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Hodge Number of Cohomology of Local Systems on the Complement of Hyperplanes in $\mathbb{P}^3$

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Abstract. The cohomology of the local system on the complement of hyperplanes has a Hodge structure as the α-invariant cohomology of a Kummer covering ramified along their hyperplanes for a generic character α. In this paper we study the case of arrangements of hyperplanes in the three dimensional complex projective space. We construct a resolution for an arrangement of hyperplanes and compute its Chow group. By computing the first Chern class of logarithmic 1-forms, we can obtain the Euler characteristic and the Hodge numbers of cohomology of local systems using the intersection set of the arrangement of hyperplanes and binomial coefficients.

1. Introduction

A finite set of hyperplanes is called an arrangement of hyperplanes. Let $\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$ be an arrangement of hyperplanes in $\mathbb{P}^N = \mathbb{P}^N(\mathbb{C})$ and $U = \mathbb{P}^N - \cup_{i=1}^n H_i$ be its complement. Let V_α be the rank one local system on U, whose monodromy around the hyperplane H_k is $\exp(2\pi i \alpha_k)$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a collection of them. The cohomology groups $H^N(U, V_\alpha)$ of are studied well as generalized hypergeometric functions, we refer to [1], [5] and [22]. In the case of rational exponents it is realized geometrically as the cohomology of a Kummer covering of $\mathbb{P}^N$ ramified along $\mathcal{A}$. When $N = 2$ a covering for a certain arrangement is well-known as a Hirzebruch's example: the surfaces obtained by a Kummer covering of $\mathbb{P}^2$ is of general type with $c_1^2 = 3c_2$ (see [12]). In general, the cohomology group $H^N(U, V_\alpha)$ has a Hodge structure as follows (see [5]).

We fix a positive integer m. Let $\pi_m : Y_m \to \mathbb{P}^N$ be the abelian covering of $\mathbb{P}^N$ ramified only along every H_i with the ramification index m and the

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Galois group $G \simeq (\mathbb{Z}/m\mathbb{Z})^{n-1}$. Then the function field K of Y_m is given by the abelian extension

$$K = \mathbb{C}(z_1/z_0, z_2/z_0, \ldots, z_N/z_0)((h_2/h_1)^{1/m}, \cdots, (h_n/h_1)^{1/m})$$

of the function field $\mathbb{C}(z_1/z_0, z_2/z_0, \ldots, z_N/z_0)$ of $\mathbb{P}^N$ where h_i is a linear form defining $H_i = \ker h_i$.

Let $\tilde{Y} \to Y_m$ be a resolution of Y_m. Then the cohomology $H^i(\tilde{Y}, \mathbb{C})$ of $\tilde{Y}$ has the action of G and a pure Hodge structure $H^i(\tilde{Y}, \mathbb{C}) = \oplus_{p+q=i} H^{p,q}$. So for a character α of G we put

$$\begin{aligned} H^i(\tilde{Y}, \mathbb{C})_\alpha &= \{\omega \in H^i(X, \mathbb{C}) \mid g^*(\omega) = \alpha(g)\omega, \quad \text{for all } g \in G\}, \\ H^{p,q}(\alpha) &= H^{p+q}(\tilde{Y}, \mathbb{C})_\alpha \cap H^{p,q} \end{aligned}$$

and then have the eigenspace decomposition

$$H^i(\tilde{Y}, \mathbb{C}) = \bigoplus_{\alpha \in G^*} H^i(\tilde{Y}, \mathbb{C})_\alpha.$$

They induce the Hodge decomposition

$$H^i(\tilde{Y}, \mathbb{C})_\alpha = \bigoplus_{p+q=i} H^{p,q}(\alpha).$$

On the other hand the cohomology $H^N(U, V_\alpha)$ is isomorphic to $H^N(\tilde{Y}, \mathbb{C})_\alpha$ for generic α. Therefore $H^N(U, V_\alpha)$ has the Hodge decomposition.

In this paper our purpose is to compute these Hodge numbers $\dim H^{p,q}(\alpha)$ when $N = 3$. It is clear that the dimension and Hodge numbers of $H^N(U, V_\alpha)$ are combinatorial. For example the dimension of $H^N(U, V_\alpha)$ for arrangement in general position is $\binom{n-2}{N}$. So we give their descriptions with the intersection set $L(\mathcal{A})$ of a arrangement $\mathcal{A}$ and binomial coefficients.

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2. The Hodge Structure of Cohomology of Local Systems on the Complement of Hyperplanes

Let $\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$ be an arrangement of hyperplanes in $\mathbb{P}^N$ and $U = M(\mathcal{A})$ be its complement. The set $L = L(\mathcal{A})$ of nonempty intersections of elements of $\mathcal{A}$ is called the intersection set of $\mathcal{A}$ and we denote by $L_p = L_p(\mathcal{A})$ the set of elements of L whose codimension in $\mathbb{P}^N$ is p. Obviously we see that $L = \cup_{p\geq 1} L_p$ and $L_1 = \mathcal{A}$.

2.1. Hodge decomposition of cohomology of local systems

Definition 2.1 (Blow ups for an arrangement). Let $\mathcal{L}$ be a subset of the intersection set $L(\mathcal{A})$ of an arrangement $\mathcal{A}$ of hyperplanes in $\mathbb{P}^N(\mathbb{C})$ and set $\mathcal{L}_p = \mathcal{L} \cap L_p(\mathcal{A})$.

$\tau : X \to \mathbb{P}^N$ is called *a blowing up of* $\mathbb{P}^N$ *along* $\mathcal{L}$ if τ is the composition of the sequence

$$X = X_{N-1} \xrightarrow{\tau_{N-1}} X_{N-2} \xrightarrow{\tau_{N-2}} \cdots \xrightarrow{\tau_2} X_1 \xrightarrow{\tau_1} X_0 = \mathbb{P}^N$$

where $X_s \overset{\tau_s}{\to} X_{s-1}$ is the blowing up along the proper transform of $\cup_{H\in\mathcal{L}_{N-s+1}} H$ under $\tau_1 \circ \cdots \circ \tau_{s-1}$. Furthermore when the total transform D of $\cup_{H\in\mathcal{A}} H$ is a normal crossing divisor, we call $\mathcal{L}$ *a singular set of* $\mathcal{A}$. The intersection of all singular sets of $\mathcal{A}$ is called *the minimal singular set of* $\mathcal{A}$.

Remark. $\mathcal{L} = L(\mathcal{A})$ is a singular set of $\mathcal{A}$, obviously. Due to [6] and [18] $\mathcal{L}$ consist of all dense edges of $\mathcal{A}$ is also a singular set of $\mathcal{A}$.

Let $\tau : X \to \mathbb{P}^N$ be a blowing up of $\mathbb{P}^N$ along a singular set $\mathcal{L}$ with a normal crossing divisor $D = \tau^{-1} \cup_{H\in\mathcal{A}} H$. Let $\pi_m : Y_m \to \mathbb{P}^N$ be the abelian covering of $\mathbb{P}^N$ ramified only along every $H \in \mathcal{A}$ with the ramification index m and the Galois group G. This induce the covering $Y \overset{\pi}{\to} X$ ramified only along D with the Galois group G. Since it is abelian, we have the eigenspace decomposition

$$\pi_* \mathcal{O}_Y = \bigoplus_{\alpha\in G^*} \mathcal{V}_\alpha.$$

In general Y has rational singularities and then let $\sigma : \tilde{Y} \to Y$ be a desingularization of X such that $\tilde{D} = (\pi\circ\sigma)^{-1} D$ is a normal crossing divisor too. Each $\mathcal{V}_\alpha$ is an invertible sheaf on X endowed with a logarithmic connection

$$\nabla_\alpha : \mathcal{V}_\alpha \to \Omega^1_X(\log D) \otimes \mathcal{V}_\alpha$$

along D induced by the Kähler differential $d : \mathcal{O}_{\tilde{Y}} \to \Omega^1_{\tilde{Y}}(\log \tilde{D})$. Then we have

$$R(\pi \circ \sigma)_* \Omega^\bullet_{\tilde{Y}}(\log \tilde{D}) = \Omega^\bullet_X(\log D) \otimes_{\mathcal{O}_X} \pi_* \mathcal{O}_Y.$$

Since the Hodge to de Rham spectral sequence for hypercohomology on $\tilde{Y}$ degenerates at E_1, the E_1-spectral sequence

$$H^q(X, \Omega^p_X(\log D) \otimes \mathcal{V}_\alpha) \Rightarrow \mathbb{H}^{p+q}(X, \Omega^\bullet_X(\log D) \otimes \mathcal{V}_\alpha)$$

degenerates at E_1 (see [7] and [8]).

On the other hand, denote $U = X \setminus D = \mathbb{P}^N \setminus \cup_{H \in \mathcal{A}} H$ and let $j : U \to X$ be the inclusion. For $\mathcal{V}_\alpha$ we have a local system $V_\alpha = Ker(\nabla_\alpha|_U)$ on U.

DEFINITION 2.2. If none of monodromies of V_α around components of D has one as eigenvalue, α is called to be *generic* for $\mathcal{L}$ (or *non-resonant* in [9]).

In this case due to [7] we know that $Rj_* V_\alpha$, $j_! V_\alpha$ and $\Omega^\bullet_X(\log D) \otimes \mathcal{V}_\alpha$ are quasi-isomorphic. Therefore there is an isomorphism

$$\mathbb{H}^i(X, \Omega^\bullet_X(\log D) \otimes \mathcal{V}_\alpha) = H^i(U, V_\alpha).$$

Furthermore it is known that

$$H^i(U, V_\alpha) = 0 \quad \text{for } i \neq N$$

(see [7], [1], [14]). Then we get the Hodge decomposition

$$H^N(U, V_\alpha) = \bigoplus_{p+q=N} H^q(X, \Omega^p_X(\log D) \otimes \mathcal{V}_\alpha)$$

of the cohomology of the local system on the complement of hyperplanes. Denote those dimensions by $h^{p,q}(\alpha)$ and called the Hodge numbers. Note that

$$\begin{aligned} H^q(X, \Omega^p_X(\log D) \otimes \mathcal{V}_\alpha) &= \overline{H^p(X, \Omega^q_X(\log D) \otimes \mathcal{V}_\alpha)} \\ &= H^p(X, \Omega^q_X(\log D) \otimes \mathcal{V}_{-\alpha}), \end{aligned}$$

because of $\overline{\mathcal{V}_\alpha} = \mathcal{V}_{\overline{\alpha}} = \mathcal{V}_{-\alpha}$.

Remark. The isomorphism

$$H^n(\tilde{Y},\mathbb{C}) \supset H^n(Y,\mathbb{C}) = H^n(X,\pi_*\mathbb{C}),$$

is compatible with the action of G, hence induces isomorphisms

$$H^n(\tilde{Y},\mathbb{C})_\alpha = H^n(X, j_! V_\alpha) = H^i(U, V_\alpha)$$

and also

$$H^{p,q}(\alpha) = H^q(X, \Omega_X^p(\log D)\otimes \mathcal{V}_\alpha).$$

2.2. Generic characters

Review our situation

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & X \\ \Big\downarrow & & \Big\downarrow{\scriptstyle\tau} \\ Y_m & \xrightarrow{\pi_m} & \mathbb{P}^N \end{array}$$

here τ is a blowing up along $\mathcal{L}$, π_m is the abelian covering ramified along $\cup_{H\in\mathcal{A}} H$ with the ramification index m and the Galois group G, π is the covering induced by π_m.

The Galois group

$$\begin{aligned} G &= Gal(Y/X) = Gal(Y_m/\mathbb{P}^N) \\ &= Gal(K/\mathbb{C}(z_1/z_0,\ldots,z_N/z_0)) \simeq (\mathbb{Z}/m\mathbb{Z})^{\oplus(n-1)} \end{aligned}$$

is isomorphic to $\mu_m^{\oplus n}/\mu_m$ here μ_m is the group of m-th root of unity. Fix a primitive m-th root of unity ζ in $\mathbb{C}$. The character group G^* of G is identified with the subset

$$B^* = \left\{ (k_H)_{H\in\mathcal{A}} \mid k_H \in \mathbb{Z}, 0 \le k_H < m, \sum_{H\in\mathcal{A}} k_H \equiv 0 (\bmod m) \right\}$$

of $\mathbb{Z}^{\mathcal{A}}$ in the following manner. An element $k = (k_H)$ of $G^* \simeq B^*$ is defined by

$$k(\sigma) = \zeta^{\sum k_H s_H} \in \mathbb{C} \quad \text{for any } \sigma = ((\zeta^{s_H}) \bmod \mu_m).$$

In addition we shall allow the identification $G^* = \frac{1}{m}B^*$, and then $\alpha \in G^*$ is

$$\alpha = (\alpha_H), \qquad \alpha_H = \frac{k_H}{m} \qquad \text{for} \quad (k_H) \in B^*.$$

For $\alpha = (\alpha_H)$, we define some numerical values as follows. We write

$$\nu(\alpha) = \sum_{H\in\mathcal{A}} \alpha_H \quad \text{and} \quad \alpha_X = \sum_{H\supset X} \alpha_H$$

for $X \in L(\mathcal{A})$. Note that $0 \le \alpha_H < 1$ and $\nu(\alpha)$ is a positive integer. Obviously, $\alpha = (\alpha_H)_{H\in\mathcal{A}}$ is generic for $\mathcal{L}$, if and only if, α_H is not zero for all H and α_X is not integer for all X in $\mathcal{L}$.

Denote the integer and decimal part of α_X by $\beta_X(\alpha)$ and $\varepsilon_X(\alpha)$ respectively, namely

$$\beta_X(\alpha) = [\alpha_X] \quad \text{and} \quad \alpha_X = \beta_X(\alpha) + \varepsilon_X(\alpha)$$

here $\beta_X(\alpha) \in \mathbb{Z}$ and $0 \le \varepsilon_X(\alpha) < 1$.

If $(\alpha_H)_H \in G^* = \dfrac{1}{m}B^*$ is generic, $-\alpha$ corresponds to an element $(1 - \alpha_H)_H$ of $\dfrac{1}{m}B^*$ and it is denoted by α^*. Rational numbers $\nu(-\alpha)$, $\alpha_X(-\alpha)$, $\beta_X(1-\alpha)$ and $\varepsilon_X(1-\alpha)$ are denoted by $\nu^*(\alpha)$, $\alpha^*_X(\alpha)$, $\beta^*_X(\alpha)$ and $\varepsilon^*_X(\alpha)$, respectively. The following is clear.

Lemma 2.1. *If $\alpha = (\alpha_H)_{H\in\mathcal{A}}$ is generic for a singular set $\mathcal{L}$ of $\mathcal{A}$ then*

$$\nu + \nu^* = n \quad \text{and} \quad \beta_X + \beta^*_X = p_X - 1$$

for $X \in \mathcal{L}$. Here n is the cardinality of $\mathcal{A}$ and p_X is the number of hyperplanes in $\mathcal{A}$ including X for $X \in L(\mathcal{A})$.

3. Facts and Results

Let $\mathcal{A}$ be an arrangement of n hyperplanes in $\mathbb{P}^N$ and U be its complement. Then the cohomology group $H^N(U, V_\alpha)$ of U for generic α has the Hodge structure; $H^N(U, V_\alpha) = \oplus_{p+q=N} H^{p,q}(\alpha)$. Denote by $\chi(\mathcal{A})$ the topological Euler characteristic of U and by $h^{p,q}(\alpha)$ the dimension of $H^{p,q}(\alpha)$. If α is generic, the dimension of $H^N(U, V_\alpha)$ is $(-1)^N\chi(\mathcal{A})$. Therefore we note that

$$(-1)^N\chi(\mathcal{A}) = \sum_{p+q=N} h^{p,q}(\alpha).$$

We shall arrange notations. Let $L = \cup_{p\ge1} L_p$ be the intersection set of $\mathcal{A}$ and $\mathcal{L} = \cup_{p\ge1}\mathcal{L}_p$ be a singular set of $\mathcal{A}$. For $X \in L(\mathcal{A})$, p_X is the number of hyperplanes in $\mathcal{A}$ including X and n_p^X (resp. m_p^X) is the number of elements of L_p (resp. $\mathcal{L}_p$) included in X.

For generic α we shall use notations defined in the preceding section; ν, ν^*, β_X, β_X^* and so on.

For integers p and q, we define the binomial coefficient

$$\binom{p}{q} = \begin{cases} \dfrac{p!}{q!(p-q)!}, & p \geq q \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Note that this vanishes when $p < q$ and when $q < 0$ and that $\binom{p}{0} = \binom{p}{p} = 1$ when $p \geq 0$. For positive integers a, b, and N, we can make sure that

$$\binom{a+b}{N} = \sum_{p+q=N} \binom{a}{p}\binom{b}{q}.$$

3.1. In general position

An arrangement $\mathcal{A}$ of hyperplanes in $\mathbb{P}^N$ is said to be in general position if $codim X = p_X$ for all X of $L(\mathcal{A})$. This means that the union of their hyperplanes is a normal crossing. In this case the topological Euler characteristic is well-known (cf. [19] and [1]).

Theorem 3.1.

$$(-1)^N \chi(\mathcal{A}) = \binom{n-2}{N}.$$

And we have the following fact for Hodge numbers.

Theorem 3.2 (Terasoma [20] Theorem 5.2.1).

$$h^{p,q}(\alpha) = \binom{\nu^*-1}{p}\binom{\nu-1}{q}.$$

3.2. $N = 2$

Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{P}^2$. We can check easily the combinatorial formula

$$\binom{n}{2} = \sum_{x \in L_2} \binom{p_x}{2}.$$

The following results are obtained by T. Oda.

Theorem 3.3 (Oda [15] Theorem 1).

$$\chi(\mathcal{A}) = \binom{n-2}{2} - \sum_{x \in L_2} \binom{p_x - 1}{2}.$$

Theorem 3.4 (Oda [15] Theorem 3, see also [12], [13]).

$$h^{p,q}(\alpha) = \binom{\nu^* - 1}{p}\binom{\nu - 1}{q} - \sum_{x \in L_2} \binom{\beta_x^*}{p}\binom{\beta_x}{q}.$$

3.3. $N = 3$

Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{P}^3$. We can easily check the following lemma.

Lemma 3.5. *We have the combinatorial Formula*

$$\binom{n}{3} = \sum_{x \in L_3} \binom{p_x}{3} - \sum_{l \in L_2} (n_3^l - 1)\binom{p_l}{3}.$$

Here n_3^l is the number of points in L_3 on l.

Main Theorems in this paper is following.

Theorem 3.6. *The topological Euler characteristic is*

$$-\chi(\mathcal{A}) = \binom{n-2}{3} - \sum_{x \in L_3} \binom{p_x - 1}{3} + \sum_{l \in L_2} \left\{ (n_3^l - 1)\binom{p_l - 1}{3} + \binom{p_l - 1}{2} \right\}.$$

Theorem 3.7. *The Hodge number is*

$$h^{p,q}(\alpha) = \binom{\nu^* - 1}{p}\binom{\nu - 1}{q} - \sum_{x \in \mathcal{L}_3} \binom{\beta_x^*}{p}\binom{\beta_x}{q} + \sum_{l \in \mathcal{L}_2} (m_3^l - 1)\binom{\beta_l^*}{p}\binom{\beta_l}{q} - \mathcal{E}^{p,q}(\alpha)$$

here we put

$$g_l = g_l(\alpha) = \nu - \beta_l - \sum_{\substack{x \in \mathcal{L}_3 \\ x \subset l}} (\beta_x - \beta_l)$$

for $l \in L_2$*, and* $\mathcal{E}^{p,q}(\alpha)$ *is given by*

$$\mathcal{E}^{p,q}(\alpha) = \sum_{l \in \mathcal{L}_2} \left\{ (g_l - 1) \binom{\beta_l^*}{p} \binom{\beta_l}{q-1} + (g_l^* - 1) \binom{\beta_l^*}{p-1} \binom{\beta_l}{q} \right\}.$$

Problem 3.8. In higher dimensional case, express Euler characteristic of the complement of hyperplanes and Hodge numbers of cohomology of local systems by the binomial coefficient like theorems above.

4. Proofs of Main Theorems

4.1. Resolution and Chow ring

In this section we shall construct the blowing up of $\mathbb{P}^3(\mathbb{C})$ and compute the strucure of its Chow ring due to [11, pp. 621–624].

Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{P}^3$, $L(\mathcal{A})$ its intersection set and $\mathcal{L}$ a subset of $\mathcal{A}$. We construct the blowing up τ along $\mathcal{L}$ which is the composition

$$X := X_2 \overset{\tau_2}{\to} X_1 \overset{\tau_1}{\to} X_0 = \mathbb{P}^3$$

of τ_1 and τ_2 as follows.

$\tau_1 : X_1 \to \mathbb{P}^3$ is the blowing up at points in $\mathcal{L}_3$. We denote by E_x the exceptional divisor over $x \in \mathcal{L}_3$, by L_x a generic line in $E_x \cong \mathbb{P}^2$ and by H the pullback of a hyperplane in $\mathbb{P}^3$. $\tau_2 : X \to X_1$ is the blowing up along the proper transforms $\hat{l}$ of $l \in \mathcal{L}_2$. We denote by F_l the exceptional divisor over $\hat{l}$, and by M_l a fiber of the $\mathbb{P}^1$-bundle $\tau_2 : F_l \to \hat{l}$. The proper transform of L_x and E_x in X is also denoted by L_x and E_x. Then we have

$$\begin{aligned} H^2(X_1) &= \mathbb{C}\{H, E_x\}_{x \in \mathcal{L}_3} \\ H^4(X_1) &= \mathbb{C}\{H^2, L_x\}_{x \in \mathcal{L}_3} \\ H^2(X) &= \mathbb{C}\{H, E_x, F_l\}_{x \in \mathcal{L}_3, l \in \mathcal{L}_2} \\ H^4(X) &= \mathbb{C}\{H^2, L_x, M_l\}_{x \in \mathcal{L}_3, l \in \mathcal{L}_2} \end{aligned}$$

and the intersection pairing of Chow ring is given by Table 1 and 2.

Table 1. $H^2 \times H^2 \to H^4$.

	H	E_x	F_l
H	H^2	0	M_l
E_y		$-\delta_{xy}L_x$	$\delta_{yl}M_l$
F_m			$\delta_{lm}F_l^2$

Table 2. $H^2 \times H^4 \to \mathbb{C}$.

	H	E_x	F_l
H^2	1	0	0
L_y	0	$-\delta_{xy}$	0
M_m	0	0	$-\delta_{lm}$
F_l^2	-1	$-\delta_{xl}$	F_l^3

In Tables 1 and 2 we use the notation for A, B in $L(\mathcal{A})$,

$$\delta_{AB} = \begin{cases} 1, & \text{if } A \subseteq B \\ 0, & \text{otherwise.} \end{cases}$$

and have relations

$$F_l^2 = -H^2 - 2(m_3^l - 1)M_l + \sum_{\substack{x \in \mathcal{L}_3 \\ x \subset l}} L_x \quad \text{and} \quad F_l^3 = 2(m_3^l - 1).$$

Now we introduce some notations and rules of computations for easy to see. Let v_X be some value associated to $X \in L(\mathcal{A})$, for example p_X, $\beta_X = \beta_X(\alpha)$ and their polynomial. We put

$$v\mathbb{E}^k := \sum_{x \in \mathcal{L}_3} v_x E_x^k \quad \text{and} \quad v\mathbb{F}^k := \sum_{l \in \mathcal{L}_2} v_l F_l^k.$$

We have following remarks by the intersection pairing of Chow ring. Note that k-th powers of $\mathbb{E}$ and $\mathbb{F}$ are denote by $\mathbb{E}^k$ and $\mathbb{F}^k$ respectively. Furthermore we can check the following expressions

$$v'\mathbb{E} \cdot v\mathbb{F}^2 = -\sum_l \left\{ \left(\sum_{x \in l} v'_x \right) v_l \right\} \quad \text{and} \quad H \cdot v\mathbb{F}^2 = -\sum_l v_l.$$

LEMMA 4.1. *We get following relations.*

$$H \cdot \mathbb{E} = 0, \quad H^2 \cdot \mathbb{F} = 0, \quad \mathbb{E}^2 \cdot \mathbb{F} = 0,$$

$$H^3 = 1, \quad \mathbb{F}^3 = 2(H - \mathbb{E}) \cdot \mathbb{F}^2.$$

REMARK. We note that

$$\mathbb{E}^2 = -\sum_{x \in \mathcal{L}_3} L_x, \quad H \cdot \mathbb{F} = \sum_{l \in \mathcal{L}_2} M_l, \quad \mathbb{E}^3 = |\mathcal{L}_3|, \quad H \cdot \mathbb{F}^2 = -|\mathcal{L}_2|.$$

4.2. The Chern classes of logarithmic 1-forms

Let $\tau : X \to \mathbb{P}^3$ be a blowing up of $\mathbb{P}^3$ along a singular set $\mathcal{L}$ with a normal crossing divisor $D = \tau^{-1} \cup_{H_i \in \mathcal{A}} H_i$. In this section we compute the Chern classes of $\Omega^1_X(\log D)$ which implies Theorem 3.6, using the above intersection pairing. First we recall the following.

PROPOSITION 4.2 ([11]). *The i-th Chern class c_i of Ω^1_X is given by*

$$\begin{aligned} c_1 &= -4H + 2\mathbb{E} + \mathbb{F} \\ c_2 &= 6H^2 - \mathbb{F}^2 - 2H \cdot \mathbb{F} \\ c_3 &= -4H^3 - 2\mathbb{E}^3 + 2H \cdot \mathbb{F}^2. \end{aligned}$$

The Chern polynomial of $\oplus_{x \in \mathcal{L}_3} \mathcal{O}_{E_x}$ is

$$C_t(\bigoplus_{x \in \mathcal{L}_3} \mathcal{O}_{E_x}) = 1 + \mathbb{E}t + \mathbb{E}^2 t^2 + \mathbb{E}^3 t^3$$

and one of $\oplus_{l \in \mathcal{L}_2} \mathcal{O}_{F_l}$ is

$$C_t(\bigoplus_{l \in \mathcal{L}_2} \mathcal{O}_{F_l}) = 1 + \mathbb{F}t + \mathbb{F}^2 t^2 + \mathbb{F}^3 t^3.$$

PROOF. We rewrite calculations in [11, pp. 621–624] with our notations. (cf. [10]) □

Proposition 4.3. *Let D_i be the proper transform of H_i by τ for $H_i \in \mathcal{A}$. Denote by h_i the i-th Chern classes of $\oplus_{H_i \in \mathcal{A}} \mathcal{O}_{D_i}$. Then we have*

$$\begin{aligned} h_1 &= nH - p\mathbb{E} - p\mathbb{F} \\ h_2 &= \binom{n+1}{2} H^2 + \binom{p+1}{2} \mathbb{E}^2 + \binom{p+1}{2} \mathbb{F}^2 \\ &\qquad\qquad - \{(nH - p\mathbb{E}) + (H - \mathbb{E})\} \cdot p\mathbb{F} \\ h_3 &= \binom{n+2}{3} - \binom{p+2}{3} \mathbb{E}^3 - \binom{p+2}{3} \mathbb{F}^3 \\ &\qquad\qquad + \{(nH - p\mathbb{E}) + 2(H - \mathbb{E})\} \cdot \binom{p+1}{2} \mathbb{F}^2. \end{aligned}$$

We recall a rule of notations, for example,

$$\binom{p+1}{2} \mathbb{E}^2 = \sum_{x \in \mathcal{L}_3} \binom{p+1}{2} E_x^2.$$

Proof. We denote

$$\mathbb{D} := \sum_{H_i \in \mathcal{A}} D_i, \qquad \mathbb{D}^{(k)} := \sum_{H_i \in \mathcal{A}} D_i^k.$$

Then we have its Chern polynomial

$$\begin{aligned} C_t\Big(\bigoplus_{H_i \in \mathcal{A}} \mathcal{O}_{D_i}\Big) &= \prod_{H_i \in \mathcal{A}} (1 + D_i t + D_i^2 t^2 + D_i^3 t^3) \\ &= 1 + \left(\sum_i D_i\right) t + \left(\sum_{i \le j} D_i \cdot D_j\right) t^2 + \left(\sum_{i \le j \le k} D_i \cdot D_j \cdot D_k\right) t^3 \\ &= 1 + \mathbb{D} t + \frac{1}{2} \left\{\mathbb{D}^2 + \mathbb{D}^{(2)}\right\} t^2 + \frac{1}{6} \left\{\mathbb{D}^3 + 3\mathbb{D} \cdot \mathbb{D}^{(2)} + 2\mathbb{D}^{(3)}\right\} t^3. \end{aligned}$$

We shall compute its coefficients which are the Chern classes h_i. we have

$$D_i = H - \sum_{\substack{x \in \mathcal{L}_3 \\ x \subset H_i}} E_x - \sum_{\substack{l \in \mathcal{L}_2 \\ l \subset H_i}} F_l.$$

Since the cardinality of $\mathcal{A}$ is n and p_X is the number of elements of $\mathcal{A}$ including X for $X \in L$, we get

$$\sum_{H_i \in \mathcal{A}} H^k = nH^k$$

$$\sum_{H_i \in \mathcal{A}} \sum_{x \subset H_i} E_x^k = \sum_x p_x E_x^k = p\mathbb{E}^k, \qquad \sum_{H_i \in \mathcal{A}} \sum_{l \subset H_i} F_l^k = \sum_l p_l F_l^k = p\mathbb{F}^k.$$

Then the first Chern class $h_1 = \mathbb{D}$ is

$$\mathbb{D} = nH - p\mathbb{E} - p\mathbb{F}$$

and by Lemma 4.1 we have

$$\begin{aligned} \mathbb{D}^2 &= n^2H^2 + p^2\mathbb{E}^2 + p^2\mathbb{F}^2 - 2(nH - p\mathbb{E}) \cdot p\mathbb{F} \\ \mathbb{D}^3 &= n^3 - p^3\mathbb{E}^3 - p^3\mathbb{F}^3 + 3(nH - p\mathbb{E}) \cdot p^2\mathbb{F}^2. \end{aligned}$$

Secondly we shall compute $\mathbb{D}^{(k)}$ $(k = 2, 3)$. By Tables 1 and 2 we obtain

$$\begin{aligned} D_i^2 &= \left\{ H - \sum_{x \subset H_i} E_x - \sum_{l \subset H_i} F_l \right\}^2 \\ &= H^2 + \sum_{x \subset H_i} E_x^2 + \sum_{l \subset H_i} F_l^2 - 2H \cdot \left(\sum_{l \subset H_i} F_l \right) \\ &\quad + 2 \left(\sum_{x \subset H_i} E_x \right) \cdot \left(\sum_{l \subset H_i} F_l \right), \end{aligned}$$

$$\begin{aligned} D_i^3 &= \left\{ H - \sum_{x \subset H_i} E_x - \sum_{l \subset H_i} F_l \right\}^3 \\ &= H^3 - \sum_{x \subset H_i} E_x^3 - \sum_{l \subset H_i} F_l^3 + 3H \cdot \left(\sum_{l \subset H_i} F_l^2 \right) \\ &\quad - 3 \left(\sum_{x \subset H_i} E_x \right) \cdot \left(\sum_{l \subset H_i} F_l^2 \right). \end{aligned}$$

We compute a sum of their last terms for all H_i as follows. Since $E_x \cdot F_l = 0$ for $x \not\subset l$ we can see

$$\mathbb{E}\cdot\mathbb{F}^k = \left(\sum_x E_x\right)\cdot\left(\sum_l F_l^k\right) = \sum_l \sum_{x\subset l} E_x \cdot F_l^k$$

and then

$$\begin{aligned}\sum_{H_i\in\mathcal{A}}\left(\sum_{x\subset H_i} E_x\right)\cdot\left(\sum_{l\subset H_i} F_l^k\right) &= \sum_{H_i\in\mathcal{A}}\sum_{l\subset H_i}\sum_{x\subset l} E_x\cdot F_l^k \\ &= \sum_l p_l \sum_{x\subset l} E_x\cdot F_l^k = \mathbb{E}\cdot p\mathbb{F}^k.\end{aligned}$$

Consequently we obtain

$$\begin{aligned}\mathbb{D}^{(2)} &= nH^2 + p\mathbb{E}^2 + p\mathbb{F}^2 - 2(H-\mathbb{E})\cdot p\mathbb{F} \\ \mathbb{D}^{(3)} &= n - p\mathbb{E}^3 - p\mathbb{F}^3 + 3(H-\mathbb{E})\cdot p\mathbb{F}^2\end{aligned}$$

Using Lemma 4.1 we can compute the Chern classes h_i. □

Therefore we can obtain the Chern classes of logarithmic 1-forms.

Theorem 4.4.

$$\begin{aligned}c_1(\Omega_X^1(\log D)) &= (n-4)H - (p-3)\mathbb{E} - (p-2)\mathbb{F} \\ c_2(\Omega_X^1(\log D)) &= \binom{n-3}{2}H^2 + \binom{p-2}{2}\mathbb{E}^2 + \binom{p-1}{2}\mathbb{F}^2 \\ &\qquad -(n-3)H\cdot(p-2)\mathbb{F} + (p-2)\mathbb{E}\cdot(p-2)\mathbb{F} \\ c_3(\Omega_X^1(\log D)) &= \binom{n-2}{3} - \binom{p-1}{3}\mathbb{E}^3 - \binom{p}{3}\mathbb{F}^3 \\ &\qquad +(n-2)H\cdot\binom{p-1}{2}\mathbb{F}^2 - (p-1)\mathbb{E}\cdot\binom{p-1}{2}\mathbb{F}^2.\end{aligned}$$

Proof. We have

$$C_t(\Omega_X^1(\log D)) = C_t(\Omega_X^1)\cdot C_t\Big(\bigoplus_{H_i\in\mathcal{A}}\mathcal{O}_{D_i}\Big)\cdot C_t\Big(\bigoplus_{x\in\mathcal{L}_3}\mathcal{O}_{E_x}\Big)\cdot C_t\Big(\bigoplus_{l\in\mathcal{L}_2}\mathcal{O}_{F_l}\Big).$$

Therefore the above propositions give direct calculations of Chern classes of $\Omega^1_X(\log D)$, with relations Lemma 4.1. □

Now we notice that the equality $(nH - p\mathbb{E}) \cdot \mathbb{F}^2 = (H - \mathbb{E}) \cdot p\mathbb{F}^2$ holds when $\mathcal{L} = L$.

DEFINITION 4.1. We define $\mathbb{R} \cdot \mathbb{F}^2$ by an equation

$$(nH - p\mathbb{E}) \cdot \mathbb{F}^2 = (H - \mathbb{E}) \cdot p\mathbb{F}^2 + \mathbb{R} \cdot \mathbb{F}^2.$$

LEMMA 4.5.

$$\mathbb{R} \cdot \mathbb{F}^2 = -\sum_{l \in \mathcal{L}_2} \sum_{\substack{x \notin \mathcal{L}_3 \\ x \subset l}} (p_x - p_l)$$

PROOF. Since $n - p_l = \sum_{x \in L_3,\ x \subset l} (p_x - p_l)$ for a line $l \in L_2$, we get

$$n - \sum_{\substack{x \in \mathcal{L}_3 \\ x \subset l}} p_x = (1 - \sum_{\substack{x \in \mathcal{L}_3 \\ x \subset l}} 1) p_l + \sum_{\substack{x \notin \mathcal{L}_3 \\ x \subset l}} (p_x - p_l).$$

Then taking a sum of these for all $l \in \mathcal{L}_2$, we get the above expression. □

For some value v_X associated to $X \in L$, we write

$$\mathbb{R} \cdot v\mathbb{F}^2 = \sum_{l \in \mathcal{L}_2} v_l \sum_{\substack{x \notin \mathcal{L}_3 \\ x \subset l}} (p_x - p_l).$$

In particular if $\mathcal{L} = L$, we get $\mathbb{R} \cdot \mathbb{F}^2 = 0$ namely

$$(nH - p\mathbb{E}) \cdot v\mathbb{F}^2 = (H - \mathbb{E}) \cdot vp\mathbb{F}^2.$$

Since $-2\binom{p}{3} + p\binom{p-1}{2} - \binom{p-1}{2} = \binom{p-1}{3}$, the following is obtained.

COROLLARY 4.6.

$$c_3(\Omega^1_X(\log D)) = \binom{n-2}{3} - \binom{p-1}{3}\mathbb{E}^3 + (H - \mathbb{E}) \cdot \binom{p-1}{3}\mathbb{F}^2$$
$$- (H - \mathbb{R}) \cdot \binom{p-1}{2}\mathbb{F}^2.$$

By the way, Gauss-Bonnet formula says that

$$c_3(\Omega_X^1(\log D)) = -\chi(\mathcal{A}).$$

Therefore, when $\mathcal{L} = L$, the above corollary implies Theorem 3.6. Note that Theorem 3.6 can be checked by topological direct argument with combinatorics.

4.3. Hodge number

The invertible sheaf $\mathcal{V}_\alpha$ is given by

$$\begin{aligned}\mathcal{V}_\alpha &= \pi^*\mathcal{O}_{\mathbb{P}}(-\nu(\alpha)) \otimes_{\mathcal{O}_X} \mathcal{O}_X\left(\sum_{x\in\mathcal{L}_3}\beta_x(\alpha)E_x + \sum_{l\in\mathcal{L}_2}\beta_l(\alpha)F_l\right)\\ &= -\nu H + \beta\mathbb{E} + \beta\mathbb{F}.\end{aligned}$$

We trace the analogue of Defintion 4.1 in order to get a relation similar to Lemma 4.5.

Definition 4.2. We define $\mathbb{R}_\alpha \cdot \mathbb{F}^2$ by the equation

$$(\nu H - \beta\mathbb{E}) \cdot \mathbb{F}^2 = (H - \mathbb{E}) \cdot \beta\mathbb{F}^2 + \mathbb{R}_\alpha \cdot \mathbb{F}^2.$$

In the same way of Lemma 4.5 we get

Lemma 4.7.

$$\mathbb{R}_\alpha \cdot \mathbb{F}^2 = -\sum_{l\in\mathcal{L}_2}\sum_{\substack{x\notin\mathcal{L}_3\\ x\subset l}}(\alpha_x - \alpha_l) + (H - \mathbb{E}) \cdot \varepsilon\mathbb{F}^2 + \varepsilon\mathbb{E} \cdot \mathbb{F}^2.$$

Note that $(\nu H - \beta\mathbb{E}) \cdot v\mathbb{F}^2 = (H - \mathbb{E}) \cdot \beta v\mathbb{F}^2 + \mathbb{R}_\alpha \cdot v\mathbb{F}^2$ for a value v_l associated to l. If $\mathcal{L} = L$ then

$$\mathbb{R}_\alpha \cdot \mathbb{F}^2 = (H - \mathbb{E}) \cdot \varepsilon\mathbb{F}^2 + \varepsilon\mathbb{E} \cdot \mathbb{F}^2.$$

And we can see easily

$$(H - \mathbb{R}_\alpha) \cdot \mathbb{F}^2 + (H - \mathbb{R}_{\alpha^*}) \cdot \mathbb{F}^2 = (H - \mathbb{R}) \cdot \mathbb{F}^2.$$

Proposition 4.8.

$$\chi(X, \mathcal{V}_\alpha) = -\binom{\nu - 1}{3} + \binom{\beta}{3}\mathbb{E}^3 - (H - \mathbb{E}) \cdot \binom{\beta}{3}\mathbb{F}^2 + (H - \mathbb{R}_\alpha) \cdot \binom{\beta}{2}\mathbb{F}^2.$$

Proof. For a line bundle L, it is well known that $\chi(X, L) = \frac{1}{6}L^3 + \frac{1}{4}c_1L^2 + \frac{1}{12}(c_1^2 + c_2)L + \frac{1}{24}c_1c_2$, where $c_i = c_i(X)$. Then the straight calculation leads the expression above. □

Proposition 4.9.

$$\begin{aligned}\chi(X, \Omega_X^1(\log D) \otimes \mathcal{V}_\alpha) = (\nu^* - 1)\binom{\nu - 1}{2} &- \beta^*\binom{\beta}{2}\mathbb{E}^3 \\ &+ (H - \mathbb{E}) \cdot \beta^*\binom{\beta}{2}\mathbb{F}^2 \\ &- (H - \mathbb{R}_\alpha) \cdot \beta\beta^*\mathbb{F}^2 + (\mathbb{R} - \mathbb{R}_\alpha) \cdot \binom{\beta}{2}\mathbb{F}^2.\end{aligned}$$

Proof. Denote i-th Chern class of $\Omega_X^1(\log D)$ by d_i. Note that the rank of $\Omega_X^1(\log D)$ is 3. We can check the following by using that Euler characteristic can be written in terms of Chern classes (see [10].);

$$\chi(\Omega_X^1(\log D) \otimes \mathcal{V}_\alpha) = \chi(\Omega_X^1(\log D)) + 3\chi(\mathcal{V}_\alpha) - 3\chi(\mathcal{O}_X) + \xi$$

here $\xi = \frac{1}{2}(d_1^2 - 2d_2 + d_1c_1 + d_1\mathcal{V}_\alpha)\mathcal{V}_\alpha$. By the straight calculation we can get that

$$\chi(\Omega_X^1(\log D)) = (n - 1)H^3, \quad \chi(\mathcal{O}_X) = \frac{1}{24}c_1c_2 = H^3$$

and this proposition. □

We unify last terms in the above propositions.

Definition 4.3. We define $\mathcal{E}^{p,q}(\alpha)$ by

$$\mathcal{E}^{p,q}(\alpha) = (H - \mathbb{R}_\alpha) \cdot \binom{\beta^*}{p}\binom{\beta}{q-1}\mathbb{F}^2 + (H - \mathbb{R}_{\alpha^*}) \cdot \binom{\beta^*}{p-1}\binom{\beta}{q}\mathbb{F}^2.$$

Note that $\mathcal{E}^{p,q}(\alpha) = \mathcal{E}^{q,p}(\alpha^*)$ and

$$\sum_{p+q=3} \mathcal{E}^{p,q}(\alpha) = (H - \mathbb{R}) \cdot \binom{p-1}{2}\mathbb{F}^2.$$

Therefore we get the Hodge numbers as follows.

Theorem 4.10.

$$\begin{aligned} h^{p,q}(\alpha) = &\binom{\nu^*-1}{p}\binom{\nu-1}{q} - \binom{\beta^*}{p}\binom{\beta}{q}\mathbb{E}^3 \\ &+ (H - \mathbb{E}) \cdot \binom{\beta^*}{p}\binom{\beta}{q}\mathbb{F}^2 - \mathcal{E}^{p,q}(\alpha). \end{aligned}$$

Proof. We know

$$\begin{aligned} \chi(X, \mathcal{V}_\alpha) &= \sum_i (-1)^i h^{0,i}(\alpha) \\ \chi(X, \Omega_X^1(\log D) \otimes \mathcal{V}_\alpha) &= \sum_i (-1)^i h^{1,i}(\alpha). \end{aligned}$$

Recall that $H^i(U, V_\alpha)$ vanishes for α is generic and i is not $N = 3$. So we have $h^{p,q}(\alpha) = 0$ when $p + q \neq 3$. We obtain

$$\begin{aligned} h^{0,3}(\alpha) &= -\chi(X, \mathcal{V}_\alpha) \\ h^{1,2}(\alpha) &= \chi(X, \Omega_X^1(\log D) \otimes \mathcal{V}_\alpha) \\ h^{3,0}(\alpha) &= \overline{h^{0,3}(\alpha^*)} = -\chi(X, \mathcal{V}_{\alpha^*}) \\ h^{2,1}(\alpha) &= \overline{h^{1,2}(\alpha^*)} = \chi(X, \Omega_X^1(\log D) \otimes \mathcal{V}_{\alpha^*}). \end{aligned}$$

Therefore two propositions above induce this theorem. □

Remark. We compare Theorem 4.10 with Collary 4.6. For a singular set $\mathcal{L}$, denote the description of $c_3(\Omega_X^1(\log D))$ in Collary 4.6 by $-\chi_{\mathcal{L}}$ and one of $h^{p,q}(\alpha)$ in Theorem 4.10 by $h^{p,q}(\alpha)_{\mathcal{L}}$. Note that they have four terms respectively. Then we have

$$-\chi_{\mathcal{L}} = \sum_{p+q=3} h^{p,q}(\alpha)_{\mathcal{L}}.$$

Furthermore the sum of i-th terms of $h^{p,q}(\alpha)_{\mathcal{L}}$ is i-th terms of $-\chi_{\mathcal{L}}$ for $1 \leq i \leq 4$.

5. Minimal Singular Sets and Examples

Recall the blowing up for an arrangement. Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{P}^3$, $\mathcal{L}$ be a singular set of $\mathcal{A}$ and $\tau : X \to \mathbb{P}^3$ be a blowing up along $\mathcal{L}$ such that the total transform D of $\cup_{H\in\mathcal{A}} H$ is a normal clossing divisor.

First minimal $\mathcal{L}_2$ consists of k-lines, $k > 2$, where k-line l is an element of L_2 such that the number of hyperplanes in $\mathcal{A}$ including l is k, we call *a singular line*.

Secondly we shall find minimal $\mathcal{L}_3$. Denote the description of $c_3(\Omega^1_X(\log D))$ in Collary 4.6 by $-\chi_{\mathcal{L}}$. We take any $x_0 \in \mathcal{L}$ and put $\mathcal{L}' = \mathcal{L} - \{x_0\}$. The difference $d(x_0) = -\chi_{\mathcal{L}} + \chi_{\mathcal{L}'}$ of them for $\mathcal{L}$ and $\mathcal{L}'$ can be write explicitly

$$d(x_0) = \binom{p_{x_0}-1}{3} - \sum_{\substack{l\in\mathcal{L}_2\\ l\supset x_0}} \binom{p_l-1}{3} - \sum_{\substack{l\in\mathcal{L}_2\\ l\supset x_0}} (p_{x_0}-p_l)\binom{p_l-1}{2}.$$

On the other hand it is well-known fact that Gauss-Bonnet formula

$$c_3(\Omega^1_X(\log D)) = -\chi(U,\mathbb{C}).$$

Since $U = X \setminus D = \mathbb{P}^3 \setminus \cup_{H\in\mathcal{A}} H$ we can see that if $\mathcal{L}$ and $\mathcal{L}'$ are singular sets of $\mathcal{A}$ then $d(x_0) = 0$.

By the explicit form above, if x_0 with $p_{x_0} = 3$ or if x_0 is included in only one singular line then $d(x_0) = 0$. If x_0 is included in two or more singular lines then $d(x_0) \neq 0$, we call *a singular point*. Consequently the minimal singular set of $\mathcal{A}$ consists of all singular lines and all points included in two or more singular lines.

5.1. 2-generic arrangements

An arrangement of hyperplanes is called to be the *p-generic*, if $p_X = k$ for all $X \in L_k(\mathcal{A}), k \leq p$. Let $\mathcal{A}$ be 2-generic arrangement of hyperplanes in $\mathbb{P}^3$. Then $\mathcal{A}$ has no singular lines. We get the combinatorial Formula

$$\binom{n}{3} = \sum_{x \in L_3} \binom{p_x}{3}.$$

The topological Euler characteristic is

$$-\chi(\mathcal{A}) = \binom{n-2}{3} - \sum_{x \in L_3} \binom{p_x - 1}{3},$$

and the Hodge numbers are

$$h^{p,q}(\alpha) = \binom{\nu^* - 1}{p}\binom{\nu - 1}{q} - \sum_{x \in \mathcal{L}_3} \binom{\beta_x^*}{p}\binom{\beta_x}{q}.$$

5.2. An arrangement without singular points

We assume that an arrangement $\mathcal{A}$ has no singular points. Namely the singular set of $\mathcal{A}$ is the set of singular lines. We have the combinatorial Formula

$$\binom{n}{3} = \sum_{l \in L_2} \left\{ \binom{p_l}{3} + (n - p_l)\binom{p_l}{2} \right\}.$$

The topological Euler characteristic is

$$-\chi(\mathcal{A}) = \binom{n-2}{3} - \sum_{l \in L_2} \left\{ \binom{p_l - 1}{3} - (n - p_l - 1)\binom{p_l - 1}{2} \right\},$$

and the Hodge numbers are

$$\begin{aligned} h^{p,q}(\alpha) = \binom{\nu^* - 1}{p}\binom{\nu - 1}{q} &- \sum_{l \in \mathcal{L}_2} \binom{\beta_l^*}{p}\binom{\beta_l}{q} \\ &- \sum_{l \in \mathcal{L}_2} \left\{ (\nu - \beta_l - 1)\binom{\beta_l^*}{p}\binom{\beta_l}{q-1} \right. \\ &\qquad \left. + (\nu^* - \beta_l^* - 1)\binom{\beta_l^*}{p-1}\binom{\beta_l}{q} \right\}. \end{aligned}$$

5.3. Generalized Ceva's arrangement

We take five points of $\mathbb{P}^3$ in general position and the arrangement $\mathcal{C}_3$ of ten hyperplanes determined by any three points is natural generalization of Ceva's configuration in $\mathbb{P}^2$ (cf. [16]). We can choose the homogeneous coordinate $[z_1, z_2, z_3, z_4]$ such that $\mathcal{C}_3$ is defined by the equation

$$z_1 z_2 z_3 z_4 \prod_{i<j} (z_i - z_j) = 0.$$

This arrangement has 15 points and 25 lines. We have combinatorial data;

- the number of 2-lines is 15,
- the number of 3-lines is 10 (that are singular lines),
- the number of 4-points is 10,
- the number of 6-points is 5 (that are singular points).

Here i-point x is in L_3 with $p_x = i$ and j-line l is in L_2 with $p_l = j$. We can take the singular set $\mathcal{L}$ consisting of 6-points and 3-lines. Therefore we obtain $-\chi(\mathcal{A}) = 6$.

Now we take $\alpha = (a, a, \ldots, a)$ such that a is a rational number, $0 < a < 1$, and $10a$ is integer. If $6a$ is not integer, then α is generic. We can compute

$$(h^{3,0}(\alpha), h^{2,1}(\alpha), h^{1,2}(\alpha), h^{0,3}(\alpha)),$$

is called the Hodge type of generic α. Just there are only the following cases;

a ;	Hodge type	a ;	Hodge type
1/10;	$(6,0,0,0)$	9/10;	$(0,0,0,6)$
2/10;	$(5,1,0,0)$	8/10;	$(0,0,1,5)$
3/10;	$(0,0,6,0)$	7/10;	$(0,6,0,0)$
4/10;	$(5,0,0,1)$	6/10;	$(1,0,0,5)$.

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