\pi_1(M_{0,1,n}) =< \alpha, \beta, \tau | [\tau, \alpha] = [\tau, \beta] = 1, [\alpha, \beta] = \tau^{-n} >.$
Class 4. $M^{-1}_{0,1,n} = M(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}; C; \begin{bmatrix} 0 \\ n \end{bmatrix})$
$(n = 0,\ 1)$
$\pi_1(M^{-1}_{0,1,n}) =< \alpha, \beta, \tau | [\tau, \alpha] = 1, \beta^{-1}\tau\beta = \tau^{-1}, [\alpha, \beta] = \tau^{-n} >.$

Here $e(M)$, $g(B)$ and $\pi_1(M)$ are the Euler number of M, the genus of B and the fundamental group of M, respectively.

Conversely, these manifolds have such GTF structures.

Furthermore these fundamental groups are not isomorphic to each other.

In particular, these manifolds are not homeomorphic to each other.

Proof. First let us consider the Class 1. By Theorem 4.1, if $\operatorname{rank} H_1(M) = 3$, then $M = M_{0,1,0}$. Thus we have only to examine the case when $\operatorname{rank} H_1(M) = 2$.

By explaining in the beginning of this section and Lemma 5.2, we can assume that $M = M(\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}, E; C; \begin{bmatrix} 0 \\ n' \end{bmatrix})$ $(s' = 1)$ or $M = M(\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}, -E; C; \begin{bmatrix} 0 \\ n' \end{bmatrix})$ $(s' = -1)$. By applying Lemma 5.1 (i) for $\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, we can see that $M(\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}, \pm E; C; \begin{bmatrix} 0 \\ n' \end{bmatrix})$ is diffeomorphic to $M(\begin{bmatrix} d' & -b' \\ -c' & a' \end{bmatrix}, \pm E; C; \begin{bmatrix} 0 \\ n' \end{bmatrix})$. Thus we can assume that $c' \geq 0$ without loss of generality. By Lemma 5.3 (i), we can assume that $0 \leq d' < c'$, provided that we assume that $d' = \pm 1$ if $c' = 0$. By Lemma 5.1 (iii), we can assume $0 \leq n' < gcd(c', d' - 1, s' - 1)$ provided that we assume that $n' \geq 0$ if $gcd(c', d' - 1, s' - 1) = 0$. Therefore, the total space M is diffeomorphic to $M(\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}, \begin{bmatrix} s' & 0 \\ 0 & s' \end{bmatrix}; C; \begin{bmatrix} 0 \\ n' \end{bmatrix})$, $c' \geq 0$, $0 \leq d' < c'$ if $c' \neq 0$, $d' = \pm 1$ if $c' = 0$, $0 \leq n' < gcd(c', d' - 1, s' - 1)$ if $gcd(c', d' - 1, s' - 1) \neq 0$, $0 \leq n'$ if $gcd(c', d' - 1, s' - 1) = 0$, $s' = \pm 1 \cdots (*)$.

Hereafter by writing this c', d', n', s' by c', d', n', s' of $(*)$, we will distinguished from c', d', n' s' of $G^{s'}_{c',d',n'}$ which will be determined as below.

Denote $G^{s'}_{c',d',n'} =< \alpha', \beta', \tau' | \tau'^{c'} = 1, \alpha'^{-1}\tau'\alpha' = \tau'^{d'}, \beta'^{-1}\tau'\beta' = \tau'^{s'}, \alpha'^{-1}\beta'^{-1}\alpha'\beta' = \tau'^{-n'} >$. Denote $G_{c',d',n'} = G^1_{c',d',n'}$. Denote $\mathcal{G} = < G^{s'}_{c',d',n'} | c' \geq 0, 0 \leq d' < c'$ if $c' \neq 0$, $d' = \pm 1$ if $c' = 0$, $0 \leq n' < gcd(c', d' -$

$1, s'-1)$ if $gcd(c', d'-1, s'-1) \neq 0$, $0 \leq n'$ if $gcd(c', d'-1, s'-1) = 0 >$. $\mathcal{G}$ is a set of presentations. For any M, $\pi_1(M)$ is isomorphic to a certain element of $\mathcal{G}$. Conversely, for any element $G^{s'}_{c',d',n'}$ of $\mathcal{G}$, this $G^{s'}_{c',d',s'}$ is the isomorphic to the fundamental group of a certain M of $(*)$.

If $G^{s'}_{c',d',n'} \in \mathcal{G}$ and $G_{c,1,0} = \pi_1(M_{c,1,0})$ $(c > 0)$ are isomorphic to each other, then $c' = c$ holds by $c' \geq 0$ and Lemma 5.5. By Lemma 5.6, $1^u 1^v \equiv d' \mod c'$, $1^{u'} 1^{v'} \equiv s' \mod c'$, thus $d' \equiv 1 \mod c'$, $s' \equiv 1 \mod c'$. Since $s' = 1$ or -1, $s' = 1$ and $d' = 1$ hold, if $c \geq 3$. By Lemma 5.7, $n' = 0$ holds. Hence if $c \geq 3$, then any element of $\mathcal{G}$ which is isomorphic to $G_{c,1,0} = \pi_1(M_{c,1,0})$ is itself. For a group G, we denote the set which consists of all elements of $\mathcal{G}$ which are isomorphic to G by the symbol $<< G >>$. Then for $c \geq 3$,

$$<< G_{c,1,0} >>=< G_{c,1,0} > (c \geq 3) \cdots (1)$$

holds. Thus, since $\pi_1(M)$ is isomorphic to a certain element of $\mathcal{G}$, if $\pi_1(M)$ is isomorphic to $G_{c,1,0} = \pi_1(M_{c,1,0})$ $(c \geq 3)$, then c', d', n', s' of $(*)$ are satisfy $c' = c$, $d' = 1$, $n' = 0$, $s' = 1$. That is, $M = M(\begin{bmatrix} a' & b' \\ c & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; C; \begin{bmatrix} 0 \\ 0 \end{bmatrix})$ $(c \geq 3)$.

Since $a' \cdot 1 - b'c = 1$, by the Euclidean algorithm, we can write $a' = 1 + kc$, $b' = 0 + k \cdot 1$ (k is an integer). By Lemma 5.3 (ii), $M(\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; C; \begin{bmatrix} 0 \\ 0 \end{bmatrix})$ $(c \geq 3)$ is diffeomorphic to $M(\begin{bmatrix} a' & b' \\ c & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; C; \begin{bmatrix} 0 \\ \bar{n} \end{bmatrix})$ $(c \geq 3)$. Again, by Lemma 5.7, $\bar{n} = 0$. Thus for $c \geq 3$, $M = M_{c,1,0}$, that is $M = M(\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; C; \begin{bmatrix} 0 \\ 0 \end{bmatrix})$.

Let us consider the case when $c < 3$. That is, $c = 1$ or 2. (Recall that $\operatorname{rank} H_1(M_{0,1,0}) = 3$.) If $\pi_1(M)$ is isomorphic to $G_{1,1,0} = \pi_1(M_{1,1,0})$ $(c = 1)$, then c', d' ,n', s' of $(*)$ satisfy the condition $c' = 1$, $d' = 0$ $n' = 0$, $s' = 1$ or -1 as above. By Lemma 5.3 (ii) and Lemma 5.7, M $(c = 1)$ is diffeomorphic to either $M(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; C; \begin{bmatrix} 0 \\ 0 \end{bmatrix})$ or $M(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}; C; \begin{bmatrix} 0 \\ 0 \end{bmatrix})$. Since $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, by Lemma 5.1 (i), these two manifolds are diffeomorphic to each other. By Lemma 5.3 (ii) and Lemma 5.7, M is diffeomorphic to $M_{1,1,0}$. That is,

$M = M(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; C; \begin{bmatrix} 0 \\ 0 \end{bmatrix})$. Simultaneously, we have

$$<< G_{1,1,0} >>=< G_{1,1,0}, G_{1,1,0}^{-1} > \cdots (2).$$

In the case when c=2, M is diffeomorphic to either $M(\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};$ $C; \begin{bmatrix} 0 \\ 0 \end{bmatrix})$ or $M(\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}; C; \begin{bmatrix} 0 \\ 0 \end{bmatrix})$, and $\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, thus M is diffeomorphic to $M_{2,1,0}$ as above, that is, $M = M(\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; C; \begin{bmatrix} 0 \\ 0 \end{bmatrix})$. We have

$$<< G_{2,1,0} >>=< G_{2,1,0}, G_{2,1,0}^{-1} > \cdots (3).$$

This completes the proof of the Class 1.

If $G^{s'}_{c',d',n'} \in \mathcal{G}$ is isomorphic to $G_{c,d,0} = \pi_1(M_{c,d,0})$ $(1 < d,\ d+1 < c,$ $c|d^2-1)$, then $c' = c$ by $c' \geq 0$ and Lemma 5.5. By Lemma 5.6, $d^u 1^v \equiv d'$ mod c', $d^{u'} 1^{v'} \equiv s' \mod c'$. Since $c|d^2-1$, $d^2 \equiv 1 \mod c$. Thus $d' \equiv 1$ or d mod c' and $s' \equiv 1$ or $d \mod c'$ hold. Since $1 < d < c-1$, $d \not\equiv -1 \mod c'$ holds. Since $s' = 1$ or -1, $s' = 1$. d' is either 1 or d. By Lemma 5.7, $n' = 0$ holds. Since $1 < d$ and $d+1 < c$ hold, $c > 3$ holds. Thus for c, d $(1 < d,$ $d+1 < c,\ c|d^2-1)$, we have $<< G_{c,d,0} >> \cap << G_{c,1,0} >>= \phi$ by (1). Hence d' does not equal to 1,

$$<< G_{c,d,0} >>=< G_{c,d,0} > (1 < d, d+1 < c,\ c|d^2-1) \cdots (4)$$

holds. Therefore, if $\pi_1(M)$ is isomorphic to $G_{c,d,0} = \pi_1(M_{c,d,0})$ $(1 < d,$ $d+1 < c,\ c|d^2-1)$, c', d', n', s' of $(*)$ satisfy the conditions $c' = c$, $d' = d$, $n' = 0$, $s' = 1$. Since $a'd - b'c = 1$, by the Euclidean algorithm, we can write $a' = a_0 + ck$, $b' = b_0 + dk$ (k is an integer, and (a_0, b_0) is the pair (a, b) whose a_0 is the smallest natural number a which satisfies $ad-bc = 1$). Therefore, for c, d $(1 < d,\ d+1 < c,\ c|d^2-1)$, $M = M_{c,d,0}$ as above. That is, $M = M(\begin{bmatrix} a_0 & b_0 \\ c & d \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; C; \begin{bmatrix} 0 \\ 0 \end{bmatrix})$ $(1 < d,\ d+1 < c,\ c|d^2-1)$. This completes the proof of the Class 2.

If $G^{s'}_{c',d',n'} \in \mathcal{G}$ is isomorphic to $G_{0,1,n} = \pi_1(M_{0,1,n})$ $(n \geq 1)$, we have $c' = 0$, $d' = 1$, $s' = 1$ as above. Since $G_{0,1,n}/[G_{0,1,n}, G_{0,1,n}] \cong \boldsymbol{Z}^2 \oplus \boldsymbol{Z}/n$, for $n \geq 1$,

$$<< G_{0,1,n} >>=< G_{0,1,n} > (n \geq 1) \cdots (5)$$

holds. Thus, if $\pi_1(M)$ is isomorphic to $G_{0,1,n} = \pi_1(M_{0,1,n})$ $(n \geq 1)$, c', d', n', s' of $(*)$ satisfy the conditions $c' = 0$, $d' = 1$, $n' = n$, $s' = 1$, and we have $M = M_{0,1,n}$ $(n \geq 1)$ as above. This completes the proof of the Class 3.

If $\pi_1(M)$ is isomorphic to $G^{-1}_{0,1,n} = \pi_1(M^{-1}_{0,1,n})$ $(n = 0, 1)$, c', d', s' of $(*)$ satisfy that $c' = 0$, $d' = 1$ or -1, $s' = 1$ or -1. We note that $n' = 0$ or -1. By Lemma 5.7, $n' = n$. Since the matrix $\begin{bmatrix} u & v \\ u' & v' \end{bmatrix}$ of Lemma 5.6 is an element of $GL(2, \boldsymbol{Z})$, $(d', s') = (-1, 1)$, $(1, -1)$ or $(-1, -1)$. Thus M is diffeomorphic to $M^1_{0,-1,n}$, $M^{-1}_{0,1,n}$ or $M^{-1}_{0,-1,n}$. Simultaneously, we have $<< G^{-1}_{0,1,n} >>\subset< G^1_{0,-1,n}, G^{-1}_{0,1,n}, G^{-1}_{0,-1,n} >$ for $n = 0$, 1. By Lemma 5.1 (i), $M^1_{0,-1,n} = M^{-1}_{0,1,n} = M^{-1}_{0,-1,n}$, thus $M = M^{-1}_{0,1,n}$, and for $n = 0$, 1,

$$<< G^{-1}_{0,1,n} >>=< G^1_{0,-1,n}, G^{-1}_{0,1,n}, G^{-1}_{0,-1,n} > (n = 0, 1) \ldots (6)$$

holds. This completes the proof of the Class 4.

If $G^{s'}_{c',d',n'} \in \mathcal{G}$ is isomorphic to $G_{c,1,n} = \pi_1(M_{c,1,n})$ $(c = 2n, 3n, 4n, 6n, n \geq 1)$, the same argument of the Class 1 shows that $c' = c$, $d' = 1$, $s' = 1$. Since $G_{c,1,n}/[G_{c,1,n}, G_{c,1,n}] \cong \boldsymbol{Z}^2 \oplus \boldsymbol{Z}/gcd(c, n)$ is isomorphic to $G_{c,1,n'}/[G_{c,1,n'}, G_{c,1,n'}] \cong \boldsymbol{Z}^2 \oplus \boldsymbol{Z}/gcd(c, n')$, $gcd(c, n) = gcd(c, n')$ holds. Let us seek c and n which satisfy that the number of n' satisfying $gcd(c, n) = gcd(c, n')$ and $1 \leq n' < c$ is at most 2. Define $k = gcd(c, n) = gcd(c, n')$, $\bar{c} = \frac{c}{k}$, $\bar{n} = \frac{n}{k}$, $\bar{n}' = \frac{n'}{k}$. Since $gcd(\bar{c}, \bar{c}) = \bar{c}$, if $\bar{c} \neq 1$, then the number of $\bar{n}'$ satisfying $gcd(\bar{c}, \bar{n}') = 1$ and $1 \leq \bar{n}' < \bar{c}$ is represented by the Euler function $\phi(\bar{c})$. If $\bar{c}$ is 1, then it is 0. Since $\phi(\bar{c})$ is the multiplicative function and if p is a prime, then $\phi(p^n) = p^n - p^{n-1}$, by denoting the canonical decomposition of $\bar{c}$ by $\bar{c} = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, $\phi(\bar{c}) = (p_1^{\alpha_1} - p_1^{\alpha_1 - 1})(p_2^{\alpha_2} - p_2^{\alpha_2 - 1}) \cdots (p_k^{\alpha_k} - p_k^{\alpha_k - 1})$ holds. There must be no prime factor of $\bar{c}$ which is larger than 3, for the sake of satisfying $\phi(\bar{c}) = 1$ or 2. Therefore, $\bar{c} = 2$, 3, $2^2 = 4$ or $2 \cdot 3 = 6$, $\bar{n} = 1$ or $\bar{c} - 1$. Hence (c, n) which we require are $(2n, n)$, $(3n, n)$, $(3n, 3n - n)$, $(4n, n)$, $(4n, 4n - n)$, $(6n, n)$ and $(6n, 6n - n)$. It means that $<< G_{c,1,n} >>\subset< G_{c,1,n}, G_{c,1,c-n} >$ $(c = 2n, 3n, 4n, 6n, n \geq 1)$. Simultaneously, it means that if $\pi_1(M)$ is isomorphic to $G_{c,1,n} = \pi_1(M_{c,1,n})$

$(c = 2n, 3n, 4n, 6n, n \geq 1)$, M is diffeomorphic to either $M_{c,1,n}$ or $M_{c,1,c-n}$. By Lemma 5.1 (ii) ($P = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$), Proposition 3.1 and Lemma 5.1 (iii), $M_{c,1,n} = M_{c,1,c-n}$ $(n|c,\ n \geq 1)$. Thus M is diffeomorphic to $M_{c,1,n}$ $(c = 2n, 3n, 4n, 6n,)$ and

$$<< G_{c,1,n} >>=< G_{c,1,n}, G_{c,1,c-n} >$$

$$(c = 2n, 3n, 4n, 6n, n \geq 1) \cdots (7)$$

holds. This completes the proof of the Class 1'.

The same argument of the Class 1' shows the Class 2'.

$$<< G_{c,d,n} >>=< G_{c,d,n}, G_{c,d,c-n} >$$

$$(1 < d, d+1 < c, c|d^2 - 1, gcd(c, d-1) = 2n, 3n, 4n, 6n, n \geq 1) \ldots (8)$$

holds. This completes the proof of all Classes.

The existing GTF structures is trivial as in the previous section. We can see that each fundamental group is not isomorphic to each other from the reason that there is no common elements in (1), (2), (3), (4), (5), (6), (7), (8). This completes the proof. □

Note that in Theorem 5.1, the assumption $g(B) = 1$ is meaningless, by Proposition 4.2.

By using a same method of [4] and [5], we can easily prove the following theorem (the case of I_1^+ and I_1^-). Hence we omit the proof of the following theorem.

THEOREM 5.2. *The same result of Theorem 5.1 holds if we change the clause "Suppose that each singular fiber is non-multiple even twin singular fiber, and that $e(M) = 2$ and $g(B) = 1$" in Theorem 5.1, to "Suppose that each singular fiber is neither multiple nor odd twin singular fiber, and that $e(M) = 2$, $g(B) = 1$ and $\sigma(M) = 0$."*

REMARK 9. Unfortunately the results of this section do not mean that all GTF's which are not treated in this section and satisfy $e(M) = 2$ and $g(B) = 1$ have a certain exotic GTF.

Finally we will show an Example. We consider the diffeomorphism type of a GTF whose invariant which is determined by Lemma 5.5 equals 3. That is, $\pi_1(M)$ has the presentation $< \alpha, \beta, \tau | \tau^3 = 1, \alpha^{-1}\tau\alpha = \tau^d, \beta^{-1}\tau\beta = \tau^s,$ $\alpha^{-1}\beta^{-1}\alpha\beta = \tau^{-n} >$ $(s = \pm 1)$. The following Example contains the new result in addition to the result of Theorem 5.1 (Theorem 5.2).

Example 5.1. A GTF whose invariant which is determined by Lemma 5.5 equals 3 and satisfying the assumption of Theorem 5.1 (Theorem 5.2) is diffeomorphic to the following one:

$$\begin{aligned}
&(1)\ M(\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; C; \begin{bmatrix} 0 \\ 0 \end{bmatrix}),\\
&(2)\ M(\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; C; \begin{bmatrix} 0 \\ 1 \end{bmatrix}),\\
&(3)\ M(\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}; C; \begin{bmatrix} 0 \\ 0 \end{bmatrix}),\\
&(4)\ M(\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}; C; \begin{bmatrix} 0 \\ 1 \end{bmatrix}).
\end{aligned}$$

Thus there are 4 types of such GTF's.

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Yatomikita Junior High School
Ooaza Kamakura 62, Yatomi-machi
Kaifu-gun, Aichi Prefecture, 496, Japan