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A Note on the Uncertainty Principle for the Dunkl Transform

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Abstract. Analogues of Heisenberg's inequality and Hardy's theorem are studied for the Dunkl transform.

1. Introduction

The uncertainty principle in harmonic analysis says "A nonzero function and its Fourier transform cannot both be sharply localized". There are two typical formulations of the uncertainty principle for the Fourier transform on the real line $\mathbb{R}$, Heisenberg's inequality and Hardy's theorem.

In order to be clear about the normalization we state our definition of the Fourier transform:

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-\sqrt{-1}\lambda x} dx.$$

Heisenberg's inequality states that for $f \in L^2(\mathbb{R})$,

$$\int_{-\infty}^{\infty} x^2 |f(x)|^2 dx \int_{-\infty}^{\infty} \lambda^2 |\hat{f}(\lambda)|^2 d\lambda \geq \frac{1}{4} \left[\int_{-\infty}^{\infty} |f(x)|^2 dx \right]^2$$

with equality only if $f(x)$ is almost everywhere equal to a constant multiple of e^{-px^2} for some $p > 0$. A proof is given in Weyl [13, Appendix 1].

Hardy [6] proved a theorem concerning the decay of f and $\hat{f}$ at infinity; let p and q be positive constants and assume that f is a function on the real line satisfying $|f(x)| \leq Ce^{-px^2}$ and $|\hat{f}(\lambda)| \leq Ce^{-q\lambda^2}$ for some positive constant C. Then (i) $f = 0$ if $pq > 1/4$; (ii) $f = Ae^{-px^2}$ for some constant A if $pq = 1/4$; (iii) there are infinitely many f if $pq < 1/4$.

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Root systems provide a rich framework for the study of harmonic analysis, which give generalizations of the classical Fourier analysis. In [4], Dunkl introduce an integral transform associated with the eigenfunctions of the Dunkl operators for a root system and proved the Plancherel theorem. In [3], de Jeu studied the Dunkl transform by completely different method and proved the inversion formula and the Plancherel theorem. The Dunkl transform has practically important properties of the classical Fourier transform; it is self-dual, it has period 4, and the Gaussian is invariant under the Dunkl transform.

In this paper we study analogues of Heisenberg's inequality and Hardy's theorem for the Dunkl transform. We give an alternative proof of Heisenberg's inequality for the Dunkl transform (Theorem 3.1), which was first proved by Rösler [10]. We prove an analogue of Hardy's theorem (Theorem 4.1) for the Dunkl transform.

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2. The Dunkl Transform

In this section, we review on results of Dunkl [4] and de Jeu [3].

Let $\mathfrak{a} = \mathbb{R}^N$ be a N-dimensional real vector space with inner product $(\cdot,\cdot)$. The norm is denoted by $|x| = (x,x)^{1/2}$. For $\alpha \in \mathfrak{a} \setminus \{0\}$ let r_α denote the orthogonal reflection with respect to the hyperplane orthogonal to α. Let $G \subset O(\mathfrak{a})$ be a finite reflection group. Let R be the corresponding root system. We will assume that R is a normalized root system, i.e. $(\alpha,\alpha) = 2$ for all $\alpha \in R$. Choose and fix a positive system $R_+ \subset R$.

A complex valued function $k : \alpha \mapsto k_\alpha$ on R which is G-invariant is called a multiplicity function. In this article we always assume that $\operatorname{Re} k_\alpha \geq 0$ for all $\alpha \in R$.

Let $\mathfrak{h} = \mathfrak{a} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification. We denote the symmetric inner product on $\mathfrak{h}$ by $(\cdot,\cdot)$ and the norm by $|x| = (x,\bar{x})^{1/2}$. For $\xi \in \mathfrak{h}$ let ∂_ξ denote the corresponding directional derivative. Define the *Dunkl operator* T_ξ by

$$(T_\xi f)(x) = (\partial_\xi f)(x) + \sum_{\alpha \in R_+} k_\alpha(\alpha,\xi)\frac{f(x) - f(r_\alpha x)}{(\alpha,x)}.$$

We have $[T_\xi, T_\eta] = 0$ for any ξ, $\eta \in \mathfrak{h}$. Given $\lambda \in \mathfrak{h}$ consider the following system of differential-difference equations on $\mathfrak{a}$:

$$T_\xi f = (\lambda, \xi) f, \quad \xi \in \mathfrak{a}. \tag{2.1}$$

THEOREM 2.1 (de Jeu [3, Theorem 2.6]). *Assume* $\operatorname{Re} k_\alpha \geq 0$ *for all* $\alpha \in R$. *Then there is a unique solution* $\operatorname{Exp}_G(\lambda, k, \cdot)$ *of* (2.1) *such that*

(i) $\operatorname{Exp}_G(\lambda, k, 0) = 1$,

(ii) $\operatorname{Exp}_G(\lambda, k, x)$ *is holomorphic in* $\lambda \in \mathfrak{h}$ *and analytic in* $x \in \mathfrak{a}$.

Indeed, de Jeu [3] proved stronger results and Opdam [8] studied Exp_G in detail.

Let

$$h(x) = \prod_{\alpha \in R_+} |(\alpha, x)|^{k_\alpha}$$

and

$$\gamma = \sum_{\alpha \in R_+} k_\alpha.$$

Then h is a homogeneous G-invariant function of degree γ. Define the normalization constant

$$c_h = \left((2\pi)^{-N/2} \int_{\mathbb{R}^N} h(x)^2 e^{-|x|^2/2} dx \right)^{-1}.$$

de Jeu [3, Corollary 4.17] proved that the constant c_h is strictly positive (see also [3, Remark 4.12]).

For $f \in L^1(\mathbb{R}^N, h^2 dx)$, let

$$\hat{f}(\lambda) = (2\pi)^{-N/2} c_h \int_{\mathbb{R}^N} f(x) \operatorname{Exp}_G(-\sqrt{-1}\lambda, k, x) h(x)^2 dx,$$

the *Dunkl transform* of f. If $k_\alpha = 0$ for all $\alpha \in R$, then it is nothing but the Fourier transform on $\mathbb{R}^N$.

We recall main results of de Jeu [3], the inversion formula and the Plancherel theorem for the Dunkl transform.

THEOREM 2.2 (de Jeu [3], Theorem 4.20, 4.26).

(1) *Assume* $\operatorname{Re} k_\alpha \geq 0$ *for all* $\alpha \in R$. *Let* $f \in L^1(\mathbb{R}^N, h^2dx)$ *and suppose that* $\hat{f} \in L^1(\mathbb{R}^N, h^2dx)$. *Then*

$$f(x) = (2\pi)^{-N/2} c_h \int_{\mathbb{R}^N} \hat{f}(\lambda) \mathrm{Exp}_G(\sqrt{-1}\lambda, k, x) h(\lambda)^2 d\lambda.$$

(2) *Assume* $k_\alpha \geq 0$ *for all* $\alpha \in R$. *If* $f \in L^1(\mathbb{R}^N, h^2dx) \cap L^2(\mathbb{R}^N, h^2dx)$, *then* $\hat{f} \in L^2(\mathbb{R}^N, h^2dx)$ *and*

$$\int_{\mathbb{R}^N} |f(x)|^2 h(x)^2 dx = \int_{\mathbb{R}^N} |\hat{f}(\lambda)|^2 h(\lambda)^2 d\lambda.$$

Remark 2.3. Let k be a complex number with $\operatorname{Re} k \geq 0$. Define

$$\mathcal{J}_k(x) = \frac{2^{k-\frac{1}{2}}\Gamma(k+\frac{1}{2})}{x^{k-\frac{1}{2}}} J_{k-\frac{1}{2}}(x),$$

where J_α is the Bessel function in the standard notation. If $N = 1$ and $G = \mathbb{Z}_2$, then

$$E_{\mathbb{Z}_2}(\sqrt{-1}\lambda, k, x) = \mathcal{J}_k(\lambda x) + \frac{\sqrt{-1}\lambda x}{2k+1} \mathcal{J}_{k+1}(\lambda x).$$

The $\mathbb{Z}_2$-invariant part of $E_{\mathbb{Z}_2}(\sqrt{-1}\lambda, k, x)$ is $\mathcal{J}_k(x)$, so the Dunkl transoform specialize to even functions is given by

$$\hat{f}(x) = \int_0^\infty f(x) \mathcal{J}_k(x) x^{2k} dx,$$

which coincides with the classical Hankel transform. We refer the reader to Dunkl [4, Section 4] and Koornwinder [7, Section 2] for details.

Remark 2.4. The Dunkl transform sometimes appears "in nature" as the spherical Fourier transform on Riemannian symmetric spaces X of the Euclidean type. Let G be the Weyl group for X and $k = m/4$, where m_α is the root multiplicity for the symmetric space. Then the restriction of the Dunkl transform to G-invariant functions coincides with the spherical Fourier transform on X. We refer the reader to de Jeu [3, Remark 4.27] and Opdam [8, Remark 6.12] for details.

3. Heisenberg's Inequality

In this section we give a proof of Heisenberg's inequality for the Dunkl transform, which was first proved by Rösler [10].

THEOREM 3.1. *Assume that $k_\alpha \geq 0$ for all $\alpha \in R$. For $f \in L^2(\mathbb{R}^N, h^2dx)$,*

$$\int_{\mathbb{R}^N} |x|^2|f(x)|^2h(x)^2dx \int_{\mathbb{R}^N} |\lambda|^2|\hat{f}(\lambda)|^2h(\lambda)^2d\lambda \geq \left(\gamma + \frac{N}{2}\right)^2 \left[\int_{\mathbb{R}^N} |f(x)|^2h(x)^2dx\right]^2$$

with equality only if $f(x)$ is almost everywhere equal to a constant multiple of $e^{-p|x|^2}$ for some $p > 0$.

PROOF. First we review on an orthogonal basis for $L^2(\mathbb{R}^n, h^2dx)$ after Dunkl [4]. Let $\{\xi_1, \dots, \xi_N\}$ be an orthonormal basis of $\mathbb{R}^N$. Define the h-Laplacian $\Delta_h = \sum_{i=1}^N T_{\xi_i}^2$. Let $\mathcal{P}_n$ denote the space of polynomial functions on $\mathbb{R}^N$ that are homogeneous of degree n. Let $\mathcal{H}_n^h = \mathcal{P}_n \cap (\ker \Delta_h)$, the space of h-harmonic polynomials. Let $d\omega$ be the normalized rotation-invariant measure on the unit sphere $S^{N-1} \subset \mathbb{R}^N$. Then we have an orthogonal decomposition $L^2(S, h^2d\omega) = \sum_{n=0}^\infty \oplus \mathcal{H}_n^h$. Let $\{p_j^{(n)}\}_{j\in J_n}$ be an orthonormal basis of $\mathcal{H}_n^h$.

Let m, n be non-negative integers and $j \in J_n$. Define

$$c_{m,n} = \left(\frac{\Gamma(N/2)m!}{\pi^{N/2}\Gamma((N/2) + \gamma + m + n)}\right)^{\frac{1}{2}}$$

and

$$\psi_{m,n,j}(x) = c_{m,n}p_j^{(n)}(x)L_m^{(n+\gamma+N/2-1)}(|x|^2)e^{-|x|^2/2}, \tag{3.1}$$

where $L_m^{(A)}$ denote the Laguerre polynomial in the standard notation. It follows from Proposition 2.4 and Theorem 2.5 of Dunkl [4] that

$$\{\psi_{m,n,j} \,:\, m,\, n = 0, 1, 2, \dots,\, j \in J_n\}$$

forms an orthonormal basis of $L^2(\mathbb{R}^N, h^2dx)$. Moreover by [4, Theorem 2.6] we have

$$\hat{\psi}_{m,n,j} = (-\sqrt{-1})^{n+2m}\psi_{m,n,j}. \tag{3.2}$$

By the recurrence relation for the Laguerre polynomial

$$tL_n^\alpha(t) = -(n+1)L_{n+1}^\alpha(t) + (\alpha+2n+1)L_n^\alpha(t) - (n+\alpha)L_{n-1}^\alpha(t),$$

we have

$$\begin{aligned} |x|^2\psi_{m,n,j}(x) = &-(m+1)\psi_{m+1,n,j}(x) \\ &+(n+\gamma+(N/2)+2m)\psi_{m,n,j}(x) \\ &-(n+\gamma+(N/2)-1+m)\psi_{m-1,n,j}(x). \end{aligned} \tag{3.3}$$

Here we put $\psi_{-1,n,j} = 0$.

Let $\langle\cdot\,,\cdot\rangle$ and $\|\cdot\|$ denote the inner product and the norm of $L^2(\mathbb{R}^N, h^2dx)$ respectively. For $f \in L^2(\mathbb{R}^N, h^2dx)$ define $a_{m,n,j} = \langle f, \psi_{m,n,j}\rangle$. We have

$$f = \sum_{m,n,j} a_{m,n,j}\psi_{m,n,j}, \tag{3.4}$$

$$\hat{f} = \sum_{m,n,j} a_{m,n,j}(-\sqrt{-1})^{n+2m}\psi_{m,n,j}. \tag{3.5}$$

By (3.3), (3.4) and (3.5) we have

$$\begin{aligned} &\||\cdot|f\|^2 + \||\cdot|\hat{f}\|^2 \\ &\quad= \sum_{m,n,j} a_{m,n,j}(\langle|\cdot|^2\psi_{m,n,j}, f\rangle + \langle|\cdot|^2(-\sqrt{-1})^{n+2m})\psi_{m,n,j}, \hat{f}\rangle \\ &\quad= 2\sum_{m,n,j} |a_{m,n,j}|^2(n+\gamma+(N/2)+2m) \\ &\quad\geq 2(\gamma+N/2)||f||^2, \end{aligned}$$

with the equality only if $a_{m,n,j} = 0$ except for $m = n = 0$.

For $f \in L^2(\mathbb{R}^N, h^2 dx)$ and $c > 0$ define $f_c(x) = f(cx)$. Since $h(cx) = c^\gamma h(x)$, it is easy to see that

$$\| |\cdot| f_c \|^2 = c^{N-2\gamma-2} \| |\cdot| f \|^2.$$

By de Jeu [3, Theorem 2.8],

$$\mathrm{Exp}_G(-\sqrt{-1} c^{-1}\lambda, k, cx) = \mathrm{Exp}(-\sqrt{-1}\lambda, k, x).$$

Thus it is easy to see that

$$\hat{f}_c(\lambda) = c^{N-2\gamma} \hat{f}(c^{-1}\lambda), \tag{3.6}$$

and hence

$$\| |\cdot| \hat{f}_c \|^2 = c^{N-2\gamma+2} \| |\cdot| \hat{f} \|^2.$$

Therefore we have

$$c^{-2} \| |\cdot| f \|^2 + c^2 \| |\cdot| \hat{f} \|^2 \geq 2(\gamma + N/2) \|f\|^2.$$

The minimum value of the left hand side as a function of c is $\| |\cdot| f \| \| |\cdot| \hat{f} \|$. □

Remark 3.2. After we wrote a draft of this article, we noticed that Theorem 3.1 was already proved by Rösler [10]. We decided to include our proof of the theorem because it is slightly simpler than that in [10].

Our proof of Theorem 3.1 is indeed similar to that of Rösler [10]. Rösler used expansions in terms of generalized Hermite polynomials and recurrence relations among them, which were given in [9]. We used expansions in terms of the basis given by Dunkl [4] and recurrence relations for the classical Laguerre polynomial. Our proof of Theorem 3.1 is inspired by de Bruijin [2] who proved Heisenberg's inequality for the Fourier transform using the Hermite polynomials (see also [5, Section 3]). Since there is a basis (3.1) of $L^2(\mathbb{R}^N, h^2 dx)$ that satisfy (3.2) and we consider variances for $|\cdot|$ that is a radial function, we could prove the theorem by using the recurrence relation for the Laguerre polynomial instead of that of the Hermite polynomial in [2].

Remark 3.3. In view of Remark 2.3 and 2.4, we obtain Heisenberg's inequality for the classical Hankel transform and the spherical Fourier transform on a Riemannian symmetric space of the Euclidean type as a corollary of Theorem 3.1. The Heisenberg's inequality for the classical Hankel transform was proved by Bowie [1].

4. Hardy's Theorem

We now state and prove an analogue of Hardy's theorem for the Dunkl transform.

Theorem 4.1. *Assume* $\operatorname{Re} k_\alpha \geq 0$ *for all* $\alpha \in R$. *Let* p *and* q *be positive constants. Suppose* f *is a measurable function on* $\mathbb{R}^N$ *satisfying*

$$|f(x)| \leq C \exp(-p|x|^2) \quad x \in \mathbb{R}^N \tag{4.1}$$

and

$$|\hat{f}(\lambda)| \leq C \exp(-q|\lambda|^2) \quad \lambda \in \mathbb{R}^N, \tag{4.2}$$

where C *is a positive constant. Then we have following results:*

(1) *If* $pq > 1/4$, *then* $f = 0$ *almost everywhere.*

(2) *If* $pq = 1/4$, *then* $f(x) = A \exp(-p|x|^2)$, *where* A *is an arbitrary constant.*

(3) *If* $pq < 1/4$, *then there are infinitely many such functions* f.

Proof. The proof follows closely that of Sitaram and Sundari [12], where analogues of Hardy's theorem were proved for certain function spaces on semisimple Lie groups.

We may assume $p = q$ without loss of generality by scaling (3.6). The main part is to prove (2), which states that f is a constant multiple of $\exp(-|x|^2/2)$ if $p = q = 1/2$. Once this is proved, the proof of (1) become self-evident. Also, the functions $\psi_{m,n,j}$ defined by (3.1) give an infinite number of examples for (3).

Assume $p = q = 1/2$. We claim that

$$|\hat{f}(\lambda)| \leq C \exp(|\lambda|^2/2) \quad \text{for all } \lambda \in \mathfrak{h}. \tag{4.3}$$

Let

$$\mathfrak{a}_+ = \{x \in \mathfrak{a} \,:\, (x, \alpha) > 0 \text{ for all } \alpha \in R_+\}.$$

We have

$$\hat{f}(\lambda) = \sum_{g \in G} \int_{\overline{\mathfrak{a}_+}} f(gx) \operatorname{Exp}_G(-\sqrt{-1}\lambda, k; gx) h(x)^2 dx. \tag{4.4}$$

Recall from de Jeu [3, Corollary 3.2] that

$$|\mathrm{Exp}_G(-\sqrt{-1}\lambda, k; x)| \le |G|^{1/2} e^{\max_{g\in G} \mathrm{Im}\,(g\lambda, x)}, \quad x \in \mathbb{R}^N. \tag{4.5}$$

By (4.1), (4.4) and (4.5), we have

$$|\hat{f}(\lambda)| \le C \int_{\overline{\mathfrak{a}_+}} \exp(-|x|^2/2 + (x,\mu)) h(x)^2 dx$$

for some positive constant C, where $\mu \in G\mathrm{Im}\,\lambda \cap \overline{\mathfrak{a}_+}$. Moreover we have

$$\begin{aligned} |\hat{f}(\lambda)| &\le C \int_{\overline{\mathfrak{a}_+}} \exp(-|x|^2/2 + (x,\mu)) h(x)^2 dx \\ &= C \exp(|\mu|^2/2) \int_{-\mu+\overline{\mathfrak{a}_+}} \exp(-|x|^2/2) h(x+\mu)^2 dx \\ &\le C' \exp(|\lambda|^2/2) \end{aligned}$$

for some positive constants C and C', which proves (4.3).

Under condition (4.1), $\hat{f}(\lambda)$ gives an entire function on $\lambda \in \mathfrak{h}$ because of (4.5). Since $\hat{f}$ satisfy estimates (4.3) and (4.2) for $q = 1/2$, it follows from [12, Lemma 2.1] that

$$\hat{f}(\lambda) = A \exp(-|\lambda|^2/2) \tag{4.6}$$

for some constant A. Thus $f(x) = A\exp(-|x|^2/2)$ by (3.2). $\square$

Remark 4.2. In view of Remark 2.3 and 2.4, we obtain Hardy's theorem for the classical Hankel transform and the spherical Fourier transform on a Riemannian symmetric space of the Euclidean type as a corollary of Theorem 4.1.

Remark 4.3. In [11] the author gives an analogue of Theorem 4.1 for the Heckman-Opdam transform, which is the trigonometric counterpart of the Dunkl transform.

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