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A Filtering Model on Default Risk

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Abstract. In this paper, we present a filtering model on a default risk related to mathematical finance. We regard as the time when a default occurs the first hitting time at zero of a one dimensional process which starts at some positive number and is not directly observed. We discuss the conditional law of the hitting time under imperfect information. We use the reference measure change technique and a new formula on a kind of conditional expectation to obtain a so-called hazard rate process. It is also discussed what the relation between the hazard rate process and the conditional law of the hitting time is like.

1. Introduction

First of all, we present a filtering model on a default risk related to mathematical finance. The model is an extension of the filtering model introduced by [7].

Fix a finite time horizon $T > 0$. Let $(\Omega, \mathcal{B}, P)$ be a complete probability space and $(\tilde{\mathcal{B}}_t)_{t\in[0,T]}$ be a weakly Brownian filtration. Let n and m be some positive integers.

B, B' and W, which are processes with values in $\mathbf{R}, \mathbf{R}^n$ and $\mathbf{R}^m$ respectively, are defined as Brownian base of $(\Omega, (\tilde{\mathcal{B}}_t), P)$. We have this Brownian base fixed.

We introduce three sorts of processes — they are one, n and m dimensional process, which are denoted by $(X_t)_{t\in[0,T]}, (Z_t)_{t\in[0,T]}$ and $(Y_t)_{t\in[0,T]}$ respectively.

Let X and Z satisfy the following stochastic differential equations.

$$(1) \qquad dX_t = dB_t + b_0(t, X_t, Z_t)dt, \qquad X_0 = x_0 > 0,$$
$$(2) \qquad dZ_t = \sigma_1(t, X_t, Z_t)dB'_t + b_1(t, X_t, Z_t)dt, \qquad Z_0 = z_0 \in \mathbf{R}^n,$$

where $b_0 : [0,T] \times \mathbf{R} \times \mathbf{R}^n \longrightarrow \mathbf{R}$, $\sigma_1 : [0,T] \times \mathbf{R} \times \mathbf{R}^n \longrightarrow \mathbf{R}^{n\times n}$ and $b_1 : [0,T] \times \mathbf{R} \times \mathbf{R}^n \longrightarrow \mathbf{R}^n$ are bounded and continuously differentiable

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functions. We assume that $b_0 \in C^{1,1,2}((0,T) \times \mathbf{R} \times \mathbf{R}^n; \mathbf{R})$ and $\dfrac{\partial b_0}{\partial z_i}$ $(i = 1, \cdots, n)$ are bounded.

We define a random time τ by

$$\tau = \inf\{t \in [0,T] | X_t = 0\}. \tag{3}$$

We set $\tau(\omega) = +\infty$ if $\inf_{t \in [0,T]} X_t(\omega) > 0$. We may consider that τ is the time when the underlying systems halt. We often call τ the default time from a financial viewpoint.

Let $(Y_t)_{t \in [0,T]}$ satisfy

$$dY_t = \sigma_2(t, Y_t) dW_t + b_2(t, X_{t \wedge \tau}, Z_{t \wedge \tau}, Y_t) dt, \qquad Y_0 = y_0 \in \mathbf{R}^m, \tag{4}$$

where $\sigma_2 : [0,T] \times \mathbf{R}^m \longrightarrow \mathbf{R}^{m \times m}$ and $b_2 : [0,T] \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^m \longrightarrow \mathbf{R}^m$ are bounded and continuously differentiable functions.

We sometimes call X, Z and Y "main system", "sub system" and "observation" respectively, following the terminology of filtering problem.

Here we assume that the diffusion part of the main system (1) is given only by a standard Brownian motion, which is independent of the other Brownian motions. We show in the appendix that under some hypotheses some general cases can be reduced to the above one by a coordinate transformation.

Let $a_i(t,x) = \sigma_i(t,x)\sigma_i(t,x)^{\mathrm{T}}$, $i = 1, 2$. We suppose that a_2 satisfies the uniform ellipticity condition, that is, for some $\varepsilon > 0$, $a_2(t,y) \geq \varepsilon I_m$ for any $t \in [0,T]$, $y \in \mathbf{R}^m$, where I_m is an m-dimensional unit matrix. Then $\sigma_2(t,y)^{-1}$ exists and satisfies

$$|\sigma_2(t,y)^{-1}\zeta| \leq \frac{1}{\sqrt{\varepsilon}} |\zeta|, \quad \zeta \in \mathbf{R}^m, \ t \in [0,T], \ y \in \mathbf{R}^m \ \text{ a.s.}$$

Denote by $(\mathcal{G}_t^\bullet)$ the right-continuous filtration generated by the process put in $\bullet$. For example,

$$\mathcal{G}_t^X = \bigcap_{t<u} \sigma\{X_s, s \leq u\}.$$

We also define the filtration $(\mathcal{F}_t)$ as

$$\mathcal{F}_t = \bigcap_{t<u} (\mathcal{G}_u^Y \vee \sigma\{\tau \wedge u\}).$$

Then each filtration satisfies the usual conditions. Apparently τ is an $(\mathcal{F}_t)$-stopping time.

Let $N_t = 1_{\{\tau \le t\}}$, that is, N_t is a default-counting process.

The subject of this paper is to discuss the existence and the explicit representation of a nonnegative $(\mathcal{G}_t^Y)$-progressively measurable process $h(t)$ such that

$$N_t - \int_0^t (1 - N_s) h(s) ds, \quad t \in [0, T],$$

is a $(P, (\mathcal{F}_t))$-martingale.

We call such a process $h(t)$ the $(\mathcal{G}_t^Y)$-hazard rate process (under P) since $h(t)$ has a connection with the distribution of the default time τ as will be discussed in section 5..

Let

$$q(t) = \int_t^\infty \frac{x_0}{\sqrt{2\pi s^3}} \exp\Big(-\frac{x_0^2}{2s}\Big) ds, \qquad t \in [0, T]$$

and let $\lambda(t) = -q(t)^{-1} \dfrac{d}{dt} q(t)$.

As for the hazard rate process, we obtain the following theorem. (See Theorem 5.1 for the exact statement.)

Theorem. *The $(\mathcal{G}_t^Y)$-hazard rate process $h(t)$ under P is given by*

$$h(t) = \frac{\hat{H}(t; Y)}{\hat{K}(t; Y)} q(t) \lambda(t),$$

where $\hat{H}(t; Y)$ and $\hat{K}(t; Y)$ are $(\mathcal{G}_t^Y)$-progressively measurable processes given in (30) *and* (31) *respectively.*

We also show some formula on a conditional expectation. The formula is utilized to prove proposition 4.1, which is the key to achieve the above theorem.

Let $B_t^a = a + B_t$ for a given $a > 0$. Let $\tau^a = \inf\{t \in [0, T] | B_t^a = 0\}$ and

$$\mathcal{F}_t^W = \bigcap_{t<u} (\mathcal{G}_u^W \vee \sigma\{\tau^a \wedge u\}).$$

We have $\mathcal{G}_t^W \subset \mathcal{F}_t^W \subset \mathcal{G}_t^{B,W}$ for $t \in [0, T]$.

Letting $N^a_t = 1_{\{\tau^a \le t\}}$, we immediately see that the process M^a defined by

$$M^a_t = N^a_t - \int_0^t (1 - N^a_{s-})\lambda(s)ds \tag{5}$$

is a $(P, (\mathcal{F}^W_t)_{t\in[0,T]})$-martingale.

Then we have the following result.

THEOREM 1.1. *Let $\alpha(t)$ and $\hat{\alpha}(t)$ be $(\tilde{\mathcal{B}}_t)$-predictable processes taking values in $\mathbf{R}$ and $\mathbf{R}^m$ respectively such that*

$$E[\int_0^T \alpha(t)^2 dt] < \infty \text{ and } \qquad E[\int_0^T |\hat{\alpha}(t)|^2 dt] < \infty.$$

Then P-a.s.

$$\begin{aligned} E[\int_0^t \alpha(s)dB_s + \int_0^t \hat{\alpha}(s)^{\mathrm{T}} dW_s | \mathcal{F}^W_t] \\ = \int_0^t E[\hat{\alpha}(s)^{\mathrm{T}} | \mathcal{F}^W_s] dW_s \\ + \int_0^t (1 - N^a_{s-})\{k_0(s;W) - q(s)^{-1}k(s;W)\} dM^a_s, \end{aligned} \tag{6}$$

where $E[\,\cdot\,|\mathcal{F}^W_s]$ stands for its predictable version, $k_0(s;W)$ and $k(s;W)$ are given in (8) *and* (9) *respectively.*

Intuitively and formally, we think of $k_0(s;W)$ and $k(s;W)$ as below;

$$\begin{aligned} k_0(s;W) &= E[\int_0^s \alpha(u)dB_u + \int_0^s \hat{\alpha}(u)^{\mathrm{T}} dW_u | \mathcal{G}^W_s, \tau^a = s], \\ k(s;W) &= E[(1 - N^a_s)\Big(\int_0^s \alpha(u)dB_u + \int_0^s \hat{\alpha}(u)^{\mathrm{T}} dW_u\Big) | \mathcal{G}^W_s]. \end{aligned}$$

Theorem 1.1 is an extension of the following proposition for the case of Brownian filtrations to $(\mathcal{F}^W_t)$. The general version of this proposition is seen in [8].

PROPOSITION 1.2. (1)*If $\hat{\alpha}(s)$ is a $\mathbf{R}^m$-valued, $(\tilde{\mathcal{B}}_s)$-progressively measurable process satisfying*

$$E[\int_0^T |\hat{\alpha}(s)|^2 ds] < \infty,$$

then P-a.s., for all $t \in [0,T]$,

$$E[\int_0^t \hat{\alpha}(s)^{\mathrm{T}} dW_s | \mathcal{G}_t^W] = \int_0^t E[\hat{\alpha}(s)^{\mathrm{T}} | \mathcal{G}_s^W] dW_s.$$

(2)*If $\alpha(s)$ is a $\mathbf{R}$-valued, $(\tilde{\mathcal{B}}_s)$-progressively measurable process satisfying*

$$E[\int_0^T \alpha(s)^2 ds] < \infty,$$

then P-a.s., for all $t \in [0,T]$,

$$E[\int_0^t \alpha(s) dB_s | \mathcal{G}_t^W] = 0.$$

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2. Preparation

Let

$$g(t,x,y) = \frac{1}{\sqrt{2\pi t}}\Big(\exp(-\frac{(x-y)^2}{2t}) - \exp(-\frac{(x+y)^2}{2t})\Big), \ t>0, \ x,y>0.$$

As is well known, g is the fundamental solution of the following heat equation with a Dirichlet boundary condition:

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t,x), \qquad t>0, x \in (0,\infty) \\ u(t,0) &= 0, \qquad t>0. \end{aligned}$$

Denote by $\mathbf{W}^k$ the space $C([0,T];\mathbf{R}^k)$ and by μ_k a k-dimensional Wiener measure. We will write $\mathbf{W}$ for $\mathbf{W}^1$.

Let $\nu_{0,x_1}^{u,x_2}(\cdot)$, $u>0, x_1>0, x_2>0$ be a probability measure on $C([0,u];\mathbf{R})$, which is the law of $(B_s)_{s\in[0,u]}$ conditioned to start from x_1, to stay in $(0,\infty)$ for $s \le u$ and to reach x_2 at time u under P, that is, for any bounded and continuous function $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ and $0 < t_1 < \cdots < t_n < u$,

$$\begin{aligned} &\int_{\mathbf{W}} \nu_{0,x_1}^{u,x_2}(d\theta) f(\theta(t_1),\cdots,\theta(t_n)) \\ &= \int_{(0,\infty)^n} \frac{g(t_1,x_1,y_1)g(t_2-t_1,y_1,y_2)\cdots g(t_n-t_{n-1},y_{n-1},y_n)g(u-t_n,y_n,x_2)}{g(u,x_1,x_2)} \\ &\qquad \times f(y_1,\cdots,y_n)dy_1\cdots dy_n. \end{aligned}$$

Hereafter we think of $\nu^{u,x_2}_{0,x_1}(d\theta), u \in [0,T]$ as the probability measure on $\mathbf{W}$ by setting $\theta(s) = x_2$ for $u < s \leq T$, $\nu^{u,x_2}_{0,x_1}(d\theta)$-a.s.

Then we see that for any bounded and continuous function $f : \mathbf{R}^n \longrightarrow \mathbf{R}$,

$$\begin{aligned} & E_{x_1}[f(B_{t_1}, \cdots, B_{t_n}), \tau > u] \\ = & \int_0^\infty dx_2 g(u, x_1, x_2) \int_{\mathbf{W}} \nu^{u,x_2}_{0,x_1}(d\theta) f(\theta(t_1), \cdots, \theta(t_n)). \end{aligned}$$

We also define $\nu^{u,0}_{0,x_1}(\cdot)$ as the limit of $\nu^{u,x_2}_{0,x_1}(\cdot)$ as $x_2 \downarrow 0$ with respect to the weak topology on probability measures.

Next, we make a few remarks about the measure ν^{u,x_2}_{0,x_1}. First, it is remarkable that the density of the finite dimensional law of three dimensional Bessel Bridge leaving x_1 at time 0 and reaching x_2 at u is given by

$$\frac{p(t_1, x_1, y_1) p(t_2 - t_1, y_1, y_2) \cdots p(t_n - t_{n-1}, y_{n-1}, y_n) p(u - t_n, y_n, x_2)}{p(u, x_1, x_2)},$$

where $p(s, x, y), s > 0, x > 0, y > 0$ stands for the transition density of three dimensional Bessel process. The well-known fact that $p(s, x, y) = x g(s, x, y) y^{-1}$ implies the following result. (Refer to Knight [6].)

LEMMA 2.1. *The law of one dimensional Brownian motion conditioned to stay in* $(0, \infty)$ *between* $x_1 > 0$ *and* $x_2 > 0$ *over period* $[0, t]$ *coincides with the one of three dimensional Bessel Bridge leaving* x_1 *at time 0 and reaching* x_2 *at* t.

Consequently we can regard $\nu^{u,x_2}_{0,x_1}(\cdot)$, $u > 0, x_1 > 0, x_2 > 0$ as the law of three-dimensional Bessel Bridge between x_1 and x_2 over $[0, u]$.

Similarly, it is not hard to see

$$\lim_{x_2 \downarrow 0} \frac{g(u - t_n, y_n, x_2)}{g(u, x_1, x_2)} = \frac{l(u - t_n, y_n)}{l(u, x_1)},$$

for $0 < t_1 < \cdots < t_n < u$, where $l(s, x) = (2\pi s^3)^{-\frac{1}{2}} x \exp(-\frac{x^2}{2s}) 1_{\{x>0\}}$ so the density of the law of $(B_{t_1}, \cdots, B_{t_n})$ under $\nu^{u,0}_{0,x_1}$ with respect to $dy_1 \cdots dy_n$, is equal to

$$\frac{g(t_1, x_1, y_1) g(t_2 - t_1, y_1, y_2) \cdots g(t_n - t_{n-1}, y_{n-1}, y_n) l(u - t_n, y_n)}{l(u, x_1)}.$$

PROPOSITION 2.2. *For any bounded continuous functional $F : \mathbf{W} \longrightarrow \mathbf{R}$,*

$$\begin{aligned} &E[N_T^a F(B^a_{\cdot\wedge\tau^a})] \\ &= \int_0^T P(\tau^a \in ds) \int_{\mathbf{W}} \nu_{0,a}^{s,0}(d\theta) F(\theta). \end{aligned}$$

PROOF. Let $0 = t_0 < t_1 < \cdots < t_N = T$ and let $f_N : \mathbf{R}^N \longrightarrow \mathbf{R}$ be a bounded continuous function.

We have

$$\begin{aligned} &E[N_T^a f_N(B^a_{t_1\wedge\tau^a}, \cdots, B^a_{t_N\wedge\tau})] \\ &= \int_0^T P(\tau^a \in ds) E[f_N(B^a_{t_1\wedge s}, \cdots, B^a_{t_N\wedge s}) | \tau^a = s] \\ &= \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} P(\tau^a \in ds) \\ &\quad \times \int_{(0,\infty)^n} f_N(x_1, \cdots, x_n, 0, \cdots, 0) \\ &\qquad\qquad \times P(B^a_{t_1} \in dx_1, \cdots, B^a_{t_n} \in dx_n, \tau^a > t_n | \tau^a = s). \end{aligned}$$

The joint distribution of $(B^a_{t_1}, \cdots, B^a_{t_n}, \tau^a)$ under P restricted to $\{t_n < \tau^a\}$ is calculated in the following way.

$$\begin{aligned} &P(B^a_{t_1} \le x_1, \cdots, B^a_{t_n} \le x_n, \tau^a > s) \\ &= \int_0^{x_1} dz_1 g(t_1, a, z_1) \int_0^{x_2} dz_2 g(t_2 - t_1, z_1, z_2) \cdots \\ &\quad \times \int_0^{x_n} dz_n g(t_n - t_{n-1}, z_{n-1}, z_n) \int_s^\infty l(u - t_n, z_n) du. \end{aligned}$$

Hence, the joint density with respect to $dx_1 \cdots dx_n ds$ is equal to

$$g(t_1, a, x_1) \cdots g(t_n - t_{n-1}, x_{n-1}, x_n) l(s - t_n, x_n).$$

We also see that for $s > t_n$,

$$\begin{aligned} &\int_{(0,\infty)^n} dx_1 \cdots dx_n g(t_1, a, x_1) \cdots g(t_n - t_{n-1}, x_{n-1}, x_n) l(s - t_n, x_n) \\ &\qquad = P(\tau^a \in ds)/ds \\ &\qquad = l(s, a). \end{aligned}$$

Therefore it follows that

$$
\begin{aligned}
&P(B^a_{t_1}\in dx_1,\cdots,B^a_{t_n}\in dx_n,\tau^a>t_n|\tau^a=s)\\
&\quad=\ \frac{g(t_1,a,x_1)g(t_2-t_1,x_1,x_2)\cdots g(t_n-t_{n-1},x_{n-1},x_n)l(s-t_n,x_n)}{l(s,a)}\\
&\qquad\quad\times dx_1\cdots dx_n.
\end{aligned}
$$

This means that the finite dimensional distribution of $(B^a_u)_{u\le s}$ under P restricted to $\{\tau^a>t_n\}$ and conditioned to $\{\tau^a=s\}$ coincides with the f.d.d. of $(B^a_u)_{u\le s}$ under $\nu^{s,0}_{0,a}$. That is,

$$
\begin{aligned}
&E[N^a_T f_N(B^a_{t_1\wedge\tau^a},\cdots,B^a_{t_N\wedge\tau^a})]\\
&\quad=\ \sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}P(\tau^a\in ds)\int_{\mathbf{W}}\nu^{s,0}_{0,a}(d\theta)f_N(\theta(t_1),\cdots,\theta(t_n),0,\cdots,0).\\
&\quad=\ \int_0^T P(\tau^a\in ds)\int_{\mathbf{W}}\nu^{s,0}_{0,a}(d\theta)f_N(\theta(t_1),\cdots,\theta(t_N)).
\end{aligned}
$$

From this equality and the monotone class argument, we can conclude that for any bounded continuous functional $F:\mathbf{W}\longrightarrow\mathbf{R}$,

$$
\begin{aligned}
&E[N^a_T F(B^a_{\cdot\wedge\tau^a})]\\
&\quad=\ \int_0^T P(\tau\in ds)\int_{\mathbf{W}^1}\nu^{s,0}_{0,a}(d\theta)F(\theta).\ \square
\end{aligned}
$$

3. Proof of Theorem 1.1

Before we begin to prove Theorem 1.1, we present some lemmas.

Lemma 3.1. *Fix $t\in[0,T]$. Let $F:[0,T]\times\mathbf{W}\times\mathbf{W}^m\longrightarrow\mathbf{R}$ be a measurable functional such that for all $t\in[0,T]$,*

$$
F(t,\cdot,\cdot):\mathbf{W}\times\mathbf{W}^m\longrightarrow\mathbf{R}
$$

is a continuous functional, $E[|F(t,B^a,W)|]<\infty$, and $F(t,B^a,W)$ is $(\mathcal{G}^{B,W}_t)$-measurable.

Then

$$(1)\qquad E[(1-N_t^a)F(t,B^a,W)|\mathcal{G}_t^W] = \int_0^\infty dxg(t,a,x)\int_{\mathbf{W}}\nu_{0,a}^{t,x}(d\theta)F(t,\theta,W).$$

$$(2)\qquad E[N_t^aF(\tau^a,B^a,W)|\mathcal{G}_t^W] = \int_0^t dsq(s)\lambda(s)\int_{\mathbf{W}}\nu_{0,a}^{s,0}(d\theta)F(s,\theta,W).$$

$$(3)\qquad E[N_t^aF(\tau^a,B^a,W)|\mathcal{F}_t^W] = \int_0^t\Big(\int_{\mathbf{W}}\nu_{0,a}^{s,0}(d\theta)F(s,\theta,W)\Big)dN_s^a.$$

PROOF. (1) By noting that $P(B_t^a\in dx|\tau^a>t)=q(t)^{-1}g(t,a,x)dx$, we have

$$\begin{aligned}
&E[(1-N_t^a)F(t,B^a,w)]\\
&\quad= E[F(t,B^a,w),\tau^a>t]\\
&\quad= \int_0^\infty dxg(t,a,x)\int_{\mathbf{W}}\nu_{0,a}^{t,x}(d\theta)F(t,\theta,w).
\end{aligned}$$

Let $\varphi(\cdot):\mathbf{W}^m\longrightarrow\mathbf{R}$ be a nonnegative $(\mathcal{G}_t^W)$-measurable functional. Then we have

$$\begin{aligned}
&E[(1-N_t^a)F(t,B^a,W)\varphi(W)]\\
&\quad= \int_{\mathbf{W}^m}\mu_m(dw)\varphi(w)E[(1-N_t^a)F(t,B^a,w)],\\
&\quad= E[\varphi(W)\int_0^\infty dxg(t,a,x)\int_{\mathbf{W}}\nu_{0,a}^{t,x}(d\theta)F(t,\theta,W)].
\end{aligned}$$

Since φ is taken arbitrarily, the first statement is proved.

(2) By Proposition 2.2 we have

$$\begin{aligned}
&E[N_t^aF(\tau^a,B^a,w)]\\
&\quad= \int_0^\infty P(\tau^a\in ds)E[N_t^aF(\tau^a,B^a,w)|\tau^a=s]\\
&\quad= \int_0^t dsq(s)\lambda(s)\int_{\mathbf{W}}\nu_{0,a}^{s,0}(d\theta)F(s,\theta,w).
\end{aligned}$$

Therefore, for any nonnegative $(\mathcal{G}_t^W)$-measurable functional $\varphi(\cdot):\mathbf{W}^m \longrightarrow \mathbf{R}$, we have

$$\begin{aligned} & E[N_t^a F(\tau^a, B^a, W)\varphi(W)] \\ = & \int_{\mathbf{W}^m} \mu_m(dw)\varphi(w)E[N_t^a F(\tau^a, B^a, w)]. \\ = & E[\varphi(W)\int_0^t dsq(s)\lambda(s)\int_{\mathbf{W}} \nu_{0,a}^{s,0}(d\theta)F(s,\theta,W)]. \end{aligned}$$

Similar to the proof of (1), this implies the conclusion.

(3) Let $\hat{\varphi}(\cdot):[0,T]\longrightarrow \mathbf{R}$ be a bounded continuous function. First, we remark that

$$\begin{aligned} & E[N_t^a F(\tau^a, B^a, w)\hat{\varphi}(\tau^a\wedge t)] \\ = & \int_0^t dsq(s)\lambda(s)E[F(s,B^a,w)\hat{\varphi}(s\wedge t)|\tau^a = s] \\ = & \int_0^t dsq(s)\lambda(s)\hat{\varphi}(s\wedge t)\int_{\mathbf{W}} \nu_{0,a}^{s,0}(d\theta)F(s,\theta,w). \end{aligned}$$

Let $C:\mathbf{W}^m \longrightarrow \mathbf{R}$ be a bounded $(\mathcal{G}_t^W)$-measurable functional. Then we have

$$\begin{aligned} & E[N_t^a F(\tau^a, B^a, W)\hat{\varphi}(\tau^a\wedge t)C(W)] \\ = & \int_{\mathbf{W}^m} \mu_m(dw)C(w)E[N_t^a F(\tau^a, B^a, w)\hat{\varphi}(\tau^a\wedge t)] \\ = & E[\hat{\varphi}(\tau^a\wedge t)C(W)N_t^a \int_{\mathbf{W}} \nu_{0,a}^{\tau^a,0}(d\theta)F(\tau^a,\theta,W)]. \\ = & E[\hat{\varphi}(\tau^a\wedge t)C(W)\int_0^t \Big(\int_{\mathbf{W}} \nu_{0,a}^{s,0}(d\theta)F(s,\theta,W)\Big)dN_s^a]. \end{aligned}$$

By the usual monotone class argument, for any bounded, $(\mathcal{F}_t^W)$-measurable random variable $\Theta:\Omega\longrightarrow \mathbf{R}$,

$$\begin{aligned} & E[N_t^a F(\tau^a, B^a, W)\Theta] \\ = & E[\Theta\int_0^t \Big(\int_{\mathbf{W}} \nu_{0,a}^{s,0}(d\theta)F(s,\theta,W)\Big)dN_s^a]. \end{aligned}$$

So the proof is complete. □

LEMMA 3.2. *Let a functional F satisfy the same condition as in the last lemma.*

For every bounded, $(\mathcal{F}_t^W)$-predictable process $f : [0,T] \times \Omega \longrightarrow \mathbf{R}$, we have

$$
\begin{aligned}
(1) \qquad & E[\int_0^t F(s; B^a, W) f(s) dN_s^a] \\
& = \int_0^t E[(1 - N_{s-}^a) f(s) \int_{\mathbf{W}} \nu_{0,a}^{s,0}(d\theta) F(s;\theta, W)] \lambda(s) ds \\
(2) \qquad & E[\int_0^t F(s; B^a, W) f(s)(1 - N_{s-}^a)\lambda(s) ds] \\
& = \int_0^t E[f(s)(1 - N_{s-}^a) q(s)^{-1} \int_0^\infty dx g(s,a,x) \\
& \quad \times \int_{\mathbf{W}} \nu_{0,a}^{s,x}(d\theta) F(s;\theta, W)] \lambda(s) ds.
\end{aligned}
$$

PROOF. (1) We note that $q(s) = P(\tau^a \geq s) = P(\tau^a \geq s | \mathcal{G}_s^W)$ for $s \in [0,T]$.

Similar to the proof of Lemma 3.1(3), monotone class argument implies that it is enough to show the result for the case such as $f(t) = \hat{\varphi}(\tau^a \wedge t) C(t)$, where $\hat{\varphi}(\cdot) : [0,T] \longrightarrow \mathbf{R}$ is a bounded continuous function and $C(t)$ is a bounded $(\mathcal{G}_t^W)$-predictable process.

Thanks to Lemma 3.1(2), we have

$$
\begin{aligned}
& E[\int_0^t F(s; B^a, W) \hat{\varphi}(\tau^a \wedge s) C(s) dN_s^a] \\
& = \quad E[E[N_t^a \hat{\varphi}(\tau^a) C(\tau^a) F(\tau^a; B^a, W) | \mathcal{G}_t^W]] \\
& = \quad E[\int_0^t ds q(s) \lambda(s) \int_{\mathbf{W}} \nu_{0,a}^{s,0}(d\theta) \hat{\varphi}(s) C(s) F(s;\theta, W)] \\
& = \quad \int_0^t E[q(s) \hat{\varphi}(s) C(s) \int_{\mathbf{W}} \nu_{0,a}^{s,0}(d\theta) F(s;\theta, W)] \lambda(s) ds \\
& = \quad \int_0^t E[E[1 - N_{s-}^a | \mathcal{G}_s^W] \hat{\varphi}(s) C(s) \int_{\mathbf{W}} \nu_{0,a}^{s,0}(d\theta) F(s;\theta, W)] \lambda(s) ds \\
& = \quad \int_0^t E[(1 - N_{s-}^a) \hat{\varphi}(\tau^a \wedge s) C(s) \int_{\mathbf{W}} \nu_{0,a}^{s,0}(d\theta) F(s;\theta, W)] \lambda(s) ds.
\end{aligned}
$$

(2) Since $\{\tau > t\}$ is an atom of $\sigma\{\tau \wedge t\}$ and independent of $(\mathcal{G}_t^W)$, the equality

$$E[(1-N_{t-}^a)\Theta|\mathcal{F}_t^W] = (1-N_{t-}^a)q(t)^{-1}E[(1-N_{t-}^a)\Theta|\mathcal{G}_t^W]$$

holds for every random variable Θ. Using Lemma 3.1(1) and the above fact, we have

$$\begin{aligned}
&E[\int_0^t F(s;B^a,W)f(s)(1-N_{s-}^a)\lambda(s)ds] \\
&= \int_0^t E[f(s)(1-N_{s-}^a)E[F(s;B^a,W)|\mathcal{F}_s^W]]\lambda(s)ds \\
&= \int_0^t E[f(s)(1-N_{s-}^a)q(s)^{-1}E[(1-N_{s-}^a)F(s;B^a,W)|\mathcal{G}_s^W]]\lambda(s)ds \\
&= \int_0^t E[f(s)(1-N_{s-}^a)q(s)^{-1} \\
&\qquad \times \int_0^\infty dxg(s,a,x)\int_{\mathbf{W}} \nu_{0,a}^{s,x}(d\theta)F(s;\theta,W)]\lambda(s)ds. \ \square
\end{aligned}$$

Now we will show Theorem 1.1.

Let $L_t = E[\int_0^t \alpha(s)dB_s + \int_0^t \hat{\alpha}(s)^{\mathrm{T}}dW_s|\mathcal{F}_t^W]$. Since

$$\begin{aligned}
L_t &= E[E[\int_0^T \alpha(s)dB_s + \int_0^T \hat{\alpha}(s)^{\mathrm{T}}dW_s|\tilde{\mathcal{B}}_t]|\mathcal{F}_t^W] \\
&= E[\int_0^T \alpha(s)dB_s + \int_0^T \hat{\alpha}(s)^{\mathrm{T}}dW_s|\mathcal{F}_t^W],
\end{aligned}$$

so L_t is a $(P,(\mathcal{F}_t^W)_{t\in[0,T]})$-square integrable martingale with $L_0 = 0$. So it follows from the $(\mathcal{F}_t^W))$-martingale representation theorem (see Kusuoka [7]) that there exist $(\mathcal{F}_s^W)$-predictable processes $a(s)$ and $\hat{a}(s)$ taking values in $\mathbf{R}^m$ and $\mathbf{R}$ respectively such that

$$E[\int_0^T |a(s)|^2 ds] < \infty, \qquad E[\int_0^T \hat{a}(s)^2\lambda(s)ds] < \infty,$$

and

$$L_t = \int_0^t a(s)^{\mathrm{T}}dW_s + \int_0^t \hat{a}(s)dM_s^a.$$

Given K_t an $(\mathcal{F}_t^W)$-martingale given by

$$K_t = \int_0^t b(s)^{\mathrm{T}} dW_s + \int_0^t \hat{b}(s) dM_s^a$$

for some $(\mathcal{F}_t^W)$-predictable, bounded processes $b(t)$ and $\hat{b}(t)$, taking values in $\mathbf{R}^m$ and $\mathbf{R}$ respectively.

The property of quadratic variation implies that

$$\begin{aligned} E[L_t K_t] &= E[[L, K]_t] \\ &= E[\int_0^t a(s)^{\mathrm{T}} b(s) ds + \int_0^t \hat{a}(s)\hat{b}(s) dN_s^a] \\ &= E[\int_0^t a(s)^{\mathrm{T}} b(s) ds + \int_0^t \hat{a}(s)\hat{b}(s)(1 - N_{s-}^a)\lambda(s) ds] \\ &= \int_0^t E[a(s)^{\mathrm{T}} b(s) + \hat{a}(s)\hat{b}(s)(1 - N_{s-}^a)\lambda(s)] ds. \end{aligned}$$

On the other hand, we note that

$$\begin{aligned} E[L_t K_t] &= E[E[\int_0^t \alpha(s) dB_s + \int_0^t \hat{\alpha}(s)^{\mathrm{T}} dW_s | \mathcal{F}_t^W] K_t] \\ &= E[\int_0^t \alpha(s) dB_s \int_0^t b(s)^{\mathrm{T}} dW_s] \\ &\quad + E[\int_0^t \hat{\alpha}(s)^{\mathrm{T}} dW_s \int_0^t b(s)^{\mathrm{T}} dW_s] \\ &\quad + E[\Big(\int_0^t \alpha(s) dB_s + \int_0^t \hat{\alpha}(s)^{\mathrm{T}} dW_s\Big) \int_0^t \hat{b}(s) dM_s^a]. \end{aligned} \tag{7}$$

The first term of (7) vanishes because B and W is independent and the second term of (7) leads to

$$\begin{aligned} &E[\int_0^t \hat{\alpha}(s)^{\mathrm{T}} b(s) ds] \\ &\quad = \int_0^t E[\hat{\alpha}(s)^{\mathrm{T}} b(s)] ds \\ &\quad = \int_0^t E[E[\hat{\alpha}(s)^{\mathrm{T}} | \mathcal{F}_s^W] b(s)] ds. \end{aligned}$$

We can choose a predictable version from equivalent processes of $E[\hat{\alpha}(s)|\mathcal{F}_s^W]$ and will also write $E[\hat{\alpha}(s)|\mathcal{F}_s^W]$ for the predictable one. (See Revuz-Yor [9].)

Since M_t^a is a $(\tilde{\mathcal{B}}_t)$-semimartingale, the last term of (7) coincides with

$$E[\int_0^t \Big(\int_0^s \hat{b}(u)dM_u^a\Big)\{\alpha(s)dB_s + \hat{\alpha}(s)^{\mathrm{T}}dW_s\}$$
$$+\int_0^t \Big(\int_0^s \alpha(u)dB_u + \int_0^s \hat{\alpha}(u)^{\mathrm{T}}dW_u\Big)\hat{b}(s)dM_s^a$$
$$+\int_0^t \alpha(s)\hat{b}(s)d[B,M^a]_s + \sum_{i=1}^m \int_0^t \hat{\alpha}^i(s)\hat{b}(s)d[W^i,M^a]_s].$$

We observe that

$$\int_0^t \Big(\int_0^s \hat{b}(u)dM_u^a\Big)\{\alpha(s)dB_s + \hat{\alpha}(s)dW_s\}$$

is a $(\tilde{\mathcal{B}}_t)$-martingale, so its mean is zero. Since the quadratic covariation processes $[B,M^a]$ and $[W^i,M^a]$ $i=1,\cdots,m$ are all zero, it follows that the last term of (7) also vanishes.

Let $\Phi : [0,T]\times \mathbf{W}\times \mathbf{W}^m \longrightarrow \mathbf{R}$ be given by

$$\Phi(s;a+B,W) = E[\int_0^s \alpha(u)dB_u + \int_0^s \hat{\alpha}(u)^{\mathrm{T}}dW_u|\mathcal{G}_s^{B,W}] \ \text{ for } s\in[0,T] \text{ a.s.}$$

For $s\in[0,T], w\in C([0,T];\mathbf{R}^m)$, let

$$k_0(s;w) = \int_{\mathbf{W}} \nu_{0,a}^{s,0}(d\theta)\Phi(s;\theta,w), \tag{8}$$

$$k(s;w) = \int_0^\infty dxg(s,a,x)\int_{\mathbf{W}} \nu_{0,a}^{s,x}(d\theta)\Phi(s;\theta,w), \tag{9}$$

If we regard $\alpha(u)$ as $\alpha(u,B,B',W)$ and so on, we have

$$\Phi(s;\theta,w) = E[\int_0^s \alpha(u,\theta,B',w)d\theta(u) + \int_0^s \hat{\alpha}(u,\theta,B',w)dw(u)].$$

Then we have

$$\begin{aligned}
&E[\Big(\int_0^t \alpha(s)dB_s + \int_0^t \hat{\alpha}(s)^{\mathrm{T}}dW_s\Big)\int_0^t \hat{b}(s)dM_s^a]\\
&= E[E[\int_0^t \alpha(s)dB_s + \int_0^t \hat{\alpha}(s)^{\mathrm{T}}dW_s|\mathcal{G}_t^{B,W}]\int_0^t \hat{b}(s)dM_s^a]\\
&= E[\int_0^t \Phi(s;a+B,W)\hat{b}(s)dM_s^a]
\end{aligned}$$

$$
\begin{aligned}
&= E[\int_0^t \Phi(s; a+B, W)\hat{b}(s)dN_s^a] \\
&\quad - E[\int_0^t \Phi(s; a+B, W)\hat{b}(s)(1-N_{s-}^a)\lambda(s)ds].
\end{aligned}
$$

Here we use Lemma 3.2 by setting $f(t) = \hat{b}(t)$ and $F = \Phi$ to obtain

$$
\begin{aligned}
&E[\int_0^t \Phi(s; a+B, W)\hat{b}(s)dN_s^a] \\
&\quad = \int_0^t E[(1-N_{s-}^a)\hat{b}(s)k_0(s; W)]\lambda(s)ds
\end{aligned}
\tag{10}
$$

and

$$
\begin{aligned}
&E[\int_0^t \Phi(s; a+B, W)\hat{b}(s)(1-N_{s-}^a)\lambda(s)ds] \\
&\quad = \int_0^t E[\hat{b}(s)(1-N_{s-}^a)q(s)^{-1}k(s; W)]\lambda(s)ds.
\end{aligned}
\tag{11}
$$

Therefore it follows

$$
\begin{aligned}
&\int_0^t E[b(s)^{\mathrm{T}}\{a(s) - E[\hat{\alpha}(s)|\mathcal{F}_s^W]\} \\
&\qquad + \hat{b}(s)(1-N_{s-}^a)\lambda(s)\{\hat{a}(s) - (k_0(s; W) - q(s)^{-1}k(s; W))\}]ds \\
&= 0
\end{aligned}
\tag{12}
$$

Since the equality (12) holds for every bounded, $(\mathcal{F}_t^W)$-predictable processes $b(s)$ and $\hat{b}(s)$,

$$
a(s) = E[\hat{\alpha}(s)|\mathcal{F}_s^W] \quad a.e.s \in [0,T], P-a.s
$$

and

$$
(1-N_{s-}^a)\{\hat{a}(s) - (k_0(s; W) - q(s)^{-1}k(s; W))\} = 0 \quad a.e.s \in [0,T], P-a.s.
$$

Hence we obtain the desired result. □

4. Calculation of the Conditional Density

Now we get down to study our filtering model. First of all, we progress by using the measure change technique, as is mostly used for nonlinear filtering problem.

Let

$$\begin{aligned} R &= \exp\Big(-\int_0^T \{b_0(t,X_t,Z_t)dB_t + \beta_1(t,X_{t\wedge\tau},Z_{t\wedge\tau},Y_t)^{\mathrm{T}}dW_t\} \\ &\quad -\frac{1}{2}\int_0^T \{b_0(t,X_t,Z_t)^2 + |\beta_1(t,X_{t\wedge\tau},Z_{t\wedge\tau},Y_t)|^2\}dt\Big), \end{aligned}$$

where $\beta_1(t,x,z,y) = \sigma_2^{-1}(t,y)b_2(t,x,z,y)$. Denote by $\tilde{P}$ an equivalent probability measure on $(\Omega,\mathcal{B})$ defined by

$$\frac{d\tilde{P}}{dP} = R \tag{13}$$

Let

$$\begin{aligned} \tilde{B}_t &= B_t + \int_0^t b_0(s,X_s,Z_s)ds, \\ \tilde{W}_t &= W_t + \int_0^t \beta_1(s,X_{s\wedge\tau},Z_{s\wedge\tau},Y_s)ds. \end{aligned}$$

We see that $(\tilde{B}_t),(B'_t)$ and $(\tilde{W}_t)$ are independent $(\tilde{P},(\tilde{\mathcal{B}}_t)_{t\in[0,T]})$-Brownian motions due to Cameron-Martin-Maruyama-Girsanov theorem, so $(\tilde{B}_t),(B'_t)$ and $(\tilde{W}_t)$ are Brownian base of $(\Omega,(\tilde{\mathcal{B}}_t),\tilde{P})$.

Then we have

$$\begin{aligned} dX_t &= d\tilde{B}_t, \\ dZ_t &= \sigma_1(t,X_t,Z_t)dB'_t + b_1(t,X_t,Z_t)dt, \\ dY_t &= \sigma_2(t,Y_t)d\tilde{W}_t. \end{aligned}$$

It follows that $(\mathcal{G}_t^X)_{t\in[0,T]}$ coincides with the natural filtration generated by $(\tilde{B}_t)_{t\in[0,T]}$. Similarly, we remark that the filtration $(\mathcal{G}_t^Y)_{t\in[0,T]}$ coincides with the natural filtration generated by $(\tilde{W}_t)_{t\in[0,T]}$ since $d\tilde{W}_t = \sigma_2(t,Y_t)^{-1}dY_t$. In other words, $(\mathcal{G}_t^Y)_{t\in[0,T]}$ can be regarded as a m-dimensional Brownian filtration, which has $(\tilde{W})$ as its base of $(\Omega,(\mathcal{G}_t^Y),\tilde{P})$.

Let $\mathcal{B}_t = \bigcap_{t<u} \sigma\{\tilde{B}_s, B'_s, \tilde{W}_s; s \le u\}$.

Hereafter we will denote by $E[\,\cdot\,]$ an expectation under the probability measure P and by $\tilde{E}[\,\cdot\,]$ under $\tilde{P}$.

The measure change from P to $\tilde{P}$ makes some computations easier since X becomes standard Brownian motion and both X and Z come to be independent of Y. However, we want to acquire the connection between default time and observed information under the original probability measure P. Indeed it is very significant to clarify the conditional Radon-Nikodym density with respect to the filtration $(\mathcal{F}_t)$. For the purpose, we take $\dfrac{dP}{d\tilde{P}}$, that is, R^{-1} instead of R and make it easier to deal with.

Let

$$
\begin{aligned}
(14)\quad \rho \equiv R^{-1} \;=\; & \exp\Big(\int_0^T \{b_0(t, X_t, Z_t)d\tilde{B}_t + \beta_1(t, X_{t\wedge\tau}, Z_{t\wedge\tau}, Y_t)^{\mathrm{T}} d\tilde{W}_t\} \\
& -\frac{1}{2}\int_0^T \{b_0(t, X_t, Z_t)^2 + |\beta_1(t, X_{t\wedge\tau}, Z_{t\wedge\tau}, Y_t)|^2\}dt\Big),
\end{aligned}
$$

and let

$$
\psi(t, x, z) = \int_0^x b_0(t, \xi, z)d\xi.
$$

Then we have

$$
\begin{aligned}
d\psi(t, X_t, Z_t) \;=\; & \frac{\partial\psi}{\partial t}(t, X_t, Z_t)dt + b_0(t, X_t, Z_t)dX_t + \nabla_z\psi(t, X_t, Z_t)^{\mathrm{T}} dZ_t \\
& + \frac{1}{2}\frac{\partial b_0}{\partial x}(t, X_t, Z_t)dt \\
& + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2\psi}{\partial z_i \partial z_j}(t, X_t, Z_t)a_1^{ij}(t, X_t, Z_t)dt,
\end{aligned}
$$

where $\nabla_z = \left(\dfrac{\partial}{\partial z_1}, \cdots, \dfrac{\partial}{\partial z_n}\right)^{\mathrm{T}}$.

Therefore

$$
\begin{aligned}
& \int_0^T b_0(t, X_t, Z_t)dX_t \\
& \quad = \psi(T, X_T, Z_T) - \psi(0, x_0, z_0) \\
& \qquad - \int_0^T \nabla_z\psi(t, X_t, Z_t)^{\mathrm{T}}[\sigma_1(t, X_t, Z_t)dB'_t + b_1(t, X_t, Z_t)dt] \\
& \qquad - \int_0^T \{\frac{\partial\psi}{\partial t}(t, X_t, Z_t) + \frac{1}{2}\frac{\partial b_0}{\partial x}(t, X_t, Z_t)dt \\
& \qquad + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2\psi}{\partial z_i \partial z_j}(t, X_t, Z_t)a_1^{ij}(t, X_t, Z_t)\}dt
\end{aligned}
$$

Substituting it into (14), we have

$$
\begin{aligned}
(15)\quad \rho \;=\; & \exp\Big(\psi(T,X_T,Z_T)-\psi(0,x_0,z_0)-\int_0^T \beta_0(t,X_t,Z_t)^{\mathrm{T}}dB'_t \\
& +\int_0^T \hat{\beta}_1(t,X_{t\wedge\tau},Z_{t\wedge\tau},Y_t)^{\mathrm{T}}dY_t \\
& -\int_0^T \{A(t,X_t,Z_t)+\frac{1}{2}|\beta_1(t,X_{t\wedge\tau},Z_{t\wedge\tau},Y_t)|^2\}dt\Big),
\end{aligned}
$$

where

$$
\begin{aligned}
\beta_0(t,x,z) &= \sigma_1(t,x,z)^{\mathrm{T}}\nabla_z\psi(t,x,z), \\
\hat{\beta}_1(t,x,z,y) &= (\sigma_2(t,y)^{-1})^{\mathrm{T}}\beta_1(t,x,z,y) \\
A(t,x,z) &= \frac{1}{2}\{b_0(t,x,z)^2+\frac{\partial b_0}{\partial x}(t,x,z) \\
&\quad +\sum_{i,j=1}^{n}\frac{\partial^2\psi}{\partial z_i\partial z_j}(t,x,z)a_1^{ij}(t,x,z)\} \\
&\quad +\nabla_z\psi(t,x,z)^{\mathrm{T}}b_1(t,x,z)+\frac{\partial\psi}{\partial t}(t,x,z).
\end{aligned}
$$

Besides, for given $\theta\in\mathbf{W}$, we think of the following stochastic differential equation:

$$
\begin{aligned}
(16)\qquad dZ_s &= \sigma_1(s,\theta(s),Z_s)dB'_s+b_1(s,\theta(s),Z_s)ds,\quad s\in[0,T], \\
Z_0 &= z_0.
\end{aligned}
$$

We write for $\Xi(\theta,z_0;B'_\cdot)(t),\ t\in[0,T]$ the strong solution of (16). (We omit z_0 hereafter.)

Now we can represent the density as

$$
\rho = e^{\psi(T,X_T,\Xi(X;B')(T))-\psi(0,x_0,z_0)}\tilde{G}(T,X,B';\tau,Y).
$$

Here for each $t\in[0,T]$ fixed, let

$$
\begin{aligned}
& \tilde{G}(t,\theta,\eta;u,y) \\
& \quad = \exp\Big(-\int_0^t \beta_0(s,\theta(s),\Xi(\theta;\eta)(s))^{\mathrm{T}}d\eta(s) \\
& \qquad\qquad +\int_0^t \hat{\beta}_1(s,\theta(s\wedge u),\Xi(\theta;\eta)(s\wedge u),y(s))^{\mathrm{T}}dy(s)
\end{aligned}
$$

$$- \int_0^t \{A(s,\theta(s),\Xi(\theta;\eta)(s))$$
$$+ \frac{1}{2}|\beta_1(s,\theta(s\wedge u),\Xi(\theta;\eta)(s\wedge u),y(s))|^2\}ds\Big),$$

for $\theta \in \mathbf{W}, \eta \in \mathbf{W}^n, u \in [0,\infty)$ and $y \in \mathbf{W}^m$.

In the rest of this section, we find out the stochastic differential equation satisfied by the density process $\rho_t = \tilde{E}[\rho|\mathcal{F}_t]$. In particular, We will make use of Theorem 1.1 to show the stochastic integral equation which the $(\mathcal{F}_t)$-conditional density satisfies.

Let $f_t = \tilde{E}[\rho|\mathcal{B}_t]$, which is a $(\tilde{P},(\mathcal{B}_t)_{t\in[0,T]})$-martingale. Because of the fact $\mathcal{F}_t \subset \mathcal{B}_t$, we have

$$\rho_t = \tilde{E}[\tilde{E}[\rho|\mathcal{B}_t]|\mathcal{F}_t] = \tilde{E}[f_t|\mathcal{F}_t].$$

We note that f_t has the following two representations.

$$f_t = 1 + \int_0^t f_{s-} b_0(s,X_s,Z_s)d\tilde{B}_s + \int_0^t f_{s-}\beta_1(s,X_{s\wedge\tau},Z_{s\wedge\tau},Y_s)^{\mathrm{T}}d\tilde{W}_s \tag{17}$$

and

$$f_t = e^{\psi(t,X_t,\Xi(X;B')(t))-\psi(0,x_0,z_0)}\tilde{G}(t,X,B';\tau,Y). \tag{18}$$

We prepare for some further notation. Notation such as $g(s,x_0,z)$, $\nu^{s,x}_{0,x_0}(\cdot)$ and so on are given in the last section.

$$H_\beta(s;y) = e^{-\psi(0,x_0,z_0)}\int_0^\infty dx g(s,x_0,x)\int_{\mathbf{W}\times\mathbf{W}^n}\nu^{s,x}_{0,x_0}(d\theta)\otimes\mu_n(d\eta) \times \beta_1(s,x,\Xi(\theta;\eta)(s),y(s))e^{\psi(s,x,\Xi(\theta;\eta)(s))}G(s,\theta,\eta;y), \tag{19}$$

$$H(s;y) = e^{-\psi(0,x_0,z_0)}\int_{\mathbf{W}\times\mathbf{W}^n}\nu^{s,0}_{0,x_0}(d\theta)\otimes\mu_n(d\eta)G(s,\theta,\eta;y), \tag{20}$$

and

$$\begin{aligned} &G(s,\theta,\eta;y)\\ =\ &\exp\Big(-\int_0^s \beta_0(u,\theta(u),\Xi(\theta;\eta)(u))^{\mathrm{T}}d\eta(u) - \int_0^s A(u,\theta(u),\Xi(\theta;\eta)(u))du\\ &+\int_0^s \hat{\beta}_1(u,\theta(u),\Xi(\theta;\eta)(u),y(u))^{\mathrm{T}}dy(u)\\ &-\frac{1}{2}\int_0^s|\beta_1(u,\theta(u),\Xi(\theta;\eta)(u),y(u))|^2du\Big),\end{aligned}$$

for $\theta \in \mathbf{W}^1, \eta \in \mathbf{W}^n$ and $y \in \mathbf{W}^m$. Moreover

$$\begin{aligned}(21)\quad & \tilde{H}_\beta(s;u,y) \\ &= e^{-\psi(0,x_0,z_0)} \int_{\mathbf{W}\times\mathbf{W}^n} \nu^{u,0}_{0,x_0}(d\theta) \otimes \mu_n(d\eta) \hat{G}(s,\theta,\eta;u,y) \\ &\quad \times \beta_1(s,0,\Xi(\theta;\eta)(u),y(s)), \\ & \hat{G}(s,\theta,\eta;u,y) \\ &= G(u,\theta,\eta;y) \exp\Big(\int_{u\wedge s}^s \hat{\beta}_1(v,0,\Xi(\theta;\eta)(u),y(v))^{\mathrm{T}} dy(v) \\ &\qquad - \frac{1}{2}\int_{u\wedge s}^s |\beta_1(v,0,\Xi(\theta;\eta)(u),y(v))|^2 dv\Big).\end{aligned}$$

The following result is our goal in this section.

PROPOSITION 4.1. *ρ_t satisfies the following a stochastic integral equation*

$$\rho_t = 1 + \int_0^t \rho_{s-}(\gamma(s)^{\mathrm{T}} d\tilde{W}_s + \kappa(s) d\tilde{M}_s), \tag{22}$$

where

$$\begin{aligned}\gamma(s) &= \rho_{s-}^{-1}\{(1-N_{s-})q(s)^{-1}H_\beta(s;Y) + N_{s-}\int_0^s \tilde{H}_\beta(s;u,Y)dN_u\} \\ \kappa(s) &= (1-N_{s-})\{\rho_{s-}^{-1}H(s;Y)-1\}.\end{aligned}$$

In particular, both $\gamma(t)$ and $\kappa(t)$ are $(\mathcal{F}_t)$-predictable processes.

PROOF. First of all, we notice that we can identify $\tilde{B}$ and $\tilde{W}$ under $\tilde{P}$ at present with B and W under P in the last two sections. Since Y and $\tilde{W}$ are equivalent under $\tilde{P}$, we will treat them equally. Let $\hat{\Phi}$ be a functional that satisfies, $\tilde{P}$-a.s., for $t \in [0,T]$,

$$\begin{aligned}\hat{\Phi}(t;X,Y) &= \tilde{E}[\int_0^t f_{s-} b_0(s,X_s,Z_s) d\tilde{B}_s \\ &\quad + \int_0^t f_{s-}\beta_1(s,X_{s\wedge\tau},Z_{s\wedge\tau},Y_s)^{\mathrm{T}} d\tilde{W}_s | \mathcal{G}_t^{X,Y}].\end{aligned}$$

It follows from (17) and (18) that

$$\begin{aligned}(23)\quad \hat{\Phi}(t;X,Y) &= \tilde{E}[f_t|\mathcal{G}_t^{X,Y}] - 1 \\ &= \int_{\mathbf{W}^n} \mu_n(d\eta) e^{\psi(t,X_t,\Xi(X;\eta)(t)) - \psi(0,x_0,z_0)} \tilde{G}(t,X,\eta;\tau,Y)\end{aligned}$$

By applying Theorem 1.1 to (17), we have

$$\begin{aligned}\rho_t &= \tilde{E}[1+\int_0^t f_{s-}b_0(s,X_s,Z_s)d\tilde{B}_s \\ &\quad + \int_0^t f_{s-}\beta_1(s,X_{s\wedge\tau},Z_{s\wedge\tau},Y_s)^{\mathrm{T}}d\tilde{W}_s|\mathcal{F}_t] \\ &= 1+\int_0^t \tilde{E}[f_{s-}\beta_1(s,X_{s\wedge\tau},Z_{s\wedge\tau},Y_s)^{\mathrm{T}}|\mathcal{F}_s]d\tilde{W}_s \\ &\quad + \int_0^t (1-N_{s-})\Big\{\int_{\mathbf{W}}\nu^{s,0}_{0,x_0}(d\theta)\hat{\Phi}(s;\theta,w) \\ &\qquad\qquad - q(s)^{-1}\int_0^\infty dx g(s,x_0,x)\int_{\mathbf{W}}\nu^{s,x}_{0,x_0}(d\theta)\hat{\Phi}(s;\theta,w)\Big\}d\tilde{M}_s.\end{aligned}$$

We have

$$\begin{aligned}&\int_{\mathbf{W}}\nu^{s,0}_{0,x_0}(d\theta)\hat{\Phi}(s;\theta,w) \\ &\quad = \int_{\mathbf{W}}\nu^{s,0}_{0,x_0}(d\theta)(1+\hat{\Phi}(s;\theta,w))-1 \\ &\quad = \int_{\mathbf{W}\times\mathbf{W}^n}\nu^{s,0}_{0,x_0}(d\theta)\otimes\mu_n(d\eta)e^{-\psi(0,x_0,z_0)}G(s,\theta,\eta;Y)-1 \\ &\quad = H(s;Y)-1.\end{aligned}$$

On the other hand, Lemma 3.1(1) implies

$$\begin{aligned}&\int_{\mathbf{W}}\nu^{s,x}_{0,x_0}(d\theta)\hat{\Phi}(s;\theta,w) \\ &\quad = \int_{\mathbf{W}}\nu^{s,x}_{0,x_0}(d\theta)(1+\hat{\Phi}(s;\theta,w))-q(s) \\ &\quad = \tilde{E}[(1-N_s)(1+\hat{\Phi}(s;X,Y))|\mathcal{G}^Y_s]-q(s) \\ &\quad = \tilde{E}[(1-N_s)\tilde{E}[f_s|\mathcal{G}^{X,Y}_s]|\mathcal{G}^Y_s]-q(s) \\ &\quad = \tilde{E}[(1-N_s)f_s|\mathcal{G}^Y_s]-q(s).\end{aligned}\tag{24}$$

Hence we have

$$\begin{aligned}\rho_t &= \tilde{E}[1+\int_0^t f_{s-}b_0(s,X_s,Z_s)d\tilde{B}_s \\ &\quad + \int_0^t f_{s-}\beta_1(s,X_{s\wedge\tau},Z_{s\wedge\tau},Y_s)^{\mathrm{T}}d\tilde{W}_s|\mathcal{F}_t]\end{aligned}\tag{25}$$

$$\begin{aligned} &= 1 + \int_0^t \tilde{E}[f_{s-}\beta_1(s, X_{s\wedge\tau}, Z_{s\wedge\tau}, Y_s)^{\mathrm{T}}|\mathcal{F}_s]d\tilde{W}_s \\ &\quad + \int_0^t (1 - N_{s-})\{H(s;Y) - q(s)^{-1}\tilde{E}[(1-N_s)f_s|\mathcal{G}_s^Y]\}d\tilde{M}_s. \end{aligned}$$

We remark that f_{s-} is replaced by f_s owing to its continuity, so we have

$$\begin{aligned} &\tilde{E}[f_{s-}\beta_1(s, X_{s\wedge\tau}, Z_{s\wedge\tau}, Y_s)|\mathcal{F}_s] \\ &= (1-N_s)\tilde{E}[f_s\beta_1(s, X_{s\wedge\tau}, Z_{s\wedge\tau}, Y_s)|\mathcal{F}_s] \\ &\quad + N_s\tilde{E}[f_s\beta_1(s, X_{s\wedge\tau}, Z_{s\wedge\tau}, Y_s)|\mathcal{F}_s] \\ &= (1-N_s)q(s)^{-1}\tilde{E}[(1-N_s)f_s\beta_1(s, X_{s\wedge\tau}, Z_{s\wedge\tau}, Y_s)|\mathcal{G}_s^Y] \\ &\quad + N_s\tilde{E}[f_s\beta_1(s, X_{s\wedge\tau}, Z_{s\wedge\tau}, Y_s)|\mathcal{F}_s]. \end{aligned}$$

Since we observe that on the set $\{\tau > s\}$,

$$f_s\beta_1(s, X_{s\wedge\tau}, Z_{s\wedge\tau}, Y_s) = \beta_1(s, X_s, \Xi(X;B')(s), Y_s)G(s, X, B', Y).$$

Hence it follows from Lemma 3.1(1) that

$$\begin{aligned} &\tilde{E}[\beta_1(s, X_s, \Xi(X;B')(s), Y_s)G(s, X, B', Y)|\mathcal{G}_t^Y] \\ &= \int_0^\infty dx g(s, x_0, x)e^{\psi(s,x,\Xi(w;\eta)(s)) - \psi(0,x_0,z_0)} \\ &\quad \times \int_{\mathbf{W}\times\mathbf{W}^n} \nu_{0,x_0}^{s,x}(d\theta) \otimes \mu_n(d\eta) \\ &\quad \times \beta_1(s, x, \Xi(\theta;\eta)(s), Y(s))G(s, \theta, \eta; Y). \end{aligned} \tag{26}$$

On the other hand, we notice that $\mathcal{F}_t \subset \mathcal{G}_t^Y \vee \mathcal{G}_{\tau\wedge t}^{X,Z}$ since $\sigma\{\tau\wedge t\} \subset \mathcal{G}_{\tau\wedge t}^{X,Z}$, where $\mathcal{G}_t^{X,Z} = \mathcal{G}_t^X \vee \mathcal{G}_t^Z$. Hence we have

$$\begin{aligned} &N_s\tilde{E}[f_s\beta_1(s, X_{s\wedge\tau}, Z_{s\wedge\tau}, Y_s)|\mathcal{F}_s] \\ &= \tilde{E}[N_s\tilde{E}[f_s|\mathcal{G}_s^Y \vee \mathcal{G}_{\tau\wedge s}^{X,Z}]\beta_1(s, X_{s\wedge\tau}, Z_{s\wedge\tau}, Y_s)|\mathcal{F}_s] \\ &= \tilde{E}[N_s e^{-\psi(0,x_0,z_0)}\beta_1(s, X_{s\wedge\tau}, Z_{s\wedge\tau}, Y_s)\hat{G}(s, X, B'; \tau, Y) \\ &\quad \times \tilde{E}[\exp\Big(\psi(t, X_t, Z_t) - \int_{\tau\wedge t}^t \beta_0(s, X_s, Z_s)^{\mathrm{T}}dB'_s \\ &\quad - \int_{\tau\wedge t}^t A(s, X_s, Z_s)ds\Big)|\mathcal{G}_t^Y \vee \mathcal{G}_{\tau\wedge t}^{X,Z}]|\mathcal{F}_t]. \end{aligned}$$

We can observe

$$\begin{aligned}
&\tilde{E}[\exp\Big(\psi(t,X_t,Z_t)-\int_{\tau\wedge t}^t \beta_0(s,X_s,Z_s)^{\mathrm{T}}dB'_s \\
&\qquad -\int_{\tau\wedge t}^t A(s,X_s,Z_s)ds\Big)|\mathcal{G}_t^Y\vee\mathcal{G}_{\tau\wedge t}^{X,Z}] \\
= \;& \tilde{E}[\exp\Big(\int_{\tau\wedge t}^t b_0(s,X_s,Z_s)d\tilde{B}_s-\frac{1}{2}\int_{\tau\wedge t}^t b_0(s,X_s,Z_s)^2ds\Big)|\mathcal{G}_t^Y\vee\mathcal{G}_{\tau\wedge t}^{X,Z}] \\
= \;& \tilde{E}[\exp\Big(\int_{\tau\wedge t}^t b_0(s,X_s,Z_s)d\tilde{B}_s-\frac{1}{2}\int_{\tau\wedge t}^t b_0(s,X_s,Z_s)^2ds\Big)|\mathcal{G}_{\tau\wedge t}^{X,Z}] \\
= \;& 1
\end{aligned}$$

since $\exp\left(\int_0^t b_0(s,X_s,Z_s)d\tilde{B}_s-\frac{1}{2}\int_0^t b_0(s,X_s,Z_s)^2ds\right)$ is a $(\tilde{P},(\mathcal{G}_t^{X,Z}))$-martingale.

Therefore, by using Lemma 3.1(3), we have

$$\begin{aligned}
(27)\qquad & N_s\tilde{E}[f_s\beta_1(s,X_{s\wedge\tau},Z_{s\wedge\tau},Y_s)|\mathcal{F}_s] \\
&= \tilde{E}[N_se^{-\psi(0,x_0,z_0)}\beta_1(s,X_{s\wedge\tau},Z_{s\wedge\tau},Y_s)\hat{G}(s,X,B';\tau,Y)|\mathcal{F}_s] \\
&= \tilde{E}[N_se^{-\psi(0,x_0,z_0)}\beta_1(s,0,\Xi(X,B')(\tau),Y_s)\hat{G}(s,X,B';\tau,Y)|\mathcal{F}_s] \\
&= \int_0^s \tilde{H}_\beta(s;u,Y)dN_u.
\end{aligned}$$

Since $\rho_{s-}\gamma(s)=E[f_{s-}\beta_1(s,X_{s\wedge\tau},Z_{s\wedge\tau},Y_s)|\mathcal{F}_s]$ and $\gamma(s)$ is $(\mathcal{F}_t)$-predictable, we can achieve from (26) and (27),

$$(28)\qquad \gamma(s)=\rho_{s-}^{-1}\{(1-N_{s-})q(s)^{-1}H_\beta(s;Y)+N_{s-}\int_0^s\tilde{H}_\beta(s;u,Y)dN_u\}.$$

As for $\kappa(s)$, since we observe that

$$(1-N_{s-})q(s)^{-1}\tilde{E}[(1-N_s)f_s|\mathcal{G}_s^Y]=(1-N_{s-})\rho_{s-},$$

we have

$$\begin{aligned}
(29)\qquad \kappa(s) &= (1-N_{s-})(H(s;Y)-q(s)^{-1}\tilde{E}[(1-N_s)f_s|\mathcal{G}_s^Y]) \\
&= (1-N_{s-})(H(s;Y)-\rho_{s-}).
\end{aligned}$$

Hence we obtain the consequence. □

5. Hazard Rate Process and Survival Probability under the Original Measure

Now we can state our main result about the hazard rate process under the original probability measure P.

THEOREM 5.1. *The $(\mathcal{G}_t^Y)$-hazard rate process under P is given by*

$$h(t) = \frac{\hat{H}(t;Y)}{\hat{K}(t;Y)} q(t)\lambda(t),$$

where

$$\hat{H}(t;y) = e^{\psi(0,x_0,z_0)} H(t;y), \tag{30}$$

$H(t;y)$ is given by (20) *and*

$$\begin{aligned} \hat{K}(t;y) &= \int_0^\infty dx g(t,x_0,x) \\ &\quad \times \int_{\mathbf{W}} \nu_{0,x_0}^{t,x}(d\theta) \otimes \mu_n(d\eta) e^{\psi(t,x,\Xi(\theta;\eta)(t))} G(t,\theta,\eta;y) \end{aligned} \tag{31}$$

for $y \in \mathbf{W}^m$.

PROOF. First, remember that $\lambda(t)$ is $(\mathcal{G}_t^Y)$-hazard rate process under $\tilde{P}$. It follows from Proposition 3.1 in Kusuoka [7] and Proposition 4.1 that the $(\mathcal{G}_t^Y)$-hazard rate process under P, $h(t)$, is given by

$$h(t) = (1+\kappa(t))\lambda(t) = \rho_{t-}^{-1} H(t;Y)\lambda(t)$$

We have

$$\begin{aligned} (1-N_t)\rho_t &= (1-N_t)q(t)^{-1}\tilde{E}[(1-N_t)f_t|\mathcal{G}_t^Y] \\ &= (1-N_t)q(t)^{-1} \\ &\quad \times \tilde{E}[(1-N_t)e^{\psi(t,X_t,Z_t)-\psi(0,x_0,z_0)}\tilde{G}(t,X,B';\tau,Y)|\mathcal{G}_t^Y] \\ &= (1-N_t)q(t)^{-1} \\ &\quad \times \tilde{E}[(1-N_t)e^{\psi(t,X_t,Z_t)-\psi(0,x_0,z_0)}G(t,X,B';Y)|\mathcal{G}_t^Y] \\ &= (1-N_t)q(t)^{-1}e^{-\psi(0,x_0,z_0)}\hat{K}(t;Y). \end{aligned}$$

The last equality follows from Lemma 3.1(1). Hence we have

$$h(t) = \frac{\hat{H}(t;Y)}{\hat{K}(t;Y)} q(t)\lambda(t). \ \square$$

REMARK 5.2. We remark that on the set $\{\tau > t\}$, we observe

$$h(t) = \lim_{x \downarrow 0} \frac{1}{2x} \frac{P(X_t \in dx|\mathcal{F}_t)}{dx}, \tag{32}$$

that is, it is an illustration of Duffie and Lando's result ([1]). In order to check it we have only to notice

$$\frac{d}{dt}q(t) = -\lim_{x \downarrow 0} \frac{g(t, x_0, x)}{2x}.$$

It is claimed in [2] that under some assumptions, the equality

$$P(\tau > T|\mathcal{F}_t) = (1 - N_t)E[\exp\Big(-\int_t^T h(u)du\Big)|\mathcal{G}_t^Y] \tag{33}$$

holds. However, in general, the equality (33) does not necessarily hold even though $h(u)$ is the $(\mathcal{G}_t^Y)$-hazard rate process under P. Refer to [3], [4] and [7].)

Here we will say that the model is standard under a probability measure P^* if $(\mathcal{G}_t^Y)$-hazard rate process under P^*, $h^*(u)$, exists and satisfies, for every $t \in [0, T]$,

$$P^*(\tau > T|\mathcal{F}_t) = (1 - N_t)E^*[\exp\Big(-\int_t^T h^*(u)du\Big)|\mathcal{G}_t^Y].$$

Although our model is proved to be standard under $\tilde{P}$, it may not be standard under the original measure P.

From now on, we investigate the relation between the survival probability with respect to P and P-hazard rate process $h(t)$.

If we let $\bar{\gamma}(t;Y) = \hat{K}(t;Y)^{-1}\hat{H}_\beta(t;Y)$, $\bar{\gamma}(t;Y)$is a $(\mathcal{G}_t^Y)$-progressively measurable process such that

$$(1 - N_{t-})\gamma(t) = (1 - N_{t-})\bar{\gamma}(t;Y).$$

Denote by $\hat{\rho}_t$ a $(\tilde{P}, (\mathcal{G}_t^Y))$-martingale defined by

$$d\hat{\rho}_t = \hat{\rho}_t\bar{\gamma}(t;Y)^{\mathrm{T}}d\tilde{W}_t, \quad \hat{\rho}_0 = 1,$$

that is
$$\hat\rho_t = \exp\Big(\int_0^t \bar\gamma(u;Y)^{\mathrm{T}} d\tilde W_u - \frac{1}{2}\int_0^t |\bar\gamma(u;Y)|^2 du\Big).$$
Now we define another probability measure Q on $(\Omega, \mathcal{B})$ by $dQ = \hat\rho_T d\tilde P$.

Let Λ be a $\mathcal{G}_T^Y$-measurable, bounded random variable.

It follows from Proposition 7 in [7] that for $t \in [0, T]$,
$$\begin{aligned} &E[\Lambda(1-N_T)|\mathcal{F}_t] \\ &\quad = (1-N_t)E^Q[\Lambda\exp\Big(-\int_t^T h(u)du\Big)|\mathcal{G}_t^Y]. \end{aligned}$$
Let $L_t = E[\frac{dQ}{dP}|\mathcal{G}_t^Y]$. Then by using the Bayes' rule,
$$\begin{aligned} &E^Q[\Lambda\exp\Big(-\int_t^T h(u)du\Big)|\mathcal{G}_t^Y] \\ &\quad = L_t^{-1}E[L_T\Lambda\exp\Big(-\int_t^T h(u)du\Big)|\mathcal{G}_t^Y]. \end{aligned}$$
Note that
$$\begin{aligned} L_t &= E[\frac{dQ}{d\tilde P}\frac{d\tilde P}{dP}|\mathcal{G}_t^Y] = E[\hat\rho_T R|\mathcal{G}_t^Y] \\ &= E[\hat\rho_T E[R|\mathcal{G}_T^Y]|\mathcal{G}_t^Y]E[R|\mathcal{G}_t^Y]^{-1}E[R|\mathcal{G}_t^Y] \\ &= \tilde E[\hat\rho_T|\mathcal{G}_t^Y]E[R|\mathcal{G}_t^Y] = \hat\rho_t E[R|\mathcal{G}_t^Y]. \end{aligned}$$
The last equality follows from the fact that $\hat\rho_t$ is a $(\tilde P, (\mathcal{G}_t^Y))$-martingale.

Let $\tilde\rho_t = \tilde E[\rho|\mathcal{G}_t^Y]$. Recalling Proposition 1.2, it is easy to see
$$\tilde\rho_t = 1 + \int_0^t \tilde E[f_s\beta_1(s, X_{s\wedge\tau}, Z_{s\wedge\tau}, Y_s)^{\mathrm{T}}|\mathcal{G}_s^Y]d\tilde W_s.$$
The calculation after this can be done as we did in the last section.

Let
$$\begin{aligned} c(t;y) &= \frac{\hat H_\beta(t;y) + \int_0^t dvq(v)\lambda(v)\hat J_\beta(t;v,y)}{\hat K(t;y) + \int_0^t dvq(v)\lambda(v)\hat J(t;v,y)}, \\ \hat J_\beta(t;v,y) &= \int_{\mathbf{W}} \nu_{0,x_0}^{v,0}(dw)\otimes\mu_n(d\eta)\hat G(t,w,\eta;v,y)\beta_1(t,0,\Xi(w;\eta)(v),y(t)) \\ \hat J(t;v,y) &= \int_{\mathbf{W}} \nu_{0,x_0}^{v,0}(dw)\otimes\mu_n(d\eta)\hat G(t,w,\eta;v,y). \end{aligned}$$

for $y \in \mathbf{W}^m$.

Direct computation of $\tilde{\rho}_s$ implies

$$\tilde{\rho}_s = \hat{K}(t; y) + \int_0^t dvq(v)\lambda(v)\hat{J}(t; v, y).$$

Notice that $c(s; Y)$ satisfies

$$c(s; Y) = \tilde{\rho}_s^{-1}\tilde{E}[f_s\beta_1(s, X_{s\wedge\tau}, Z_{s\wedge\tau}, Y_s)|\mathcal{G}_s^Y],$$

and we have

$$\tilde{\rho}_t = 1 + \int_0^t \tilde{\rho}_s c(s; Y)^{\mathrm{T}} d\tilde{W}_s,$$

which has the following explicit solution:

$$\tilde{\rho}_t = \exp\Big(\int_0^t c(u; Y)^{\mathrm{T}} d\tilde{W}_u - \frac{1}{2}\int_0^t |c(u; Y)|^2 du\Big).$$

Cameron-Martin-Maruyama-Girsanov theorem implies $W_t^P = \tilde{W}_t - \int_0^t c(s; Y)ds$ is a $(P, \mathcal{G}_t^Y)$-Brownian motion and

$$E[R|\mathcal{G}_t^Y] = \exp\Big(-\int_0^t c(u; Y)^{\mathrm{T}} dW_u^P - \frac{1}{2}\int_0^t |c(u; Y)|^2 du\Big).$$

Therefore

$$L_t = \exp\Big(\int_0^t \{\bar{\gamma}(u; Y) - c(u; Y)\} dW_u^P - \frac{1}{2}\int_0^t |\bar{\gamma}(u; Y) - c(u; Y)|^2 du\Big).$$

Now we can present the following.

Proposition 5.3. *Let $t, s \in [0, T]$ with $t < s$. For any $(\mathcal{G}_s^Y)$-measurable, bounded random variable Λ,*

$$\begin{aligned}
&E[\Lambda(1 - N_s)|\mathcal{F}_t] \\
&= (1 - N_t)E[\Lambda \exp\Big(-\int_t^s h(u)du\Big) \\
&\quad \times \exp\Big(\int_t^s \{\bar{\gamma}(u; Y) - c(u; Y)\}^{\mathrm{T}} dW_u^P \\
&\qquad - \frac{1}{2}\int_t^s |\bar{\gamma}(u; Y) - c(u; Y)|^2 du\Big)|\mathcal{G}_t^Y],
\end{aligned}$$

where (W_t^P) *is a* $(P, (\mathcal{G}_t^Y))$*-Brownian motion.*

Remark 5.4. We attempt to give a simple illustration of the above result from a financial view. Let r_t be a $(\mathcal{G}_t^Y)$-adapted, non-negative process that means the risk-free instantaneous interest rate. We suppose that P is a so-called equivalent martingale measure. If we consider a defaultable security whose payoff at the terminal time T is given by $1_{\{\tau > T\}}$ and that pays no coupon before T, its price at time t, $P(t,T)$, is defined by the last proposition:

$$\begin{aligned}
&P(t,T) \\
&= (1-N_t)E[\exp\Big(-\int_t^T \{r_u + h(u)\}du\Big) \\
&\qquad \times \exp\Big(\int_t^T \{\bar{\gamma}(u;Y) - c(u;Y)\}^{\mathrm{T}} dW_u^P \\
&\qquad\qquad - \frac{1}{2}\int_t^T |\bar{\gamma}(u;Y) - c(u;Y)|^2 du\Big)|\mathcal{G}_t^Y].
\end{aligned}$$

Roughly speaking, the credit spread can be explained by the hazard rate plus some fluctuation caused by the difference between the filtrations $(\mathcal{F}_t)$ and $(\mathcal{G}_t^Y)$.

Appendix

Here we use a differential geometric approach to present a way of transformation from a more general case to our simple case which is defined in section 1.

Let $\tilde{Z}_t = (\tilde{Z}_t^0, \tilde{Z}_t')^{\mathrm{T}} = (\tilde{Z}_t^0, \tilde{Z}_t^1, \cdots, \tilde{Z}_t^d)^{\mathrm{T}}$ satisfy the following stochastic differential equations:

$$d\tilde{Z}_t^i = \sum_{k=0}^d \tilde{\sigma}^{ik}(\tilde{Z}_t)d\tilde{B}_t + \tilde{b}^i(t, \tilde{Z}_t)dt, \quad i = 0, 1, \cdots, d,$$

where $\tilde{\sigma}^{ij}$ and $\tilde{b}^i$ are bounded and smooth functions. We have

$$d\langle \tilde{Z}^i, \tilde{Z}^j\rangle_t = \tilde{a}^{ij}(\tilde{Z}_t)dt,$$

where

$$\tilde{a}^{ij}(z) = \sum_{k=0}^{d} \tilde{\sigma}^{ik}(z)\tilde{\sigma}^{kj}(z), \quad i,j = 0,1,\cdots,d.$$

We assume that the matrix $(\tilde{a}^{ij})_{i,j=0,\cdots,d}$ is non-singular. We often take $z \in \mathbf{R}^{1+d}$ as $(z^0, z') \in \mathbf{R} \times \mathbf{R}^d$ and so forth.

PROPOSITION A.1. *Assume that $\tilde{a}^{00}$ is twice continuously differentiable and let $g(z') = \dfrac{1}{\sqrt{\tilde{a}^{00}(0,z')}}$.*

Besides, let $Y_t^0 = g(\tilde{Z}_t')\tilde{Z}_t^0$ and $Y_t^i = \tilde{Z}_t^i$, $i = 1,\cdots,d$.

Then there exists a function $\hat{a}^{00}(y^0, y')$ such that $\hat{a}^{00}(0, y') = 1$ for all $y' \in \mathbf{R}^d$ and

$$d\langle Y^0, Y^0\rangle_t = \hat{a}^{00}(Y_t^0, Y_t')dt.$$

PROOF. Ito's formula implies

$$\begin{aligned}
dY_t^0 &= g(\tilde{Z}_t')d\tilde{Z}_t^0 + \tilde{Z}_t^0 dg(\tilde{Z}_t') + d\langle \tilde{Z}^0, g(\tilde{Z}')\rangle_t \\
&= g(\tilde{Z}_t')d\tilde{Z}_t^0 + \tilde{Z}_t^0\Big\{\sum_{i=1}^{d} \frac{\partial g}{\partial z^i}(\tilde{Z}_t')d\tilde{Z}_t^i + \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 g}{\partial z^i z^j}(\tilde{Z}_t')d\langle \tilde{Z}^i, \tilde{Z}^j\rangle_t\Big\} \\
&\quad + \sum_{i=1}^{d} \frac{\partial g}{\partial z^i}(\tilde{Z}_t')d\langle \tilde{Z}^0, \tilde{Z}^i\rangle_t \\
&= g(Y_t')\Big\{\sum_{k=0}^{d} \tilde{\sigma}^{0k}((g(Y_t')^{-1}Y_t^0, Y_t'))d\tilde{B}_t + \tilde{b}^0(t,(g(Y_t')^{-1}Y_t^0, Y_t'))dt\Big\} \\
&\quad + g(Y_t')^{-1}Y_t^0\Big\{\sum_{i=1}^{d} \frac{\partial g}{\partial y^i}(Y_t')\Big(\sum_{k=0}^{d} \tilde{\sigma}^{ik}(g(Y_t')^{-1}Y_t^0, Y_t')d\tilde{B}_t \\
&\qquad\qquad + \tilde{b}^i(t,(g(Y_t')^{-1}Y_t^0, Y_t'))dt\Big) \\
&\quad + \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 g}{\partial y^i \partial y^j}(Y_t')\tilde{a}^{ij}(g(Y_t')^{-1}Y_t^0, Y_t')dt\Big\} \\
&\quad + \sum_{i=1}^{d} \frac{\partial g}{\partial y^i}(Y_t')\tilde{a}^{0i}(g(Y_t')^{-1}Y_t^0, Y_t')dt.
\end{aligned}$$

This means that Y is a diffusion process. Besides, we have

$$\begin{aligned}
& d\langle Y^0, Y^0\rangle_t \\
&\quad = \{g(Y_t')^2\tilde{a}^{00}(g(Y_t')^{-1}Y_t^0, Y_t') \\
&\qquad + 2g(Y_t')g(Y_t')^{-1}Y_t^0\sum_{i=1}^d \frac{\partial g}{\partial y^i}(Y_t')\tilde{a}^{0i}(g(Y_t')^{-1}Y_t^0, Y_t') \\
&\qquad + (g(Y_t')^{-1}Y_t^0)^2\sum_{i,j=1}^d \frac{\partial g}{\partial y^i}(Y_t')\frac{\partial g}{\partial y^j}(Y_t')\tilde{a}^{ij}(g(Y_t')^{-1}Y_t^0, Y_t')\}dt.
\end{aligned}$$

Let

$$\begin{aligned}
(34)\qquad & \hat{a}(y^0, y') \\
&\quad = g(y')^2\tilde{a}^{00}(g(y')^{-1}y^0, y') \\
&\qquad + 2g(y')g(y')^{-1}y^0\sum_{i=1}^d \frac{\partial g}{\partial y^i}(y')\tilde{a}^{0i}(g(y')^{-1}y^0, y') \\
&\qquad + (g(y')^{-1}y^0)^2\sum_{i,j=1}^d \frac{\partial g}{\partial y^i}(y')\frac{\partial g}{\partial y^j}(y')\tilde{a}^{ij}(g(y')^{-1}y^0, y').
\end{aligned}$$

Then it is easy to see $d\langle Y^0, Y^0\rangle_t = \hat{a}^{00}(Y_t)dt$ and $\hat{a}^{00}(0, y') = 1$, so we obtain the result. □

Adding to the last proposition, we observe that the property that $Y_t^0 = 0$ if and only if $\tilde{Z}_t^0 = 0$ is also valid. Therefore it is not different in essence to take up Y instead of $\tilde{Z}$. That is the reason we may assume without loss of generality that

$$\tilde{a}^{00}(0, z') = 1 \quad \text{for any } z' \in \mathbf{R}^d. \tag{35}$$

Define the function $H : \mathbf{R}^{1+d} \times \mathbf{R}^{1+d} \longrightarrow \mathbf{R}$ by

$$H(x, y) = \sum_{i,j=0}^d \tilde{a}^{ij}(x)y^i y^j. \tag{36}$$

We consider the following Hamiltonian system:

$$\begin{cases} \dfrac{d}{dt}x(t) &= \nabla_y H(x(t), y(t)), \quad x(0) = u \\ \dfrac{d}{dt}y(t) &= -\nabla_x H(x(t), y(t)), \quad y(0) = v, \end{cases} \tag{37}$$

and let $x(t;u,v)$ and $y(t;u,v)$ be solutions of the above equations.

Let

$$\begin{aligned}\Phi(z^0,z') &= x(z^0;(0,z'),(1,0))\\ \Psi(z^0,z') &= y(z^0;(0,z'),(1,0)).\end{aligned}$$

PROPOSITION A.2. *Assume that $\Phi:\mathbf{R}^{1+d}\longrightarrow\mathbf{R}^{1+d}$ is a diffeomorphism. Define $(1+d)$-dimensional process $Z_t=(Z_t^0,Z_t')=(Z_t^0,Z_t^1,\cdots,Z_t^d)$ as*

$$Z_t=\Phi^{-1}(\tilde{Z}_t).$$

Then $(0,\tilde{Z}_t')=\Phi^{-1}((0,\tilde{Z}_t'))$ and there exists a $(1+d)$-dimensional $(\mathcal{G}_t^{\tilde{B}})$-Brownian motion $B=(B^0,B^1,\cdots,B^d)$ and a matrix-valued function $\Gamma(z)$ in the form of

$$\Gamma(z)=\left(\begin{array}{c|c}\frac{1}{2} & \mathbf{0}^{\mathrm{T}}\\ \hline \mathbf{0} & \hat{\Gamma}(z)\end{array}\right)$$

such that the martingale part of Z vanishing at time 0 is given by

$$\int_0^t\Gamma(Z_s)dB_s.$$

PROOF. From the definition of Φ and the hypothesis of diffeomorphism, it follows that $(0,z')=\Phi(0,z')$, so the first statement is apparent.

Let $d\langle Z^i,Z^j\rangle_t=a^{ij}(Z_t)dt$. By the definition of Z_t, we notice that

$$\tilde{Z}_t^i=\Phi^i(Z_t^0,Z_t'),\quad i=0,1,\cdots,d.$$

Since it follows from Ito's formula that

$$d\tilde{Z}_t^i=\sum_{k=0}^d\frac{\partial\Phi^i}{\partial z^k}(Z_t^0,Z_t')dZ_t^k+\frac{1}{2}\sum_{k,l=0}^d\frac{\partial^2\Phi^i}{\partial z^k\partial z^l}(Z_t^0,Z_t')a^{kl}(Z_t)dt,$$

we have

$$d\langle\tilde{Z}^i,\tilde{Z}^j\rangle_t=\sum_{k,l=0}^d\frac{\partial\Phi^i}{\partial z^k}(Z_t^0,Z_t')a^{kl}(Z_t)\frac{\partial\Phi^j}{\partial z^l}(Z_t^0,Z_t')dt.$$

Denoting by $\tilde{A}, A$ and D the $(1+d)$-dimensional square matrices $(\tilde{a}^{ij})$, (a^{ij}) and $\left(\dfrac{\partial \Phi^i}{\partial z^k}\right)$ respectively, it follows that

$$\tilde{A} = DAD^{\mathrm{T}}.$$

This equality implies $A^{-1} = D^{\mathrm{T}}\tilde{A}^{-1}D$, thus

$$a_{ij}(Z_t) = \sum_{k,l=0}^{d} \frac{\partial \Phi^k}{\partial z^i}(Z_t^0, Z_t')\tilde{a}_{kl}(\Phi(Z_t^0, Z_t'))\frac{\partial \Phi^l}{\partial z^j}(Z_t^0, Z_t'),$$

where (a_{ij}) is the (i,j)-component of A^{-1} and so is $\tilde{a}_{ij}$. In short, super-indices are used for an original matrix, while sub-indices for the inverse.

Now we need the following lemma.

LEMMA A.3.

$$\sum_{i,j=0}^{d} \frac{\partial \Phi^i}{\partial z^0}(Z_t^0, Z_t')\tilde{a}_{ij}(\Phi(Z_t^0, Z_t'))\frac{\partial \Phi^j}{\partial z^k}(Z_t^0, Z_t') = 4\delta_{0k}.$$

PROOF. We define the function $S : \mathbf{R} \times \mathbf{R}^{1+d} \times \mathbf{R}^{1+d} \longrightarrow \mathbf{R}$ by

$$S(t,u,v) = \int_0^t \Big\{ \frac{d}{ds}x(s;u,v)^{\mathrm{T}} y(s;u,v) - H(x(s;u,v), y(s;u,v)) \Big\} ds.$$

We can check that S satisfies the following properties that

$$\nabla_u S(t,u,v) = \nabla_u x(s;u,v)^{\mathrm{T}} y(s;u,v) - v, \tag{38}$$

$$\nabla_v S(t,u,v) = \nabla_v x(s;u,v)^{\mathrm{T}} y(s;u,v), \tag{39}$$

where

$$\nabla_u f(u) = \Big(\frac{\partial f^i}{\partial u^j}\Big)_{i,j=0,1,\cdots,d},$$

for a differentiable function $f = (f^0, f^1, \cdots, f^d)^{\mathrm{T}}$.

From (36) and (37) it follows that

$$\begin{aligned} S(t,u,v) = \int_0^t \{ & \nabla_y H(x(s;u,v), y(s;u,v))^{\mathrm{T}} y(s;u,v) \\ & - H(x(s;u,v), y(s;u,v))\} ds \end{aligned} \tag{40}$$

$$\begin{aligned}
&= \int_0^t \{2H(x(s;u,v),y(s;u,v)) \\
&\qquad - H(x(s;u,v),y(s;u,v))\}ds \\
&= \int_0^t H(x(s;u,v),y(s;u,v))ds \\
&= tH(u,v).
\end{aligned}$$

The last equality holds because of the property of Hamilton type equation. Next let

$$\tilde{S}(z^0, z') = S(z^0, (0, z'), (1, 0)).$$

Due to (40) and the assumption (35), it is not hard to see

$$\tilde{S}(z^0, z') = z^0 H((0, z'), (1, 0)) = z^0,$$

that is,

$$\frac{\partial \tilde{S}}{\partial z^0}(z^0, z') = 1, \quad \frac{\partial \tilde{S}}{\partial z^k}(z^0, z') = 0.$$

On the other hand, by noticing $H(\Phi(z^0, z'), \Psi(z^0, z')) = H((0, z'), (1, 0)) = 1$, we observe

$$\begin{aligned}
\frac{\partial \tilde{S}}{\partial z^0}(z^0, z') &= \frac{\partial S}{\partial z^0}(z^0, (0, z'), (1, 0)) \\
&= \frac{\partial \Phi}{\partial z^0}(z^0, z')^{\mathrm{T}} \Psi(z^0, z') - H(\Phi(z^0, z'), \Psi(z^0, z')) \\
&= \frac{\partial \Phi}{\partial z^0}(z^0, z')^{\mathrm{T}} \Psi(z^0, z') - 1,
\end{aligned}$$

and (38) implies that for each $k = 1, \cdots .d$,

$$\frac{\partial \tilde{S}}{\partial z^k}(z^0, z') = \frac{\partial \Phi}{\partial z^k}(z^0, z')^{\mathrm{T}} \Psi(z^0, z').$$

Therefore it follows from these results that

$$\frac{\partial \Phi}{\partial z^0}(z^0, z')^{\mathrm{T}} \Psi(z^0, z') = 2, \tag{41}$$

$$\frac{\partial \Phi}{\partial z^k}(z^0, z')^{\mathrm{T}} \Psi(z^0, z') = 0, \quad k = 1, \cdots .d. \tag{42}$$

Moreover we have

$$\begin{aligned}\frac{\partial\Phi^i}{\partial z^0}(z^0,z') &= \frac{\partial}{\partial y^i}H(\Phi(z^0,z'),\Psi(z^0,z'))\\ &= 2\sum_{j=0}^{d}\tilde{a}^{ij}(\Phi(z^0,z'))\Psi^j(z^0,z')),\end{aligned}$$

so

$$\Psi^j(z^0,z') = \frac{1}{2}\sum_{i=0}^{d}\tilde{a}_{ij}(\Phi(z^0,z'))\frac{\partial\Phi^i}{\partial z^0}(z^0,z').$$

Substituting this for (41) and (42), we get the desired conclusion. □

This lemma asserts that $a^{00}=\frac{1}{4}$ and $a^{0i}=0, i=1,\cdots,d$, that is,

$$A(z) = \left(\begin{array}{c|c}\frac{1}{4} & \mathbf{0}^{\mathrm{T}}\\ \hline \mathbf{0} & \hat{A}(z)\end{array}\right)$$

since $a_{00}=4$ and $a_{0i}=0, i=1,\cdots,d$.

If we let M^i be a martingale part of Z^i with $M_0=0$, then

$$d\langle M^i,M^j\rangle_t = d\langle Z^i,Z^j\rangle_t.$$

Hence from Theorem V.3.9 in Revuz-Yor [9], it follows that there exists a $(1+d)$-dimensional Brownian motion B such that

$$M_t = \int_0^t \Gamma_s dB_s,$$

where Γ is the matrix such that $A=\Gamma\Gamma^{\mathrm{T}}$. □

In particular, we notice

$$dM_t^0 = \frac{1}{2}dB_t^0,\quad dM_t^i = \sum_{k=1}^{d}\Gamma^{ik}(Z_t)dB_t^k,\ i=1,\cdots,d.$$

Provided that Z_t has a drift coefficient vector

$$b(t,Z_t) = (b^0(t,Z_t),\cdots,b^d(t,Z_t))^{\mathrm{T}},$$

we observe by comparing both drift terms that

$$\tilde{b}^i(t,z) = \sum_{k=0}^{d} \frac{\partial \Phi^i}{\partial z^k}(z^0,z')b^k(t,z) + \frac{1}{2}\sum_{k,l=0}^{d} \frac{\partial^2 \Phi^i}{\partial z^k \partial z^l}(z^0,z')a^{kl}(z),$$

so it follows that in terms of matrix,

$$b = D^{-1}(\tilde{b} - c),$$

where

$$c^i = \frac{1}{2}\sum_{k,l=0}^{d} \frac{\partial^2 \Phi^i}{\partial z^k \partial z^l}(D^{-1}\tilde{A}(D^{\mathrm{T}})^{-1})^{kl}.$$

After letting $X_t = 2Z_t$, from the last proposition we can achieve a standardized representation (X_t, Z'_t):

$$dX_t = 2dZ_t = dB_t^0 + b_0(t, X_t, Z'_t)dt$$

and

$$dZ'_t = \sigma_1(X_t, Z'_t)dB'_t + b_1(t, X_t, Z'_t)dt,$$

where

$$\begin{aligned}
b_0(t,x,z') &= 2b^0(t, \frac{1}{2}x, z'), \\
\sigma_1(x,z') &= \left(\Gamma^{ij}(\frac{1}{2}x, z')\right)_{i,j=1,\cdots,d}, \\
b_1(t,x,z') &= \left(b^1(t, \frac{1}{2}x, z'), \cdots, b^d(t, \frac{1}{2}x, z')\right)^{\mathrm{T}}
\end{aligned}$$

and $B' = (B^1, \cdots, B^d)^{\mathrm{T}}$. Moreover we can observe that $\tilde{Z}_t^0 = 0$ is equivalent to $X_t = 0$.

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