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# *Stable-homotopy Seiberg-Witten Invariants for Rational Cohomology $K3\#K3$'s*

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**Abstract.** We show that if $X$ is a closed spin 4-manifold which has the same rational cohomology ring as $K3\#K3$, then the stable-homotopy Seiberg-Witten invariant is non-trivial for every spin structure on X. As an application we obtain a generalized adjunction inequality for such manifolds.

## 1. Introduction and Statement

The Seiberg-Witten invariant is usually a subtle invariant of smooth 4-manifolds. There exist, however, some 4-manifolds for which some parts of their Seiberg-Witten invariants are determined only by their cohomology rings: J. W. Morgan and Z. Szabó showed that if $X$ is a closed spin manifold which has the same rational cohomology ring as a K3 surface, then the Seiberg-Witten invariant is odd for every spin structure on $X$ [7]. D. Ruberman and S. Strle showed a similar result for 4-manifolds which has the same rational homology groups as $T^4$ [8].

Our aim is to give some more examples. S. Bauer and the first author independently introduced the stable-homotopy version of Seiberg-Witten invariant [1] [2]. We shall consider this version of Seiberg-Witten invariant.

Theorem 1 (See Theorem 19). *If $X$ is a closed spin 4-manifold which has the same rational cohomology ring as $K3\#K3$, then the stable-homotopy Seiberg-Witten invariant is non-trivial for every spin structure on X.*

As an application we obtain a generalized adjunction inequality (Theorem 22).

H. Matsue pointed out to the authors that the generalized adjunction inequality implies the non-existence of a spin closed 4-manifold with signature$= -32$ and $b_2 < 44$ [3].

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We shall discuss some generalizations in [4].

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## 2. Some Stable Homotopy Sets

In this section we prepare some notations and lemmas to formulate and prove our main theorem.

### 2.1. A geometric construction

In this subsection we use the following notation. Fix an integer $n \geq 1$.

1. Let $V_0$ and $V_1$ be two finite dimensional complex vector spaces satisfying
$$\dim_{\mathbf{C}} V_0 - \dim_{\mathbf{C}} V_1 = 2n.$$
We regard them as $S^1$-spaces with the standard $S^1$-action.

2. Let $W_0$ and $W_1$ be two finite dimensional real vector spaces satisfying
$$\dim_{\mathbf{R}} W_0 - \dim_{\mathbf{R}} W_1 = 2 - 4n.$$
We regard them as $S^1$-spaces with the trivial $S^1$-action.

3. Let $\mathrm{Map}(S(V_0 \oplus W_0), S(V_1 \oplus W_1))^{S^1}$ be the set of $S^1$-equivariant smooth maps from the sphere $S(V_0 \oplus W_0)$ to the sphere $S(V_1 \oplus W_1)$.

4. Let $[S(V_0 \oplus W_0), S(V_1 \oplus W_1)]^{S^1}$ be the quotient of $\mathrm{Map}(S(V_0 \oplus W_0), S(V_1 \oplus W_1))^{S^1}$ divided by the equivalence relation defined by the $S^1$-equivariant smooth homotopy.

We shall construct a map
$$\delta : [S(V_0 \oplus W_0), S(V_1 \oplus W_1)]^{S^1} \times [S(V_0 \oplus W_0), S(V_1 \oplus W_1)]^{S^1} \to \mathbf{Z}_2$$
which satisfies the cocycle condition
$$\delta(f_0, f_1) + \delta(f_1, f_2) = \delta(f_0, f_2). \tag{1}$$

For two elements $f_0$ and $f_1$ of $\mathrm{Map}(S(V_0 \oplus W_0), S(V_1 \oplus W_1))^{S^1}$, let $F(f_0, f_1)$ be the set of smooth $S^1$-equivariant maps

$$\tilde{f} : S(V_0 \oplus W_0) \times [0,1] \to V_1 \oplus W_1$$

which satisfy the following three conditions.

1. $\tilde{f}(p, 0) = f_0(p)$ and $\tilde{f}(p, 1) = f_1(p)$.

2. The restriction of $\tilde{f}$ on $S(0 \oplus W_0) \times [0,1]$ does not vanish.

3. The map $\tilde{f}$ is transverse to the zero section.

LEMMA 2. *The set $F(f_0, f_1)$ is not empty.*

PROOF. This lemma follows from a standard transversality argument. Since we shall use a similar argument later, we write down the argument in some details. Obviously we have $\tilde{f}$ which satisfies the first of the three conditions. Let $M$ be the zero set $\tilde{f}^{-1}(0)$, which is a compact set contained in $S(V_0 \oplus W_0) \times (0,1)$. Note that

$$\dim(S(0 \oplus W_0) \times [0,1]) - \dim W_1 = 2 - 4n < 0.$$

It implies that we can $S^1$-equivariantly perturb $\tilde{f}$ so that

1. the support of the perturbation is contained in a small neighborhood of $M \cap (S(0 \oplus W_0) \times [0,1])$, and

2. the perturbed map $\tilde{f}'$ does not vanish on $S(0 \oplus W_0)$.

Let $M'$ be the zero set $\tilde{f}'^{-1}(0)$, which is a compact set contained in $(S(V_0 \oplus W_0) \setminus S(0 \oplus W_0)) \times (0,1)$. Since the $S^1$-action is free on the neighborhood of $M'$, we can $S^1$-equivariantly perturb $\tilde{f}'$ further so that

1. the support of the perturbation is contained in a small neighborhood of $M'$, and

2. the perturbed map $\tilde{f}''$ is transverse to the zero section.

Then the perturbed map $\tilde{f}''$ is an element of $F(f_0, f_1)$. $\square$

We denote by $B$ the space $(S(V_0 \oplus W_0) \setminus S(0 \oplus W_0)) \times (0,1)$, which is $S^1$-equivariantly diffeomorphic to $S(V_0) \times W_0 \times (0,1)$. Let $\tilde{f}$ be an element of $F(f_0, f_1)$, and $M$ the zero set $\tilde{f}^{-1}(0)$, which is a 2-dimensional smooth closed submanifold of $B$. The $S^1$-action on $B$ is free. Let $\bar{B}$ denote the quotient space $B/S^1$, which is diffeomorphic to $\bar{B} \cong P(V_0) \times W_0 \times (0,1)$, and $\bar{M}$ the quotient space $M/S^1$, which is a 1-dimensional smooth closed submanifold of $\bar{B}$. Let $E$ denote the vector bundle over $\bar{B}$ defined to be $E := B \times_{S^1} (V_1 \oplus W_1)$. The map $\tilde{f}$ descends to a section $\bar{f}$ of $E$. The zero set of $\bar{f}$ is $\bar{M}$.

Let $\nu(\bar{M})$ be the normal bundle of $\bar{M}$ in $\bar{B}$. Since $\bar{f}$ is transverse to the zero section, we have a bundle isomorphism $\alpha_{\bar{f}} : \nu(\bar{M}) \to E|\bar{M}$.

Since $H_1(\bar{B}, \mathbf{Z}) = 0$, there exists an immersion $i_D : D \to \bar{B}$ from a compact orientable surface $D$ with boundary to $\bar{B}$ such that the image of the boundary $\partial D$ is equal to $\bar{M}$. (Since $\bar{B}$ is a simply connected manifold with dimension larger than $2n$, we could take $D$ as an embedded surface which is diffeomorphic to a disjoint union of 2-disks.) We identify $\partial D$ with $\bar{M}$. Let $\nu(D)$ be the normal bundle of $D$. Note that the restriction of $\nu(D) \oplus \mathbf{R}$ over $\bar{M}$ is canonically isomorphic to $\nu(\bar{B})$, and hence isomorphic to $E|\bar{M}$ via $\alpha_{\bar{f}}$.

We write $E|D$ for the pullback $i_D^* E$ for simplicity. We denote by $[\nu(D) \oplus \mathbf{R}, E|D, \alpha_{\bar{f}}]$ the element of the relative $KO$-group $KO(D, \bar{M})$ defined by this triple. We define $\delta(f_0, f_1, \tilde{f}, D) \in \mathbf{Z}_2$ by

$$\delta(f_0, f_1, \tilde{f}, D) := \langle w_2([\nu(D) \oplus \mathbf{R}, E|D, \alpha_{\bar{f}}]), [D, \bar{M}] \rangle.$$

We show that $\delta(f_0, f_1, \tilde{f}, D)$ does not depend on the choice of $\tilde{f}$ nor $D$.

Suppose that $\tilde{f}'$ is another element of $F(f_0, f_1)$. Let $M'$ be the zero set $\tilde{f}'^{-1}(0)$, and $\bar{M}'$ the quotient space $M'/S^1$. Let $\imath_{D'} : D' \to \bar{B}$ be an immersed compact orientable surface with boundary $\partial D' = \bar{M}'$.

Lemma 3. $\delta(f_0, f_1, \tilde{f}, D) = \delta(f_0, f_1, \tilde{f}', D')$

Proof. We use a homotopy connecting $\tilde{f}$ and $\tilde{f}'$. By using an argument parallel to the proof of Lemma 2, we can find a smooth $S^1$-equivariant map

$$\hat{f} : S(V_0 \oplus W_0) \times [0,1] \times [0,1] \to V_1 \oplus W_1$$

which satisfies the following three conditions.

1. $\hat{f}(p,t,0) = \tilde{f}(p,t)$ and $\hat{f}(p,t,1) = \tilde{f}'(p,t)$.
2. The restriction of $\hat{f}$ on $S(0 \oplus W_0) \times [0,1] \times [0,1]$ does not vanish.
3. The map $\hat{f}$ is transverse to the zero section.

Noticing

$$\dim(S(0 \oplus W_0) \times [0,1] \times [0,1]) - \dim W_1 = 3 - 4n < 0,$$

we may find such $\hat{f}$ as in the proof of Lemma 2. Let $N$ denote the zero set $\tilde{f}^{-1}(0)$, and $\bar{N}$ the quotient $N/S^1$. Then $\bar{N}$ satisfies the following properties.

1. $\bar{N}$ is a compact 2-dimensional submanifold of $\bar{B} \times [0,1]$ with boundary $\bar{M} \times \{0\} \coprod \bar{M}' \times \{1\}$.
2. There is a canonical bundle isomorphism $\alpha_{\bar{N}} : \nu(\bar{N}) \to \hat{E}|\bar{N}$ over $\bar{N}$, where $\hat{E}$ is the pull-back of $E$.

Since the fiber and the base space of $\hat{E}$ are both orientable, $\bar{N}$ is an orientable compact surface with boundary.

Note that $D$ and $D'$ can be regarded as immersed submanifolds in $\bar{B} \times \{0\}$ and $\bar{B} \times \{1\}$ respectively. Then $D \cup \bar{N} \cup D'$ is a 2-dimensional closed manifold with corner $\bar{M} \times \{0\} \coprod \bar{M}' \times \{1\}$. We can deform $D$ and $D'$ to some immersed submanifolds $\hat{D}$ and $\hat{D}'$ of $\bar{B} \times (-1,0]$ and $\bar{B} \times [1,2)$ respectively so that $\hat{N} := \hat{D} \cup \bar{N} \cup \hat{D}'$ is a 2-dimensional smooth closed immersed submanifold in $\bar{B} \times (-1,2)$ (To get $\hat{D}$, push the interior of $D$ down in the last coordinate. To get $\hat{D}'$, push the interior of $D'$ up.) Since $\bar{N}$ is orientable, $\hat{N}$ is also orientable.

From the definitions of $\delta := \delta(f_0, f_1, \tilde{f}, D)$ and $\delta' := \delta(f_0, f_1, \tilde{f}', D')$, we have

$$\delta - \delta' = \langle w_2(\nu(\hat{N})), [\hat{N}] \rangle - \langle w_2(\hat{E}), [\hat{N}] \rangle.$$

Let $a$ be the generator of

$$H^2(\bar{B} \times (-1,2), \mathbf{Z}_2) = H^2(P(V_0), \mathbf{Z}_2) \cong \mathbf{Z}_2.$$

Then the second term is expressed in terms of $a$:

$$\langle w_2(\hat{E}), [\hat{N}] \rangle = (\dim_{\mathbf{C}} V_1) \langle a, [\hat{N}] \rangle$$

The first term is calculated as follows. The orientability of the closed surface $\hat{N}$ implies $\langle w_2(T\hat{N}), [\hat{N}]\rangle = 0$. Hence we have

$$\begin{aligned}\langle w_2(\nu(\hat{N})), [\hat{N}]\rangle &= \langle w_2(T\hat{N} \oplus \nu(\hat{N})), [\hat{N}]\rangle \\ &= \langle w_2(T(\bar{B} \times (-1,2))), [\hat{N}]\rangle \\ &= (\dim_{\mathbf{C}} V_0)\langle a, [\hat{N}]\rangle\end{aligned}$$

Since $\dim_{\mathbf{C}} V_0 - \dim_{\mathbf{C}} V_1 = 2n$, we obtain

$$\delta - \delta' = (\dim_{\mathbf{C}} V_0 - \dim_{\mathbf{C}} V_1)\langle a, [\hat{N}]\rangle = 0 \quad \in \mathbf{Z}_2. \ \square$$

PROPOSITION 4. *$\delta(f_0, f_1, \tilde{f}, D)$ depends only on the $S^1$-equivariant homotopy classes $[f_0]$ and $[f_1]$.*

PROOF. From the previous lemma, $\delta(f_0, f_1, \tilde{f}, D)$ depends only on $f_0$ and $f_1$. We write $\delta(f_0, f_1)$ for $\delta(f_0, f_1, \tilde{f}, D)$. From the definition of $\delta(f_0, f_1)$, we have the cocycle condition (1). Hence, to prove the proposition, it suffices to show $\delta(f_0, f_1) = 0$ for $f_0$ and $f_1$ satisfying $[f_0] = [f_1]$. When $[f_0] = [f_1]$, there exists $\tilde{f}$ in $F(f_0, f_1)$ which does not vanish everywhere. It implies $\delta(f_0, f_1) = 0$. $\square$

From the above proposition, $\delta$ descends to the map

$$\delta : [S(V_0 \oplus W_0), S(V_1 \oplus W_1)]^{S^1} \times [S(V_0 \oplus W_0), S(V_1 \oplus W_1)]^{S^1} \to \mathbf{Z}_2$$

which satisfies the cocycle condition (1).

**2.2.** $\{S(\mathbf{C}^{2n}), S(\mathbf{R}^{4n-2})\}^{S^1}$

Let $C(\mathbf{C}^{2n}, \mathbf{R}^{4n-2})^{S^1}$ be the category defined as follows.

1. An object of $\mathbf{C}(\mathbf{C}^{2n}, \mathbf{R}^{4n-2})^{S^1}$ is $(V_0, V_1, W_0, W_1, o)$ which satisfies the following conditions.
   (a) $V_0$ and $V_1$ are two finite dimensional complex vector spaces satisfying $\dim_{\mathbf{C}} V_0 - \dim_{\mathbf{C}} V_1 = 2n$.
   (b) $W_0$ and $W_1$ are two finite dimensional real vector spaces satisfying $\dim_{\mathbf{R}} W_0 - \dim_{\mathbf{R}} W_1 = 2 - 4n$.

(c) $o$ is one of the two orientations of $W_0 \oplus W_1$.

2. A morphism from $(V_0, V_1, W_0, W_1, o)$ to $(V_0', V_1', W_0', W_1', o')$ is $(V, \lambda_0, \lambda_1, W, \mu_0, \mu_1)$, which satisfies the following conditions.
   (a) $V$ is a finite dimensional complex vector space.
   (b) $W$ is a finite dimensional real vector space.
   (c) $\lambda_i : V_i \oplus V \to V_i'$ $(i = 0, 1)$ is an isomorphism as complex vector space.
   (d) $\mu_i : W_i \oplus W \to W_i'$ $(i = 0, 1)$ is an isomorphism as real vector space.
   (e) $o'$ is the orientation on
   $$W_0' \oplus W_1' \cong (W_0 \oplus W_1) \oplus (W \oplus W)$$
   that comes from the orientation $o$ on $W_0 \oplus W_1$ and the orientation of the complex structure on $W \oplus W \cong W \otimes \mathbf{C}$.

REMARK 5. The definition would make sense only when we fix an orientation of the direct sum explicitly. We, however, do not do this because we shall actually deal with only the mod 2 version of the construction.

We define a functor $\iota$ from $C(\mathbf{C}^{2n}, \mathbf{R}^{4n-2})^{S^1}$ to the category of set as follows.

1. $\iota : (V_0, V_1, W_0, W_1, o) \longmapsto [S(V_0 \oplus W_0), S(V_1 \oplus W_1)]^{S^1}$.
2. $\iota : (V, \lambda_0, \lambda_1, W, \mu_0, \mu_1) \longmapsto ([f] \mapsto [f * \mathrm{id}_{S(V \oplus W)}])$.

A word of explanation for the right-hand side of the latter: Suppose that $\phi = (V, \lambda_0, \lambda_1, W, \mu_0, \mu_1)$ is a morphism from $(V_0, V_1, W_0, W_1, o)$ to $(V_0', V_1', W_0', W_1', o')$. The map

$$\iota_\phi : [S(V_0 \oplus W_0), S(V_1 \oplus W_1)]^{S^1} \to [S(V_0' \oplus W_0'), S(V_1' \oplus W_1')]^{S^1}$$

is defined as follows. Note that $S(V_i' \oplus W_i')$ is $S^1$-equivariantly diffeomorphic to the join of $S(V_i \oplus W_i)$ and $S(V \oplus W)$. The diffeomorphism is induced from (the inverse of) $\lambda_i$ and $\mu_i$. By using this diffeomorphism, we identify each other. We now define the map $\iota_\phi$ by the join with the identity map on $S(V \oplus W)$: $\iota_\phi([f]) = [f * \mathrm{id}_{S(V\oplus W)}]$.

Since join satisfies the association relation, $\iota$ becomes a functor.

It is easy to see that $(C(\mathbf{C}^{2n}, \mathbf{R}^{4n-2})^{S^1}, \iota)$ is an inductive system and we can define its inductive limit.

Definition 6.

$$\{S(\mathbf{C}^{2n}), S(\mathbf{R}^{4n-2})\}^{S^1} := \varinjlim [S(V_0 \oplus W_0), S(V_1 \oplus W_1)]^{S^1}$$

Lemma 7. *Suppose that $\phi = (V, \lambda_0, \lambda_1, W, \mu_0, \mu_1)$ is a morphism from $(V_0, V_1, W_0, W_1)$ to $(V_0', V_1', W_0', W_1')$. Let $[f_0]$ and $[f_1]$ be two elements of $[S(V_0 \oplus W_0), S(V_0 \oplus W_1)]^{S^1}$. Then we have*

$$\delta(\iota_\phi[f_0], \iota_\phi[f_1]) = \delta([f_0], [f_1]) \in \mathbf{Z}_2.$$

Proof. Let $\tilde{f}$ be an element of $F(f_0, f_1)$ by which $\delta([f_0], [f_1])$ can be calculated. Then we can use the join $\tilde{f} * \mathrm{id}_{S(V\oplus W)}$ to calculate $\delta(\iota_\phi[f_0], \iota_\phi[f_1])$. From this construction the lemma immediately follows. □

From this lemma we have a well-defined map

$$\delta : \{S(\mathbf{C}^{2n}), S(\mathbf{R}^{4n-2})\}^{S^1} \times \{S(\mathbf{C}^{2n}), S(\mathbf{R}^{4n-2})\}^{S^1} \to \mathbf{Z}_2,$$

which satisfies the cocycle condition.

Since $S(W_1)$ is connected, the constant maps from $S(V_0 \oplus W_0)$ to $S(W_1)$ are homotopic to each other. Note that any morphism preserves the class of constant maps.

Definition 8. We denote by

$$0 \in [S(V_0 \oplus W_0), S(V_1 \oplus W_1)]^{S^1}$$

the homotopy class of the constant maps from $S(V_0 \oplus W_0)$ to $S(W_0)$. We also denote their inductive limit by the same notation 0.

Definition 9. We define

$$\delta : \{S(\mathbf{C}^{2n}), S(\mathbf{R}^{4n-2})\}^{S^1} \to \mathbf{Z}_2$$

to be $\delta([f]) = \delta([f], 0)$.

To identify the element 0 we can use the following lemma.

Lemma 10. *Let $f$ be an element of* $\mathrm{Map}(S(V_0 \oplus W_0), S(V_1 \oplus W_1))^{S^1}$*. We denote by $D(V_0 \oplus W_0)$ the disk in $V_0 \oplus W_0$ such that $\partial D(V_0 \oplus W_0) = S(V_0 \oplus W_0)$. Suppose that there is an $S^1$-equivariant smooth map*

$$\hat{f} : D(V_0 \oplus W_0) \to (V_1 \oplus W_1) \setminus \{0\}$$

*such that the composition*

$$\begin{array}{rcl} S(V_0 \oplus W_0) & \to & D(V_0 \oplus W_0) \\ & \stackrel{\hat{f}}{\to} & (V_1 \oplus W_1) \setminus \{0\} \to S(V_1 \oplus W_1) \end{array}$$

*is equal to $f$. Then we have $[f] = 0$.*

Proof. By restricting $\hat{f}$ on the spheres with various radius, we can construct an $S^1$-equivariant smooth homotopy from $f$ to a constant path. □

Remark 11.

1. It is known that $\{S(\mathbf{C}^{2n}), S(\mathbf{R}^{4n-2})\}^{S^1}$ has a natural group structure. The element 0 is the unit with respect to this group structure.
2. By a similar construction we can define $\{S(\mathbf{C}^{m}), S(\mathbf{R}^{n})\}^{S^1}$ for any non-negative integers $m$ and $n$.

We show that $\delta$ is bijective.

Lemma 12. *For $n \geq 2$ the map*

$$\delta : \{S(\mathbf{C}^{2n}), S(\mathbf{R}^{4n-2})\}^{S^1} \to \mathbf{Z}_2$$

*is bijective.*

PROOF. It is known that there is a natural bijection $\mathbf{Z}_2 \to \{S(\mathbf{C}^{2n}), S(\mathbf{R}^{4n-2})\}^{S^1}$ [1]. This bijection is given by the composition

$$\begin{aligned}
\pi_1(SO) &\xrightarrow{J} \pi_1^S \\
&= \varinjlim [S^{4n-2} * S(W_0), S(W_1)] \\
&\xrightarrow{c*} \varinjlim [\mathbf{CP}^{2n-1} * S(W_0), S(W_1)] \\
&= \varinjlim [S(\mathbf{C}^{2n} \oplus W_0), S(W_1)]^{S^1} \\
&\to \varinjlim [S(V_0 \oplus W_0), S(V_1 \oplus W_1)]^{S^1} \\
&= \{S(\mathbf{C}^{2n}), S(\mathbf{R}^{4n-2})\}^{S^1},
\end{aligned}$$

where $J$ is the J-homomorphism, and $c^*$ is the map induced from the collapsing map

$$c : \mathbf{CP}^{2n-1} \to \mathbf{CP}^{2n-1}/\mathbf{CP}^{2n-2} \cong S^{4n-2}.$$

Since the second Stiefel-Whitney class is responsible for $\pi_1(SO) \cong \mathbf{Z}_2$, the definition of $\delta$ implies that the composition

$$\pi_1(SO) \to \{S(\mathbf{C}^{2n}), S(\mathbf{R}^{4n-2})\}^{S^1} \to \mathbf{Z}_2$$

is an isomorphism, and so is $\delta$. □

## 2.3. Examples

We calculate the invariant $\delta$ for two examples, which we shall use in the next subsection.

Let $\mathbf{H}$ be the quaternion field, and $\operatorname{Im}\mathbf{H}$ the imaginary part of $\mathbf{H}$. We regard $\mathbf{H}$ as a complex vector space by the right multiplication of $S^1 = \{\cos\theta + i\sin\theta\}_{0\le\theta<2\pi}$.

We take

$$V_0 = \mathbf{H}^n, \quad V_1 = 0, \quad W_0 = 0, \quad W_1 = (\operatorname{Im}\mathbf{H})^n \oplus \mathbf{R}^{n-2}.$$

Let $S(V_0 \oplus W_0) = S(\mathbf{H}^n)$ and $S(V_1 \oplus W_1) = S((\operatorname{Im}\mathbf{H})^n \oplus \mathbf{R}^{n-2})$ be the $4n-1$-sphere in $\mathbf{H}^n$ and the $4n-3$-sphere in $(\operatorname{Im}\mathbf{H})^n \oplus \mathbf{R}^{n-2}$ defined by the equations

$$\begin{aligned}
S(\mathbf{H}^n) &= \{(q_0, \cdots, q_{n-1}) \mid |q_0|^4 + \cdots + |q_{n-1}|^4 = 2\} \\
S((\operatorname{Im}\mathbf{H})^n \oplus \mathbf{R}^{n-2}) &= \{(r_0, \cdots, r_{n-1}, s_0, \cdots, s_{n-3}) \mid |r_0|^2 + \cdots + |r_{n-1}|^2 \\
&\quad + s_0^2 + \cdots + s_{n-3}^2 = 2\}.
\end{aligned}$$

LEMMA 13. *Let* $f_0$ *and* $f_1$ *be the elements of* $\mathrm{Map}(S(\mathbf{H}^2), S((\mathrm{Im}\,\mathbf{H})^2))^{S^1}$ *defined by*

$$f_0(q_0, q_1) = (i, i), \quad f_1(q_0, q_1) = (q_0 i \overline{q_0}, q_1 i \overline{q_1}).$$

*Then we have* $\delta([f_0], [f_1]) = 1$*. In particular, since* $f_0$ *is a constant map, we have* $\delta([f_1]) = 1$*.*

PROOF. We can take an element $\tilde{f}$ of $F(f_0, f_1)$ to be

$$\tilde{f}(q_0, q_1, t) := (1 - t) f_0(q_0, q_1) + t f_1(q_0, q_1).$$

If $(q_0, q_1, t)$ is an element of $M = \tilde{f}^{-1}(0)$, then we have $q_0 i \overline{q_0} = q_1 i \overline{q_1} = -ci$ for $c = (1 - t)/t > 0$. This equation implies

$$\begin{aligned} M \;\; = \;\; & \{(q_0, q_1, \frac{1}{2}) \,|\, q_0 = j\cos\phi_0 + k\sin\phi_0,\; q_1 = j\cos\phi_1 + k\sin\phi_1, \\ & (0 \le \phi_0, \phi_1 < 2\pi)\}. \end{aligned}$$

Let $U$ be the open neighborhood of $M$ given by

$$U := \{(q_0, q_1, t) \in S(\mathbf{H}^2) \times (0, 1) \mid q_0 = c_0 + ic_i + jc_j + kc_k, (c_j, c_k) \neq (0, 0)\}.$$

Note that the $S^1$ action is free on $U$. A slice $S$ of $U$ for the $S^1$-action is given by

$$S := \{(q_0, q_1, t) \in S(\mathbf{H}^2) \times (0, 1) \mid q_0 = c_0 + ic_i + jc_j, c_j > 0\}.$$

The quotient space $\bar{M} = M/S^1$ is diffeomorphic to the intersection

$$M \cap S = \{(j, q_1, \frac{1}{2}) \mid q_1 = j\cos\phi_1 + k\sin\phi_1 \quad (0 \le \phi_1 < 2\pi)\}.$$

We write $M_S$ for this intersection. Instead of considering the quotient space, we shall consider the slice.

To find a disk $D_S$ which is bounded by the circle $M_S$, we notice that $M_S$ is on the 2-sphere

$$\begin{aligned} & S(j\mathbf{R} \oplus (j\mathbf{R} + k\mathbf{R})) \times \{\frac{1}{2}\} = \\ & \qquad \{(q_0, q_1, \frac{1}{2}) \mid q_0 \in j\mathbf{R}, q_1 \in j\mathbf{R} + k\mathbf{R}, |q_0|^4 + |q_1|^4 = 2\}. \end{aligned}$$

Then we may choose $D_S$ as the closure of either one of the connected components of $S(j\mathbf{R}\oplus(j\mathbf{R}+k\mathbf{R}))\times\{\frac{1}{2}\}\setminus M_S$. Although we do not need this, an explicit choice of $D_S$ may be given as follows:

$$\begin{aligned} D_S \quad := \quad & \{(2^{\frac{1}{4}}j\sqrt{\cos\psi},\, 2^{\frac{1}{4}}q_1\sqrt{\sin\psi},\, \frac{1}{2})\,|\,0\leq\psi\leq\frac{\pi}{4}, \\ & q_1 = j\cos\phi_1 + k\sin\phi_1 \quad (0\leq\phi_1<2\pi,)\}. \end{aligned}$$

Let $E_S$ be the restriction to $S$ of the trivial bundle $E=(S(\mathbf{H}^2)\times(0,1))\times(\mathrm{Im}\,\mathbf{H})^2$. The restriction of $\tilde{f}$ to $S$ is a section of $E_S$. The derivative of $\tilde{f}$ gives the isomorphism

$$\alpha_{M_S}: \nu(M_S)\to E_S|M_S.$$

We would like to calculate $\alpha_{M_S}$ in terms of the trivializations of $\nu(M_S)$ and $E_S|M_S$ that extend to those of $\nu(D_S)\oplus\mathbf{R}$ and $E_S|D_S$, where $\nu(D_S)$ is the normal bundle of $D_S$ in $S$.

From our construction, $E_S$ is identified with the trivial bundle $S\times(\mathrm{Im}\,\mathbf{H})^2$. We give a trivialization of $\nu(D_S)$ as follows.

Note that $S$ is the product of an open subset of a 7-sphere in $(\mathbf{R}+i\mathbf{R}+j\mathbf{R})\oplus\mathbf{H}$ and the interval $(0,1)$. On the other hand $D_S$ is a closed domain of a 2-sphere in $(j\mathbf{R}\oplus j\mathbf{R}+k\mathbf{R})\times\{1/2\}$. Hence we have the natural trivialization

$$\nu(D_S)\cong D_S\times(\mathbf{C}^2\oplus\mathbf{R}\partial_t).$$

To identify $\alpha_{M_S}$, we use the simple calculation:

$$d\tilde{f} = (\partial_t\tilde{f}) + t((dq_0)i\overline{q_0} + q_0 i d\overline{q_0}, (dq_1)i\overline{q_1} + q_1 i d\overline{q_1}).$$

The partial derivative $\partial_t$ at a point of $M$ is given by

$$\partial_t((1-t)f_0 + tf_1) = -f_0 + f_1 = -2f_0 = (-2i,-2i).$$

Here we used the relation $f_1=-f_0$ satisfied on $M$.

It is also straightforward to calculate the derivative along the other part of $\nu(D_S)|M_S$. The directional derivative along $(u_0,u_1)\in\mathbf{C}^2$ is equal to

$$\alpha_{M_S}(u_0,u_1) = (u_0 i\overline{j} + j i\overline{u_0}, u_1 i\overline{q_1} + q_1 i\overline{u_1}) \tag{2}$$

$$= (-2k\overline{u_0}k, 2q_1\overline{u_1}i). \tag{3}$$

Here we used the property that $\overline{q_1}$ is contained in $j\mathbf{R} + k\mathbf{R}$.

The remaining 1-dimensional direction is the normal of $M_S$ in $D_S$. It is not hard to calculate the derivative along this direction. We do not, however, need this calculation. If we use the trivialization $E_S = S \times (\operatorname{Im}\mathbf{H})^2$, the subspace of $(\operatorname{Im}\mathbf{H})^2$ spanned by the image of $\nu(D_S)|M_S$ is identified with $\langle (j\mathbf{R} + k\mathbf{R})^2, \mathbf{R}(i,i)\rangle$. Note that this image does not depend on the point where we take the derivation. Since $\alpha_{M_S}$ is an isomorphism, the image of the normal of $M_S$ in $D_S$ should lie in a complement of this constant image.

The only topological twist of the map $\alpha_{M_S}$, with respect to the above trivializations, comes from the second term of (2), where the map depends on $q_1$. This twist corresponds to the image of $1 \in \pi_1(SO(2)) = \mathbf{Z}$ by the surjective map onto $\pi_1(SO(6)) = \mathbf{Z}_2$. Since the second Stiefel-Whitney class is responsible for $\pi_1(SO)$ (cf. the proof of Lemma 12), this implies that $\delta([f_0],[f_1]) = 1 \in \mathbf{Z}_2$. □

LEMMA 14. *Suppose that* $n \geq 3$. *Let* $f_0$ *and* $f_1$ *be the elements of* $\operatorname{Map}(S(\mathbf{H}^n), S((\operatorname{Im}\mathbf{H})^n \oplus \mathbf{R}^{n-2}))^{S^1}$ *defined by*

$$\begin{aligned} f_0(q_0,\cdots,q_{n-1}) &= \left(\frac{2}{n-2}\right)^{\frac{1}{4}}(0,\cdots,0,i,\cdots,i), \\ f_1(q_0,\cdots,q_{n-1}) &= (q_0 i\overline{q_0},\cdots,q_{n-1}i\overline{q_{n-1}},0,\cdots,0). \end{aligned}$$

*Then* $\delta([f_0],[f_1]) = 0$, *and so* $\delta([f_1]) = 0$.

PROOF. We can take an element $\tilde{f}$ of $F(f_0,f_1)$ to be

$$\tilde{f}(q_0,\cdots,q_{n-1},t) := (1-t)f_0(q_0,\cdots,q_{n-1}) + tf_1(q_0,\cdots,q_{n-1}).$$

Then the zero set $M = \tilde{f}^{-1}(0)$ is empty, which implies $\delta([f_0],[f_1]) = 0$. □

**2.4.** $\{S(\mathbf{H}^n), S(\tilde{\mathbf{R}}^{4n-2})\}^{Pin_2}$

We use the following notations.

1. $Sp_1 = \{q \in \mathbf{H} \,|\, |q| = 1\}$.
2. $Pin_2 = \{\cos\theta + i\sin\theta\}_{0\leq\theta<2\pi} \cup \{j\cos\phi + k\sin\phi\}_{0\leq\phi<2\pi} \subset Sp_1$.
3. We regard $\mathbf{H}$ as a right $Pin_2$ module by the right multiplication.
4. We regard $\operatorname{Im}\mathbf{H}$ as a $Pin_2$ module by the conjugation.

5. Let $\tilde{\mathbf{R}}$ be the non-trivial 1-dimensional real representation of $Pin(2)/S^1 = \{\pm 1\}$.

Note that $\mathrm{Im}\,\mathbf{H}$ is isomorphic to $\tilde{\mathbf{R}}^3$ as $Pin(2)$-module.

In this subsection let $V_0, V_1, W_0, W_1$ be four finite dimensional right $Pin_2$ modules which satisfy the following conditions.

1. Any irreducible submodule of $V_0$ or $V_1$ is isomorphic to $\mathbf{H}$. In other words they are quaternionic vector spaces.
2. $\dim_{\mathbf{H}} V_0 - \dim_{\mathbf{H}} V_1 = n$.
3. Any irreducible submodule of $W_0$ or $W_1$ is isomorphic to $\tilde{\mathbf{R}}$.
4. $\dim_{\mathbf{R}} W_0 - \dim_{\mathbf{R}} W_1 = 2 - 4n$.
5. An orientation $o$ on $W_0 \oplus W_1$ is given.

We write $\mathrm{Map}(S(V_0 \oplus W_0), S(V_1 \oplus W_1))^{Pin_2}$ and $[S(V_0 \oplus W_0), S(V_1 \oplus W_1)]^{Pin_2}$ for the obviously defined sets. Similarly we define the category $C(S(\mathbf{H}^n), S(\tilde{\mathbf{R}}^{4n-2}))^{Pin_2}$ and the inductive limit

$$\{S(\mathbf{H}^n), S(\tilde{\mathbf{R}}^{4n-2})\}^{Pin_2} := \varinjlim [S(V_0 \oplus W_0), S(V_1 \oplus W_1)]^{Pin_2}.$$

Remark 15.

1. By a similar construction we can define $\{S(\mathbf{H}^k), S(\tilde{\mathbf{R}}^l)\}^{Pin_2}$ for any non-negative integers $k$ and $l$.
2. There is no natural group structure on the set $\{S(\mathbf{H}^k), S(\tilde{\mathbf{R}}^l)\}^{Pin_2}$.

The main proposition in this section is:

Proposition 16. *The image of the following composition:*

$$\{S(\mathbf{H}^n), S(\tilde{\mathbf{R}}^{4n-2})\}^{Pin_2} \to \{S(\mathbf{C}^{2n}), S(\mathbf{R}^{4n-2})\}^{S^1} \xrightarrow{\delta} \mathbf{Z}_2$$

*is* $1$ *if* $n = 2$, *and* $0$ *if* $n \geq 3$.

Proof. We denote the composition by the same notation $\delta$. Notice that the map $f_1$, defined in Lemma 13 for $n = 2$ and in Lemma 14 for

$n \geq 3$, is $Pin_2$-equivariant. These lemmas imply that the element $[f_1]$ of $\{S(\mathbf{H}^n), S(\tilde{\mathbf{R}}^{4n-2})\}^{Pin_2}$ satisfies the assertion. Let $[f_0]$ be any element of $\{S(\mathbf{H}^n), S(\tilde{\mathbf{R}}^{4n-2})\}^{Pin_2}$. To prove the proposition it suffices to show $\delta([f_0],[f_1]) = 0$ by the cocycle condition (1) of $\delta$. We can assume that the $Pin_2$ equivariant representatives $f_0$ and $f_1$ are two elements of $\mathrm{Map}(S(V_0 \oplus W_0), S(V_1 \oplus W_1))^{Pin_2}$ for same $V_0, V_1, W_0$ and $W_1$.

By using a transversality argument, as in the proof of Lemma 2, we can show that $F(f_0, f_1)$ has a $Pin_2$-equivariant element $\tilde{f}$. The only points we have to be careful are that the action of $Pin_2/S^1$ is free on $S(W_0)$, and that the action of $Pin_2$ is free on $S(V_0 \oplus W_0) \setminus S(W_0)$. We use this $\tilde{f}$ to calculate $\delta(f_0, f_1)$.

Let $M$ be the zero set $\tilde{f}^{-1}(0)$, and $\bar{M}$ the quotient $M/S^1$. We also use other notations $B$, $\bar{B}$ etc. in Section 1.1.

The involutive action of $Pin_2/S^1 = \{\pm 1\}$ is free on $\bar{B}$. Let $g$ be the action of the non-trivial element $-1$. The action of $g$ exchanges the connected components of $\bar{M}$ mutually.

We shall show that $g$ does not fix any connected component of $\bar{M}$. Then, by considering the quotient by $g$, we can take $D$ so that $D$ is $g$-invariant and that the contribution of the components of $D$ to $\delta = \delta(f_0, f_1, \tilde{f}, D)$ is divided into pairs, which implies that $\delta$ is 0 in $\mathbf{Z}_2$.

Let $C$ be a connected component of $\bar{M}$ and $C/g$ its projection into $\bar{M}/g$. Since $\bar{B}$ is simply connected, we have the isomorphism $H^1(\bar{B}/g, \mathbf{Z}_2) \cong \mathbf{Z}_2$. Let $b$ be its non-trivial element. Note that $C$ is $g$-invariant if and only if $\langle b, [C/g] \rangle = 1$.

The isomorphism $\alpha_C : \nu(C) \to E|C$ is $g$-equivariant and descends to an isomorphism

$$\hat{\alpha}_C : \nu(C)/g \to (E|C)/g.$$

It implies the relation $\langle w_1(\nu(C)/g) - w_1((E|C)/g), [C/g] \rangle = 0$. Since the tangent bundle of the circle $C/g$ is trivial, $w_1(\nu(C)/g) = w_1(\nu(C)/g \oplus TC/g)$ is the restriction of $w_1(T\bar{B}/g)$. Hence we have

$$\langle w_1(T\bar{B}/g) - w_1(E/g), [C/g] \rangle = 0. \tag{4}$$

We calculate $w_1(T\bar{B}/g)$ and $w_1(E/g)$.

1. The action of $j \in Pin_2$ preserves the orientation of $S(V_0)$ and reverses the orbit of the $S^1$ action. Hence the $g$-action on $P(V_0)$ reverses

its orientation. On the other hand the $g$-action on $W_0$ preserves its orientation if and only if $\dim_{\mathbf{R}} W_0$ is even. Since $\bar{B} \cong P(V_0) \times W_0 \times (0,1)$, these two observations imply

$$w_1(T\bar{B}/g) = (1 + \dim_{\mathbf{R}} W_0)b.$$

2. The action of $j \in Pin_2$ preserves the orientation of $V_1$. On the other hand the $g$-action on $W_1$ preserves its orientation if and only if $\dim_{\mathbf{R}} W_1$ is even. Since the fiber of $E$ is $V_1 \oplus W_0$, these two observations imply
$$w_1(E/g) = (\dim_{\mathbf{R}} W_1)b.$$

Recall that we are assuming $\dim_{\mathbf{R}} W_0 - \dim_{\mathbf{R}} W_1 = 2 - 4n$. Hence we have

$$w_1(T\bar{B}/g) - w_1(E/g) = (1 + \dim_{\mathbf{R}} W_0 - \dim_{\mathbf{R}} W_1)b = b$$

and the relation (4) implies the relation

$$\langle b, [C/g]\rangle = 0 \in \mathbf{Z}_2,$$

and that the component $C$ is not $g$-invariant. □

## 3. Stable-homotopy Seiberg-Witten Invariants for Spin Structures

Let $X$ be an oriented closed 4-manifold with the first Betti number $b_1(X) = 0$. Let $c$ be a $Spin^c$-structure of $X$. We write $m$ and $n$ for

$$m := \frac{c_1(c)^2 - \mathrm{sign}(X)}{2}, \quad \text{and} \quad n := b^+,$$

where $\mathrm{sign}(X)$ is the signature of the intersection form of $X$, and $b^+$ is the dimension of a maximal positive-definite subspace $H^+(X)$ of $H^2(X, \mathbf{R})$.

We assume that $m$ is non-negative. Let $o_X$ be an orientation of $H^+(X)$.

We can define the stable-homotopy Seiberg-Witten invariant $SW(X, c, o_X)$ of $(X, c, o_X)$ [2], [1]. This invariant is an element of some stable homotopy set defined in Section 1.2:

$$SW(X, c, o_X) \quad \in \quad \{S(\mathbf{C}^m), S(\mathbf{R}^n)\}^{S^1}.$$

REMARK 17.

1. The invariant is defined as follows. Fix a Riemannian metric on $X$. Let $S = S^0 \oplus S^1$ be the spinor bundle for the $Spin^c$ structure $c$. We put

$$\begin{aligned} \tilde{V}_0 &:= \Gamma(S^+), \\ \tilde{V}_1 &:= \Gamma(S^1), \\ \tilde{W}_0 &:= \operatorname{Ker}(\Gamma(\wedge^1 T^*X) \stackrel{d^*}{\to} \Gamma(\wedge^0 T^*X)), \\ \tilde{W}_1 &:= \Gamma(\wedge^+ T^*X), \end{aligned}$$

where $\wedge^+ T^*X$ is the self-dual component of $\wedge^2 T^*X$. The Seiberg-Witten equation gives rise to an $S^1$-equivariant map

$$SW : \tilde{V}_0 \oplus \tilde{W}_0 \to \tilde{V}_1 \oplus \tilde{W}_1.$$

A finite dimensional approximation of this map gives the definition of $SW(X, c, o_X)$.

2. When the first Betti number of $X$ is positive, we have a generalization of the above construction [2], [1].

*Example* 18. Let $Y$ be a connected sum of some copies of $\overline{\mathbf{CP}^2}$. We denote $c$ the unique $Spin^c$-structure on $Y$ which satisfies $c_1(c)^2 = \operatorname{sign}(Y)$, for which we have $m = 0$. Since $m = n = 0$, we can take $o_Y$ as the standard orientation which comes from the positive orientation of a point. Note that $Y$ has a Riemannian metric of positive scalar curvature. For this Riemannian metric the only solution of the Seiberg-Witten equation is the unique reducible solution [5] [6]. Moreover the linearlization of the Seiberg-Witten equation at the reducible solution is an isomorphism. It implies that $SW(Y, c, o_Y)$ is equal to the identity id in

$$\{S(\mathbf{C}^0), S(\mathbf{R}^0)\}^{S^1} = \lim [S(V_0 \oplus W_0), S(V_0 \oplus W_0)]^{S^1}.$$

Suppose that $X$ is an oriented closed spin 4-manifold with non-positive signature and with $b_1(X) = 0$. Then the intersection form of $X$ is isomorphic to

$$2kE_8 \oplus lH$$

over $\mathbf{R}$ for some non-negative integer $k$ and $l$, where $E_8$ is the Cartan matrix of $E_8$, and $H$ is the hyperbolic unimodular matrix with rank 2 (we use the convention that $E_8$ is negative definite).

Let $s$ be a spin structure of $X$. We can define the stable-homotopy Seiberg-Witten invariants $SW(X,s)^{spin}$ [2]. This invariant is an element of some stable homotopy sets defined in Section 2.3:

$$SW(X,s)^{spin} \quad \in \quad \{S(\mathbf{H}^k), S(\tilde{\mathbf{R}}^l)\}^{Pin_2}.$$

From now on we assume that $k=n$ and $l=4n-2$. If $n=2$, then the rational cohomology ring of $X$ is isomorphic to that of $K3\#K3$, and if $n \geq 3$, then it is isomorphic to that of $E_1\#E_2$, where $E_1$ and $E_2$ are spin simply-connected elliptic surfaces but, at least one of which is not a $K3$ surface.

The following is the main theorem of this paper.

THEOREM 19. *If $X$ has the same rational cohomology ring as $K3\#K3$, then we have*

$$\delta(SW(X,s)) = 1 \in \mathbf{Z}_2.$$

*for any spin structure $s$.*

PROOF. Since $\delta(SW(X,s)) = \delta(SW^{spin}(X,s))$, Proposition 16 immediately implies the theorem. □

From Proposition 16 we also obtain the other cases $n \geq 3$.

THEOREM 20. *Let $X$ be a spin 4-manifold with $b_1(X)=0, b_2^+(X) = 4n-2, b_2^-(X)=20n-2$ and $n \geq 3$. Then we have*

$$\delta(SW(X,s)) = 0 \in \mathbf{Z}_2$$

*for any spin structure $s$.*

COROLLARY 21. *Suppose that $X$ is an oriented closed spin 4-manfold which has the same rational cohomology ring as $K3\#K3$. Let $s$ be a spin structure on $X$. Suppose that $Y$ is a connected sum of some copies of $\overline{\mathbf{CP}^2}$. Let $c$ be the $Spin^c$ structure on $Y$ which satisfies $c_1(c)^2 = \mathrm{sign}(Y)$. We write $s\#c$ for the $Spin^c$ structure on the connected sum $X\#Y$ which is* *induced from $s$ and $c$. Then the Seiberg-Witten equation on $(X\#Y, s\#c)$ has a solution for any Riemannian metric.*

Proof. We use the connected sum formula proved by Bauer [1]: the Seiberg-Witten invariant of connected sum is given by the operation of join $*$. From Theorem 19 and Example 18, we have

$$\begin{aligned} SW(X\#Y, s\#c, o_X \oplus o_Y) &= SW(X, s, o_X) * SW(Y, c, o_Y) \\ &= SW(X, s, o_X) * \mathrm{id} = SW(X, s, o_X), \end{aligned}$$

From Lemma 10, if there is no solution, then $SW(X, s, o_X)$ must be 0. This contradicts Theorem 19. □

As an application of the above corollary, we obtain the following generalized adjunction inequality.

Theorem 22. *Suppose $X$ is an oriented closed spin 4-manifold which has the same rational cohomology ring as $K3\#K3$. Let $\Sigma$ be an embedded oriented closed surface with genus $g(\Sigma)$. Then we have the inequality*

$$\max\{2g(\Sigma) - 2, 0\} \geq [\Sigma] \cdot [\Sigma].$$

Proof. We follow the arguments of [5].

We denote by $n$ the self-intersection number $[\Sigma]\cdot[\Sigma]$. When $n$ is negative, we have nothing to prove.

We assume that $n$ is non-negative. Let $Y$ be the connected sum of $n$ copies of $\overline{\mathbf{CP}^2}$. Let $c$ be the $Spin^c$ structure on $Y$ which satisfies $c_1(c) = -n$. The Poincaré dual of $c_1(c)$ is represented by an embedded oriented 2-sphere $\Sigma'$.

We can make a connected sum of $\Sigma$ and $\Sigma'$ inside $X\#Y$. The normal bundle of this connected sum $\Sigma\#\Sigma'$ is trivial.

Corollary 21 and a standard argument of "stretching neck" [5] [6] imply that there exist a translation invariant solution of the Seiberg-Witten equation on $(\Sigma\#\Sigma') \times \mathbf{R}$, for the restricted $Spin^c$ structure, and any Riemannian metric on $\Sigma\#\Sigma'$. From the argument of Kronheimer and Mrowka in [5], this implies the inequality $\max\{2g(\Sigma) - 2, 0\} \geq \langle e(\nu), [\Sigma]\rangle = n$. □

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