J. Math. Sci. Univ. Tokyo
8 (2001), 275–316.

Irrationality of Fast Converging Series of Rational Numbers

By Daniel Duverney

Abstract. We say that the series of general term $u_n \neq 0$ is fast converging if $\log |u_n| \leq c2^n$ for some $c < 0$. We prove irrationality results and compute irrationality measures for some fast converging series of rational numbers, by using Mahler's transcendence method in the form introduced by Loxton and Van der Poorten. With very weak assumptions on sequence u_n, this method allows to obtain only irrationality results.

1. Introduction

Let (u_n) be a sequence of complex numbers. We say that $\sum_{n=0}^{+\infty} u_n$ is a *fast converging series* if

$$|u_n| \leq Ch^{2^n} \tag{1.1}$$

for some constants $C \in]0, +\infty[$ and $h \in]0, 1[$.

In this paper, we will be interested in the irrationality properties of fast converging series of rational numbers of the form

$$S = \sum_{n=0}^{+\infty} \frac{a_n}{b_n u_n}, \tag{1.2}$$

with $a_n \in \mathbb{Z} \setminus \{0\}$, $b_n \in \mathbb{Z} \setminus \{0\}$, $u_n \in \mathbb{N} \setminus \{0\}$ and satisfying

$$\begin{cases} \lim\limits_{n \to +\infty} u_n = +\infty & \\ cu_n^2 \leq u_{n+1} \leq c' \ u_n^2 & \text{for some positive constants } c \text{ and } c' \\ a_n = O(u_n^\alpha) & \text{for some constant } \alpha \in]0, 1[\\ b_n = O(u_n^\varepsilon) & \text{for every } \varepsilon > 0. \end{cases} \tag{1.3}$$

1991 *Mathematics Subject Classification.* 11J72, 11J82.

Key words: Fast converging series; irrationality; irrationality measures; Mahler's transcendence method.

It is well-known since Liouville [15] that one can prove irrationality and transcendence results by approximating real numbers by sequences of rationals with good convergence properties. More precisely, the basic (and elementary) result in *diophantine approximation theory*, with respect to irrationality problems, is the following

THEOREM 1.1. *Let $\alpha \in \mathbb{R}$. Suppose that there exists a sequence P_n/Q_n of rational numbers satisfying*

$$0 < \left| \alpha - \frac{P_n}{Q_n} \right| \leq \frac{\varepsilon(n)}{Q_n}$$

with $\lim\limits_{n \to +\infty} \varepsilon(n) = 0$. *Then α is irrational.*

For a proof, see for example [5, Theorem 1.5].

In some elementary cases, Theorem 1.1 allows to prove the irrationality of α when α is a series of rational numbers and P_n/Q_n is the *partial sum of order n*. The most classical proofs of irrationality using this method date back to Fourier [10], who gave the now standard elementary proof of the irrationality of e, and to Liouville himself [15], who proved the irrationality of

$$\theta = \sum_{n=0}^{+\infty} \frac{1}{m^{n^2}}, \qquad m \in \mathbb{N} \setminus \{0,1\}. \tag{1.4}$$

However, these two proofs rely heavily on two facts :

- First, the numerators in the series are all equal to 1.

- Second, every denominator in the series divides the following one : $n!$ divides $(n+1)!$, and m^{n^2} divides $m^{(n+1)^2}$.

Hence, these two series are *Engel series* ([5, Chapter 2], [22, Chapter 4], for example), for which theorem 1.1 allows to obtain an irrationality criterion.

However, it seems impossible to obtain a *general criterion* for series converging so slowly. On the contrary, conditions (1.3), which rest *only* on the speed of convergence of the series (and *not* on the arithmetical properties of its terms), allow to obtain such a criterion (Theorem 3.1 below).

The scope of the paper is as follows. In section 2, we will present a brief review of the history of fast converging series and their *irrationality* *properties*. As in this paper we are not interested in their transcendence properties, we will only mention Mahler's method [20] when it works. In section 3, we will present our new results, including irrationality statements and computation of irrationality measures. Section 4 will be devoted to the proof of our main irrationality statement (Theorem 3.1.). In section 5, we will prove corollaries to Theorem 3.1. Finally, in section 6, we will prove theorem 3.2, which gives irrationality measures for fast converging series.

2. A Brief History of Fast Converging Series

2.1. Sylvester and Lucas

The oldest result on the irrationality of fast converging series seems to be due to Sylvester [24], who proved in 1880

THEOREM 2.1. *Let* $\alpha \in]0,1]$. *Then* α *can be expanded, in a unique way, in a series of the form*

$$\alpha = \sum_{n=0}^{+\infty} \frac{1}{u_n}, \tag{2.1}$$

with $u_n \in \mathbb{N} \setminus \{0,1\}$ *and* $u_{n+1} \geq u_n^2 - u_n + 1$ *for every* $n \in \mathbb{N}$. *Moreover,* α *is rational if and only if ,*

$$u_{n+1} = u_n^2 - u_n + 1 \tag{2.2}$$

for every $n \geq N_0$.

For a proof of Theorem 2.1, see [24], [22, Chapter 4], [11, Chapter 1], [5, Exercice 2.9]. As a matter of fact, it is very easy to verify that α in (2.1) is rational when (2.2) is satisfied ; indeed

$$u_{n+1} = u_n^2 + u_n + 1 \Leftrightarrow \frac{1}{u_{n+1} - 1} = \frac{1}{u_n - 1} - \frac{1}{u_n},$$

so that, in this case,

$$\sum_{n=N_0}^{+\infty} \frac{1}{u_n} = \sum_{n=N_0}^{+\infty} \left(\frac{1}{u_n - 1} - \frac{1}{u_{n+1} - 1} \right) = \frac{1}{u_{N_0} - 1} \in \mathbb{Q}.$$

Sylvester's theorem 2.1 shows that fast converging series of rational numbers can achieve rational values. Another example can be obtained from a formula given by Lucas in 1878 ([17], [18, p. 184]) :

$$\sum_{n=0}^{+\infty} \frac{x^{2^n}}{1-x^{2^{n+1}}} = \frac{x}{1-x} \qquad (|x|<1). \tag{2.3}$$

In the formula (2.3) as in Sylvester's theorem, the terms of the series cancel each other, because

$$\frac{x^{2^n}}{1-x^{2^{n+1}}} = \frac{x^{2^n}}{1-x^{2^n}} - \frac{x^{2^{n+1}}}{1-x^{2^{n+1}}},$$

which proves (2.3).

For $x = a/b$, with $a \in \mathbb{Z}$, $b \in \mathbb{N} \setminus \{0\}$ and $|a| < b$, we obtain a fast converging series whose sum is rational :

$$\sum_{n=0}^{+\infty} \frac{a^{2^n} b^{2^n}}{b^{2^{n+1}} - a^{2^{n+1}}} = \frac{a}{b-a}. \tag{2.4}$$

It is interesting to observe, following Lucas, that (2.3) also gives a fast converging series whose sum is an *irrational quadratic number* ; indeed, if we take $x = 1/\Phi$, where

$$\Phi = \frac{1+\sqrt{5}}{2} \tag{2.5}$$

is the golden number, we obtain

$$\sum_{n=0}^{+\infty} \frac{1}{F_{2^n}} = \frac{7-\sqrt{5}}{2}, \tag{2.6}$$

where F_n is Fibonacci sequence, defined by

$$F_0 = 0, \qquad F_1 = 1, \qquad F_{n+1} = F_n + F_{n-1} \quad (n \geq 1). \tag{2.7}$$

Naturally, in (2.6) sequence F_n can be replaced by any Lucas sequence [23, p. 41].

2.2. Golomb, Erdös and Strauss

Now we have to jump over more than 80 years, exactly until 1963. At this time, Golomb [12] proved the irrationality of the sum of the reciprocals of the Fermat numbers

$$S_1 = \sum_{n=0}^{+\infty} \frac{1}{2^{2^n}+1}. \tag{2.8}$$

The proof is very complicated for such a result (in fact, S_1 is transcendental by Mahler's theorem ([19], [20, p.5], [5, Exercice 12.13]), but contains some interesting remarks. It rests on the formula

$$\sum_{n=0}^{+\infty} \frac{x^{2^n}}{1+x^{2^n}} + \sum_{n=0}^{+\infty} \frac{x^{2^n}}{1-x^{2^n}} = \frac{2x}{1-x} \quad (|x|<1), \tag{2.9}$$

which is a direct consequence of (2.3). Taking $x = \dfrac{1}{2}$ in (2.9), Golomb obtains

$$S_1 = 2 - S_2, \quad S_2 = \sum_{n=0}^{+\infty} \frac{1}{2^{2^n}-1}. \tag{2.10}$$

He then proves that the expansion of S_2 in base 2 is not periodic, which proves that S_2, and therefore S_1, is irrational. However, this part of the proof is too complicated, because the partial sums of the series giving S_2 are sufficient to prove its irrationality as indicated in section 1 .

Golomb's paper motivated an important work of Erdös and Strauss [7] ; these authors studied irrationality of fast converging series of the form

$$S_3 = \sum_{n=0}^{+\infty} \frac{1}{u_n}, \tag{2.11}$$

where $u_n \in \mathbb{N} \setminus \{0\}$. They called such series Ahmes series (Ahmes was the aegyptian mathematician who wrote the Rhindt papyrus more than 3000 years ago). They didn't completely succeeded in giving a criterion of irrationality without arithmetical conditions on the u_n's. For example, they proved

Theorem 2.2. *Let u_n be an increasing sequence of positive integers satisfying*

$$\limsup \frac{u_{n+1}}{u_n^2} \geq 1 \tag{2.12}$$

(2.13) $LCM(u_0, u_1, \dots, u_n)/u_{n+1}$ *is bounded.*
Then the series S_3 is rational if and only if $u_{n+1} = u_n^2 - u_n + 1$ for all $n \geq n_0$.

As an application, Erdös and Strauss obtained the following remarkable generalization of Golomb's result.

Theorem 2.3. *Let $a \in \mathbb{N} \setminus \{0, 1\}$, and let $b_n \in \mathbb{Z}$, such that the series $\sum \left| b_n a^{-2^n} \right|$ is convergent and $a^{2^n} + b_n \neq 0$ for every $n \geq 0$. Then*

$$S_4 = \sum_{n=0}^{+\infty} \frac{1}{a^{2^n} + b_n} \tag{2.14}$$

is irrational.

Erdös' works on this subject led him to set the following question ([8, p. 64], [9, p. 105]) :
(2.15) *Is it true that if $\dfrac{u_{n+1}}{u_n^2} \longrightarrow 1$ then $\displaystyle\sum_{n=0}^{+\infty} \frac{1}{u_n}$ is irrational unless $u_{n+1} = u_n^2 - u_n + 1$ for $n \geq n_0$?*
We will give a partial answer to this question in Corollary 3.2.

2.3. Recent results

In 1993 Badea [2] generalized Sylvester's results and proved the following

Theorem 2.4. *Let a_n, u_n be sequences of positive integers such that the series*

$$S_5 = \sum_{n=0}^{+\infty} \frac{a_n}{u_n}$$

is convergent. Suppose that

$$u_{n+1} \geq \frac{a_{n+1}}{a_n}(u_n^2 - u_n) + 1. \tag{2.16}$$

Then S_5 is a rational number if and only if

$$u_{n+1} = \frac{a_{n+1}}{a_n}(u_n^2 - u_n) + 1 \tag{2.17}$$

for every $n \geq N$.

In fact, in [2] Theorem 1.5 appeared as a corollary of a more general result whose proof is based on the fact that any non increasing sequence $k_{n+1} \leq k_n$ of positive integers must be constant for $n \geq N$; this proof is similar to the proof of Sylvester's theorem 1.2, althought it is more complicated.

Motivated by Badea's work, Hančl gave in 1996 another criterion of irrationality for fast converging series of rational numbers [13]. As an application, he obtained

THEOREM 2.5. *Let k be a positive integer, and let u_n be a sequence of positive integers such that $u_1 > 2$ and*

$$ku_{n-1}^2 - (3k-1)u_{n-1} < u_n < ku_{n-1}^2 - ku_{n-1} \tag{2.18}$$

for every $n \geq n_0$. Then the number

$$S_6 = \sum_{n=1}^{+\infty} \frac{k^n}{u_n}$$

is irrational.

It should be noted that, for the first time, Badea considered fast converging series of rational numbers with numerators different from 1. However, in Badea's as well as in Hančl's results, the numerators must be *positive.*

Recently, I proved in [6] the following

THEOREM 2.6. *Let $a_n \in \mathbb{Z} \setminus \{0\}$, $b_n \in \mathbb{Z} \setminus \{0\}$, $u_n \in \mathbb{N} \setminus \{0\}$ satisfy*

$$\begin{cases} \lim\limits_{n \longrightarrow +\infty} u_n = +\infty \\ u_{n+1} = \beta u_n^2 + O(u_n^\gamma), \ \beta \in \mathbb{Q}_+^*, \ 0 \leq \gamma < 2 \\ \log|a_n| = o(2^n), \ \log|b_n| = o(2^n). \end{cases} \tag{2.19}$$

Then $S_7 = \sum_{n=0}^{+\infty} \dfrac{a_n}{b_n u_n}$ *is rational if and only if*

$$u_{n+1} = \beta u_n^2 - \frac{a_{n+1}b_n}{a_n b_{n+1}} u_n + \frac{a_{n+2}b_{n+1}}{\beta a_{n+1} b_{n+2}} \tag{2.20}$$

for $n \geq n_0$.

It is clear that Theorem 2.5 is a direct consequence of Theorem 2.6.

The method used for proving theorem 2.6 is quite different from the one used by Badea and Hančl. It comes directly from Mahler's transcendence method in the form of Loxton and Van der Poorten [16], and turns out to be rather similar to the method of Erdös and Strauss. Mahler's transcendence method allows to obtain irrationality results in some cases, as it has been recently observed (see [3], [4], [1]).

The new results presented in section 3 are extensions of Theorem 2.6, basically obtained by following the same ideas.

2.4. More fast converging series with rational sums

Let us give some other amusing examples of fast converging series of rational numbers with rational sums. We will use Theorem 2.6. Suppose that a_n, b_n and u_n in series S_7 satisfy (2.20) for $n \geq 1$ and for some $\beta \in \mathbb{Q}_+^*$. Then, by arguing the same way as for Sylvester series, we have

$$u_{n+1} - \frac{a_{n+2}b_{n+1}}{\beta a_{n+1}b_{n+2}} = \beta u_n \left(u_n - \frac{a_{n+1}b_n}{\beta a_n b_{n+1}} \right).$$

Hence

$$\frac{a_{n+1}}{b_{n+1}} \frac{1}{u_{n+1} - \dfrac{a_{n+2}b_{n+1}}{\beta a_{n+1}b_{n+2}}} = \frac{a_n}{b_n} \left(\frac{1}{u_n - \dfrac{a_{n+1}b_n}{\beta a_n b_{n+1}}} - \frac{1}{u_n} \right).$$

Therefore

$$\sum_{n=1}^{+\infty} \frac{a_n}{b_n u_n} = \sum_{n=1}^{+\infty} \left(\frac{a_n}{b_n} \frac{1}{u_n - \dfrac{a_{n+1}b_n}{\beta a_n b_{n+1}}} - \frac{a_{n+1}}{b_{n+1}} \frac{1}{u_{n+1} - \dfrac{a_{n+2}b_{n+1}}{\beta a_{n+1}b_{n+2}}} \right),$$

and we obtain

$$\sum_{n=1}^{+\infty} \frac{a_n}{b_n u_n} = \frac{a_1}{b_1} \frac{1}{u_1 - \dfrac{a_2 b_1}{\beta a_1 b_2}}. \tag{2.21}$$

Example 2.1. Let $\beta = 1, a_n = 2^n, b_n = 1, u_n = \alpha^{2^n} + 1$, where $\alpha \in \mathbb{N} \setminus \{0, 1\}$. It is easy to check that $u_{n+1} = u_n^2 - 2u_n + 2$. Therefore, (2.20) is satisfied and we get, by using (2.21),

$$\sigma_1(\alpha) = \sum_{n=1}^{+\infty} \frac{2^n}{\alpha^{2^n} + 1} = \frac{2}{\alpha^2 - 1} \qquad (\alpha \in \mathbb{N} \setminus \{0, 1\}). \tag{2.22}$$

Example 2.2. Let $\alpha \in \mathbb{Z} \setminus \{0\}$, and consider polynomial $P(X) = X^2 - \alpha X - 1$. Let $\omega > 1$ be a root of P. Then the other root is $-1/\omega$. Define the general Lucas sequence V_n [22, p. 41] by

$$V_n = \omega^n + \left(-\frac{1}{\omega}\right)^n. \tag{2.23}$$

Note that $V_n \in \mathbb{N}$ and that $V_2 = \left(\omega - \dfrac{1}{\omega}\right)^2 + 2 = \alpha^2 + 2$. In the case where $\alpha = 1$, ω is the golden number Φ and V_n is the classical Lucas sequence

$$L_n = \Phi^n + \left(-\frac{1}{\Phi}\right)^n. \tag{2.24}$$

Let $\beta = 1$, $a_n = 4^n$, $b_n = 1$, $u_n = V_{2^n} + 2$. One checks easily that, for $n \geq 1$,

$$\begin{aligned} u_n^2 - 4u_n + 4 &= (\omega^{2^n} + \omega^{-2^n} + 2)^2 - 4(\omega^{2^n} + \omega^{-2^n} + 2) + 4 \\ &= \omega^{2^{n+1}} + \omega^{-2^{n+1}} + 2 = u_{n+1}. \end{aligned}$$

Hence (2.21) applies and we get

$$\sigma_2(\alpha) = \sum_{n=1}^{+\infty} \frac{4^n}{V_{2^n} + 2} = \frac{4}{V_2 - 2} = \frac{4}{\alpha^2}. \tag{2.25}$$

Example 2.3. With the notations of example 2.2, let $\beta = 1$, $a_n = (-2)^n$, $b_n = 1$, $u_n = V_{2^n} - 1$. Then

$$\begin{aligned} u_n^2 + 2u_n - 2 &= (\omega^{2^n} + \omega^{-2^n} - 1)^2 + 2(\omega^{2^n} + \omega^{-2^n} - 1) - 2 \\ &= \omega^{2^{n+1}} + \omega^{-2^{n+1}} - 1 = u_{n+1}. \end{aligned}$$

Therefore, by using (2.21), we have

$$\sigma_3(\alpha) = \sum_{n=1}^{+\infty} \frac{(-2)^n}{V_{2^n} - 1} = \frac{-2}{V_2 + 1} = \frac{-2}{\alpha^2 + 3}. \tag{2.26}$$

Remark 2.1. Formulas (2.22) and (2.25) are well-known (see for example [21, p.140] and [14, Theorem 3]. Very likely, it is the same for the formula (2.26), but I don't know any reference. Series σ_1, σ_2 and σ_3 will appear naturally in Corollary 3.5 below.

It is interesting remarking that formulas (2.22), (2.25), and (2.26) come from sums of rational fractions.

Theorem 2.7. *For every $x \in \mathbb{C}$, with $|x| < 1$,*

$$\sum_{n=1}^{+\infty} \frac{2^n x^{2^n}}{1 + x^{2^n}} = \frac{2x^2}{1 - x^2}. \tag{2.27}$$

$$\sum_{n=1}^{+\infty} \frac{4^n x^{2^n}}{(1 + x^{2^n})^2} = \frac{4x^2}{(1 - x^2)^2}. \tag{2.28}$$

$$\sum_{n=1}^{+\infty} \frac{(-2)^n x^{2^n}}{x^{2^{n+1}} - x^{2^n} + 1} = \frac{-2x^2}{x^4 + x^2 + 1}. \tag{2.29}$$

Proof. For proving (2.27), observe that the function

$$f(x) = \sum_{n=1}^{+\infty} \frac{2^n x^{2^n}}{1 + x^{2^n}}$$

is analytic in $\mathcal{D} = \{x \in \mathbb{C} / \, |x| < 1\}$. As (2.27) holds for every $x = 1/\alpha$ with $\alpha \in \mathbb{N} \setminus \{0, 1\}$ by (2.22), (2.27) holds in $\mathcal{D}$ by analytic continuation.

The proof of (2.28) and (2.29) is the same ; observe that (2.28) and (2.29) hold for every $x = \dfrac{-1}{\omega} = \dfrac{1}{2}(\alpha - \sqrt{\alpha^2+4})$ when $\alpha \in \mathbb{N} \setminus \{0\}$ by (2.25) and (2.26). When $\alpha \longrightarrow +\infty$, $x \longrightarrow 0$, which proves (2.28) and (2.29) by analytic continuation. □

REMARK 2.2. One can obtain other formulas from (2.27), (2.28), (2.29) by term-by-term derivation. For example, by deriving (2.27) we obtain (2.28). But (2.29) seems of a different nature. If we differentiate it term-by-term, we get

$$\sum_{n=1}^{+\infty} \frac{(-4)^n x^{2^n}(1-x^{2^{n+1}})}{(x^{2^{n+1}} - x^{2^n} + 1)^2} = \frac{2x^2(x^2-1)}{(x^4+x^2+1)^2}. \tag{2.30}$$

Example 2.4. If we replace, in (2.28), (2.29), and (2.30), x by $1/\alpha$ with $\alpha \in \mathbb{N} \setminus \{0,1\}$, we obtain other fast converging series of rational numbers with rational sums. However, these series do not satisfy the assumptions of theorem 3.1 below, contrary to (2.22), (2.25), and (2.26) : the numerators a_n are too large.

3. Presentation of the results

Our main result will be the following

THEOREM 3.1. *Let $a_n \in \mathbb{Z} \setminus \{0\}$, $b_n \in \mathbb{Z} \setminus \{0\}$, $u_n \in \mathbb{N} \setminus \{0\}$ be sequences satisfying conditions* (1.3) *with $\alpha < \dfrac{1}{7}$. Let*

$$S = \sum_{n=0}^{+\infty} \frac{a_n}{b_n u_n}. \tag{3.1}$$

Then, if S is rational, there exist sequences $p_n \in \mathbb{N} \setminus \{0\}$, $q_n \in \mathbb{N} \setminus \{0\}$, depending only on u_n (and not on a_n and b_n), such that

$$u_{n+1} = \frac{p_n}{q_n} u_n^2 - \frac{a_{n+1} b_n}{a_n b_{n+1}} u_n + \frac{a_{n+2} b_{n+1} q_{n+1}}{a_{n+1} b_{n+2} p_{n+1}} \tag{3.2}$$

for every $n \geq N(\alpha)$ *and, for every* $\mu \in]3\alpha, 1-4\alpha[$,

$$\left\{ \begin{array}{l} p_n = O(u_n^{\mu-2} u_{n+1}), \quad q_n = O(u_n^{\mu}) \\ \left| \dfrac{u_{n+1}}{u_n^2} - \dfrac{p_n}{q_n} \right| \leq \dfrac{1}{q_n u_n^{\mu}}. \end{array} \right. \tag{3.3}$$

As for Sylvesters', Erdös and Strauss, and Badea's results, there is reciprocal to this theorem. Indeed, if (3.2) is satisfied, then

$$u_{n+1} - \frac{a_{n+2} b_{n+1} q_{n+1}}{a_{n+1} b_{n+2} p_{n+1}} = \frac{p_n}{q_n} u_n \left(u_n - \frac{a_{n+1} b_n q_n}{a_n b_{n+1} p_n} \right).$$

Hence

$$\frac{a_{n+1}}{b_{n+1}} \cdot \frac{1}{u_{n+1} - \dfrac{a_{n+2} b_{n+1} q_{n+1}}{a_{n+1} b_{n+2} p_{n+1}}} = \frac{a_n}{b_n} \left(\frac{1}{u_n - \dfrac{a_{n+1} b_n q_n}{a_n b_{n+1} p_n}} - \frac{1}{u_n} \right).$$

Therefore $\dfrac{a_n}{b_n u_n}$ can be written as a *difference* of consecutive terms of the same sequence, and we have, with $N = N(\alpha)$,

$$\begin{aligned} \sum_{n=N}^{+\infty} \frac{a_n}{b_n u_n} &= \sum_{n=N}^{+\infty} \left(\frac{a_n}{b_n} \frac{1}{u_n - \dfrac{a_{n+1} b_n q_n}{a_n b_{n+1} p_n}} - \frac{a_{n+1}}{b_{n+1}} \frac{1}{u_{n+1} - \dfrac{a_{n+2} b_{n+1} q_{n+1}}{a_{n+1} b_{n+2} p_{n+1}}} \right) \\ &= \frac{a_N}{b_N} \frac{1}{u_N - \dfrac{a_{N+1} b_N q_N}{a_N b_{N+1} p_N}} \in \mathbb{Q}. \end{aligned}$$

Proving theorem 3.1 will be more difficult. This will be done in section 5.

In Theorem 3.1, unfortunately, the sequence p_n/q_n is not explicitely known. However, this restriction can be overcome in some cases. For example, consider the entire function

$$f(x) = \sum_{n=0}^{+\infty} \frac{x^n}{u_n}, \tag{3.4}$$

with $u_n \in \mathbb{N} \smallsetminus \{0\}$ and

$$
\text{(3.5)} \qquad \begin{cases} \lim\limits_{n \longrightarrow +\infty} u_n = +\infty \\ c u_n^2 \leq \ u_{n+1} \leq c' u_n^2 \qquad \text{for some constants } c > 0 \text{ and } c' > 0. \end{cases}
$$

Then almost all values of $f(x)$ at rational points are irrational ; more precisely we have

Corollary 3.1. *Let $(u_n) \in \mathbb{N} \smallsetminus \{0\}$ satisfy (3.5), and let $f(x)$ be defined by (3.4). Then $f(r)$ is irrational for every $r \in \mathbb{Q}^*$, except perhaps for one value of r.*

As a second example, we can give a partial answer to question (2.15).

Corollary 3.2. *Let $u_n \in \mathbb{N} \smallsetminus \{0\}$ satisfy $\lim\limits_{n \longrightarrow +\infty} u_n = +\infty$ and*

$$
\text{(3.6)} \qquad \sum_{n=0}^{+\infty} \left(\frac{u_{n+1}}{u_n^2} - 1 \right) < \infty.
$$

Suppose that $a_n \in \{-1, 1\}$ for every $n \in \mathbb{N}$. Then $\sum_{n=0}^{+\infty} \dfrac{a_n}{u_n} \in \mathbb{Q}$ if and only if

$$
u_{n+1} = u_n^2 - \frac{a_{n+1}}{a_n} u_n + \frac{a_{n+2}}{a_{n+1}}
$$

for every $n \geq N$.

This contains, as a special case, Theorem 2.3 of Erdös and Strauss, as well as its alternate case, namely

Corollary 3.3. *Let $a \in \mathbb{N} \smallsetminus \{0, 1\}$, and let $b_n \in \mathbb{Z}$ such that the series $\sum |b_n| \, a^{-2^n}$ is convergent and $a^{2^n} + b_n \neq 0$ for every $n \geq 0$. Let $\varepsilon = \pm 1$. Then*

$$
S_8 = \sum_{n=0}^{+\infty} \frac{\varepsilon^n}{a^{2^n} + b_n} \notin \mathbb{Q}.
$$

Theorem 3.1 also allows us to generalize Theorem 2.6 to the case where $\beta \notin \mathbb{Q}$:

COROLLARY 3.4. *Let $a_n \in \mathbb{Z} \setminus \{0\}$, $b_n \in \mathbb{Z} \setminus \{0\}$, $u_n \in \mathbb{N} \setminus \{0\}$ satisfy*

$$\left\{\begin{array}{l}\lim_{n \longrightarrow +\infty} u_n = +\infty,\\ u_{n+1} = \beta u_n^2 + O(u_n^\gamma),\ \beta \in \mathbb{R}_+^*,\ 0 \leq \gamma < 2,\\ \log|a_n| = o(2^n),\ \log|b_n| = o(2^n).\end{array}\right. \tag{3.8}$$

Then $S_9 = \sum_{n=0}^{+\infty} \frac{a_n}{b_n u_n}$ is rational if and only if β is rational and

$$u_{n+1} = \beta u_n^2 - \frac{a_{n+1} b_n}{a_n b_{n+1}} u_n + \frac{a_{n+2} b_{n+1}}{\beta a_{n+1} b_{n+2}}. \tag{3.9}$$

As a special case of corollary 3.4, we obtain irrationality results on series containing linear recurring sequences with subscripts in geometric progression.

COROLLARY 3.5. *Suppose that $d \in \mathbb{N} \setminus \{0\}$, $a_n \in \mathbb{Z} \setminus \{0\}$, $b_n \in \mathbb{Z}$, with*

$$\log|a_n| = o(2^n)\ ,\ \log|b_n| = o(2^n)\ \text{if } b_n \neq 0. \tag{3.10}$$

Suppose that $v_n \in \mathbb{N}$ is a linear recurring sequence satisfing

$$\left\{\begin{array}{l} v_{n+d} = \sum_{h=1}^{d} \alpha_h v_{n+d-h},\\ v_n = \sum_{h=1}^{d} A_h \omega_h^n,\end{array}\right. \tag{3.11}$$

with $\alpha_h \in \mathbb{Z}$, $A_h \in \mathbb{R}^*$, $\omega_h \in \mathbb{R}^*$ for $h = 1, \dots, d$, $\alpha_d \neq 0$, and

$$|\omega_1| > |\omega_2| > \cdots > |\omega_d|\,, \quad |\omega_1| > 1. \tag{3.12}$$

Assume that $v_{2^n} + b_n \neq 0$ for every $n \in \mathbb{N} \setminus \{0\}$. Then

$$S_{10} = \sum_{n=1}^{+\infty} \frac{a_n}{v_{2^n} + b_n},$$

is irrational, except if there exist rational numbers p and q, a rational integer α, and $i \in \{1, 2, 3\}$, such that $S_{10} = p + q\sigma_i(\alpha)$; in these cases S_{10} is rational by (2.22), (2.25) and (2.26).

Example 3.1. Let $\alpha \in \mathbb{N} \smallsetminus \{0,1\}$. For $v_n = \alpha^n$, we obtain under hypothesis (3.10)

$$S_{11} = \sum_{n=1}^{+\infty} \frac{a_n}{\alpha^{2^n} + b_n} \notin \mathbb{Q},$$

except if $a_n = k2^n$ and $b_n = 1$ for every $n \geq N$, where k is a non zero constant natural number.

This generalizes corollary 3.3, under a slightly stronger hypothesis on b_n.

Example 3.2. If we consider Fibonacci sequence, which satisfies

$$F_n = \frac{1}{\sqrt{5}} \left(\Phi^n - \left(-\frac{1}{\Phi} \right)^n \right), \tag{3.13}$$

where Φ is the golden number, we obtain, under hypothesis (3.10) :

$$S_{12} = \sum_{n=1}^{+\infty} \frac{a_n}{F_{2^n} + b_n} \notin \mathbb{Q}.$$

Note that transcendence and algebraic independance results on series like S_{12} can be obtained by Mahler's method, but with very strong regularity hypothesis on a_n and b_n ([20, pp. 13 and 99], [21], [25]).

As a last corollary of Theorem 3.1, we can give precise results on the irrationality of $f(r)$ when u_n satisfies (3.8) (compare to Corollary 3.1).

Corollary 3.6. *Suppose that* $u_n \in \mathbb{N} \smallsetminus \{0\}$ *satisfies*

$$\begin{cases} \lim_{n \to +\infty} u_n = +\infty \\ u_{n+1} = \beta u_n^2 + O(u_n^\gamma), \ \beta \in \mathbb{R}_+^*, \gamma \in]0,2[. \end{cases} \tag{3.14}$$

Let f be the entire function defined by

$$f(x) = \sum_{n=0}^{+\infty} \frac{x^n}{u_n}. \tag{3.15}$$

Then $f(r)$ is irrational for every $r \in \mathbb{Q}^*$, except when $r \in \mathbb{Z}$ and when there exist $\eta \in \mathbb{N} \setminus \{0\}$, $\delta \in \mathbb{N} \setminus \{0\}$ such that, for every large n,

$$\left\{\begin{array}{l} \eta | r \\ u_n = \delta v_n, \ \textit{with } v_n \in \mathbb{N} \\ v_{n+1} = \eta v_n^2 - r v_n + \dfrac{r}{\eta}. \end{array}\right. \tag{3.16}$$

Corollaries 3.1 to 3.6 will be proved in section 5.

Now it it natural to ask if irrationality measures can be given for fast converging series, that is, if one can prove some of them are not Liouville numbers. We will give a positive answer under the hypothesis of corollary 3.4. However, for technical reasons which will appear in the proof, we will have to suppose that β in (3.8) is not a Liouville number.

THEOREM 3.2. *Let $a_n \in \mathbb{Z} \setminus \{0\}$, $b_n \in \mathbb{Z} \setminus \{0\}$, $u_n \in \mathbb{N} \setminus \{0\}$ satisfy*

$$\left\{\begin{array}{l} \lim\limits_{n \longrightarrow +\infty} u_n = +\infty \\ u_{n+1} = \beta u_n^2 + O(u_n^\gamma), \ \beta \in \mathbb{R}_+^*, 0 \leq \gamma < 2 \\ \log |a_n| = o(2^n), \ \log |b_n| = o(2^n). \end{array}\right. \tag{3.17}$$

Assume that β is not a Liouville number, which means that there exist $K > 0$ and $\lambda \geq 2$ such that, for every rational $A/B \neq \beta$,

$$\left|\beta - \frac{A}{B}\right| \geq \frac{K}{|B|^\lambda}. \tag{3.18}$$

Assume moreover that, if $\beta \in \mathbb{Q}$,

$$u_{n+1} \neq \beta u_n^2 - \frac{a_{n+1} b_n}{a_n b_{n+1}} u_n + \frac{a_{n+2} b_{n+1}}{\beta a_{n+1} b_{n+2}} \tag{3.19}$$

for every $n \geq N$. Then, for every $\varepsilon > 0$, there exists $q_0 = q_0(\varepsilon) \in \mathbb{N}$ such that, for every rational p/q satisfying $|q| \geq q_0$,

$$\left|\sum_{n=0}^{+\infty} \frac{a_n}{b_n u_n} - \frac{p}{q}\right| \geq \frac{1}{|q|^{\tau+\varepsilon}}, \tag{3.20}$$

with $\tau = 4\dfrac{2\lambda+\omega}{\omega}$, $\omega = \inf(2-\gamma, 1)$.

REMARK 3.1. In the case where $\beta \in \mathbb{Q}$, we can take $K = 1$ and $\lambda = 2$ in (3.18). Hence

$$\tau = 4\frac{4+\omega}{\omega}.$$

So theorem 2 of [6] gives a better irrationality measure (for example 9 instead of 20 when $\omega = 1$). This is due to the fact that the general construction used to prove Theorem 3.2 is not so well suited to the case $\beta \in \mathbb{Q}$ than the one used in [6]. In particular, for rational β, one could use $\lambda = 1$ in (3.18). As it will appear in the proof, the condition $\lambda \geq 2$ in Theorem 3.2 will only make the proof simpler, and is not necessary.

Example 3.3. Theorem 3.2 applies to the general series S_{10} of Corollary 3.5, because here $\beta = \omega_k$ is an algebraic number and we can take $\lambda = 2+\varepsilon$ by Roth's Theorem. For instance, in the special case of series S_{12} in Example 3.2, we have $\beta = \sqrt{5}$ (whence $\lambda = 2$) and $\gamma = 0$ by (3.13). Therefore, $\omega = 1$ and $\tau = 20$.

4. Proof of Theorem 3.1

4.1. Lemmas

In what follows, sequences a_n, b_n and u_n, satisfy the hypothesis of Theorem 3.1. However, note that the upper bound $u_{n+1} \leq c' u_n^2$ is not necessary in Lemmas 4.1 to 4.5.

LEMMA 4.1. *There exists* $A > 0$ *such that*

$$u_0 u_1 \ldots u_{n-1} \leq A^n u_n, \quad (n \geq 1).$$

PROOF. Put $A = \max(u_0/u_1, 1/c)$. Then Lemma 4.1 follows by induction, because $u_{n+1} \geq c u_n^2$. □

LEMMA 4.2. *There exists* $\theta > 1$ *and* $B > 0$ *such that*

$$u_n \geq B\theta^{2^n} \quad (n \geq 0).$$

PROOF. Put $v_n = cu_n$. Then $v_{n+1} \geq v_n^2$ for every $n \in \mathbb{N}$. As $\lim_{n \longrightarrow +\infty} v_n = +\infty$, we can choose N such that $v_N > 1$; by induction, one sees that $v_n \geq (v_N)^{2^{n-N}}$ for $n \geq N$, which is Lemma 4.2 with $\theta = (v_N)^{2^{-N}} > 1$. □

LEMMA 4.3. $\displaystyle\sum_{k=n+1}^{+\infty} \frac{a_k}{b_k u_k} = O(u_n^{2(\alpha-1)})$.

PROOF. Let $M \in \mathbb{N}$ such that $u_n \geq c^{-2}$ for every $n \geq M$. Induction on j shows that $u_{n+j} \geq c u_n^{j+1}$ for every $n \geq M$ and $j \geq 1$ (note that $u_n^j \geq u_n \geq c^{-2}$). As $a_n = O(u_n^\alpha)$, there exists a constant $D > 0$ such that

$$\left| \sum_{k=n+1}^{+\infty} \frac{a_k}{b_k u_k} \right| \leq \sum_{k=n+1}^{+\infty} \frac{|a_k|}{u_k} \leq D \sum_{k=n+1}^{+\infty} \frac{1}{u_k^{1-\alpha}}$$

$$\leq D c^{\alpha-1} \sum_{m=2}^{+\infty} \frac{1}{u_n^{(1-\alpha)m}} = D c^{\alpha-1} \frac{1}{u_n^{2(1-\alpha)}} \frac{1}{1 - \dfrac{1}{u_n^{1-\alpha}}}.$$

As $\lim_{n \longrightarrow +\infty} u_n = +\infty$, Lemma 4.3 is proved.

The following lemma is similar to Dirichlet's Theorem on diophantine approximation ([5, Theorem 1.6], for example). □

LEMMA 4.4. *Let $\mu \in]0,1[$. For every $n \geq N_0(\mu)$ there exists $(p_n, q_n) \in \mathbb{N}^2$, $q_n \neq 0$, $p_n \neq 0$, such that*

$$\left| \frac{u_{n+1}}{u_n^2} - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n u_n^\mu} \tag{4.1}$$

$$q_n = O(u_n^\mu) \tag{4.2}$$

$$p_n = O(u_n^{\mu-2} u_{n+1}). \tag{4.3}$$

PROOF. Denote, as usual, by $[x]$ the integral part of x. Put $Q_n = [u_n^\mu] + 1$, and consider the numbers

$$\alpha_i = i \frac{u_{n+1}}{u_n^2} - \left[i \frac{u_{n+1}}{u_n^2} \right], \quad i = 0, 1, \ldots, Q_n. \tag{4.4}$$

Two cases can occur.

First case : There exist $i < j$ such that $\alpha_i = \alpha_j$. Then

$$\frac{u_{n+1}}{u_n^2} = \frac{p_n}{q_n},$$

with $q_n = j - i$ and $p_n = \left[j\frac{u_{n+1}}{u_n^2}\right] - \left[i\frac{u_{n+1}}{u_n^2}\right]$. Clearly $q_n \leq Q_n \leq 2u_n^{\mu}$ for large n ; thus (4.1) and (4.2) are fulfiled. Moreover

$$p_n = q_n\frac{u_{n+1}}{u_n^2} \leq 2u_n^{\mu}\frac{u_{n+1}}{u_n^2},$$

which proves (4.3).

Second case : The numbers α_i are all distinct for $i = 0, 1, \ldots, Q_n$. Let us divide the interval $[0, 1]$ into Q_n intervals with length $1/Q_n$. By the pigeon-hole principle, at least one of these intervals contains two distinct α_i's. Therefore there exist $i < j$ such that $|\alpha_i - \alpha_j| \leq 1/Q_n$. If we put, as before,

$$q_n = j - i \quad \text{and} \quad p_n = \left[j\frac{u_{n+1}}{u_n^2}\right] - \left[i\frac{u_{n+1}}{u_n^2}\right],$$

we see that

$$\left|q_n\frac{u_{n+1}}{u_n^2} - p_n\right| \leq \frac{1}{Q_n} \leq \frac{1}{u_n^{\mu}},$$

and Lemma 4.4 is proved. Observe that $p_n \neq 0$ for large n because of (4.1) and the fact that $u_{n+1} \geq cu_n^2$. $\square$

LEMMA 4.5. *Let $\mu \in]0, 1[$, and let p_n and q_n be defined by Lemma 4.4. Then*

$$\frac{p_n u_n}{u_{n+1}} - \frac{q_n}{u_n} = O(u_n^{-1-\mu}).$$

PROOF. By (4.1), we have

$$q_n u_{n+1} = p_n u_n^2 + O(u_n^{2-\mu}).$$

Therefore

$$\frac{p_n u_n}{u_{n+1}} - \frac{q_n}{u_n} = q_n \left(\frac{p_n u_n}{p_n u_n^2 + O(u_n^{2-\mu})} - \frac{1}{u_n} \right)$$
$$= \frac{q_n O(u_n^{2-\mu})}{p_n u_n^3 + O(u_n^{3-\mu})} \sim \frac{q_n}{p_n} O(u_n^{-1-\mu})$$

As $u_{n+1} \geq c u_n^2$, we have by (4.1)

$$\frac{p_n}{q_n} \geq \frac{c}{2} \quad \text{for large } n.$$

This proves Lemma 4.5. □

4.2. Proof of Theorem 3.1

Suppose that a_n, b_n, u_n satisfy the hypothesis of Theorem 3.1. Let S be defined by (3.1).

Let $\mu \in]0,1[$, and let p_n and q_n be defined by Lemma 4.4 (we will choose the value of μ later).

We put

$$A_n = \left(p_n b_n b_{n+1} a_n u_n - a_{n+1} q_n b_n^2\right) \left(\sum_{k=n}^{+\infty} \frac{a_k}{b_k u_k} \right) - p_n b_{n+1} a_n^2. \tag{4.5}$$

An easy computation shows that

$$\begin{cases} A_n &= a_n a_{n+1} b_n \left(\dfrac{p_n u_n}{u_{n+1}} - \dfrac{q_n}{u_n} \right) + R_n - S_n, \\ R_n &= p_n b_n b_{n+1} a_n u_n \sum_{k=n+2}^{+\infty} \dfrac{a_k}{b_k u_k}, \\ S_n &= a_{n+1} q_n b_n^2 \sum_{k=n+1}^{+\infty} \dfrac{a_k}{b_k u_k}. \end{cases} \tag{4.6}$$

By Lemmas 4.3 and 4.4, we have for every $\varepsilon \in]0,1[$

$$S_n = O(u_{n+1}^{\alpha}) O(u_n^{\mu}) O(u_n^{2\varepsilon}) O(u_n^{2(\alpha-1)}).$$

As $u_n = O(u_{n+1}^{1/2})$, we get, by keeping one u_n^{-1} in the last factor,

$$S_n = O(u_{n+1}^{2\alpha+(\mu/2)+\varepsilon-(1/2)} u_n^{-1}). \tag{4.7}$$

Similarly :

$$R_n = O(u_n^{\mu-2} u_{n+1}) O(u_n^{\varepsilon}) O(u_{n+1}^{\varepsilon}) O(u_n^{\alpha}) u_n O(u_{n+1}^{2(\alpha-1)})$$

$$R_n = O(u_{n+1}^{(5\alpha/2)+(\mu/2)+(3\varepsilon/2)-1} u_n^{-1}). \tag{4.8}$$

By using (4.7), (4.8), and Lemma 4.5, we obtain

$$\begin{aligned} A_n = {} & O(u_{n+1}^{(3\alpha/2)-(\mu/2)+(\varepsilon/2)} u_n^{-1}) + O(u_{n+1}^{(5\alpha/2)+(\mu/2)+(3\varepsilon/2)-1} u_n^{-1}) \\ & + O(u_{n+1}^{2\alpha+(\mu/2)+\varepsilon-(1/2)} u_n^{-1}). \end{aligned} \tag{4.9}$$

We will choose μ in such a way that each of the three numbers $\dfrac{3\alpha}{2} - \dfrac{\mu}{2}$, $\dfrac{5\alpha}{2} + \dfrac{\mu}{2} - 1$, $2\alpha + \dfrac{\mu}{2} - \dfrac{1}{2}$ is *negative.* Put $\mu = t\alpha$; then we must have

$$t > 3, \quad (t+5)\alpha < 2, \quad (t+4)\alpha < 1.$$

The third condition implies the second one. Hence we have to find t such that

$$3 < t < \frac{1}{\alpha} - 4,$$

which is possible only if $\alpha < 1/7$, and is equivalent to find μ satisfying

$$\mu \in]3\alpha, 1-4\alpha[. \tag{4.10}$$

By (4.10), the three numbers $\dfrac{3\alpha}{2} - \dfrac{\mu}{2}$, $\dfrac{5\alpha}{2} + \dfrac{\mu}{2} - 1$, $2\alpha + \dfrac{\mu}{2} - \dfrac{1}{2}$ are negative. Now if we choose ε small enough, each of the three numbers $\dfrac{3\alpha}{2} - \dfrac{\mu}{2} + \dfrac{\varepsilon}{2}$, $\dfrac{5\alpha}{2} + \dfrac{\mu}{2} + \dfrac{3\varepsilon}{2} - 1$, $2\alpha + \dfrac{\mu}{2} + \varepsilon - \dfrac{1}{2}$ is negative. Therefore, by putting

$$\delta = \max\left(\frac{3\alpha}{2} - \frac{\mu}{2} + \frac{\varepsilon}{2}, \frac{5\alpha}{2} + \frac{\mu}{2} + \frac{3\varepsilon}{2} - 1, 2\alpha + \frac{\mu}{2} + \varepsilon - \frac{1}{2}\right), \tag{4.11}$$

we have by (4.9)

$$A_n = O(u_{n+1}^{\delta} u_n^{-1}), \quad \text{with } \delta < 0 \text{ for } \varepsilon < \varepsilon_0(\alpha, \mu). \tag{4.12}$$

Now we observe that

$$\sum_{k=n}^{+\infty} \frac{a_k}{b_k u_k} = S - \sum_{k=0}^{n-1} \frac{a_k}{b_k u_k}. \tag{4.13}$$

Define the following *rational integers* :

$$\begin{cases} K_n & = b_0 b_1 \dots b_{n-1} u_0 u_1 \dots u_{n-1} \\ B_n & = K_n (p_n b_n b_{n+1} a_n u_n - a_{n+1} q_n b_n^2) \\ C_n & = K_n p_n b_{n+1} a_n^2 + B_n \sum_{k=0}^{n-1} \dfrac{a_k}{b_k u_k}. \end{cases} \tag{4.14}$$

Multiply A_n by K_n in (4.5) ; by using (4.12), (4.13) and (4.14), we obtain

$$B_n S - C_n = K_n O(u_{n+1}^{\delta} u_n^{-1}). \tag{4.15}$$

But by using Lemma 4.1 we can obtain an upper bound for K_n ; for $\varepsilon \in]0,1[$, there exists $\nu = \nu(\varepsilon) > 0$ such that

$$|K_n| \leq \nu^n u_0^{1+\varepsilon} \dots u_{n-1}^{1+\varepsilon} \leq \nu^n A^{2n} u_n^{1+\varepsilon} \leq \nu^n A^{2n} u_n^{1+\varepsilon}.$$

Hence

$$K_n = O((\nu A^2)^n u_n u_{n+1}^{\varepsilon/2}). \tag{4.16}$$

Therefore (4.15) can be written as

$$B_n S - C_n = O((\nu A^2)^n u_{n+1}^{\delta + (\varepsilon/2)}). \tag{4.17}$$

For $\varepsilon < \varepsilon_1(\alpha, \mu)$, we have $\delta + \dfrac{\varepsilon}{2} < 0$. If we choose such an ε and fix it, we obtain, by using Lemma 4.2,

$$\lim_{n \to +\infty} (B_n S - C_n) = 0.$$

Assume that S is rational ; as B_n, C_n are integers, this implies

$$B_n S - C_n = 0 \quad \text{for} \quad n \geq N(\alpha). \tag{4.18}$$

Now consider the determinant

$$\Delta_n = \begin{vmatrix} B_n & -C_n \\ B_{n+1} & -C_{n+1} \end{vmatrix}. \tag{4.19}$$

By (4.18) we have $\Delta_n = 0$ for $n \geq N(\alpha)$. But if we multiply the first column of Δ_n by $\sum_{k=0}^{n-1} \dfrac{a_k}{b_k u_k}$ and add it to the second one, we obtain by using (4.14)

$$\Delta_n = \begin{vmatrix} B_n & -K_n p_n b_{n+1} a_n^2 \\ B_{n+1} & -K_{n+1} p_{n+1} b_{n+2} a_{n+1}^2 - B_{n+1} \dfrac{a_n}{b_n u_n} \end{vmatrix}.$$

We replace B_n and B_{n+1} by using the second equality in (4.14), and develop Δ_n. We get

$$\begin{aligned} \Delta_n = K_n K_{n+1} \frac{a_{n+1} b_n}{u_n} \big(& -a_n a_{n+1} b_{n+1} b_{n+2} p_n p_{n+1} u_n^2 \\ & + a_{n+1}^2 b_n b_{n+2} p_{n+1} q_n u_n + a_n a_{n+1} b_{n+1} b_{n+2} p_{n+1} q_n u_{n+1} \\ & - a_n a_{n+2} b_n b_{n+1}^2 q_n q_{n+1} \big) . \end{aligned}$$

The term in the brackets is zero for $n \geq N(\alpha)$; if we divide it by $a_n a_{n+1} b_{n+1} b_{n+2} p_{n+1} q_n$, we obtain (3.2), and Theorem 3.1 is proved.

Remark 4.1. The method used for proving Theorem 3.1 is a weak form of Mahler's transcendence method in the form introduced by Loxton and Van der Poorten. This will be apparent in the special case where

$$u_{n+1} = \beta u_n^2 + \tau_n, \quad \beta \in \mathbb{Q}_+^*, \tag{4.20}$$

first studied in [6].

If we introduce

$$\begin{cases} f_n(x) & = \beta + \tau_n x^2 \\ \varphi_n(x) & = \sum_{k=n}^{+\infty} \dfrac{a_k x^{2^{k-n}}}{b_k (f_k \circ f_{k-1} \circ \cdots \circ f_{n+1})(x)}, \end{cases} \tag{4.21}$$

we easily see that

$$\sum_{k=n}^{+\infty} \frac{a_k}{b_k u_k} = \varphi_n\left(\frac{1}{u_n}\right). \tag{4.22}$$

Now we can compute explicitly the [1/1] Padé-approximants to φ_n, namely find μ_n, ν_n, ρ_n satisfying

$$(\mu_n x + \nu_n)\varphi_n - \rho_n x = O(x^3). \tag{4.23}$$

By (4.21) we see that

$$\varphi_n(x) = \frac{a_n}{b_n}x + \frac{a_{n+1}}{b_{n+1}\beta}x^2 + O(x^4), \tag{4.24}$$

so that (4.23) is equivalent to

$$\begin{cases} \mu_n \dfrac{a_n}{b_n} + \nu_n \dfrac{a_{n+1}}{b_{n+1}\beta} = 0 \\ \nu_n \dfrac{a_n}{b_n} - \rho_n = 0. \end{cases} \tag{4.25}$$

If we put $\beta = \eta/\delta$ and look for *integers* μ_n, ν_n, ρ_n we can take

$$\begin{cases} \mu_n &= -a_{n+1}\delta b_n^2 \\ \nu_n &= \eta b_n b_{n+1} a_n \\ \rho_n &= \eta b_{n+1} a_n^2. \end{cases} \tag{4.25}$$

This explains the number A_n defined in [6], formula (10). In the present paper, we have only modified the definition of A_n be replacing η by p_n and δ by q_n in order to obtain more general results (see formula (4.5)).

5. Proof of Corollaries 3.1 to 3.6

5.1. Proof of Corollary 3.1

Suppose that $f(r) \in \mathbb{Q}$ and $f(r') \in \mathbb{Q}$, with $r = a/b$, $r' = a'/b'$, $a, a', b, b' \in \mathbb{Z} \smallsetminus \{0\}$, $r \neq r'$. By Theorem 3.1 we have for $n \geq N$

$$u_{n+1} = \frac{p_n}{q_n}u_n^2 - \frac{a}{b}u_n + \frac{a}{b}\frac{q_{n+1}}{p_{n+1}} \tag{5.1}$$

(5.2) $$u_{n+1} = \frac{p_n}{q_n} u_n^2 - \frac{a'}{b'} u_n + \frac{a'}{b'} \frac{q_{n+1}}{p_{n+1}}.$$

Substracting (5.2) to (5.1), we obtain

(5.3) $$u_n = \frac{q_{n+1}}{p_{n+1}}.$$

Therefore, if we replace in (5.1) q_{n+1}/p_{n+1} by u_n and q_n/p_n by u_{n-1}, we get for $n \geq N+1$

(5.4) $$u_{n+1} = \frac{u_n^2}{u_{n-1}}.$$

Hence $\lim_{n\to+\infty} \dfrac{u_{n+1}}{u_n^2} = 0$, contrary to the assumption.

5.2. Proof of Corollary 3.2

Suppose that $\sum_{n=0}^{+\infty} \dfrac{a_n}{u_n} \in \mathbb{Q}$, $a_n \in \{-1, 1\}$. By Theorem 3.1 we can write for large n

$$u_{n+1} = \frac{p_n}{q_n} u_n^2 - \frac{a_{n+1}}{a_n} u_n + \frac{a_{n+2}}{a_{n+1}} \frac{q_{n+1}}{p_{n+1}}.$$

For every n, put $\dfrac{p_n}{q_n} = \dfrac{p'_n}{q'_n}$ where p'_n and q'_n are prime to each other. Then we have

(5.5) $$u_{n+1} = \frac{p'_n}{q'_n} u_n^2 - \frac{a_{n+1}}{a_n} u_n + \frac{a_{n+2}}{a_{n+1}} \frac{q'_{n+1}}{p'_{n+1}}.$$

As $\dfrac{a_{n+1}}{a_n} \in \{-1, 1\}$ and $u_n \in \mathbb{N}$, p'_{n+1} must divide q'_n. This implies $p'_{n+1} \leq q'_n$ for $n \geq N$, that is

(5.6) $$p'_{n+1} \leq \frac{q'_n}{p'_n} p'_n \quad \text{for } n \geq N.$$

Therefore, by induction we have

(5.7) $$p'_n \leq p'_N \prod_{k=N}^{n-1} \frac{q'_k}{p'_k} \quad \text{for } n \geq N.$$

But we have by (3.3), for every $\mu \in]0,1[$,

$$\frac{u_{n+1}}{u_n^2} - \frac{p'_n}{q'_n} = O(u_n^{-\mu}).$$

Hence

$$1 - \frac{p'_n}{q'_n} = \left(1 - \frac{u_{n+1}}{u_n^2}\right) + O(u_n^{-\mu}).$$

By (3.6), this implies that the series $\sum \left(1 - \frac{p'_n}{q'_n}\right)$ is convergent : therefore $\lim_{n\to+\infty} \frac{p'_n}{q'_n} = 1$ and the infinite product $\prod \frac{p'_n}{q'_n}$ is convergent, so that p'_n is *bounded* by (5.7). As $\lim_{n\to+\infty} \frac{p'_n}{q'_n} = 1$, q'_n is also *bounded*, which implies

$$\frac{p'_n}{q'_n} = 1 \quad \text{for } n \geq N_1, \tag{5.8}$$

and completes the proof of Corollary 3.2.

5.3. Proof of Corollary 3.3

We apply Corollary 3.2. Here

$$\begin{cases} u_n & = a^{2^n} + b_n \\ a_n & = \varepsilon^n. \end{cases} \tag{5.9}$$

As the series $\sum b_n a^{-2^n}$ is convergent, we have

$$u_n \sim a^{2^n}, \quad b_n = o(a^{2^n}). \tag{5.10}$$

Moreover

$$\frac{u_{n+1}}{u_n^2} - 1 = -\frac{2b_n a^{2^n}}{u_n^2} + \frac{b_{n+1}}{u_n^2} - \frac{b_n^2}{u_n^2}. \tag{5.11}$$

By (5.10), each of the three series in the right hand side of (5.11) is convergent. Hence $\sum \left(\frac{u_{n+1}}{u_n^2} - 1\right)$ is convergent. By Corollary 3.2, we have for $n \geq N_1$

$$u_{n+1} = u_n^2 - \varepsilon u_n + \varepsilon. \tag{5.12}$$

Multiply (5.11) by u_n^2, and substract it from (5.12), we obtain

$$u_n = a^{2^n} + b_n = \varepsilon(2b_n a^{2^n} - b_{n+1} + b_n^2) + 1.$$

Hence

$$\varepsilon b_{n+1} - \varepsilon b_n^2 = a^{2^n}(2\varepsilon b_n - 1) + 1 - b_n. \tag{5.13}$$

Following [7], observe that (5.13) implies

$$b_{n+1} = a^{2^n}(b_n(2 + b_n a^{-2^n}) - \varepsilon - \varepsilon b_n a^{-2^n} + \varepsilon a^{-2^n}).$$

As $b_n = o(a^{-2^n})$, for every $n \geq N_2 \geq N_1$ satisfying $b_n \neq 0$, we therefore have

$$|b_{n+1}| \geq a^{2^n} |b_n|. \tag{5.14}$$

But $b_n = 0 \Rightarrow b_{n+1} = \varepsilon(1 - a^{2^n}) \neq 0$, so that (5.14) holds for every $n \geq N_3$, with $N_3 = N_2$ or $N_3 = N_2 + 1$. By induction, we get for every $k \geq 0$

$$|b_{N_3+k}| \geq |b_{N_3}| \, a^{2^{N_3+k} - 2^{N_3}}.$$

Therefore $\lim_{k\to+\infty} |b_{N_3+k}| \, a^{-2^{N_3+k}} \neq 0$, and this contradiction proves Corollary 3.3.

5.4. Proof of Corollary 3.4

By lemma 4.2, we have for every $\alpha > 0$

$$a_n = O(u_n^\alpha), \quad b_n = O(u_n^\alpha). \tag{5.15}$$

Suppose that $S_9 \in \mathbb{Q}$; then, by Theorem 3.1,

$$u_{n+1} = \frac{p_n}{q_n} u_n^2 - \frac{a_{n+1} b_n}{a_n b_{n+1}} u_n + \frac{a_{n+2} b_{n+1} q_{n+1}}{a_{n+1} b_{n+2} p_{n+1}} \tag{5.16}$$

for $n \geq N(\alpha)$. But by hypothesis

$$u_{n+1} = \beta u_n^2 + O(u_n^\gamma), \quad \text{with } 0 \leq \gamma < 2. \tag{5.17}$$

Substracting (5.17) to (5.16), we obtain

$$q_n\beta - p_n = q_n O(u_n^{\gamma-2}) - \frac{a_{n+1}b_n}{a_n b_{n+1}}\frac{q_n}{u_n} + \frac{a_{n+2}b_{n+1}q_{n+1}}{a_n b_{n+2}p_{n+1}}\frac{q_n}{u_n^2}. \tag{5.18}$$

In (4.10), we choose

$$\mu = (1-\alpha)3\alpha + \alpha(1-4\alpha) = 4\alpha - 7\alpha^2.$$

By (3.3), we have

$$q_n = O(u_n^{4\alpha-7\alpha^2}). \tag{5.19}$$

Therefore (5.18) implies, by using $u_{n+1} = O(u_n^2)$,

$$q_n\beta - p_n = O(u_n^{\gamma-2+4\alpha-7\alpha^2}) + O(u_n^{7\alpha-7\alpha^2-1}) + O(u_n^{18\alpha-21\alpha^2-2}). \tag{5.20}$$

Hence, by multiplying (5.20) by q_{n+1}, we obtain

$$\begin{aligned} q_nq_{n+1}\beta - p_nq_{n+1} = O(u_n^{\gamma-2+12\alpha-21\alpha^2}) &+ O(u_n^{15\alpha-21\alpha^2-1}) \\ &+ O(u_n^{26\alpha-35\alpha^2-2}). \end{aligned} \tag{5.21}$$

Similarly, replace n by $n+1$ in (5.20), and multiply by q_n ; we get

$$\begin{aligned} q_nq_{n+1}\beta - p_{n+1}q_n = O(u_n^{2\gamma-4+12\alpha-21\alpha^2}) &+ O(u_n^{18\alpha-21\alpha^2-2}) \\ &+ O(u_n^{40\alpha-49\alpha^2-4}). \end{aligned} \tag{5.22}$$

Now we choose α so small that each of the numbers $\gamma-2+12\alpha-21\alpha^2$, $15\alpha-21\alpha^2-1$, $26\alpha-35\alpha^2-2$, $2\gamma-4+12\alpha-21\alpha^2$, $18\alpha-21\alpha^2-2$, $40\alpha-49\alpha^2-4$ is *negative*. With such a choice of α, (5.21) and (5.22) imply

$$\lim_{n\to+\infty}(q_nq_{n+1}\beta - p_nq_{n+1}) = \lim_{n\to+\infty}(q_nq_{n+1}\beta - p_{n+1}q_n) = 0$$

Therefore

$$\begin{cases} \beta = \lim_{n\to+\infty} p_n/q_n \\ \lim_{n\to+\infty}(p_nq_{n+1} - p_{n+1}q_n) = 0. \end{cases} \tag{5.23}$$

As $p_n, q_n, p_{n+1}, q_{n+1}$ are integers, this means that $p_nq_{n+1} - p_{n+1}q_n = 0$ for n large enough. Hence, for $n \geq N_1(\alpha)$

$$\frac{p_n}{q_n} = \frac{p_{n+1}}{q_{n+1}} = \lim_{n\to+\infty}\frac{p_n}{q_n} = \beta. \tag{5.24}$$

So $\beta \in \mathbb{Q}$, and Corollary 3.4 is proved by using (5.24) and (5.16).

5.5. Proof of Corollary 3.5

Put $u_n = v_{2^n} + b_n = A_1\omega_1^{2^n} + \sum_{h=2}^d A_h\omega_h^{2^n} + b_n$. We then have

$$(5.25)\qquad \begin{cases} u_n = A_1\omega_1^{2^n} + t_n \\ t_n = \displaystyle\sum_{h=2}^{d} A_h\omega_h^{2^n} + b_n. \end{cases}$$

Observe that, by (3.10) and (3.12),

$$(5.26)\qquad \begin{cases} t_n \sim A_2\omega_2^{2^n} & \text{if } d \geq 2 \text{ and } |\omega_2| > 1 \\ t_n = O(|\omega_1|^{\varepsilon 2^n}) & (\text{for every } \varepsilon > 0) \text{ otherwise.} \end{cases}$$

We see immediatly that

$$(5.27)\qquad u_{n+1} - \frac{1}{A_1}u_n^2 = t_{n+1} - 2t_n\omega_1^{2^n} - \frac{1}{A_1}t_n^2.$$

As $u_n \sim A_1\omega_1^{2^n}$, we see, by using (5.26), that there exists $\gamma \in]0,2[$ such that $u_{n+1} - \frac{1}{A_1}u_n^2 = O(u_n^\gamma)$. Hence Corollary 3.4 applies ; so, if $S_{10} \in \mathbb{Q}$, we have $A_1 \in \mathbb{Q}$ and, for every $n \geq N_0$,

$$u_{n+1} = \frac{1}{A_1}u_n^2 - \frac{a_{n+1}}{a_n}u_n + A_1\frac{a_{n+2}}{a_{n+1}}.$$

Comparing this equality to (5.27) yields

$$(5.28)\qquad t_{n+1} - 2t_n\omega_1^{2^n} - \frac{1}{A_1}t_n^2 = -\frac{a_{n+1}}{a_n}(A_1\omega_1^{2^n} + t_n) + A_1\frac{a_{n+2}}{a_{n+1}}.$$

Now we distinguish four cases.

First case : $d \geq 2$ and $|\omega_2| > 1$. By taking the equivalents in (5.28), we obtain in virtue of (5.26)

$$-2A_2\omega_2^{2^n}\omega_1^{2^n} \sim -\frac{a_{n+1}}{a_n}A_1\omega_1^{2^n}.$$

Therefore $\omega_2^{2^n} \sim \dfrac{a_{n+1}}{a_n}\dfrac{A_1}{2A_2}$.

This is impossible because $|\omega_2| > 1$ and $\log |a_n| = o(2^n)$.

Second case : $d = 1$. In this case $t_n = b_n \in \mathbb{Z}$. We write (5.28) in the following form :

$$\omega_1^{2^n} \left(A_1 \frac{a_{n+1}}{a_n} - 2t_n \right) = \frac{1}{A_1} t_n^2 - t_{n+1} - \frac{a_{n+1}}{a_n} t_n + A_1 \frac{a_{n+2}}{a_{n+1}}. \tag{5.29}$$

Suppose that the *rational number* $A_1 \dfrac{a_{n+1}}{a_n} - 2t_n$ is different from 0 ; in this case, denoting by $\delta > 0$ the denominator of A_1, we have

$$\left| A_1 \frac{a_{n+1}}{a_n} - 2t_n \right| \geq \frac{1}{\delta \, |a_n|},$$

which implies by (5.29) :

$$\omega_1^{2^n} \leq \delta \, |a_n| \left| \frac{1}{A_1} t_n^2 - t_{n+1} - \frac{a_{n+1}}{a_n} t_n + A_1 \frac{a_{n+2}}{a_{n+1}} \right|.$$

By using (5.26), we get $\omega_1^{2^n} = O(|\omega_1|^{\varepsilon 2^n})$ for every $\varepsilon > 0$, a contradiction. Hence (5.29) implies, for every $n \geq N \geq N_0$,

$$\begin{cases} A_1 \dfrac{a_{n+1}}{a_n} = 2t_n \\ \dfrac{1}{A_1} t_n^2 - t_{n+1} - \dfrac{a_{n+1}}{a_n} t_n + A_1 \dfrac{a_{n+2}}{a_{n+1}} = 0. \end{cases} \tag{5.30}$$

Replacing t_n and t_{n+1} from the first equality into the second yields

$$\frac{1}{2} \frac{a_{n+2}}{a_{n+1}} = \left(\frac{1}{2} \frac{a_{n+1}}{a_n} \right)^2.$$

Therefore, for every $n \geq N$,

$$\frac{1}{2} \frac{a_{n+1}}{a_n} = \left(\frac{1}{2} \frac{a_{N+1}}{a_N} \right)^{2^{n-N}} = \left(\left(\frac{1}{2} \frac{a_{N+1}}{a_N} \right)^{2^{-N}} \right)^{2^n}. \tag{5.31}$$

So, as $\log|a_n| = o(2^n)$, $\left|\frac{1}{2}\frac{a_{N+1}}{a_N}\right| > 1$ is impossible. Taking the inverses in (5.31), we see that $\left|\frac{1}{2}\frac{a_{N+1}}{a_N}\right| < 1$ is also impossible. Therefore $\frac{1}{2}\left|\frac{a_{N+1}}{a_N}\right| = 1$. By (5.31), this implies

$$\frac{a_{n+1}}{a_n} = 2 \quad \text{for every } n \geq N+1. \tag{5.32}$$

Thus $t_n = A_1$ for every $n \geq N+1$ by using (5.30), which means that $u_n = A_1(\omega_1^{2^n}+1)$. As $A_1 \in \mathbb{Q}$ and $u_n \in \mathbb{Z}$, $\omega_1 \in \mathbb{Z}$; as $|\omega_1| > 1$, then $\omega_1 \in \mathbb{Z} \smallsetminus \{-1, 0, 1\}$, and

$$S_{10} = \sum_{n=1}^{N} \frac{a_n}{A_1\omega_1^{2^n} + b_n} + \frac{a_N}{A_1}\sum_{n=1}^{+\infty} \frac{2^n}{(\omega_1^{2^N})^{2^n}+1} \tag{5.33}$$

Third case : $d \geq 2$ and $|\omega_2| = 1$. Let $P(X) = X^d - \sum_{h=1}^{d} \alpha_h X^{d-h}$ be the characteristic polynomial of v_n, defined in (3.11). As $\omega_2 = \pm 1$ is a root of P, we have

$$P(X) = (X-\omega_2)Q(X), \qquad Q \in \mathbb{Z}(X), \tag{5.34}$$

and $\omega_1, \omega_3, \ldots, \omega_d$ are the roots of Q. But $A_1, A_2, \ldots, A_d$ are the roots of the system

$$\begin{cases} v_0 = & A_1 + A_2 + \cdots + A_d \\ v_1 = & A_1\omega_1 + A_2\omega_2 + \cdots + A_d\omega_d \\ \vdots & \\ v_{d-1} = & A_1\omega_1^{d-1} + A_2\omega_2^{d-1} + \cdots + A_d\omega_d^{d-1}. \end{cases}$$

Hence, by Cramer's formula,

$$A_2 = \frac{\begin{vmatrix} 1 & v_0 & 1 & \cdots & 1 \\ \omega_1 & v_1 & \omega_3 & \cdots & \omega_d \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \omega_1^{d-1} & v_{d-1} & \omega_3^{d-1} & \cdots & \omega_d^{d-1} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \omega_1 & \omega_2 & \omega_3 & \cdots & \omega_d \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \omega_1^{d-1} & \omega_2^{d-1} & \omega_3^{d-1} & \cdots & \omega_d^{d-1} \end{vmatrix}}. \tag{5.35}$$

So A_2 is a symmetric rational fraction in $\omega_1, \omega_3, \dots, \omega_d$ with integer coefficients. Therefore $A_2 \in \mathbb{Q}$. Write $A_2 = r/s$, $r, s \in \mathbb{Z}$, $s \neq 0$. We then have

$$S_{10} = \sum_{n=1}^{+\infty} \frac{s a_n}{s A_1 \omega_1^{2^n} + s A_3 \omega_3^{2^n} + \cdots + s A_d \omega_d^{2^n} + r + s b_n}.$$

Hence, if $d = 2$ we are led back to the second case. If $d \geq 3$, as $|\omega_3| < |\omega_2| = 1$, we are led to the fourth case.

Fourth case : $d \geq 2$ and $|\omega_2| < 1$. As $\omega_1 \omega_2 \dots \omega_d = -\alpha_d \in \mathbb{Z} \setminus \{0\}$ and $1 > |\omega_2| > |\omega_3| > \cdots > |\omega_d|$, we cannot have $|\omega_1 \omega_2| < 1$. Therefore $|\omega_1 \omega_2| \geq 1$. We have by (5.29), for every $\varepsilon > 0$,

$$A_1 a_{n+1} - 2 t_n a_n = O(|\omega_1|^{(-1+\varepsilon)2^n}). \tag{5.36}$$

Put $t'_n = t_n - b_n$; then $t'_n \sim A_2 \omega_2^{2^n}$ by (5.35), and (5.36) yields

$$A_1 a_{n+1} - 2 a_n b_n - 2 t'_n a_n = O(|\omega_1|^{(-1+\varepsilon)2^n}). \tag{5.37}$$

As $A_1 \in \mathbb{Q}$ and $|\omega_2| < 1$, this implies

$$A_1 a_{n+1} - 2 a_n b_n = 0. \tag{5.38}$$

for every large n, whence

$$-2 a_n = O(|\omega_2|^{-2^n}) O(|\omega_1|^{(-1+\varepsilon)2^n}). \tag{5.39}$$

Suppose that $|\omega_1 \omega_2| > 1$. By choosing ε small enough, (5.39) implies $a_n = 0$ for every large n, a contradiction. Hence $|\omega_1 \omega_2| = 1$. This implies that $d = 2$, otherwise $|\omega_1 \omega_2 \dots \omega_d|$ would be less than 1. Therefore the characteristic polynomial of sequence v_n is of the form

$$P(X) = X^2 + eX \pm 1, \qquad e \in \mathbb{Z}.$$

As $|\omega_1| > 1$, P has no rational root. Let $\sigma \neq Id$ be the morphism of conjugaison in quadratic field $\mathbb{K} = \mathbb{Q}(\omega_1)$. As $\sigma(\omega_1) = \omega_2$, $v_n \in \mathbb{Z}$, and $A_1 \in \mathbb{Q}$, we have

$$v_n = A_1 \omega_1^n + A_2 \omega_2^n \Rightarrow v_n = A_1 \omega_2^n + \sigma(A_2) \omega_1^n.$$

Therefore $A_2 = A_1$ and

$$t_n = A_1 \omega_2^{2^n} + b_n. \tag{5.40}$$

Replace in (5.29) and use (5.38) ; we get, as $|\omega_1 \omega_2| = 1$,

$$-2A_1 = \frac{1}{A_1}(A_1 \omega_2^{2^n} + b_n)^2 - (A_1 \omega_2^{2^{n+1}} + b_{n+1}) - \frac{a_{n+1}}{a_n}(A_1 \omega_2^{2^n} + b_n) + A_1 \frac{a_{n+2}}{a_{n+1}}.$$

As $A_1 \in \mathbb{Q}$ and $|\omega_2| < 1$, this implies

$$-2A_1 = \frac{1}{A_1} b_n^2 - b_{n+1} - \frac{a_{n+1}}{a_n} b_n + A_1 \frac{a_{n+2}}{a_{n+1}}. \tag{5.41}$$

By (5.38), we have $\dfrac{a_{n+1}}{a_n} = \dfrac{2bn}{A_1}$. Replacing in (5.41), we get

$$b_{n+1} = \frac{1}{A_1} b_n^2 - 2A_1. \tag{5.42}$$

Put $b_n = A_1 c_n$; (5.42) becomes

$$c_{n+1} = c_n^2 - 2 = f(c_n); \qquad f(x) = x^2 - 2. \tag{5.43}$$

It is easy to see that $f(x) > x$ for every $x > 2$. Therefore, if $c_0 > 2$, then c_n is increasing ; if c_n has a limit ℓ, then $\ell = \ell^2 - 2$, which is impossible because $c_0 > 2$. Hence $\lim_{n \to +\infty} c_n = +\infty$ and for large n we have, by (5.43), $c_{n+1} \geq \frac{1}{2} c_n^2$. By Lemma 4.2, $b_n = A_1 c_n \geq A_1 C' \theta^{2^n}$ with $\theta > 1$, a contradiction. Therefore $c_0 \leq 2$. If $c_0 < -2$, then $c_1 > 2$ and we can argue the same way and get a contradiction, whence

$$c_0 \in [-2, 2]. \tag{5.44}$$

As $f(x) \in [-2, 2]$ for every $x \in [-2, 2]$, we have

$$c_n \in [-2, 2], \quad \forall n \in \mathbb{N}. \tag{5.45}$$

But $c_n = b_n/A_1$ and $b_n \in \mathbb{Z}$. So (5.45) implies that c_n takes only a finite number of values. Therefore there exist $N \in \mathbb{N}$ and $k \in \mathbb{N} \setminus \{0\}$ such that $c_{N+k} = c_N$. By using (5.43), we immediately get by induction

$$c_{n+k} = c_n = f^k(c_n) \qquad \text{for every } n \geq N. \tag{5.46}$$

But f^k is clearly a monic polynomial with integer coefficients and degree 2^k. Moreover $f^1(0) = -2$; $f^2(0) = f(f(0)) = f(-2) = 2$; $f^3(0) = f(f^2(0)) = f(2) = 2 \ldots$. Hence

$$f^k(x) = x^{2^k} + \cdots \pm 2; \qquad \forall k \in \mathbb{N} \setminus \{0\}. \tag{5.47}$$

By (5.46), we see that c_n is a *rational* root of an equation with integer coefficients of the form

$$x^{2^k} + \cdots \pm 2 = 0. \tag{5.48}$$

Therefore we have only four possibilities :

$$c_n = -2,\ c_n = 1,\ c_n = -1,\ c_n = 2. \tag{5.49}$$

But if $c_n = -2$, then $c_{n+1} = 2$, and if $c_n = 1$, then $c_{n+1} = -1$. Therefore $c_n = -1$ or $c_n = 2$ for every $n \geq N+1$, and

$$b_n = -A_1 \text{ or } b_n = 2A_1 \quad \text{for every } n \geq N+1. \tag{5.50}$$

In the first case, we have $a_{n+1} = -2a_n$ by (5.38), so that

$$S_{10} = \sum_{n=1}^{N} \frac{a_n}{A_1(\omega_1^{2^n} + \omega_1^{-2^n}) + b_n} + \frac{a_N}{A_1} \sum_{n=1}^{+\infty} \frac{(-2)^n}{(\omega_1^{2^N})^{2^n} + (\omega_1^{-2^N})^{2^n} - 1}. \tag{5.51}$$

In the second case, we have $a_{n+1} = 4a_n$ by (5.38), and we get

$$S_{10} = \sum_{n=1}^{N} \frac{a_n}{A_1(\omega_1^{2^n} + \omega_1^{-2^n}) + b_n} + \frac{a_N}{A_1} \sum_{n=1}^{+\infty} \frac{4^n}{(\omega_1^{2^N})^{2^n} + (\omega_1^{-2^N})^{2^n} + 2}. \tag{5.52}$$

The proof of Corollary 3.5 is complete.

5.6. Proof of Corollary 3.6

Let $a \in \mathbb{Z} \setminus \{0\}$, $b \in \mathbb{Z} \setminus \{0\}$. By Corollary 3.4, $f(\frac{a}{b}) \notin \mathbb{Q}$ if $\beta \in \mathbb{R}_+^* \setminus \mathbb{Q}$. So, if $f(\frac{a}{b}) \in \mathbb{Q}$, by (3.9) there exist $(\eta, \delta) \in \mathbb{N}^2$ such that, for $n \geq N$,

$$u_{n+1} = \frac{\eta}{\delta} u_n^2 - \frac{a}{b} u_n + \frac{\delta}{\eta} \frac{a}{b}. \tag{5.53}$$

We can suppose

$$GCD(\eta, \delta) = 1 \quad ; \quad GCD(a, b) = 1. \tag{5.54}$$

Let $d = GCD(a, \eta)$, $\eta = d\eta'$, $a = da'$. Then (5.53) becomes

$$bu_{n+1} = \frac{\eta b}{\delta} u_n^2 - au_n + \frac{\delta a'}{\eta'}. \tag{5.55}$$

The fraction $\delta a'/\eta'$ is irreducible ; therefore $\eta' | \delta$ by (5.55). But $GCD(\eta', \delta) = 1$; hence $\eta' = 1$, $\eta = d$, $a = a'\eta$, so that we can write (5.55) in the form

$$bu_{n+1} = \frac{\eta b}{\delta} u_n^2 - a'\eta u_n + \delta a'. \tag{5.56}$$

Let p be any prime divisor of δ ; by (5.56), $p | u_n$ for every $n \geq N$. Put $\delta = p\delta'$, $u_n = pu'_n$, and replace in (5.56) ; we obtain

$$bu'_{n+1} = \frac{\eta b}{\delta'} u'^2_n - a'\eta u'_n + \delta' a'. \tag{5.57}$$

Hence we see by induction that $\delta | u_n$. Put

$$u_n = \delta v_n, \tag{5.58}$$

and replace in (5.56). We get

$$bv_{n+1} = \eta b v_n^2 - a'\eta v_n + a'. \tag{5.59}$$

As $GCD(a',b)=1$ by (5.54), (5.59) implies that $b|(\eta v_n-1)$. Put

$$\eta v_n - 1 = bk_n \quad \text{for } n \geq N. \tag{5.60}$$

Now multiply (5.59) by η, and replace ηv_n by using (5.60). We obtain

$$bk_{n+1} = (b^2+2b-a'\eta)k_n. \tag{5.61}$$

By (5.60), b and η are prime to each other, as well as b and a' by (5.54). Therefore

$$GCD(b^2+2b-a'\eta, b) = 1. \tag{5.62}$$

Suppose that $b>1$; let p be a prime divisor of b. By (5.61) and (5.62), $p|k_n$ for every $n \geq N$. More precisely, by putting

$$\begin{cases} k_n = k_n' p^{\alpha(n)},\ b = p^{\alpha} b' \\ p \nmid k_n',\ p \nmid b' \\ \alpha(n) \geq 1,\ \alpha \geq 1 \end{cases} \tag{5.63}$$

we have by (5.61) and (5.62)

$$\alpha + \alpha(n+1) = \alpha(n), \tag{5.64}$$

which is impossible because $\alpha(n) \geq 1$.

Therefore $b=1$ and (5.59) becomes

$$v_{n+1} = \eta v_n^2 - a'\eta v_n + a', \tag{5.65}$$

which proves Corollary 3.6.

6. Proof of Theorem 3.2

6.1. A lemma on irrationality measures

We give a complete proof of a classical lemma allowing to compute irrationality measures.

LEMMA 6.1. *Let $\alpha \in \mathbb{R}$. Suppose that there exist constants $a > 0$, $b > 0$, $h \geq 1$, a function $g : \mathbb{N} \longrightarrow \mathbb{R}_+^*$, increasing for $n \geq N$ and satisfying $\lim g(n)_{n\to+\infty} = +\infty$, and a sequence C_n/B_n of rational numbers such that*

$$|B_n C_{n+1} - B_{n+1} C_n| \neq 0 \quad \text{for every } n \geq N \tag{6.1}$$

$$|B_n| = O(g(n)^a) \tag{6.2}$$

$$|B_n \alpha - C_n| = O(g(n)^{-1}) \tag{6.3}$$

$$g(n+1) \leq b(g(n))^h \quad \text{for every } n \geq N. \tag{6.4}$$

Then, for every $\varepsilon > 0$, there exists $q_0 = q_0(\varepsilon) \in \mathbb{N}$ such that, for every rational p/q satisfying $|q| \geq q_0$

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{|q|^{m+\varepsilon}}, \tag{6.5}$$

where $m = ah^2 + 1$.

See [5, Theorem 9.7] for an effective version of lemma 6.1 (there is a misprint in this book ; the condition "$b \leq 1$" should be replaced by "$b > 0$").

PROOF OF LEMMA 6.1. By (6.2) and (6.3), there exist $k > 0$ and $\ell > 0$ such that, for every $n \geq N$

$$\begin{cases} |B_n| \leq k(g(n))^a \\ |B_n \alpha - C_n| \leq \ell/g(n) \end{cases} \tag{6.6}$$

Choose q_1 such that

$$\frac{q_1 \ell}{g(N)} \geq \frac{1}{2}. \tag{6.7}$$

Now let $(p, q) \in \mathbb{Z} \times \mathbb{Z}$, with $|q| \geq q_1$. Let ν be the least integer satisfying

$$\frac{|q|\,\ell}{g(\nu)} < \frac{1}{2}. \tag{6.8}$$

It is possible to find such a ν because $\lim_{n \to +\infty} g(n) = +\infty$. By (6.7), we have $\nu \geq N+1$. As ν is the least integer satisfying (6.8), we have $g(\nu - 1) \leq 2\,|q|\,\ell$, which implies, by (6.4), $g(\nu) \leq b(2\,|q|\,\ell)^h$. Using (6.4) again, we get

$$g(\nu) < g(\nu+1) \leq b^{h+1}(2\,|q|\,\ell)^{h^2}. \tag{6.9}$$

Now we consider the determinant

$$\Delta_\nu = \begin{vmatrix} B_\nu & C_\nu \\ B_{\nu+1} & C_{\nu+1} \end{vmatrix},$$

which is not zero by (6.1) ; this means that the vectors (B_ν, C_ν) and $(B_{\nu+1}, C_{\nu+1})$ form a basis of $\mathbb{R}^2$. Therefore, one of the two determinants

$$\begin{vmatrix} B_\nu & C_\nu \\ q & p \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} B_{\nu+1} & C_{\nu+1} \\ q & p \end{vmatrix}$$

is not zero. Put $s = \nu$ or $\nu + 1$, such that

$$\delta_s = \begin{vmatrix} B_s & C_s \\ q & p \end{vmatrix} \neq 0.$$

As $\delta_s \in \mathbb{Z}$, $|\delta_s| \geq 1$, that is $|pB_s - qC_s| \geq 1$. Therefore $1 \leq |q(B_s\alpha - C_s) - B_s(q\alpha - p)|$, which implies

$$1 \leq |q|\,|B_s\alpha - C_s| + |B_s|\,|q\alpha - p|\,.$$

By using (6.6), we get

$$1 \leq \frac{|q|\,\ell}{g(s)} + kg(s)^a\,|q\alpha - p|\,.$$

Hence, by (6.8), $\dfrac{1}{2} < kg(s)^a\,|q\alpha - p|$.

Using (6.9), we finally get

$$|q\alpha - p| > \frac{1}{2kb^{a(h+1)}(2\,|q|\,\ell)^{ah^2}},$$

which proves lemma 6.1 by choosing $q_0 = q_0(\varepsilon)$ such that $2^{ah^2} k b^{a(h+1)} \ell^{ah^2} \leq q_0^{\varepsilon}$. $\square$

6.2. Proof of Theorem 3.2

By (5.15), we have $a_n = O(u_n^\alpha)$ and $b_n = O(u_n^\alpha)$ for every $\alpha > 0$. Put

$$\left\{\begin{array}{l} \omega_\alpha = \inf(2-\gamma, 1-3\alpha) \\ \mu = \dfrac{\omega_\alpha}{\lambda} - \alpha. \end{array}\right. \tag{6.10}$$

As $\lambda \geq 2$ we see that, for α small enough

$$3\alpha < 7\alpha < \mu < \frac{1}{2} < 1 - 4\alpha, \tag{6.11}$$

so that (4.10) is fulfiled.

Now we follow the proof of Theorem 3.1 until (4.7).As $\mu < \frac{1}{2}$, we see that (4.11) becomes, for α small enough, and $\varepsilon = \alpha$,

$$\delta = -\frac{\mu}{2} + 2\alpha, \tag{6.12}$$

so that (4.17) can be written as, because $u_{n+1} = O(u_n^2)$ and $\varepsilon = \alpha$,

$$B_n S - C_n = O(u_n^{-\mu+6\alpha}). \tag{6.13}$$

Similarly we have by (4.16), with ε replaced by α,

$$K_n = O(u_n^{1+2\alpha}). \tag{6.14}$$

Now we have to find an upper bound for B_n. By (4.14) and (3.3) we have

$$|B_n| \leq |K_n|\, O(u_n^{4\alpha}) O(u_n^{\mu}) u_n.$$

Hence, by (6.14),

$$B_n = O(u_n^{2+\mu+6\alpha}). \tag{6.15}$$

In Lemma 6.1, we therefore choose

$$\left\{\begin{array}{l} g(n) = u_n^{\mu-6\alpha} \\ a = \dfrac{2+\mu+6\alpha}{\mu-6\alpha} \\ h = 2 \\ b = (2\beta)^{\mu-6\alpha}. \end{array}\right. \tag{6.16}$$

It remains to prove that (6.1) holds. Assume that there exist infinitely many n such as $B_nC_{n+1} - B_{n+1}C_n = 0$. Then a look at (4.19) shows that (5.16), and hence (5.18), hold for infinitely many n. Therefore we have for infinitely many n, as $p_{n+1}/q_{n+1} \to \beta$,

$$\beta - \frac{p_n}{q_n} = O(u_n^{\gamma-2}) + O(u_n^{-1+3\alpha}) + O(u_n^{-2+6\alpha}).$$

So, for α small enough, we get

$$\beta - \frac{p_n}{q_n} = O(u_n^{-\omega_\alpha}). \tag{6.17}$$

By (3.18) this implies, if $p_n/q_n \neq \beta$, $K/q_n^\lambda = O(u_n^{-\omega_\alpha})$, whence $K = O(u_n^{-\omega_\alpha+\lambda\mu}) = O(u_n^{-\alpha})$. Therefore $K = 0$, which is impossible, and $p_n/q_n = \beta$ for infinitely many n ; this means that $\beta \in \mathbb{Q}$. So (6.1) holds by (3.19). By lemma 6.1 and (6.16), we get (6.5) with $m = 4(2+\mu+6\alpha)/(\mu-6\alpha)$, which proves Theorem 3.2 by choosing α small enough.

Acknowledgment. The author thanks Professors Masanori Kadsurada, Kumiko Nishioka, Taka-aki Tanaka and Michel Waldschmidt for useful remarks and pointing out errors in a first version of this work.

References

[1] Amou, M. and M. Katsurada, Irrationality results for values of generalized Tschakaloff series, J. Number Theory **76** (1999), 155–169.

[2] Badea, C., A Theorem on irrationality of infinite series and applications, Acta Arith. **LXIII.4** (1993), 313–323.

[3] Duverney, D., Propriétés arithmétiques des solutions de certaines équations fonctionnelles de Poincaré, J. Th. Nombres Bordeaux **8** (1996), 443–447.

[4] Duverney, D., Sur les propriétés arithmétiques de la fonction de Tschakaloff, Periodica Math. Hungarica **35 (3)** (1997), 149–157.

[5] Duverney, D., *Théorie des Nombres*, Dunod, 1998.

[6] Duverney, D., Sur les séries de nombres rationnels à convergence rapide, C.R. Acad. Sci. Paris, t. 328, Ser. I (1999), 553–556.

[7] Erdös, P. and E. G. Straus, On the irrationality of certain Ahmes series, J. Indian Math. Soc. **27** (1963), 129–133.

[8] Erdös, P. and R. L. Graham, *Old and new problems and results in Combinatorial Number Theory*, Monographie n°28 de l'Enseignement Mathématique, Genève, 1980.

[9] Erdös, P., *On the irrationality of certain series : problems and results*, in New advances in Transcendence theory, edited by A. Baker, Cambridge University Press, 1988, p. 102–109.

[10] Fourier, C., *Mélanges d'Analyse Algébrique*, Stainville, 1815.

[11] Galambos, J., *Representation of Real Numbers by Infinite Series*, Lecture Notes in Math 502, Springer, 1976.

[12] Golomb, J. W., On the sum of the reciprocals of the Fermat numbers and related irrationalities, Can. J. Math. **15** (1963), 475–478.

[13] Hančl, J., Irrationality of quick convergent series, J. Théorie Nombres Bordeaux **8** (1996), 275–282.

[14] Jennings, D., *Some reciprocal summations identities, ...* in *Applications of Fibonacci Numbers*, vol. 7, Bergum G. E. et al. editors, Kluwer Academic Publishers, 1998.

[15] Liouville, J., *Sur des classes très étendues de quantités dont la valeur n'est ni algébrique ni même réductible à des irrationnelles algébriques*, C. R. Acad. Sci. Paris vol. 18, 1844, p. 883–885.

[16] Loxton, J. H. and A. J. Van der Poorten, *Transcendence and algebraic indenpendence by a method of Mahler*, in Transcendence Theory; advances and applications, edited by Baker A. and Masser D. W., Academic Press (1977), 211–226.

[17] Lucas, E., Théorie des fonctions numériques simplement périodiques, Amer. J. Math. **1** (1878), 185–240 and 289–321.

[18] Lucas, E., *Théorie des nombres*, Gauthiers-Villars, Paris (1901); reprinted by A. Blanchard, Paris, 1961.

[19] Mahler, K., Remarks on a paper by W. Schwarz, J. Number theory **1** (1969), 512–521.

[20] Nishioka, Ku., Mahler functions and transcendence, Lecture Notes in Math. **1631** (1996), Springer.

[21] Nishioka, Ku., Algebraic independence of reciprocal sums of binary recurrences, Mh. Math. **123** (1997), 135–148.

[22] Perron, O., *Irrationalzahlen*, Chelsea, 1948.

[23] Ribenboim, P., *The Book of Prime Numbers Records*, Springer, 1988.

[24] Sylvester, J., On a point in the theory of vulgar fractions, Amer. J. Math. **3** (1880), 332–335.

[25] Tanaka, T., Algebraic independence results related to linear recurrences, Osaka J. Math. **36** (1999), 203–227.

(Received September 13, 2000)

24, Place du Concert
F-59800 LILLE FRANCE
E-mail: dduverney@nordnet.fr