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# *Families of Representations of Punctured Torus Bundles*

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**Abstract.** We construct a complex surface of irreducible $SL(6,\mathbb{C})$ representations of the fundamental group of a once punctured torus bundle over $S^1$. It contains the Mangum-Shanahan curve of $SL(3,\mathbb{C})$ representations [MS]. As a corollary, we show that almost all punctured torus bundle groups are linear.

## 1. Introduction

In the study of three dimensional manifolds, no one could have doubts as to the importance of irreducible representations of the fundamental groups. For instance, the Casson invariant of integral homology 3-spheres is defined by counting algebraically the number of conjugacy classes of irreducible representations from the fundamental group to the Lie group $SU(2)$. Representing knot groups into well known groups to get various information of given knots has played an important role in the knot theory. Further it is known that irreducible representations of certain 3-manifold groups in $SL(2,\mathbb{C})$ give information on embedded incompressible surfaces.

From these historical backgrounds, it seems natural to ask whether irreducible representations in other Lie groups exist. In [MS], Mangum and Shanahan construct a complex curve of $SL(3,\mathbb{C})$ representations of the fundamental group of a once punctured torus bundle over $S^1$. It is irreducible for all but finitely many values of the parameter. In particular, they show that these representations are different from those obtained by composing representations in $SL(2,\mathbb{C})$ with the unique irreducible representation of $SL(2,\mathbb{C})$ in $SL(3,\mathbb{C})$. Moreover infinitely many of these representations are conjugate to $SU(3)$ representations.

In this paper, following Mangum-Shanahan's recipe, we construct a complex two-parameters family $\rho_{q,t}$ $(q,t\in\mathbb{C})$ of $SL(6,\mathbb{C})$ representations of the

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fundamental group of a once punctured torus bundle. We use here the Krammer representation of the braid group $B_4$ [Kr] to construct the family. In our setting, it is irreducible for all but finitely many curves in $\mathbb{C}^2$ and contains the Mangum-Shanahan curve of $SL(3,\mathbb{C})$ representations. As a corollary of the construction, we see that almost all punctured torus bundle groups are linear. We hope that this complex surface contributes to an understanding of geometry and topology on once punctured torus bundles.

## 2. The Surface of Representations

In this section, we construct a complex surface of irreducible $SL(6,\mathbb{C})$ representations of the fundamental group of a once punctured torus bundle over $S^1$. The idea of the construction is due to Mangun and Shanahan [MS]. In order to obtain six dimensional representations, we use the Krammer representation (see [Kr]) of the braid group.

Let $M = M_f$ be an oriented punctured torus bundle over $S^1$ with monodromy $f$. Namely, we form the mapping torus $M_f$ by taking $T \times [0,1]$ and glueing $T \times \{0\}$ and $T \times \{1\}$ via $f$, where $T$ is a once punctured torus. Then the fundamental group of $M$ has a presentation of the form

$$\langle x, y, z \mid zxz^{-1} = f_*(x),\ zyz^{-1} = f_*(y)\rangle,$$

where $f_*$ is the induced homomorphism on the fundamental group of the fiber $T$.

We first review the construction of a homomorphism

$$\iota : \pi_1 M \to B_4/Z_4,$$

where $B_4$ is the four strand braid group and $Z_4$ is its center (see [MS]). We adopt Artin's presentation of $B_4$:

$$\langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2,\ \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3,\ [\sigma_1, \sigma_3] = 1\rangle.$$

As is known (see [B]), the free group $F_2$ of rank two injects into $B_4$, and the generators are mapped to $\sigma_1\sigma_3^{-1}$ and $\sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}$. In particular, the image is a normal subgroup of $B_4$, so that $B_4$ acts on $F_2$ by conjugation. Thus we have a homomorphism

$$h : B_4 \to \mathrm{Aut}(F_2).$$

Further Dyer, Formanek and Grossman show in [DFG] that the kernel of $h$ is $Z_4$ and the image of an injective homomorphism

$$\bar{h} : B_4/Z_4 \hookrightarrow \mathrm{Aut}(F_2)$$

is $\mathrm{Aut}^+(F_2)$. Here $\mathrm{Aut}^+(F_2)$ denotes an index two subgroup of $\mathrm{Aut}(F_2)$, which is the preimage of $SL(2,\mathbb{Z})$ under a natural surjective homomorphism from $\mathrm{Aut}(F_2)$ to $GL(2,\mathbb{Z})$. We therefore obtain an isomorphism

$$\bar{h} : B_4/Z_4 \xrightarrow{\cong} \mathrm{Aut}^+(F_2).$$

We now identify $F_2$ with the fundamental group of $T$ (namely, $F_2 \cong \langle x, y \rangle$). Then the desired homomorphism $\iota : \pi_1 M \to B_4/Z_4$ is defined by

$$\begin{aligned} \iota(x) &= [\sigma_1 \sigma_3^{-1}] \\ \iota(y) &= [\sigma_2 \sigma_1 \sigma_3^{-1} \sigma_2^{-1}] \\ \iota(z) &= \bar{h}^{-1}(f_*). \end{aligned}$$

Next let us recall that the Krammer representation

$$k : B_4 \to GL\left(6, \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]\right)$$

of $B_4$ (see [Kr]) is defined by

$$k(\sigma_1) = \begin{pmatrix} q^2 t & 0 & 0 & 0 & 0 & 0 \\ q(q-1)t & 0 & 0 & q & 0 & 0 \\ q(q-1)t & 0 & 0 & 0 & q & 0 \\ 0 & 1 & 0 & 1-q & 0 & 0 \\ 0 & 0 & 1 & 0 & 1-q & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$k(\sigma_2) = \begin{pmatrix} 1-q & q & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & q^2(q-1)t & 0 & 0 \\ 0 & 0 & 1 & q(q-1)^2 t & 0 & 0 \\ 0 & 0 & 0 & q^2 t & 0 & 0 \\ 0 & 0 & 0 & q(q-1)t & 0 & q \\ 0 & 0 & 0 & 0 & 1 & 1-q \end{pmatrix}$$

and

$$k(\sigma_3)=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-q & q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & q^3(q-1)t \\ 0 & 0 & 0 & 1-q & q & 0 \\ 0 & 0 & 0 & 1 & 0 & q^2(q-1)t \\ 0 & 0 & 0 & 0 & 0 & q^2t \end{pmatrix},$$

where $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ is the ring of Laurent polynomials on two variables. From a straightforward computation, we see that the image of the generator of the center $Z_4$ is given by

$$k\left((\sigma_1\sigma_2\sigma_3)^4\right) = q^8t^2I,$$

where $I$ denotes the identity matrix. Thereby if we define a modified representation via

$$\alpha(\sigma_i) = -q^{-2/3}t^{-1/6}k(\sigma_i),$$

then we can easily check that $\alpha\left((\sigma_1\sigma_2\sigma_3)^4\right) = I$ and $\det\alpha(\sigma_i) = 1$. Hence we obtain a representation

$$\bar{\alpha} : B_4/Z_4 \to SL\left(6, \mathbb{Z}[q^{\pm 1/3}, t^{\pm 1/6}]\right).$$

The composition of $\iota$ and $\bar{\alpha}$ yields a complex surface $\rho_{q,t}$ $(q, t \in \mathbb{C})$ of representations of $\pi_1 M$ in $SL(6, \mathbb{C})$:

$$\rho_{q,t} : \pi_1 M \to SL(6, \mathbb{C}).$$

A direct computation shows that

$$\rho_{q,t}(x)=\begin{pmatrix} q^2t & 0 & 0 & 0 & 0 & 0 \\ q(q-1)t & 0 & 0 & 0 & q & q(1-q) \\ q(q-1)t & 0 & 0 & 1 & q-1 & -(q-1)^2 \\ 0 & 0 & 1 & 0 & 1-q & 1-q \\ 0 & \bar{q} & 1-\bar{q} & -1+\bar{q} & -\bar{q}(q-1)^2 & -\bar{q}(q-1)^2 \\ 0 & 0 & 0 & 0 & 0 & \bar{q}^2\bar{t} \end{pmatrix}$$

and

$$\rho_{q,t}(y)=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & q^2 \\ 0 & q^2t & q^2(q-1)t & q^2(1-q)t & -q^2(q-1)^2t & -q^2(q-1)^2t \\ 0 & q(q-1)t & q(q-1)^2t & w_1(q,t) & -q(q-1)^3t & w_3(q,t) \\ 0 & 0 & q^2t & 0 & q^2(1-q)t & q^2(1-q)t \\ 0 & 0 & q(q-1)t & -\bar{q}^3(q-1)\bar{t} & w_1(q,t) & -q(q-1)^2t \\ \bar{q}^2 & \bar{q}^2(q-1) & \bar{q}(q-1) & w_2(q,t) & qw_2(q,t) & -\bar{q}(q-1)^2 \end{pmatrix},$$

where we have put $\bar{q} = q^{-1}$, $\bar{t} = t^{-1}$ and

$$\begin{aligned} w_1(q,t) &= \bar{q}^2\bar{t} - qt + 2q^2t - q^3t, \\ w_2(q,t) &= (1-q)(1-q^2t+q^3t)\bar{q}^4\bar{t}, \\ w_3(q,t) &= (1-q)(-1+qt-2q^2t+q^3t). \end{aligned}$$

REMARK 2.1. As for a calculation of the image of the generator $z$, see [MS] Remark 4. We remark here that there is no need for us to calculate $\rho_{q,t}(z)$ explicitly in the discussions below.

LEMMA 2.2. *For all but finitely many curves in $\mathbb{C}^2$, the representation $\rho_{q,t}$ $(q,t \in \mathbb{C})$ is irreducible.*

PROOF. We show that the non-zero common invariant subspace of matrices $\rho_{q,t}(x)$ and $\rho_{q,t}(y)$ coincides with the vector space $\mathbb{C}^6$. Because if they have no non-trivial common invariant subspace, then $\rho_{q,t}$ becomes an irreducible representation.

To this end, we first transform basis of $\mathbb{C}^6$ so that $\rho_{q,t}(x)$ is diagonalized. The non-singular matrix

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{-1-qt+q^2t+q^3t^2}{(-1+q)t} \\ q & 0 & -q^2 & -1 & \frac{(-1+q)q^3t}{(1+qt)(-1+q^2t)} & -1+q+q^2t \\ -1+q & 1 & q & -1 & \frac{q^2t-2q^3t+q^4t-q^4t^2+q^5t^2}{(1+qt)(-1+q^2t)} & -1+q+q^2t \\ 0 & 1 & -q & 1 & \frac{(-1+q)q^2t}{(1+qt)(-1+q^2t)} & 1 \\ 1 & 0 & 1 & 1 & \frac{(1-2q+q^2+q^2t+q^3t)qt}{(1+qt)(-1+q^2t)} & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

realizes this change of basis, if $(q,t) \in \mathbb{C}^2$ is not contained in curves

$$(1+qt)(-1+q^2t) = 0, \quad (-1+q)t = 0 \quad \text{and} \quad 1+q = 0.$$

In fact, a computation shows that

$$X = P^{-1}\rho_{q,t}(x)P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -q^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -q & 0 & 0 \\ 0 & 0 & 0 & 0 & q^{-2}t^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & q^2t \end{pmatrix}$$

and

$$Y = P^{-1}\rho_{q,t}(y)P = (\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_6),$$

where

$$\mathbf{y}_1 = \begin{pmatrix} \frac{2-qt+q^2t+q^3t-q^4t+q^3t^2+q^5t^2}{q(1+q)^2t} \\ \frac{(-1+q)(-1+qt)(1+q^3t)}{q(1+q)^2t} \\ \frac{(1+t)(1-q^2t)(1+q^3t)}{q^2(1+q)^2t(1+qt)} \\ \frac{(1-q^2t)(1+q^2t)}{(1+q)^2t} \\ \frac{(-1+q)(-1+q^2t)}{q^3t} \\ 0 \end{pmatrix},$$

$$\mathbf{y}_2 = \begin{pmatrix} \frac{(-1+q)(-1+qt)(1+q^3t)}{q^2(1+q)^2t} \\ \frac{1+q^2-qt+q^2t+q^3t-q^4t+2q^5t^2}{q^2(1+q)^2t} \\ \frac{(1+t)(1-q^2t)(1+q^3t)}{q^3(1+q)^2t(1+qt)} \\ \frac{(-1+q^2t)(1+q^2t)}{(1+q)^2t} \\ \frac{(-1+q)(-1+q^2t)}{q^4t} \\ 0 \end{pmatrix},$$

$$\mathbf{y}_3 = \begin{pmatrix} \frac{(1-qt)(1+qt)}{q(1+q)t} \\ \frac{(-1+qt)(1+qt)}{(1+q)t} \\ 0 \\ \frac{1+q^3t^2}{(1+q)t} \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{y}_4 = \begin{pmatrix} \frac{(1-qt)(1+q^3t)(-1+q^4t)}{q^2(1+q)t(-1+q^2t)} \\ \frac{(1-qt)(1+q^3t)(-1+q^4t)}{q^2(1+q)t(-1+q^2t)} \\ \frac{1+t-q^2t+q^3t+q^3t^2-q^4t^2+q^6t^2+q^6t^3}{q^3(1+q)t(1+qt)} \\ 0 \\ \frac{(1-q)(1+q)(1+q^3t)}{q^4t} \\ 0 \end{pmatrix},$$

$$\mathbf{y}_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{(-1+q)q^2t}{(1+qt)(-1+q^2t)} \end{pmatrix} \quad \text{and} \quad \mathbf{y}_6 = \begin{pmatrix} \frac{(-1+qt)(1+q^2t)(-1+q^4t)}{q^2(1+q)t} \\ \frac{(-1+qt)(1+q^2t)(-1+q^4t)}{q^2(1+q)t} \\ \frac{(1+t)(1-q^2t)(1+q^2t)(1+q^3t)}{q^3(1+q)t(1+qt)} \\ 0 \\ \frac{(-1+q^2t)(1-q+q^3+q^2t-q^4t+q^5t)}{(-1+q)q^4t} \\ 0 \end{pmatrix}.$$

Let $\{e_1, \cdots, e_6\}$ be the new basis of $\mathbb{C}^6$ with respect to the matrix $P$ and $V$ denote a non-zero common invariant subspace of $X$ and $Y$. We then show that $V$ coincides with $\mathbb{C}^6$.

From the form of the diagonal matrix $X$, the subspace $V$ contains at least one of $e_3, e_4, e_5, e_6$ or $ue_1 + ve_2$, where $(u, v) \neq (0, 0)$.

(i) At first we suppose that $V$ contains one of $e_3, e_4, e_5$ or $e_6$. Then we see from the form of $\mathbf{y_3}, \mathbf{y_4}, \mathbf{y_5}$ and $\mathbf{y_6}$ that $V$ contains other three vectors provided we remove finitely many curves in $\mathbb{C}^2$ (in fact, we can easily show that $e_3 \in V \Rightarrow e_4 \in V \Rightarrow e_5 \in V \Rightarrow e_6 \in V \Rightarrow e_3 \in V$). Further from the form of $\mathbf{y_4}$ or $\mathbf{y_6}$ (respectively $\mathbf{y_3}$), the vector $e_1 + e_2$ (respectively $q^{-1}e_1 - e_2$) is contained in $V$. We have assumed $q \neq -1$, so that $V$ contains $e_1$ and $e_2$. Hence we can conclude $V = \mathbb{C}^6$.

(ii) Next we consider the case where $V$ contains the vector $ue_1 + ve_2$. Since we have removed the curve $1 + q = 0$, either $u - v$ or $u + qv$ is non-zero. From the form of $\mathbf{y_1}$ and $\mathbf{y_2}$, we see that $V$ contains $e_4$ (respectively $e_3$) when $u - v \neq 0$ (respectively $u + qv \neq 0$). Thus combining with discussion in (i), we can conclude $V = \mathbb{C}^6$ in this case. The proof is over. □

Lemma 2.3. *The complex surface $\rho_{q,t}$ contains the Mangum-Shanahan curve $\rho_q$ of representations of $\pi_1 M$ in $SL(3, \mathbb{C})$.*

Proof. Let us recall that the Mangum-Shanahan curve $\rho_q$ is constructed from the Burau representation of the braid group $B_4$. On the other hand, if we specialize the Krammer representation at $t = 1$, it is given by the symmetric square of the Burau representation (see [Kr] Proposition 3.2). Therefore the assertion immediately follows. □

To sum up we have the following theorem.

Theorem 2.4. *Let $M$ be an oriented punctured torus bundle over $S^1$. Then there exists a complex two-parameters family of representations $\rho_{q,t}$ of $\pi_1 M$ in $SL(6, \mathbb{C})$. Moreover $\rho_{q,t}$ is irreducible for all but finitely many curves in $\mathbb{C}^2$ and contains the Mangum-Shanahan curve $\rho_q$ of $SL(3, \mathbb{C})$ representations.*

## 3. Linearity of Punctured Torus Bundle Groups

As a corollary of the construction of the representation $\rho_{q,t}$, we see the linearity of punctured torus bundle groups.

Corollary 3.1. *Let $M_f$ be an oriented punctured torus bundle over $S^1$. Assume that the monodromy $f$ is given by $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ on $F_2^{abel} \cong \mathbb{Z} \oplus \mathbb{Z}$. If $|a+d| > 2$ (namely $M_f$ admits the hyperbolic structure), then the fundamental group $\pi_1 M_f$ is linear.*

Proof. If no non-trivial power of $f$ induces an inner automorphism on $F_2$, then $\iota$ is an injective homomorphism (see [MS] Remark 1). The assumption on $f$ implies that $f^n$ cannot be an inner automorphism. Further the Krammer representation $k$ is faithful (see [Kr]), so that the modified representation $\bar{\alpha}$ is also faithful. Assigning to $q, t$ any algebraically independent non-zero complex values, we have an injective homomorphism $\pi_1 M_f \hookrightarrow SL(6,\mathbb{C})$. This completes the proof. $\square$

Remark 3.2. It is known that the fundamental group of hyperbolic 3-manifolds has a faithful representation in $GL(4,\mathbb{R})$ (see [K] Problem 3.33 (A)).

Let $M$ be the complement of the figure eight knot, which is one of well known punctured torus bundles over $S^1$. We adopt the following presentation of $\pi_1 M$:

$$\pi_1 M = \langle x, y, z \mid zxz^{-1} = xy,\ zyz^{-1} = yxy \rangle.$$

In this setting, the automorphism $f_*$ on $F_2 = \langle x, y \rangle$ is given by

$$\begin{aligned} x &\mapsto xy \\ y &\mapsto yxy. \end{aligned}$$

Thus the monodromy of $M$ is just $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Since this matrix is a hyperbolic element of $SL(2,\mathbb{Z})$, we may conclude from Corollary 3.1 that $\pi_1 M$ is linear.

Finally we consider the complement of the trefoil knot. Its fundamental group is described by

$$\langle x, y, z \mid zxz^{-1} = xy,\ zyz^{-1} = x^{-1} \rangle.$$

Then the trace of the monodromy is 1, so that Corollary 3.1 gives no information on the linearity of this group. In fact, a computation reveals that the sixth power of the monodromy induces the inner automorphism

$$\begin{aligned} x &\mapsto [x,y]x[x,y]^{-1} \\ y &\mapsto [x,y]y[x,y]^{-1}. \end{aligned}$$

Remark 3.3. As is known, the fundamental group of the trefoil knot is isomorphic to the braid group $B_3$. Further from the classical result, the group $B_3$ is known to be linear (see [B]).

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