J. Math. Sci. Univ. Tokyo
8 (2001), 317–327.

Analytic Discs Attached to Half Spaces of $\mathbb{C}^n$ *and Extension of Holomorphic Functions*

By Luca Baracco and Giuseppe Zampieri

Abstract. Let M be a real hypersurface of $\mathbb{C}^n$, M^+ a closed half space with boundary M, z_o a point of M. We prove that the existence of a disc A tangent to M at z_o, attached to M^+ but not to M (i.e.$\partial A \subset M^+$ but $\partial A \not\subset M$), entails extension of holomorphic functions from the interior of M^+ to a full neighborhood of z_o. This result covers a result in [9], where the disc A is assumed to lie on one side M^+ of M. The basic idea, which underlies to the whole paper, is due to A. Tumanov [8] and consists in attaching discs to manifolds with boundary. Further, holomorphic extendability by the aid of tangent discs attached to M and of "defect 0" is a particular case of a general theorem of "wedge extendibility" of CR–functions by A. Tumanov.

§1. Introduction

We treat the problem of extension of holomorphic functions from one side, say M^+ of a real hypersurface $M \subset \mathbb{C}^N$ to a full neighborhood of a point $z_o \in M$. We prove that if there exists an analytic disc A attached to M^+ but not to M (i.e. verifying $\partial A \subset M^+$ but not $\partial A \subset M$), and tangent to M at z_o, then any holomorphic function f in $\overset{\circ}{M}{}^+ \cap B$ for a ball $B \supset A$, extends at z_o. This generalizes the former result of [9] where the full disc, instead of only its boundary, was assumed to belong to M^+.

The problem of extension of CR functions on M to either side of M, was completely solved by Trepreau and Tumanov in [4] and [5]. They characterize this extension by means of "minimality" that is absence of complex hypersurfaces in M. A CR function f on M is always a "jump" $f = f^+ - f^-$ of holomorphic functions f^+, f^- on $\overset{\circ}{M}{}^+$ and $\overset{\circ}{M}{}^-$ respectively. This is related to our problem but we have a gain in generality. Thus for instance M can be non–minimal and nevertheless a tangent disc, which maybe leaves

1980 *Mathematics Subject Classification (1985 Revision).* Primary 58G; Secondary 32F.

M far from z_o, can provide extension for holomorphic functions on $\overset{\circ}{M}{}^+ \cap B$ for $B \supset A$. On the other hand if there are points $z_\nu \in A \setminus M$ in any neighborhood of z_o, then extension takes place in the sense of germs (cf. [1] for a more precise statement).

We owe our general method to the theory of infinitesimal deformation of analytic discs by Tumanov. Let us explain it in further detail and point out the novelty of our contribution. Starting from an initial analytic disc A, we produce a family of discs A_η through z_o smoothly depending on a real parameter η, with the same CR data as A such that $\dot{A}$, the derivative of A_η in η i.e. the so–called "infinitesimal deformation" of A, is transversal to $T_{z_o}M$ and "points" to the negative side M^- of M. In proving this, we notice that the normal component of $\dot{A}$ has holomorphic extension from the boundary to inside regardless of the assumption that A has "defect" 1 as was assumed in the former papers. Transversality of $\dot{A}$, in combination with tangency of A provides, via Taylor expansion in η, a disc which is actually transversal to M. By attaching a new family of discs through nearby points, whose CR components are just translations of the formers, we then fill a full neighborhood of z_o in M^-. Cauchy's formula in this family of discs provides the desired extension of any holomorphic function from $\overset{\circ}{M}{}^+ \cap B$ (with $B \supset A$) to a full neighborhood of z_o.

We are deeply indebted to A. Tumanov for many invaluable advice.

§2. Statement and Proof

Let M be a real hypersurface of $\mathbb{C}^n$, M^+ one of the two closed half spaces of boundary M (M^+ locally on one side of M), $A = A(\tau)$ $\tau \in \Delta$ an analytic disc of $\mathbb{C}^n$, $z_o = A(1)$ a point of $\partial A \cap M$. Let $T^{\mathbb{C}}M = TM \cap iTM$ be the complex tangent bundle to M and let $\{B\}$ denote the system of spheres of center z_0.

Let $C^{k,\alpha}$ denote the functions whose derivatives up to order k satisfy a Lipschitz condition with exponent $\alpha, 0 < \alpha < 1$.

Theorem 1. *Let M be a $C^{2,\alpha}$ hypersurface, and let A be a disc $C^{1,\alpha}$ up to the boundary and small (in $C^{1,\alpha}$–norm). Suppose*

(1) $\partial_\tau A(1) \in T^{\mathbb{C}}_{z_0}M$
(2) $\partial A \subset M^+$

(3) $\partial A \cap \overset{\circ}{M}{}^+ \neq \emptyset$.

Then if B contains $\bar{A}$ there is B' such that any holomorphic function on $B \cap \overset{\circ}{M}{}^+$ extends to B'.

Proof. We choose complex coordinates $z = x + iy,\ z = (z_1, z')$ such that $z_0 = 0$, and M is described (at z_0) by:

$$y_1 = h(x_1, z'), \quad h(0) = 0,\ \partial h(0) = 0. \tag{4}$$

We set $A(\tau) = (w_1(\tau), w'(\tau)), (w_1(\tau) = u(\tau) + iv(\tau))$, define

$$\zeta(\tau) = -h(w(\tau), w'(\tau)) + v(\tau) \quad \tau \in \Delta$$

and call the "ζ-component" of A. Note that by (2) $\zeta(\tau) \leq 0\ \forall \tau \in \partial\Delta$, by (3) $\zeta(\tau_1) < 0$ for some $\tau_1 \in \partial\Delta$, and finally $\zeta(1) = 0$ because $A(1) = 0$. For a small real parameter η we seek a family of analytic discs through z_0 $A_\eta = (u_\eta + iv_\eta, w')$ ($C^{1,\alpha}$ up to the boundary) and with ζ component:

$$\zeta_{A_\eta}(\tau) = (1 - \eta)\zeta_A(\tau)$$

and z' components

$$z' \circ A_\eta(\tau) = w'(\tau).$$

This is obtained as the holomorphic continuation from $\partial\Delta$ to Δ of the solution $u_\eta + iv_\eta$ of the system:

$$\begin{cases} v_\eta(\tau) = h(u_\eta(\tau), w'(\tau)) + (1 - \eta)\zeta(\tau), \\ v_\eta(\tau) = T_1(u_\eta(\tau)), \end{cases} \tag{5}$$

where T_1 is the Hilbert transform normalized by $T_1 u(1) = 0$. (5) is in turn equivalent to:

$$\begin{cases} u_\eta = -T_1\left(h(u_\eta(\tau), w'(\tau)) + (1 - \eta)\zeta(\tau)\right) \\ v_\eta(\tau) = T_1(u_\eta(\tau)). \end{cases} \tag{6}$$

The first of (6) is the celebrated "Bishop equation". The introduction of "ζ-component" with $\zeta \leq 0$ in Bishop equation corresponds to "attach discs to manifolds with boundary". This idea is due to Tumanov [8]. We need the following variant of [8] Prop. 2.2 p.635.

LEMMA 2. *Let h be $C^{k,\alpha}(\mathbb{C}^n,\mathbb{R})$. Then $\forall w \in C^{k-1,\alpha}(\partial\Delta,\mathbb{C}^{n-1})$ and $\zeta \in C^{k-1,\alpha}(\partial\Delta,\mathbb{R})$ small, there exists an unique solution u_η of (6) in $C^{k-1,\alpha}(\partial\Delta,\mathbb{R})$. Thus $u_\eta + iv_\eta$ is holomorphically extendible to Δ, it belongs to $C^{k-1,\alpha}(\bar{\Delta},\mathbb{C})$, and with the notation $A_\eta = (u_\eta + iv_\eta, w')$ we have:*

$$A_\eta \in C^{k-1} \quad \partial_\tau A_\eta \in C^{k-1}. \tag{7}$$

PROOF. Let

$$\begin{aligned} F : C^{k-1,\alpha}(\partial\Delta,\mathbb{R}) \times C^{k-1,\alpha}(\partial\Delta,\mathbb{C}^{n-1}) \times C^{k-1,\alpha}(\partial\Delta,\mathbb{R}) \times \mathbb{R} \times \mathbb{R} \\ \longrightarrow C^{k-1,\alpha}(\partial\Delta,\mathbb{R}) \\ (u, w', \zeta, x, \eta) \longmapsto u + T_1(h(u, w') + (1-\eta)\zeta) - x. \end{aligned}$$

We have when the functions w' are holomorphic

$$\begin{cases} A_\eta \text{ extends holomorphically to } \Delta \\ u_\eta(1) = x \\ v_\eta = h(u_\eta, w) + (1-\eta)\zeta \text{ on } \partial\Delta, \end{cases} \tag{8}$$

if and only if

$$F = 0.$$

If h is $C^{k,\alpha}$, then F is C^1 (as application between functional spaces) and its differential evaluated at $(u_0, w'_0, \zeta_0, x_0, \eta_0)$ is given by

$$F'(u, w', x, \eta) = u + T_1(\partial_x hu + \partial_{w'} hw' + \partial_{\bar{w}'} h\bar{w}' + (1-\eta_0)\zeta - \eta\zeta) - x$$

We have

$$\begin{cases} F(0,0,0,0) = 0 \\ F'(u) = u + T_1(\partial_x hu). \end{cases} \tag{9}$$

Thus the equation $F = 0$ has solution u_η in $C^{k-1,\alpha}(\partial\Delta, \mathbb{R})$. We note that $\partial_\eta u_\eta$ is solution of

$$\partial_\eta u_\eta + T_1(\partial_x h \partial_\eta u_\eta - \zeta) = 0 \tag{10}$$

Let us rewrite (10) as $G(\partial_\eta u_\eta) = 0$. Since $\partial_{x_1} h$ and ζ are $C^{k-1,\alpha}$, then G is C^1 as application of $C^{k-1,\alpha}(\partial\Delta, \mathbb{R})$ into itself. In particular $\partial_\eta u_\eta \in C^{k-1}(\partial\Delta, \mathbb{R})$ i.e. $\partial_\theta \partial_\eta u_\eta$ is $C^{k-2}(\partial\Delta, \mathbb{R})$. As for $\partial_\theta \partial_\eta v_\eta$ ($v_\eta = T_1 u_\eta$) we observe that

$$\begin{aligned}
\partial_\theta \partial_\eta v_\eta &= \partial_\theta \partial_\eta (h(u_\eta, w) + (1-\eta)\zeta) \\
&= \partial_\theta (\partial_{x_1} h \partial_\eta u_\eta - \zeta) \\
&= \partial_{x_1} h \partial_\theta \partial_\eta u_\eta + \partial_{x_1}^2 h \partial_\theta u_\eta \partial_\eta u_\eta - \partial_\theta \zeta.
\end{aligned}$$

Thus it also belongs to C^{k-2}. Finally for $\tau = |\tau| e^{i\theta}$, Cauchy-Riemann equations yield

$$\partial_\theta (u_\eta + i v_\eta) = i e^{i\theta} \partial_\tau (u_\eta + i v_\eta) \tag{11}$$

Clearly from (11), (7) follows. □

End of proof of Theorem 1

For $r = y_1 - h$ we claim that

$$\Re e < \partial r \circ A, \partial_\tau \partial_\eta A_\eta > |_{\tau=1} < 0. \tag{12}$$

In fact we first find a real function λ on $\partial\Delta$, $\lambda(1) = 1$ such that

$$< \lambda \partial r \circ A, \partial_\eta A_\eta > \text{ extends holomorphically to } \Delta.$$

To this end it is enough to solve the equation

$$\lambda \Im m \partial_1 r A = T_1(\Re e \lambda \partial_1 r A) + 1$$

and to remark that $\partial_\eta A_\eta = (\partial_\eta (u_\eta + i v_\eta), 0)$, whence:

$$< \lambda \partial r \circ A, \partial_\eta A_\eta > = < \lambda \partial_1 r \circ A, \partial_\eta A_\eta > .$$

Let us denote by ϕ the above holomorphic function. We have $\Re e\phi_{|\partial\Delta} = -\zeta\lambda$ It follows

$$\begin{cases} \Re e\phi_{|\partial\Delta} \geq 0 \\ \Re e\phi(1) = 0 \\ \Re e\phi(\tau_1) > 0. \end{cases}$$

Then Hopf's lemma applies and gives (12) (recall that we have chosen $\lambda(1) = 1$). We apply now Lemma 2 and get the Taylor expansion:

$$\partial_\tau A_\eta = \partial_\tau A + \eta\partial_\tau\partial_\eta A_\eta|_{\eta=0} + o(\eta). \tag{13}$$

Since $< \partial r \circ A, \partial_\tau A > |_{\tau=1} = 0$, then we get from (12),(13):

$$\Re e < \partial r \circ A, \partial_\tau A_\eta > |_{\tau=1} < 0 \quad \forall\eta \text{ small}. \tag{14}$$

Thus A_η is transversal to M and "points outside $\overset{\circ}{M}{}^{+}$". Then we get the conclusion of the proof of Theorem 1 according to the following slight variant of results by [8].

LEMMA 3. *Let $\overset{\circ}{M}{}^{+}$ be defined by $r < 0$ and let D be an analytic disc in $C^{1,\alpha}(\bar{\Delta}, \mathbb{C})$, small, with $D(1) = z_0 \in M$, and satisfying*

$$\begin{cases} \partial D \hookrightarrow M^+ \\ \Re e < \partial r \circ A, \partial_\tau D > |_{\tau=1} < 0. \end{cases} \tag{15}$$

Then if B is a sphere of center z_o with $B \supset \bar{D}$, there is $B' \subset B$ such that any holomorphic function on $B \cap \overset{\circ}{M}{}^{+}$ extends to B'.

PROOF. Let $z_0 = 0$, $M : y_1 = h((x_1, z')$, $r = y_1 - h$ with $h(0,0) = 0$, $\partial h(0) = 0$. Consider the discs

$$D_\beta = (u_\beta + iv_\beta, w' - w'(\beta)) \quad \beta \in [0,1],\ \beta \to 1$$

where u_β is a solution of

$$u_\beta = T_\beta(h(u_\beta, w' - w'(\beta)) - \zeta) \tag{16}$$

(Here ζ is the "ζ-component" of D.) By an easy variant of Lemma 2 the equation (16) has an unique solution and we have that

$$(17) \qquad D_\beta,\ \partial_\tau D_\beta \ \text{ are } \ C^1 \text{ with respect to } \eta.$$

We get from (15),(17):

$$(18) \qquad \Re e < \partial r(z_0), \partial_\tau D_\beta(1) > \leq c < 0 \quad \forall \beta \text{ small.}$$

Since $D_\beta(1) \in M$ for all β, then

$$(19) \qquad \Re e < \partial r(z_0), D_\beta(1) - z_0 >= o(\beta)$$

From (18) and (19) we get

$$v_\beta(\beta) > 0.$$

Set $n = -\partial r(z_o)$, denote by I the interval tn for $0 < t < c$, and observe that for D_β small and suitable c:

$$(20.\text{a}) \qquad z + \partial \bar{D}_\beta \subset \overset{\circ}{M}{}^+ \quad \forall z \text{ in a neighborhood of } I,$$

$$(20.\text{b}) \qquad z + \bar{D}_\beta \subset \overset{\circ}{M}{}^+ \quad \forall z \text{ in a neighborhood of } cn.$$

Also define the domain $V = \cup(z + D_\beta)$ for z in a neighborhood of I. Remark that V contains a neighborhood B' of z_o as soon as $c > v_\beta(\beta)$. Also, since D is $C^{1,\alpha}$–small, we can suppose that $V \cap \overset{\circ}{M}{}^+$ is connected and that $V \subset B$ (because $\bar{D}$ is contained in B, and the D_β's are close to D).

For f holomorphic in $\overset{\circ}{M}{}^+ \cap B$, we then define on V:

$$F(z) = \frac{1}{2\pi i} \int_{|\tau|=1} \frac{f(z + D_\beta(\tau) - i v_\beta(\beta) e_1)}{\tau - \beta} d\tau.$$

We have that F is holomorphic and coincides with f on the open subset defined by (20.b). Then F and f coincide on the whole domain $V \cap \overset{\circ}{M}{}^+$. Thus f is extended to V (which contains B'). This concludes the proof of Lemma 3. □

The proof of Theorem 1 is also complete. □

Corollary 4. *Let M be $C^{2,\alpha}$, let A be $C^{1,\alpha}$ in $\bar{\Delta}$ and let $z_0 = A(1) \in M$. Suppose*

$$\begin{cases} \partial_\tau A(1) \in T^{\mathbb{C}}_{z_0} M \\ A \subset M^+ \\ A \cap \overset{\circ}{M}{}^+ \neq \emptyset. \end{cases}$$

Then for any B with $B \cap A \cap \overset{\circ}{M}{}^+ \neq \emptyset$ there exists $B' \subset B$ such that holomorphic functions extend from $B \cap \overset{\circ}{M}{}^+$ to B'.

Proof. For any choice of $\Delta_1 \subset \Delta$ and of $\Phi : \Delta \to \Delta_1$, analytic diffeomorphism $C^{1,\alpha}$ up to the boundary, we define a new disc A_1 by $A_1 = A \circ \Phi$. Since $A \subset M^+$ then $\partial A_1 \subset M^+$.

Let $z_1 = A(\tau_1) \in \overset{\circ}{M}{}^+$ and take $\Delta_1 \subset \Delta$ with $\tau_1 \in \Delta_1$; this gives a disc A_1 such that $z_1 \in \partial A_1$. Also A_1 can be supposed to be small. Then Theorem 1 applies. □

Remark 5. Corollary 4 has the following consequences:

(i) ([4]) Let $\overset{\circ}{M}{}^+$ be pseudoconvex. Then any disc A tangent to M and attached to M^+ (i.e. verifying $\partial A \subset M^+$) must be in M. In fact first one sees that $A \subset M^+$ (otherwise the pseudoconvexity of $\overset{\circ}{M}{}^+$ is violated) and then one applies Corollary 4.

(ii) Assume there is a germ at z_0 of an analytic hypersurface S contained in M. Then according to [6], there is not the extension property for germs of holomorphic functions from $\overset{\circ}{M}{}^+$ to $\mathbb{C}^n$. It follows that any analytic disc $A \subset M^+$ tangent to M at z_0 verify $A \cap B \subset M$ for some B. Note that if one attaches a small disc to S through z_o with the same $T^{\mathbb{C}}_{z_0} S$-components as $A \cap B$, this must coincide with $A \cap B$ (by the uniqueness of the solution to Bishop's equation); and this must be contained in S because S is complex. Hence in fact $A \cap B \subset S$.

Corollary 6. *Let M be $C^{2,\alpha}$, let A be $C^{1,\alpha}$ in $\bar{\Delta}$ and let $z_0 = A(1) \in$*

M and assume

$$\begin{cases} \partial_\tau A \subset T^{\mathbb{C}}_{z_0} M \\ \partial A \subset M^+ \end{cases}.$$

Then any holomorphic function in $B \cap \overset{\circ}{M^+}$ *(with* $B \supset \bar{A}$ *)which extends holomorphically to a full neighborhood of some point* $z_1 \in \partial A$ *it also extends to a neighborhood* B' *of* z_0*.*

Proof. By Theorem 1 it is not restrictive to assume $z_1 \in M$. Then it is immediate to find a new domain $\tilde{M}^+ \supset M^+$ which coincides with M^+ at z_0 such that

$$\begin{cases} f \text{ extends to } \tilde{M}^+ \\ \partial A \cap \overset{\circ}{M^+} \neq \emptyset. \end{cases}$$

Then Theorem 1 applies. □

We consider now the case of $codM = l > 1$. We choose coordinates $z = (z', z''), z' = (z_1, .., z_l)$ such that M is defined by $y' = H(x', z''), H = (h_1, .., h_l)^t$, and also write $r_j = y_j - h_j$. A wedge W with edge M is a domain of type

$$W = \{(z', z'') \in B; y' - h(x', z'') \in \Gamma\}$$

where B is a neighborhood of z_0 and Γ is an open cone of $\mathbb{R}^l_{y'}$. Theorem 1 can be easily generalized as follows:

Proposition 7. *Let* M *be* $C^{2,\alpha}$, $codM > 1$*, let* A *be an analytic disc in* $C^{1,\alpha}(\bar{\Delta}, \mathbb{C})$*, and let* W *be a wedge with edge* M*. Assume:*

$$\begin{cases} \partial A \subset W \cup M \\ \partial_\tau A(1) \in T^{\mathbb{C}}_{z_0} M \\ \partial A \cap W \neq \emptyset \end{cases}$$

Then if $B \supset \bar{A}$*, there exists* $B' \subset B$ *such that any holomorphic function on* $B \cap W$ *extends to* B'*.*

PROOF. We choose coordinates such that M is defined by $y_j = h_j(x', z'')$, put $w''(\tau) = z'' \circ A(\tau)$, $\zeta'(\tau) = y' \circ A - H \circ A$ and find a solution $u' = u'_\eta$ of the system:

$$u'(\tau) = -T_1(H(u', w'') + (1-\eta)\zeta'(\tau)) \quad \tau \in \partial\Delta.$$

By putting $A_\eta = (u'_\eta + iT_1(u'_\eta), w'')$, we get a family of discs which depend C^1 on η. Also $\partial_\tau A_\eta$ depend C^1 on η; thus we get again

$$\partial_\tau A_\eta = \partial_\tau A + \eta \partial_\eta \partial_\tau A + o(\eta). \tag{21}$$

Let $\lambda(\tau), \tau \in \partial\Delta$ be a real $l \times l$ matrix which solves

$$\lambda \Re e \partial' r = -T_1 \lambda(\Im m \partial' r).$$

It follows that $\lambda\partial' r$ extends holomorphically to Δ. Let $\theta'_0 \in \mathbb{R}^l$ verify $\forall \tau$

$$\theta'_0 \lambda(A(\tau)) \in \Gamma^\circ \quad (\Gamma^o \text{ denoting the polar of } \Gamma).$$

Then

$$\Re e < \theta'_0 \lambda \partial r, \partial_\tau \partial_\eta A_\eta > (1) < 0. \tag{22}$$

In fact

$$< \theta'_0 \lambda \partial r, \partial_\eta A_\eta > = < \theta'_0 \lambda \partial' r, \partial_\eta A_\eta >$$

extends holomorphically. Moreover

$$\Re e < \theta'_0 \lambda \partial r, \partial_\eta A_\eta > = -\theta'_0 \lambda \zeta'.$$

Now $\partial A \subset W \cup M$ implies $\zeta'(\tau) \in \Gamma' \cup \{0\}$ for $\Gamma' \subset\subset \Gamma$, and thus also $\lambda\zeta' \in \Gamma$ since λ is close to $\mathrm{id}_{l\times l}$. It follows

$$-\theta'_0 \lambda \zeta'(\tau) \geq 0 \quad \forall \tau \in \partial\Delta$$
$$-\theta'_0 \lambda \zeta'(\tau) > 0 \quad \text{at } \tau = \tau_1.$$

Then Hopf's Lemma gives (21). By using (20) we get a disc A_η transversal to M which points to $n \overset{\text{def.}}{=} \partial_\tau A_\eta$ with $< \theta' \lambda \partial' r, n >< 0$. If we let θ'_o describe the whole Γ^o, we conclude $n \in -\Gamma$.

Also any holomorphic function f on $W \cap B$ with $B \supset A$, is extended to the n–direction. Note on the other hand that the convex hull of Γ and n is $\mathbb{R}^l$. Then by the Airapetyan–Henkin "edge of the wedge" Theorem (cf. e.g. [6, Th. 1.2]), f is extended indeed to a full neighborhood of z_o. $\square$

References

[1] Baracco, L. and G. Zampieri, Analytic discs attached to manifolds with boundary, Publ. Res. Inst. Math. Sci. **33** (1997), 687–694.

[2] Bogges, A., CR Manifolds and the tangential Cauchy–Riemann complex, Studies in Adv. Math. CRC Press (1991).

[3] Baouendi, M. S., Rothschild, L. P. and J. M. Trepreau, On the geometry of analytic discs attached to real manifolds, J. Diff. Geometry **39** (1994), 379–405.

[4] Trepreau, J. M., Sur le prolongement holomorphe des fonctions C–R définies sur une hypersurfsce réelle de classe C^2 dans $\mathbb{C}^n$, Invent. Math. **83** (1986), 583–592.

[5] Tumanov, A., Extension of CR functions into a wedge from a manifold of finite type, Math. Ussr-Sb. **64** (1989), 129–140.

[6] Tumanov, A., Connections and propagation of analyticity for CR functions, Duke Math. Jour. **73 1** (1994), 1–24.

[7] Tumanov, A., On the propagation of extendibility of CR functions, "Complex Analysis and Geometry", Lect. Notes in Math., Marcel–Dekker (1995), 479–498.

[8] Tumanov, A., Extending CR functions from manifolds with boundaries, Math. Res. Letters **2** (1995), 629–642.

[9] Zampieri, G., Analytic discs in a domain of $\mathbb{C}^n$ tangent to the boundary, Preprint (1996).

(Received July 17, 2000)
(Revised February 25, 2001)

Luca Baracco and Giuseppe Zampieri
Universita Degli Studi di Padova
Dipartmento di Matematica Pura ed Applicata
via Belzoni, 7
I-35131 Padova, Italy