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Umbilical Points of the Graphs of Homogeneous Polynomials of Degree 3

By Naoya ANDO

Abstract. Let P_o^3 be the set of the homogeneous polynomials of degree 3 such that on their graphs, the origin $o := (0,0,0)$ of $\mathbf{R}^3$ is isolated as an umbilical point, and $P_o^{3,1/2}, P_o^{3,-1/2}$ the sets of the elements of P_o^3 such that on their graphs, the index of o is equal to $1/2, -1/2$, respectively. In this paper, it is seen that P_o^3 is divided into $P_o^{3,1/2}$ and $P_o^{3,-1/2}$ by the cone obtained from a rectangular torus in the vector space of the homogeneous polynomials of degree 3.

1. Introduction

Let P^3 be the set of the homogeneous polynomials in two real variables x, y of degree 3. The set P^3 may be considered as a 4-dimensional vector space. For two elements $f, g \in P^3$, set

$$\langle f, g\rangle := \frac{1}{\pi}\int_0^{2\pi} \tilde{f}(\theta)\tilde{g}(\theta)d\theta,$$

where

$$\tilde{f}(\theta) := f(\cos\theta, \sin\theta), \qquad \tilde{g}(\theta) := g(\cos\theta, \sin\theta). \tag{1.1}$$

It is seen that $\langle\ ,\ \rangle$ is an inner product on P^3. Let H^1, H^3 be the sets of the spherical harmonic functions in two real variables of degree $1, 3$, respectively. Then H^3 and the set

$$(x^2+y^2)H^1 := \{(x^2+y^2)h_1\ ;\ h_1 \in H^1\}$$

may be considered as two-dimensional subspaces of P^3. For $n = 1, 3$, let h_{nc}, h_{ns} be the homogeneous polynomials of degree 3 defined by

$$h_{nc}(x,y) := (x^2+y^2)^{\frac{3-n}{2}}\mathrm{Re}(z^n), \qquad h_{ns}(x,y) := (x^2+y^2)^{\frac{3-n}{2}}\mathrm{Im}(z^n),$$

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where $z = x + \sqrt{-1}y$. Then for $n = 1, 3$, it is seen that $\{h_{nc}, h_{ns}\}$ is an orthonormal base of $(x^2+y^2)^{\frac{3-n}{2}}H^n$ with the inner product $\langle\ ,\ \rangle$, and for $h_1 \in (x^2+y^2)H^1$ and for $h_3 \in H^3$, $\langle h_1, h_3\rangle = 0$ holds. Therefore $\{h_{3c}, h_{3s}, h_{1c}, h_{1s}\}$ is an orthonormal base of P^3. Set

$$\mathsf{S}^3 := \{\, X_1h_{3c} + X_2h_{3s} + X_3h_{1c} + X_4h_{1s} \\ ;\ X_1^2 + X_2^2 + X_3^2 + X_4^2 = 1\,\}.$$

Let P_o^3 be the set of the homogeneous polynomials of degree 3 such that on their graphs, the origin $o := (0, 0, 0)$ of $\mathbf{R}^3$ is isolated as an umbilical point, and $P_o^{3,1/2}, P_o^{3,-1/2}$ the sets of the elements of P_o^3 such that on their graphs, the index of o(see [3, pp. 137]) is equal to $1/2, -1/2$, respectively. Then the following hold([1]):

$$P_o^{3,1/2}, P_o^{3,-1/2} \neq \emptyset, \qquad P_o^3 = P_o^{3,1/2} \sqcup P_o^{3,-1/2}.$$

In this paper, the following theorem is proved.

THEOREM 1.1. *The following hold*:

(1) $P_o^{3,1/2} \cap \mathsf{S}^3 = \{\sum_{i=1}^4 X_i^2 = 1,\ 0 \leqq X_1^2 + X_2^2 < 1/10\}$;

(2) $P_o^{3,-1/2} \cap \mathsf{S}^3 = \{\sum_{i=1}^4 X_i^2 = 1,\ 1/10 \leqq X_1^2 + X_2^2 \leqq 1\} \setminus C_{1/\sqrt{10},3/\sqrt{10}}$, *where*

$$C_{1/\sqrt{10},3/\sqrt{10}} := \left\{X_1 = \frac{1}{\sqrt{10}}\cos 3\rho,\ \ X_2 = \frac{1}{\sqrt{10}}\sin 3\rho,\right. \\ \left. X_3 = \frac{3}{\sqrt{10}}\cos\rho, \quad X_4 = \frac{3}{\sqrt{10}}\sin\rho, \quad \rho \in [0, 2\pi)\right\}.$$

Roughly speaking, *the set P_o^3 is divided into $P_o^{3,1/2}$ and $P_o^{3,-1/2}$ by the cone obtained from the rectangular torus*

$$\{X_1^2 + X_2^2 = 1/10,\ X_3^2 + X_4^2 = 9/10\}$$

in S^3.

REMARK 1.2. The set $P^3 \setminus P_o^3$ is represented as

$$P^3 \setminus P_o^3 = \{\, c\{(\cos\rho)x + (\sin\rho)y\}^3\ ;\ c \in \mathbf{R},\ \rho \in [0, \pi)\,\}$$

(see [1]). The following holds:

$$\begin{aligned}\{(\cos\rho)x+(\sin\rho)y\}^3 &= \frac{\cos 3\rho}{4}h_{3c}(x,y)+\frac{\sin 3\rho}{4}h_{3s}(x,y)\\ &\quad+\frac{3\cos\rho}{4}h_{1c}(x,y)+\frac{3\sin\rho}{4}h_{1s}(x,y).\end{aligned}$$

Therefore it is seen that $\mathsf{S}^3 \cap (P^3 \setminus P_o^3)$ is the simply closed curve $C_{1/\sqrt{10},3/\sqrt{10}}$ in (2) of Theorem 1.1. Then it follows that one of (1) and (2) in Theorem 1.1 implies the other.

This paper is organized as follows. Although Theorem 1.1 has already been stated, our main theorem in this paper is stated in Section 2.. Theorem 1.1 is a corollary of our main theorem, Theorem 2.1. After notation and terms are prepared in Section 3., Theorem 2.1 is proved in Section 4..

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2. Main Theorem

Let P_o^k be the set of the homogeneous polynomials of degree $k \geqq 3$ such that on their graphs, the origin o of $\mathbf{R}^3$ is isolated as an umbilical point, and f an element of P_o^k and $\tilde{f}$ the function on $\mathbf{R}$ obtained from f as in (1.1). A real number at which $\dfrac{d\tilde{f}}{d\theta} = 0$ is called *a root of* f and the set of the roots of f is represented by R_f. For each root $\theta_0 \in R_f$, the straight line

$$L(\theta_0) := \{\ (x,y) \in \mathbf{R}^2\ ;\ x\sin\theta_0 - y\cos\theta_0 = 0\ \}$$

on $\mathbf{R}^2$ through o is called *a root line of* f. The natural coordinates (x,y) on the xy-plane may be considered as coordinates on the graph G_f of f. Then a root line is considered not only as a subset of $\mathbf{R}^2$ but also as a subset of G_f. The set of the root lines of f is represented by $\tilde{R}_f$. If $\dfrac{d\tilde{f}}{d\theta} \not\equiv 0$, then $\sharp\tilde{R}_f \leqq k$ holds. Let $\mathrm{Umb}(\mathsf{G}_f)$ be the set of the umbilical points of G_f, and for each $L \in \tilde{R}_f$, let $\mathrm{Umb}(\mathsf{G}_f; L)$ be the set of the umbilical points of G_f on $L \setminus \{o\}$. Then the following hold([2]):

(1) $\mathrm{Umb}(\mathsf{G}_f) = \{o\} \sqcup \bigsqcup_{L\in\tilde{R}_f} \mathrm{Umb}(\mathsf{G}_f; L)$;

(2) $\sharp\,\mathrm{Umb}(\mathsf{G}_f; L) = 0$ or 2.

Therefore if $\dfrac{d\tilde{f}}{d\theta} \not\equiv 0$, then $\sharp\,\mathrm{Umb}(\mathsf{G}_f) \in \{2i+1\}_{i=0}^{k}$ holds. Particularly, if $k = 3$, then $\sharp\,\mathrm{Umb}(\mathsf{G}_f) = 1, 3, 5$ or 7 holds.

Let S^3 be as in Section 1.. Then S^3 may be considered as a Riemannian manifold isometric to a unit 3-sphere $\{x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$ in $\mathbf{R}^4$. Let $\mathsf{S}^{3,1}$, $\mathsf{S}^{3,3}$ be the subsets of S^3 defined by

$$\mathsf{S}^{3,1} := \mathsf{S}^3 \cap (x^2 + y^2) H^1, \qquad \mathsf{S}^{3,3} := \mathsf{S}^3 \cap H^3.$$

It is seen that $\mathsf{S}^{3,1}$ and $\mathsf{S}^{3,3}$ are great circles of S^3 with $\mathsf{S}^{3,1} \cap \mathsf{S}^{3,3} = \emptyset$. For a pair $(h_1, h_3) \in \mathsf{S}^{3,1} \times \mathsf{S}^{3,3}$, there exists the only minimal geodesic segment $\gamma_{h_1 h_3}$ in S^3 the ends of which are h_1 and h_3. Each element of $\gamma_{h_1 h_3}$ is represented as

$$f_{\varphi, h_1, h_3} := (\cos\varphi) h_1 + (\sin\varphi) h_3, \qquad \varphi \in [0, \pi/2]. \tag{2.1}$$

Then it is seen that for any element $f \in \mathsf{S}^3$, there exists a triplet $(\varphi, h_1, h_3) \in [0, \pi/2] \times \mathsf{S}^{3,1} \times \mathsf{S}^{3,3}$ satisfying $f = f_{\varphi, h_1, h_3}$.

The purpose of this paper is to describe the relation among the two numbers $\sharp\tilde{R}_{f_{\varphi,h_1,h_3}}$ and $\sharp\,\mathrm{Umb}\,(\mathsf{G}_{f_{\varphi,h_1,h_3}})$, and the index $\mathrm{ind}_o\,(\mathsf{G}_{f_{\varphi,h_1,h_3}})$ of o for a triplet $(\varphi, h_1, h_3) \in [0, \pi/2] \times \mathsf{S}^{3,1} \times \mathsf{S}^{3,3}$ satisfying $f_{\varphi,h_1,h_3} \in P_o^3$. Our main theorem in this paper is the following.

THEOREM 2.1. *For a pair* $(h_1, h_3) \in \mathsf{S}^{3,1} \times \mathsf{S}^{3,3}$, *a case of just one of the following types happens*:

Type 1(*General Type*).

There exists the number $\varphi_1 \in (0, \arctan 1/3)$ *satisfying*

$$\begin{aligned}
&(\sharp\tilde{R}_{f_{\varphi,h_1,h_3}},\ \sharp\,\mathrm{Umb}\,(\mathsf{G}_{f_{\varphi,h_1,h_3}}),\ \mathrm{ind}_o\,(\mathsf{G}_{f_{\varphi,h_1,h_3}}))\\
&\qquad = \begin{cases} (1,\ 3,\ 1/2) & \textit{for } \varphi \in [0, \varphi_1), \\ (2,\ 5,\ 1/2) & \textit{for } \varphi = \varphi_1, \\ (3,\ 7,\ 1/2) & \textit{for } \varphi \in (\varphi_1, \arctan 1/3), \\ (3,\ 1,\ -1/2) & \textit{for } \varphi \in [\arctan 1/3, \pi/2]. \end{cases}
\end{aligned}$$

Type 2(*Exceptional Type*).

There exists the number $\varphi_2 \in (0, \arctan 1/3)$ *satisfying*

$$(\sharp \tilde{R}_{f_{\varphi,h_1,h_3}},\ \sharp\,\mathrm{Umb}\,(\mathsf{G}_{f_{\varphi,h_1,h_3}}),\ \mathrm{ind}_o\,(\mathsf{G}_{f_{\varphi,h_1,h_3}}))$$
$$= \begin{cases} (1,\ 3,\ 1/2) & \text{for } \varphi \in [0, \varphi_2], \\ (3,\ 7,\ 1/2) & \text{for } \varphi \in (\varphi_2, \arctan 1/3), \\ (3,\ 1,\ -1/2) & \text{for } \varphi \in [\arctan 1/3, \pi/2]. \end{cases}$$

Type 3(Singular Type).

An element $f_{\arctan 1/3, h_1, h_3}$ *does not belong to* P_o^3, *and the following holds*:

$$(\sharp \tilde{R}_{f_{\varphi,h_1,h_3}},\ \sharp\,\mathrm{Umb}\,(\mathsf{G}_{f_{\varphi,h_1,h_3}}),\ \mathrm{ind}_o\,(\mathsf{G}_{f_{\varphi,h_1,h_3}}))$$
$$= \begin{cases} (1,\ 3,\ 1/2) & \text{for } \varphi \in [0, \arctan 1/3), \\ (3,\ 1,\ -1/2) & \text{for } \varphi \in (\arctan 1/3, \pi/2]. \end{cases}$$

REMARK 2.2. Theorem 2.1 says

$$\sharp\,\mathrm{Umb}(\mathsf{G}_f) = \begin{cases} 1, & \text{if } f \in P_o^{3,-1/2}, \\ 3, 5 \text{ or } 7, & \text{if } f \in P_o^{3,1/2}, \end{cases}$$

and that the number $\sharp\,\mathrm{Umb}(\mathsf{G}_f)$ for $f \in P_o^{3,1/2}$ may actually be equal to each of the integers $3, 5, 7$. Moreover Theorem 2.1 says that for any $L \in \tilde{R}_f$, the following holds:

$$\sharp\,\mathrm{Umb}(\mathsf{G}_f; L) = \begin{cases} 0, & \text{if } f \in P_o^{3,-1/2}, \\ 2, & \text{if } f \in P_o^{3,1/2}. \end{cases}$$

3. Preliminaries

Let f be an element of P_o^k and r a positive constant such that on $0 < x^2 + y^2 \leqq r^2$, there exists no umbilical point of G_f, and r_0 the supremum of such numbers as r. For any $\theta_0 \in R_f$, there exists the continuous function $\phi_{r,\theta_0,\theta_0}$ on $\mathbf{R}$ such that

(1) $\phi_{r,\theta_0,\theta_0}(\theta_0) = \theta_0$,

(2) for any $\theta \in \mathbf{R}$, $\cos\phi_{r,\theta_0,\theta_0}(\theta)\dfrac{\partial}{\partial x} + \sin\phi_{r,\theta_0,\theta_0}(\theta)\dfrac{\partial}{\partial y}$ is in the principal directions at $(r\cos\theta, r\sin\theta)$

(see [1]). It is said that *the sign of a root* θ_0 *is positive*(*resp. negative*) if there exists a positive number $\varepsilon > 0$ such that for any $r \in (0, r_0)$ and for any $\theta \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon) \setminus \{\theta_0\}$,

$$\{\theta - \phi_{r,\theta_0,\theta_0}(\theta)\}(\theta - \theta_0) > 0(\text{resp.} < 0)$$

holds. A root θ_0 is said to be *related*(*resp. non-related*) if the sign of θ_0 is either positive or negative(resp. neither positive nor negative). If $\theta_0 \in R_f$ is related, then the sign of θ_0 is denoted by $\mathrm{sign}(\theta_0)$.

Suppose $\dfrac{d\tilde{f}}{d\theta} \not\equiv 0$. Then for a root $\theta_0 \in R_f$, there exists a positive integer m satisfying $\dfrac{d^{m+1}\tilde{f}}{d\theta^{m+1}}(\theta_0) \neq 0$. The minimum of such integers as m is called *the multiplicity of* θ_0 and denoted by $\mu(\theta_0)$. A root $\theta_0 \in R_f$ is related(resp. non-related) if and only if $\mu(\theta_0)$ is an odd(resp. even) integer(see [1]). For a related root θ_0, it is said that *the c-sign of* θ_0 *is positive*(*resp. negative*) if $\tilde{f}(\theta_0)\dfrac{d^{\mu(\theta_0)+1}\tilde{f}}{d\theta^{\mu(\theta_0)+1}}(\theta_0) \leqq 0(\text{resp.} > 0)$, and the c-sign of θ_0 is denoted by $\mathrm{c} - \mathrm{sign}(\theta_0)$. The following hold.

PROPOSITION 3.1 ([1], [2]). *If a related root* θ_0 *satisfies* $\mathrm{c} - \mathrm{sign}(\theta_0) = +$, *then* $\mathrm{sign}(\theta_0) = +$ *holds.*

PROPOSITION 3.2 ([2]). *For a related root* θ_0 *with* $\mathrm{c} - \mathrm{sign}(\theta_0) = -$, $\mathrm{sign}(\theta_0) = +$(*resp.* $= -$) *is equivalent to* $\sharp\,\mathrm{Umb}(\mathsf{G}_f; L(\theta_0)) = 0$(*resp.* $= 2$).

Let N_+(resp. N_-) be the number of the root lines determined by the roots with positive(resp. negative) sign. Then $\mathrm{ind}_o(\mathsf{G}_f)$ is represented as

$$\mathrm{ind}_o(\mathsf{G}_f) = 1 - \frac{N_+ - N_-}{2},$$

and moreover the following holds([1]):

$$N_+ - N_- \in \{k - 2i\}_{i=0}^{[k/2]}.$$

Particularly, if $k = 3$, then we obtain $\mathrm{ind}_o(\mathsf{G}_f) \in \{1/2, -1/2\}$.

We want to construct a map $\mathrm{D.Q}_f$ from R_f to $\tilde{\mathbf{R}} := \mathbf{R} \cup \{\infty\}$. Let θ_0 be a root at which $\tilde{f}(\theta_0) \neq 0$. Then we set

$$\mathrm{D.Q}_f(\theta_0) := \frac{d^2\tilde{f}}{d\theta^2}(\theta_0) \Big/ \tilde{f}(\theta_0).$$

Let θ_0 be a root at which $\tilde{f}(\theta_0) = 0$. Then from $f \in P_o^k$, we obtain $\frac{d^2\tilde{f}}{d\theta^2}(\theta_0) \neq 0$. Therefore we set $\mathrm{D.Q}_f(\theta_0) := \infty$. The value $\mathrm{D.Q}_f(\theta_0)$ is called *the determinant quotient at* θ_0.

PROPOSITION 3.3 ([2]). *For* $\theta_0 \in R_f$, $\mathrm{Umb}(\mathsf{G}_f; L(\theta_0)) \neq \emptyset$ *is equivalent to*

$$\mathrm{D.Q}_f(\theta_0) \in (-k, k(k-2)). \tag{3.1}$$

In addition, if (3.1) *holds, then* $\sharp\,\mathrm{Umb}(\mathsf{G}_f; L(\theta_0)) = 2$ *holds.*

Particularly, if $k = 3$, then we see that $\mathrm{Umb}(\mathsf{G}_f; L(\theta_0)) \neq \emptyset$ is equivalent to $\mathrm{D.Q}_f(\theta_0) \in (-3, 3)$.

For $(h_1, h_3) \in \mathsf{S}^{3,1} \times \mathsf{S}^{3,3}$, there exists a pair of numbers $(\alpha, \beta) \in [0, 2\pi/3) \times [0, 2\pi)$ satisfying

$$h_3 = (\sin 3\alpha)h_{3c} + (\cos 3\alpha)h_{3s}, \qquad h_1 = (\sin\beta)h_{1c} + (\cos\beta)h_{1s}.$$

Noticing $\tilde{h}_{nc}(\theta) = \cos n\theta$, $\tilde{h}_{ns}(\theta) = \sin n\theta$ for $n = 1, 3$, we represent $\tilde{f}_{\varphi,h_1,h_3}$ as

$$\tilde{f}_{\varphi,h_1,h_3}(\theta) = \sin\varphi \sin 3(\theta + \alpha) + \cos\varphi \sin(\theta + \beta).$$

Choosing suitable orthogonal coordinates on $\mathbf{R}^2$, we may represent $\tilde{f}_{\varphi,h_1,h_3}$ as

$$\tilde{f}_{\varphi,h_1,h_3}(\theta) = \sin\varphi \sin 3(\theta + \alpha) + \cos\varphi \sin\theta, \tag{3.2}$$

where $\alpha \in [0, \pi/3]$. From now on, by $f_{\varphi,\alpha}$ we denote f_{φ,h_1,h_3}. In addition, by $\mathrm{D.Q}_{\varphi,\alpha}$(resp. $R_{\varphi,\alpha}, \mathsf{G}_{\varphi,\alpha}$) we denote $\mathrm{D.Q}_{f_{\varphi,\alpha}}$(resp. $R_{f_{\varphi,\alpha}}, \mathsf{G}_{f_{\varphi,\alpha}}$). The following hold:

$$\frac{d\tilde{f}_{\varphi,\alpha}}{d\theta}(\theta) = 3\sin\varphi \cos 3(\theta + \alpha) + \cos\varphi \cos\theta, \tag{3.3}$$

$$\frac{d^2\tilde{f}_{\varphi,\alpha}}{d\theta^2}(\theta) = -9\sin\varphi \sin 3(\theta + \alpha) - \cos\varphi \sin\theta. \tag{3.4}$$

For $a \in \mathbf{R}$, $b \in \mathbf{R}\backslash\{0\}$ and for $\infty \in \tilde{\mathbf{R}}$, we promise that the following hold:

$$a + \infty = \infty + a = \infty,$$
$$a/\infty = 0, \quad \infty/a = b \times \infty = b/0 = \infty.$$

Of course we respect addition and multiplication in $\mathbf{R}$. Then by (3.2) and by (3.4), we may represent $\mathrm{D.Q}_{\varphi,\alpha}(\theta_0)$ at $\theta_0 \in R_{\varphi,\alpha}$ as

$$\mathrm{D.Q}_{\varphi,\alpha}(\theta_0) = -1 - \frac{8}{1 + \cot\varphi \dfrac{\sin\theta_0}{\sin 3(\theta_0+\alpha)}}. \tag{3.5}$$

For $x_1, x_2 \in \tilde{\mathbf{R}}$ with $x_1 \neq x_2$, we define *the generalized closed interval* $[x_1, x_2]$ as follows:

(1) If $x_1, x_2 \in \mathbf{R}$ with $x_1 < x_2$, then $[x_1, x_2]$ represents the usual closed interval determined by x_1, x_2;

(2) If $x_1, x_2 \in \mathbf{R}$ with $x_1 > x_2$, then $[x_1, x_2]$ represents the set $\tilde{\mathbf{R}} \setminus (x_2, x_1)$, where (x_2, x_1) is the usual open interval determined by x_2, x_1;

(3) If $x_1 \in \mathbf{R}$ and if $x_2 = \infty$, then $[x_1, x_2]$ represents the set $\{x \geqq x_1\} \cup \{\infty\}$;

(4) If $x_1 = \infty$ and if $x_2 \in \mathbf{R}$, then $[x_1, x_2]$ represents the set $\{x \leqq x_2\} \cup \{\infty\}$.

We similarly define *the generalized open interval* (x_1, x_2) and *the generalized semi-closed intervals* $[x_1, x_2), (x_1, x_2]$ for $x_1, x_2 \in \tilde{\mathbf{R}}$ with $x_1 \neq x_2$. We introduce into each generalized interval I the total order $\preceq$ as follows:

Total Order $\preceq$.

If $\infty \notin I$, then the total order $\preceq$ corresponds with the usual total order $\leqq$ in $\mathbf{R}$. Suppose $\infty \in I$. Then for any $c \in \mathbf{R} \setminus I$,

(1) if $I \cap \{x > c\} \neq \emptyset$, then for $y_1, y_2 \in I$ with $c < y_1 \leqq y_2$, the following holds:

$$y_1 \preceq y_2 \preceq \infty;$$

(2) if $I \cap \{x < c\} \neq \emptyset$, then for $y_3, y_4 \in I$ with $y_3 \leqq y_4 < c$, the following holds:

$$\infty \preceq y_3 \preceq y_4.$$

Let Φ be a map between two generalized intervals. Then Φ is said to be *increasing*(*resp. decreasing*) if Φ preserves(resp. reverses) the order relation.

4. Proof of Theorem 2.1

Type A. $\alpha = 0$.

We obtain $f_{\varphi,0} \in P_o^3$ for any $\varphi \in [0, \pi/2]$. We see that

(1) if $\varphi \in [0, \arctan 1/9]$, then $\tilde{R}_{\varphi,0} = \{L(\pi/2)\}$ holds;

(2) if $\varphi \in (\arctan 1/9, \pi/2]$, then the following holds:

$$\tilde{R}_{\varphi,0} = \{L(\pi/2),\ L(\theta_0^{(2)}(\varphi)),\ L(-\theta_0^{(2)}(\varphi))\},$$

where $\theta_0^{(2)}$ is the decreasing map from $(\arctan 1/9, \pi/2]$ onto $[\pi/6, \pi/2)$ satisfying $\dfrac{d\tilde{f}_{\varphi,0}}{d\theta}(\theta_0^{(2)}(\varphi)) = 0$ for any $\varphi \in (\arctan 1/9, \pi/2]$.

By (3.5), we obtain

$$\mathrm{D.Q}_{\varphi,0}(\pi/2) = -1 - \frac{8}{1 - \cot\varphi}.$$

Then we see that the map $\varphi \mapsto \mathrm{D.Q}_{\varphi,0}(\pi/2)$ is increasing from $[0, \pi/2]$ onto the generalized interval $[-1, -9]$ and we obtain

$$\mathrm{D.Q}_{\arctan 1/3,0}(\pi/2) = 3.$$

Therefore by Proposition 3.3, we obtain

$$\sharp\, \mathrm{Umb}(\mathsf{G}_{\varphi,0}; L(\pi/2)) = \begin{cases} 2 & \text{for } \varphi \in [0, \arctan 1/3), \\ 0 & \text{for } \varphi \in [\arctan 1/3, \pi/2]. \end{cases}$$

By Proposition 3.1 and by Proposition 3.2, we obtain

$$\mathrm{sign}(\pi/2) = \begin{cases} + & \text{for } \varphi \in [0, \arctan 1/9] \cup [\arctan 1/3, \pi/2], \\ - & \text{for } \varphi \in (\arctan 1/9, \arctan 1/3). \end{cases}$$

The determinant quotients at $\pm\theta_0^{(2)}(\varphi)$ for $\varphi \in (\arctan 1/9, \pi/2]$ are represented as

$$\mathrm{D.Q}_{\varphi,0}(\pm\theta_0^{(2)}(\varphi)) = -1 - \frac{8}{1 + d_0^{(2)}(\varphi)}, \tag{4.1}$$

where

$$d_0^{(2)}(\varphi) := \frac{\cot\varphi}{4\cos^2\theta_0^{(2)}(\varphi) - 1}. \tag{4.2}$$

From $\dfrac{d\tilde{f}_{\varphi,0}}{d\theta}(\theta_0^{(2)}(\varphi)) = 0$ and from (4.2), we obtain

$$\tan\varphi = -\frac{1}{3(4\cos^2\theta_0^{(2)}(\varphi) - 3)} \tag{4.3}$$

$$= \frac{1}{d_0^{(2)}(\varphi)(4\cos^2\theta_0^{(2)}(\varphi) - 1)}. \tag{4.4}$$

Therefore we obtain

$$d_0^{(2)}(\varphi) = -3\frac{4\cos^2\theta_0^{(2)}(\varphi) - 3}{4\cos^2\theta_0^{(2)}(\varphi) - 1}. \tag{4.5}$$

Then it is seen that $d_0^{(2)}$ is decreasing from $(\arctan 1/9, \pi/2]$ onto the generalized interval $[0, -9)$. Therefore we see from (4.1) that the map $\varphi \mapsto \mathrm{D.Q}_{\varphi,0}(\theta_0^{(2)}(\varphi))$ is decreasing from $(\arctan 1/9, \pi/2]$ onto $[-9, 0)$. From (4.1), (4.3) and from (4.5), we obtain

$$\mathrm{D.Q}_{\arctan 1/3,0}(\pm\theta_0^{(2)}(\arctan 1/3)) = -3.$$

Therefore we obtain

$$\sharp\,\mathrm{Umb}\,(\mathsf{G}_{\varphi,0}; L(\pm\theta_0^{(2)}(\varphi))) = \begin{cases} 2 & \text{for } \varphi \in (\arctan 1/9, \arctan 1/3), \\ 0 & \text{for } \varphi \in [\arctan 1/3, \pi/2]. \end{cases}$$

Since $\mathrm{D.Q}_{\varphi,0}(\pm\theta_0^{(2)}(\varphi)) < 0$, we obtain

$$\mathrm{sign}\,(\theta_0^{(2)}(\varphi)) = \mathrm{sign}\,(-\theta_0^{(2)}(\varphi)) = +.$$

We see that the type $\alpha = 0$ corresponds to Type 2 in Theorem 2.1 and that φ_2 in Type 2 is equal to $\arctan 1/9$.

Type B. $\alpha = \pi/3$.

It is seen that $f_{\arctan 1/3,\pi/3}(x, y)$ is equal to $\dfrac{2}{5}\sqrt{10}y^3$, which is not an element of P_o^3. If $\varphi \in [0, \pi/2] \setminus \{\arctan 1/3\}$, then $f_{\varphi,\pi/3} \in P_o^3$ holds. We see that

(1) if $\varphi \in [0, \arctan 1/3)$, then $\tilde{R}_{\varphi,\pi/3} = \{L(\pi/2)\}$ holds;

(2) if $\varphi \in (\arctan 1/3, \pi/2]$, then the following holds:

$$\tilde{R}_{\varphi,\pi/3} = \{L(\pi/2),\ L(\theta^{(2)}_{\pi/3}(\varphi)),\ L(-\theta^{(2)}_{\pi/3}(\varphi))\},$$

where $\theta^{(2)}_{\pi/3}$ is the increasing map from $(\arctan 1/3, \pi/2]$ onto $(0, \pi/6]$ satisfying $\dfrac{d\tilde{f}_{\varphi,\pi/3}}{d\theta}(\theta^{(2)}_{\pi/3}(\varphi)) = 0$ for any $\varphi \in (\arctan 1/3, \pi/2]$.

The determinant quotient at $\pi/2$ is represented as

$$\mathrm{D.Q}_{\varphi,\pi/3}(\pi/2) = -1 - \frac{8}{1 + \cot\varphi}.$$

Then we see that the map $\varphi \mapsto \mathrm{D.Q}_{\varphi,\pi/3}(\pi/2)$ is decreasing from $[0, \pi/2]$ onto $[-9, -1]$ and we obtain

$$\mathrm{D.Q}_{\arctan 1/3,\pi/3}(\pi/2) = -3.$$

Therefore we obtain

$$\sharp\,\mathrm{Umb}(\mathsf{G}_{\varphi,\pi/3}; L(\pi/2)) = \begin{cases} 2 & \text{for } \varphi \in (0, \arctan 1/3), \\ 0 & \text{for } \varphi \in [\arctan 1/3, \pi/2]. \end{cases}$$

Since $\mathrm{D.Q}_{\varphi,\pi/3}(\pi/2) < 0$, we obtain $\mathrm{sign}(\pi/2) = +$.

The determinant quotients at $\pm\theta^{(2)}_{\pi/3}(\varphi)$ for $\varphi \in (\arctan 1/3, \pi/2]$ are represented as

$$\mathrm{D.Q}_{\varphi,\pi/3}(\pm\theta^{(2)}_{\pi/3}(\varphi)) = -1 - \frac{8}{1 - d^{(2)}_{\pi/3}(\varphi)},$$

where

$$d^{(2)}_{\pi/3}(\varphi) := \frac{\cot\varphi}{4\cos^2\theta^{(2)}_{\pi/3}(\varphi) - 1}.$$

As in Type A, we obtain

$$d^{(2)}_{\pi/3}(\varphi) = 3\frac{4\cos^2\theta^{(2)}_{\pi/3}(\varphi) - 3}{4\cos^2\theta^{(2)}_{\pi/3}(\varphi) - 1}.$$

Therefore we see that the map $\varphi \mapsto \mathrm{D.Q}_{\varphi,\pi/3}(\theta^{(2)}_{\pi/3}(\varphi))$ is increasing from $(\arctan 1/3, \pi/2]$ onto $(\infty, -9]$, and this implies

$$\mathrm{sign}\,(\pm\theta^{(2)}_{\pi/3}(\varphi)) = +, \qquad \mathrm{Umb}\,(\mathsf{G}_{\varphi,\pi/3}; L(\pm\theta^{(2)}_{\pi/3}(\varphi))) = \emptyset.$$

We see that the type $\alpha = \pi/3$ corresponds to Type 3 in Theorem 2.1.

Type C. $\alpha \in (0, \pi/3)$.

We see that $f_{\varphi,\alpha} \in P^3_o$ holds for any $\varphi \in [0, \pi/2]$ and for any $\alpha \in (0, \pi/3)$, and that for any $\alpha \in (0, \pi/3)$, there exists the increasing map $\theta^{(1)}_\alpha$ from $[0, \pi/2]$ onto $[\pi/2, 5\pi/6 - \alpha]$ satisfying $\dfrac{d\tilde{f}_{\varphi,\alpha}}{d\theta}(\theta^{(1)}_\alpha(\varphi)) = 0$ for any $\varphi \in [0, \pi/2]$. In addition, we see that for any $\alpha \in (0, \pi/3)$, there exist the number $\varphi(\alpha) \in (0, \pi/2)$ and the number $\theta(\alpha) \in (\pi/3 - \alpha, \pi/2 - \alpha)$ satisfying

(1) $\tilde{R}_{\varphi,\alpha} = \{L(\theta^{(1)}_\alpha(\varphi))\}$ for $\varphi \in [0, \varphi(\alpha))$;

(2) $\tilde{R}_{\varphi(\alpha),\alpha} = \{L(\theta^{(1)}_\alpha(\varphi(\alpha))),\ L(\theta(\alpha))\}$;

(3) $\tilde{R}_{\varphi,\alpha} = \{L(\theta^{(i)}_\alpha(\varphi))\}^3_{i=1}$ for $\varphi \in (\varphi(\alpha), \pi/2]$, where $\theta^{(2)}_\alpha$(resp. $\theta^{(3)}_\alpha$) is the increasing(resp. decreasing) map from $[\varphi(\alpha), \pi/2]$ onto $[\theta(\alpha), \pi/2 - \alpha]$(resp. $[\pi/6 - \alpha, \theta(\alpha)]$) satisfying $\dfrac{d\tilde{f}_{\varphi,\alpha}}{d\theta}(\theta^{(i)}_\alpha(\varphi)) = 0$ for any $\varphi \in [\varphi(\alpha), \pi/2]$ and for $i = 2, 3$.

Proposition 4.1. *A root $\theta(\alpha)$ of $f_{\varphi(\alpha),\alpha}$ is non-related.*

For $\alpha \in (0, \pi/3)$ and for $\varphi \in [0, \pi/2]$, we set

$$\Delta^{(1)}_\alpha(\varphi) := \mathrm{D.Q}_{\varphi,\alpha}(\theta^{(1)}_\alpha(\varphi)).$$

Firstly, we want to study the determinant quotient at $\theta^{(1)}_\alpha(\varphi)$.

Lemma 4.2. *The following hold*:

(1) *For $\alpha \in (0, \pi/6)$,*

  (a) *for any $\varphi \in [0, (\theta^{(1)}_\alpha)^{-1}(2\pi/3 - \alpha)]$, $\Delta^{(1)}_\alpha(\varphi) \in [-1, 0)$ holds,*

  (b) *$\Delta^{(1)}_\alpha$ is decreasing from $[(\theta^{(1)}_\alpha)^{-1}(2\pi/3 - \alpha), \pi/2]$ onto $[-9, -1]$;*

(2) *For* $\alpha \in [\pi/6, \pi/3)$, $\Delta_\alpha^{(1)}$ *is decreasing from* $[0, \pi/2]$ *onto* $[-9, -1]$.

Proof. For $\alpha \in (0, \pi/3)$ and for $\varphi \in [0, \pi/2]$, we set

$$d_\alpha^{(1)}(\varphi) := \cot\varphi \frac{\sin\theta_\alpha^{(1)}(\varphi)}{\sin 3(\theta_\alpha^{(1)}(\varphi) + \alpha)}.$$

The following holds:

$$\Delta_\alpha^{(1)}(\varphi) = -1 - \frac{8}{1 + d_\alpha^{(1)}(\varphi)}.$$

If $\alpha \in [\pi/6, \pi/3)$, then $d_\alpha^{(1)}$ is decreasing from $[0, \pi/2]$ onto the generalized interval $[0, \infty]$, and therefore $\Delta_\alpha^{(1)}$ is decreasing from $[0, \pi/2]$ onto $[-9, -1]$. Suppose $\alpha \in (0, \pi/6)$. Then noticing $\dfrac{d^2 \tilde{f}_{\varphi,\alpha}}{d\theta^2}(\theta_\alpha^{(1)}(\varphi)) \neq 0$, we obtain $d_\alpha^{(1)}(\varphi) \in [\infty, -9)$ and $\Delta_\alpha^{(1)}(\varphi) \in [-1, 0)$ for $\varphi \in [0, (\theta_\alpha^{(1)})^{-1}(2\pi/3 - \alpha)]$. It is easily seen that $d_\alpha^{(1)}$ is decreasing from $[(\theta_\alpha^{(1)})^{-1}(2\pi/3 - \alpha), \pi/2]$ onto $[0, \infty]$. Hence we have proved Lemma 4.2. $\square$

From Lemma 4.2, we obtain

Proposition 4.3. *For* $\alpha \in (0, \pi/3)$ *and for* $\varphi \in [0, \pi/2]$, $\operatorname{sign}(\theta_\alpha^{(1)}(\varphi)) = +$ *holds.*

We shall prove

Lemma 4.4. *For any* $\alpha \in (0, \pi/3)$, *the following hold*:
(1) $\varphi(\alpha) < \arctan 1/3$; (2) $\Delta_\alpha^{(1)}(\varphi(\alpha)) \in (-3, 0)$.

Proof. For $\alpha \in (0, \pi/3)$ and for $\varphi \in [\arctan 1/3, \pi/2]$, $\sharp \tilde{R}_{\varphi,\alpha} = 3$ holds. Therefore we obtain $\varphi(\alpha) < \arctan 1/3$. The following hold:

$$\begin{aligned}\left|\frac{\sin\theta_\alpha^{(1)}(\varphi(\alpha))}{\sin 3(\theta_\alpha^{(1)}(\varphi(\alpha)) + \alpha)}\right| &= \frac{\sqrt{1 - \cos^2\theta_\alpha^{(1)}(\varphi(\alpha))}}{\sqrt{1 - \cos^2 3(\theta_\alpha^{(1)}(\varphi(\alpha)) + \alpha)}} \\ &= \frac{\sqrt{1 - 9\tan^2\varphi(\alpha)\cos^2 3(\theta_\alpha^{(1)}(\varphi(\alpha)) + \alpha)}}{\sqrt{1 - \cos^2 3(\theta_\alpha^{(1)}(\varphi(\alpha)) + \alpha)}} \\ &> 1.\end{aligned}$$

Therefore we obtain $|d_\alpha^{(1)}(\varphi(\alpha))| > 3$. Then by Lemma 4.2, we obtain $\Delta_\alpha^{(1)}(\varphi(\alpha)) \in (-3, 0)$. Hence we have proved Lemma 4.4. $\square$

From Lemma 4.2 and from Lemma 4.4, we obtain

Proposition 4.5. *For* $\varphi \in [0, \varphi(\alpha))$, $\sharp\,\mathrm{Umb}\,(\mathsf{G}_{\varphi,\alpha}; L(\theta_\alpha^{(1)}(\varphi))) = 2$ *holds.*

For $\alpha \in (0, \pi/3)$, $\varphi \in [\varphi(\alpha), \pi/2]$ and for $i = 2, 3$, we set

$$\Delta_\alpha^{(i)}(\varphi) := \mathrm{D.Q}_{\varphi,\alpha}(\theta_\alpha^{(i)}(\varphi)).$$

We need to study the determinant quotient at a root $\theta_\alpha^{(i)}(\varphi)(i = 2, 3)$.

Lemma 4.6. *The following hold*:

(1) *The map* $\Delta_\alpha^{(2)}$ *is increasing from* $[\varphi(\alpha), \pi/2]$ *onto the generalized interval* $[0, -9]$;

(2) *For* $\alpha \in (0, \pi/6]$, $\Delta_\alpha^{(3)}$ *is decreasing from* $[\varphi(\alpha), \pi/2]$ *onto* $[-9, 0]$;

(3) *For* $\alpha \in (\pi/6, \pi/3)$,

  (a) $\Delta_\alpha^{(3)}$ *is decreasing from* $[\varphi(\alpha), (\theta_\alpha^{(3)})^{-1}(0)]$ *onto* $[-9, 0]$,

  (b) $\Delta_\alpha^{(3)}(\varphi) \in (3, -9]$ *holds for* $\varphi \in ((\theta_\alpha^{(3)})^{-1}(0), \pi/2]$.

Proof. For $\alpha \in (0, \pi/3)$, $\varphi \in [\varphi(\alpha), \pi/2]$ and for $i = 2, 3$, we set

$$d_\alpha^{(i)}(\varphi) := \cot\varphi \frac{\sin\theta_\alpha^{(i)}(\varphi)}{\sin 3(\theta_\alpha^{(i)}(\varphi) + \alpha)}.$$

Then by $\dfrac{d\tilde{f}_{\varphi,\alpha}}{d\theta}(\theta_\alpha^{(i)}(\varphi)) = 0$, we obtain

$$d_\alpha^{(i)}(\varphi) = -\frac{\tan\theta_\alpha^{(i)}(\varphi)}{\tan 3(\theta_\alpha^{(i)}(\varphi) + \alpha)}.$$

We set

$$\delta_\alpha(\theta) = -\frac{\tan\theta}{\tan 3(\theta + \alpha)}.$$

We see that δ_α is increasing from $[\pi/3-\alpha, \pi/2-\alpha]$ onto $[\infty, 0]$. Noticing $d_\alpha^{(i)}(\varphi(\alpha)) = -9$ for $i = 2, 3$, we see that $d_\alpha^{(2)}$ is increasing from $[\varphi(\alpha), \pi/2]$ onto $[-9, 0]$ and that $d_\alpha^{(3)}$ is decreasing from $[\varphi(\alpha), (\theta_\alpha^{(3)})^{-1}(\pi/3-\alpha)]$ onto $[\infty, -9]$. Therefore $\Delta_\alpha^{(2)}$ is increasing from $[\varphi(\alpha), \pi/2]$ onto the generalized interval $[0, -9]$ and $\Delta_\alpha^{(3)}$ is decreasing from $[\varphi(\alpha), (\theta_\alpha^{(3)})^{-1}(\pi/3-\alpha)]$ onto $[-1, 0]$. We have already proved (1) in Lemma 4.6.

If $\alpha \in (0, \pi/6]$, then δ_α is increasing from $[\pi/6-\alpha, \pi/3-\alpha]$ onto $[0, \infty]$. Therefore $d_\alpha^{(3)}$ is decreasing from $[(\theta_\alpha^{(3)})^{-1}(\pi/3-\alpha), \pi/2]$ onto $[0, \infty]$ and $\Delta_\alpha^{(3)}$ is decreasing from $[(\theta_\alpha^{(3)})^{-1}(\pi/3-\alpha), \pi/2]$ onto $[-9, -1]$. We have already proved (2) in Lemma 4.6.

If $\alpha \in (\pi/6, \pi/3)$, then δ_α is increasing from $[0, \pi/3-\alpha]$ onto $[0, \infty]$. Therefore $d_\alpha^{(3)}$ is decreasing from $[(\theta_\alpha^{(3)})^{-1}(\pi/3-\alpha), (\theta_\alpha^{(3)})^{-1}(0)]$ onto $[0, \infty]$ and $\Delta_\alpha^{(3)}$ is decreasing from $[(\theta_\alpha^{(3)})^{-1}(\pi/3-\alpha), (\theta_\alpha^{(3)})^{-1}(0)]$ onto $[-9, -1]$. There exists the negative number $c_\alpha^{(3)} < 0$ such that δ_α is a map from $[\pi/6-\alpha, 0]$ onto $[c_\alpha^{(3)}, 0]$. Suppose $c_\alpha^{(3)} \leqq -3$. Then there exists a number $\theta_0 \in (\pi/6-\alpha, 0)$ satisfying

$$\tan\theta_0 = 3\tan 3(\theta_0+\alpha). \tag{4.6}$$

If we set

$$F_1(\theta) := \tan\theta, \qquad F_2(\theta) := 3\tan 3(\theta+\alpha),$$

then we obtain

$$F_1(0) = 0, \qquad F_2(0) = 3\tan 3\alpha, \tag{4.7}$$

$$F_1'(\theta) = \frac{1}{\cos^2\theta}, \qquad F_2'(\theta) = \frac{9}{\cos^2 3(\theta+\alpha)}. \tag{4.8}$$

From (4.8), we obtain

$$F_1'(\theta) < F_2'(\theta) \tag{4.9}$$

for $\theta \in (\pi/6-\alpha, 0]$. From (4.7) and from (4.9), we obtain $F_1(\theta) > F_2(\theta)$ for any $\theta \in (\pi/6-\alpha, 0]$, which contradicts (4.6). Therefore we obtain $c_\alpha^{(3)} > -3$, $d_\alpha^{(3)}(\varphi) \in (-3, 0]$ and $\Delta_\alpha^{(3)}(\varphi) \in (3, -9]$ for $\varphi \in [(\theta_\alpha^{(3)})^{-1}(0), \pi/2]$. Hence we have proved (3) in Lemma 4.6. $\square$

Lemma 4.4 says $\varphi(\alpha) < \arctan 1/3$ for $\alpha \in (0, \pi/3)$. We shall compute the value $\Delta_\alpha^{(i)}(\arctan 1/3)$ for $\alpha \in (0, \pi/3)$. By (3.3), we obtain

$$\cos 3(\theta_\alpha^{(i)}(\arctan 1/3) + \alpha) = -\cos\theta_\alpha^{(i)}(\arctan 1/3).$$

Therefore we obtain

$$\left|\frac{\sin\theta_\alpha^{(i)}(\arctan 1/3)}{\sin 3(\theta_\alpha^{(i)}(\arctan 1/3)+\alpha)}\right| = 1.$$

By (3.5), we obtain $|\Delta_\alpha^{(i)}(\arctan 1/3)| = 3$. Therefore by Lemma 4.2 and by Lemma 4.6, we obtain

LEMMA 4.7. *For any* $\alpha \in (0, \pi/3)$*, the following hold*:

(1) $\Delta_\alpha^{(1)}(\arctan 1/3) = \Delta_\alpha^{(3)}(\arctan 1/3) = -3$;

(2) $\Delta_\alpha^{(2)}(\arctan 1/3) = 3$.

From Lemma 4.2, Lemma 4.6 and from Lemma 4.7, we obtain

PROPOSITION 4.8. *Let* $\varphi(\alpha)$ *be as above.*

(1) *For* $\varphi \in [\varphi(\alpha), \arctan 1/3)$ *and for* $L \in \tilde{R}_{\varphi,\alpha}$, $\sharp\,\mathrm{Umb}(\mathsf{G}_{\varphi,\alpha}; L) = 2$ *holds*;

(2) *For* $\varphi \in [\arctan 1/3, \pi/2]$, $\mathrm{Umb}(\mathsf{G}_{\varphi,\alpha}) = \{o\}$ *holds.*

From Lemma 4.6, Lemma 4.7 and from Proposition 4.8, we obtain

PROPOSITION 4.9. *For any* $\alpha \in (0, \pi/3)$*, the following holds*:

$$(\,\mathrm{sign}\,(\theta_\alpha^{(2)}(\varphi)), \mathrm{sign}\,(\theta_\alpha^{(3)}(\varphi))) = \begin{cases} (+,+) & \textit{for } \varphi \in [\arctan 1/3, \pi/2], \\ (-,+) & \textit{for } \varphi \in (\varphi(\alpha), \arctan 1/3). \end{cases}$$

From Proposition 4.1, Proposition 4.3, Proposition 4.5, Proposition 4.8 and from Proposition 4.9, we see that the type $\alpha \in (0, \pi/3)$ corresponds to Type 1 in Theorem 2.1 and that φ_1 in Type 1 is equal to $\varphi(\alpha)$.

Hence we have proved Theorem 2.1.

References

[1] Ando, N., An isolated umbilical point of the graph of a homogeneous polynomial, Geom. Dedicata **82** (2000), 115–137.

[2] Ando, N., The behavior of the principal distributions around an isolated umbilical point, to appear in J. Math. Soc. Japan.

[3] Hopf, H., Lectures on differential geometry in the large, Notes by J. M. Gray, Stanford Univ., 1954. Reprinted in Lecture Notes in Math., vol. 1000, Springer-Verlag.

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Graduate School of Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914, Japan
E-mail: andou@ms.u-tokyo.ac.jp