J. Math. Sci. Univ. Tokyo
8 (2001), 357–363.

Regularity of Weak Solutions of Semilinear Elliptic Differential Equations

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Abstract. We consider elliptic systems of semilinear differential equations with nonlinearity of polynomial growth. We then prove that any locally p-integrable weak solution is smooth if $p \gg 1$. We give a lower bound, optimal in the general setting, of the exponent p.

Introduction

Let M be an open domain in $\boldsymbol{R}^n$. Let $P(x,D)$ be an elliptic differential operator with smooth coefficients. Consider a polynomial $F(x,u)$ in u with smooth coefficients in x defined on M and the nonlinear differential equation

$$P(x,D)u + F(x,u) = 0. \tag{0.1}$$

If $p \geq \deg F$ and u is a locally p-integrable function, $F(x,u(x))$ defines a locally integrable function, and we say that u is a weak solution to (0.1) if the equation holds in the distribution sense. The weak solutions in this sense are not smooth in general notwithstanding the ellipticity of the equation. It is a basic problem to see if the weak solutions u are smooth or not, and this is discussed in several particular situations in many papers. The literature on singular solutions to equations of type (0.1) is huge (and the results are deep). However, in these literatures, we can not find any regularity theorem proved for (0.1) in a general setting. (Cf. however [T2, Chapter 14] for a systematic discussion on methods of establishing regularity of solutions to nonlinear elliptic equations.)

The purpose of this paper is to establish a regularity theorem in a simple, general setting (i.e., without any technical assumption) for semilinear

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2000 *Mathematics Subject Classification*. 35B65, 35D10, 35J60.
Key words: Weak solution, semilinear elliptic differential equation.
This work was supported by the Grant-In-Aid for the Sciences of the Sumitomo Foundation 1996-1997.

elliptic equations with polynomial nonlinearity, and we prove the following quite general result.

Let $m = \operatorname{ord} P$. We always assume $m \geq 1$. Let F be a continuous (resp. smooth[1]) function defined on $M \times \boldsymbol{C}$. Let γ be a non-negative real number, and assume

$$|F(x,u)| \leq A|u|^\gamma + B, \quad \text{with } A,\ B \geq 0.$$

We then have

Theorem 0.1. *Let $p \geq \max\{1, \gamma\}$ and assume*

$$p > (\gamma - 1)n/m. \tag{0.2}$$

If a function u in $\mathcal{L}_{p,\mathrm{loc}}(M)$ satisfies

$$P(x,D)u + F(x,u) = 0 \quad \text{in } \mathcal{D}'(M), \tag{0.3}$$

then u is in $\mathcal{L}_q^m$ for any $q > 1$ in M (resp. smooth in M in the case where F is smooth).

Corollary 0.2. *Let $\gamma \geq 1$. Assume one of the following:* (a) $m \geq n$, *or* (b) $m < n$ *and* $\gamma < n/(n-m)$. *If $u \in \mathcal{L}_{\gamma,\mathrm{loc}}(M)$ and satisfies* (0.3), *u is in $\mathcal{L}_q^m$ for any $q > 1$ (resp. smooth) in M.*

$\mathcal{L}_q^m$ denotes the sheaf of germs of functions u which are locally in the Sobolev space of order m in the sense of L_q (i.e., $D^\alpha u$ is locally q-integrable if $|\alpha| \leq m$).

It is immediate to extend Theorem 0.1 to overdetermined elliptic systems. See Section 1 for the precise statement. Note however that we need a microlocal point of view in order to treat the overdetermined case.

As we shall see in Section 3, if $p < (\gamma - 1)n/m$, we can not expect in general to establish regularity of locally p-integrable weak solutions of equation (0.1). Hence (0.2) is the optimal bound of the exponent p (except for the equality case).

Corollary 0.2 is a quantitative description of the well-known (but not proved in the literature) fact that the solutions to elliptic differential equations are smooth if the nonlinearity is weak.

[1]In this paper, a smooth function means a function of class C^∞.

1. Main Result

Let M be an open domain in $\boldsymbol{R}^n$.

Notation. $\mathcal{D}'$ denotes the sheaf of distributions on M. For $p \geq 1$, $\mathcal{L}_{p,\mathrm{loc}}$ denotes the sheaf of locally p-integrable functions, and $\mathcal{L}_{p+0,\mathrm{loc}}$ the union of $\mathcal{L}_{q,\mathrm{loc}}$, $q > p$. $\mathcal{L}_{\infty,\mathrm{loc}}$ denotes the sheaf of locally bounded functions.

Let $P(x, D)$ be an $n_1 \times n_0$ matrix of differential operators on M with smooth coefficients. In the following m denotes the order (the maximum of the orders of matrix elements) of P. Let $F(x, u)$ be a smooth function (valued in $\boldsymbol{C}^{n_1}$) defined on $M \times \boldsymbol{C}^{n_0}$.

For an n_0-vector valued unknown function u in x, we consider the differential equation

$$P(x, D)u + F(x, u) = 0. \tag{1.1}$$

Let T^*M denote the cotangent bundle of M, $\pi : T^*M \to M$ the projection. In order to make a precise definition of the ellipticity of P, let us set $E = M \times \boldsymbol{C}^{n_0}$ and $E' = M \times \boldsymbol{C}^{n_1}$, and consider them (trivial) vector bundles on M.

Definition 1.1. We say that a matrix of differential operators P of order m is elliptic if its m-symbol $\sigma_m(P)$ yields an injective bundle map $\pi^*E \to \pi^*E'$ outside the zero section of T^*M.

The following theorem is then well known.

Theorem 1.2. *Let P be an elliptic matrix of differential operators of order m in the sense of Definition 1.1. Let $F(x, u)$ be a smooth* (*resp. continuous*) *function in x and u. Let u be a locally bounded function which satisfies equation* (1.1)*. Then u is smooth* (*resp. in $\mathcal{L}_q^m$ for any $q > 1$*) *in M.*

We now assume that there exist a non-negative number γ, $A \in \mathcal{L}_{\infty,\mathrm{loc}}(M)$ and $B \in \mathcal{L}_{n+0,\mathrm{loc}}(M)$ such that

$$|F(x, u)| \leq A(x)|u|^\gamma + B(x). \tag{1.2}$$

Here $|\cdot|$ denotes an hermitian norm of $\boldsymbol{C}^{n_1}$ (resp. of $\boldsymbol{C}^{n_0}$) in the left-hand (resp. right-hand) side.

If $p \geq \max\{1, \gamma\}$, $F(x, u)$ defines a locally integrable section of $\boldsymbol{C}^{n_1}$ for any locally p-integrable function u. The principal result of this paper is

Theorem 1.3. *Let P be an elliptic matrix of differential operators of order m in the sense of Definition 1.1. Let $F(x, u)$ be a smooth* (*resp. continuous*) *function in x and u satisfying* (1.2). *Let $p \geq \max\{1, \gamma\}$ and assume*

$$p > (\gamma - 1)n/m.$$

Let u be a locally p-integrable function which satisfies equation (1.1). *Then u is smooth* (*resp. in $\mathfrak{L}_q^m$ for any $q > 1$*) *in M.*

2. Proof

For $p \geq 1$ and $s \in \boldsymbol{R}$, let $\mathfrak{L}_p^s$ denote the sheaf of germs of distributions which are locally in the Sobolev space of order s in the sense of L_p. $\mathfrak{L}_p^0$ is denoted simply by $\mathfrak{L}_p$.

Let us first recall the following theorem of Meyer.

Lemma 2.1 [M, Theorem 1]. *Let $p > 1$ ($p \neq \infty$), and $s \in \boldsymbol{R}$. Let M be a smooth real manifold of dimension n. Let $F(x, y)$ be a smooth function in $(x, y) \in M \times \boldsymbol{C}^N$. Let $u \in \mathfrak{L}_p^s(M)^{\oplus N}$. If $s > n/p$, then $F(x, u)$ is in $\mathfrak{L}_p^s(M)$.*

For the later use of Theorem 1.2, we recall its proof.

Proof of Theorem 1.2. Let u be a locally bounded $\boldsymbol{C}^{n_0}$-valued function. Then $F(x, u)$ is locally in $\mathfrak{L}_\infty$ and in particular in $\mathfrak{L}_p$, and it follows from the ellipticity of P that u is a section of $\mathfrak{L}_p^m$ for any $p > 1$ (by L_p boundedness of the left microlocal parametrix of P [T1, Chapter XI, Theorem 2.5]). Assume F to be smooth. By Lemma 2.1, $F(x, u)$ is in $\mathfrak{L}_p^m$ if $p > n/m$. It then follows again from the ellipticity of P that u is in $\mathfrak{L}_p^{2m}$ for $p > n/m$. Repeating this argument, we see that

$$u \in \bigcap_{k \in \boldsymbol{Z}} \Gamma(M, \mathfrak{L}_p^k(E)) = \Gamma(M, E)(= \text{the space of smooth sections of } E). \ \square$$

We now prove Theorem 1.3.

Let us recall the following classical fact. We give a simple proof to it.

LEMMA 2.2. *Let M be a domain of dimension n. Then $\mathcal{L}_{1,\mathrm{loc}}(M)$ is contained in $\mathcal{L}_p^s(M)$ if $s+n(1-1/p)<0$ and $p>1$.*

PROOF. This follows from the identity $f=f*\delta(x)$ for $f\in\mathcal{D}'(\boldsymbol{R}^n)$ and the fact that the delta function belongs to $\mathcal{L}_p^s(\boldsymbol{R}^n)$ if $s+n(1-1/p)<0$ and $p>1$. □

PROOF OF THEOREM 1.3. We may assume $\gamma\geq 1$.

Case I: $p>\gamma$. We may assume in (1.2) that $B\in\mathcal{L}_{n'}(M)$ with $n'>n$. Let

$$q=\max\{n/m,1\}.$$

Let u be a locally p-integrable $\boldsymbol{C}^{n_0}$-valued function satisfying (1.1). We may assume $p/\gamma\leq n'$, by making p smaller if necessary. If $p>\gamma$, $F(x,u(x))$ is locally p/γ-integrable, and it follows from the ellipticity of P that u is in $\mathcal{L}_{p/\gamma}^m$. We now prove that, if $p>\gamma$ and $p>(\gamma-1)q$ (which is the hypothesis of Theorem 1.3), u is locally in $\mathcal{L}_{q'}^m$ for some $q'>q$. If $p/\gamma>q$, this is trivial. Suppose $p/\gamma\leq q$. Let us take $\delta>0$ so that

$$(\gamma-p/q)(1+\delta)<1 \quad\text{and}\quad (1+\delta)q<n'.$$

By the Sobolev embedding theorem, if $p/\gamma<q$, we have

$$\mathcal{L}_{p/\gamma}^m\subset\mathcal{L}_{p/(\gamma-p/q)}\subset\mathcal{L}_{(1+\delta)p}.$$

If $p/\gamma=q$, $\mathcal{L}_{p/\gamma}^m$ is a subsheaf of $\mathcal{L}_r$ for any $r>1$; therefore, in particular,

$$\mathcal{L}_{p/\gamma}^m\subset\mathcal{L}_{(1+\delta)p}.$$

Thus u is locally $(1+\delta)p$-integrable on M, and $F(x,u)$ is locally $(1+\delta)p/\gamma$-integrable. It then follows again from the ellipticity of P that

$$u\in\mathcal{L}_{(1+\delta)p/\gamma}^m(M,E).$$

We can repeat this argument as long as $(1+\delta)^Np/\gamma\leq q$ (by replacing p by $(1+\delta)^Np$), and we have the claim above.

By the Sobolev embedding theorem, u is then continuous.

Thus we can assume that u is a continuous function and satisfies (1.1). It then follows from Theorem 1.2 that u is in $\mathfrak{L}_q^m$ for any $q > 1$ and is smooth in the case where F is smooth.

Case II: $p = \gamma$. Let u be a locally γ-integrable function on M and satisfy (1.1). Since $F(x, u(x))$ is a locally integrable function on M, it follows from Lemma 2.2 and L_p boundedness of the left microlocal parametrix of P that u is in $\mathfrak{L}_{q,\mathrm{loc}}(M)$ if $n(1 - 1/q) < m$. Hence u is locally q-integrable for any $q > 1$ if $m \geq n$, and for any $q < n/(n - m)$ if $m < n$. If we assume $\gamma < n/(n - m)$ in the case $m < n$, we are reduced to establishing the regularity of u in the case I. □

Remark. By the proof above, it is sufficient to assume that the nonlinear term F satisfies (1.2) with $B \in \mathfrak{L}_{q+0,\mathrm{loc}}(M)$, where

$$q = \max\{n/m, 1\}.$$

3. An Example

Let U be an open ball in $\boldsymbol{R}^n$ with centre at 0. Let m be a positive even integer, and $\gamma > 1$. Let $\triangle = D_1^2 + \cdots + D_n^2$, $D_i = \partial/\partial x_i$, and consider the differential equation

$$\triangle^k u = u^\gamma, \tag{3.1}$$

with $k = m/2$, on U. If

$$(n - 2j)(\gamma - 1) \neq m \quad \text{for } j = 1, \ldots, k,$$

this equation has a spherically symmetric solution on $U \setminus \{0\}$ of the form

$$u = C\|x\|^{-m/(\gamma-1)} \tag{3.2}$$

with $C \in \boldsymbol{C}$, $C \neq 0$. For any $p < (\gamma - 1)n/m$, u is p-integrable on $U \setminus \{0\}$. If $(n - m)\gamma > n$, since $\gamma < (\gamma - 1)n/m$, u^γ is integrable on $U \setminus \{0\}$.

Suppose now $(n-m)\gamma > n$. Let us denote by the same u the $\mathcal{L}_{\gamma,\mathrm{loc}}$ function on U given by extending u to the whole U. If we can find $p > 1$ so that

$$p < (\gamma - 1)n/m \quad \text{and} \quad p \geq n/(n-m), \tag{3.3}$$

since u is p-integrable and the right-hand side u^γ is integrable, it follows from a theorem of Bochner [B] that u satisfies (3.1) in $\mathcal{D}'(U)$. Since (3.3) is possible if (and only if) $(n-m)\gamma > n$, we always get a non-trivial spherically symmetric solution u on the whole U from (3.2).

Since this u has a singularity of $\|x\|^{-\alpha}$ at $x = 0$, we have

PROPOSITION 3.1. *If $(n-m)\gamma > n$, we have a discontinuous $\mathcal{L}_{\gamma,\mathrm{loc}}$ solution of the differential equation* (3.1) *which is locally p-integrable for any $p < (\gamma - 1)n/m$.*

Compare this with Theorem 0.1, from which it follows that a weak solution of (3.1) is of class $C^{m-\epsilon}$ for any $\epsilon > 0$ (and smooth if γ is an integer) if it is locally p-integrable for any $p > (\gamma - 1)n/m$.

Acknowledgement. The author would like to thank Mitsuru Sugimoto for discussions.

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(Received December 5, 2000)

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