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Left Distributive Quasigroups and Gyrogroups

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Abstract. The connection between gyrogroups and some types of left distributive quasigroups is established by means of isotopy considerations. Any quasigroup of reflection is isotopic to some gyrocommutative gyrogroup and any left distributive quasigroup satisfying some specific condition is isotopic to some nongyrocommutative gyrogroup. The geometry of reductive homogeneous spaces and the semidirect product for homogeneous loops are used to produce local exact decompositions of groups and local gyrocommutative gyrogroups.

1. Introduction

A gyrogroup is roughly the set of relativistically admissible velocities together with their binary Einstein composition law in Special Theory of Relativity. Such an algebraic structure turns out to be noncommutative and nonassociative and this noncommutativity-nonassociativity is generated by the Thomas precession (or Thomas gyration) that occurs in some physical phenomena. The identification of the Thomas precession with the left inner mapping of the loop theory is of prime significance for both of the gyrogroup and the loop theories. Since their discovery by A.A. Ungar [20, 21, 23] gyrogroups became a subject of intensive investigations in their physical and geometrical meaning [20, 23, 24, 25] as well as in their loop-theoretical interpretation [21, 22, 23, 11, 18, 9, 6]. Indeed, in 1988 A.A. Ungar found a famous physical example of a formerly well known algebraic structure called left Bruck loop. In [25] gyrogroups were split up into gyrocommutative and nongyrocommutative ones. It is now clear that not only left Bruck loops algebraically describe gyrogroups but, more generally, homogeneous loops do so (in fact most of the characteristic properties of gyrogroups are properties of homogeneous loops with the automorphic inverse property).

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The basic role of homogeneous loops in the gyrogroup theory is stressed in [11, 6].

Concerned in the further loop-theoretical interpretation of gyrogroups we link, by isotopy considerations, gyrogroups with left distributive quasigroups satisfying some additional conditions. Thus in Section 3 quasigroups of reflection (i.e. left distributive quasigroups with the left law of keyes) were found to be isotopic equivalents of gyrocommutative gyrogroups (incidentally, for the global algebraic theory of such gyrogroups, the mapping $x \mapsto x^2$ happens to be a permutation) while left distributive quasigroups (with a left identity) satisfying the conditon $(*)$ isotopically describe nongyrocommutative gyrogroups. Thus the search of gyrocommutative or nongyrocommutative gyrogroups could be reduced, under some conditions, to the one of corresponding left distributive quasigroups. In Section 4 we consider the geometry of reductive symmetric spaces and its connection with gyrogroups (this is attuned to the present subject since by the O. Loos definition [12], a symmetric space can be seen as the smooth (or differentiable) counterpart of a left distributive quasigroup satisfying the left law of keyes). Here the emphasis is more on the geometry of local smooth gyrogroups whereas the geometry of global smooth gyrocommutative gyrogroups was already investigated by W. Krammer and H.K. Urbantke in [9]. In this section we also point out that the concept of the semidirect product for quasigroups and loops could be used to produce, at least for the case of homogeneous loops, examples of exact decompositions of groups and local gyrocommutative gyrogroups. In contrast of gyrocommutative gyrogroups, nongyrocommutative gyrogroups likely do not allow any geometry of symmetric spaces. In Section 2 we recall some useful notions of loop theory and some results from gyrogroup theory.

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2. Preliminaries

A groupoid $(G,.)$ is a set G with a binary operation $(.)$. If $a \in G$, the *left translation* L_a and *right translation* R_a by a are defined by $L_a b = a.b$ and $R_a b = b.a$, for any $b \in G$. If, for any $a \in G$, L_a and R_a are permutations of G (i.e. bijections of G onto itself), then G is called a *quasigroup.* In which case L_a^{-1} and R_a^{-1} have sense and one defines $L_a^{-1}b = a\backslash b$ and $R_a^{-1}b = b/a$. A groupoid $(G,\star)$ is said to be *isotopic* to a quasigroup $(G,.)$ whenever they are some permutations α, β, γ of G such that $a \star b = \gamma^{-1}(\alpha a.\beta b)$ for any a, b in G (the triple α, β, γ is then called an *isotopy*). It turns out that $(G,\star)$ is also a quasigroup.

For fundamentals on quasigroup and loop theory we refer to [1, 3, 13].

Throughout this paper we shall deal with a special class of quasigroups, namely the class of left distributive quasigroups. A quasigroup $(G,.)$ is said to be *left distributive* if its left translations L_a, $a \in G$, are its automorphisms, that is

$$a.bc = ab.ac$$

for all a, b, c in G (in order to reduce the number of brackets we use juxtaposition in place of dot whenever applicable). If there exists in G an element e such that $e.a = a$, $\forall a \in G$, then e is called the *left identity* of $(G,.)$. Likewise is defined the right identity of $(G,.)$. If the left and right identity of $(G,.)$ coincide, then e is called the (two-sided) *identity element* of $(G,.)$. A *loop* is a quasigroup with an identity element.

A loop $(G,.)$ is called a *left A-loop* (or a *left special loop*) if its *left inner mappings* $l_{a,b} = L_{a.b}^{-1} L_a L_b$ are its automorphisms. If, moreover, $(G,.)$ has the *left inverse property (LIP)* $L_a^{-1} = L_{a^{-1}}$ (here a^{-1} denotes the *inverse* of a), then $(G,.)$ is called a *homogeneous loop.* Homogeneous loops were investigated by M. Kikkawa in [7], where among others their relation with the geometry of reductive homogeneous spaces is established . One of the most studied classes of loops is the one of Bol loops. A loop $(G,.)$is called a *left Bol loop*, if

$$L_a L_b L_a = L_{a.ba} \quad (left\ Bol\ identity)$$

for every a, b in G. Bol loops are studied by D.A. Robinson in [14] (see also [1, 4, 15]). Left Bol loops are known to possess, for instance, the left inverse property (LIP). A left Bol loop with the *automorphic inverse* *property* $(a.b)^{-1} = a^{-1}.b^{-1}$ is called a *left Bruck loop*. A featuring property of left Bruck loops is the *left Bruck identity*

$$ab.ab = a.(bb.a). \tag{1}$$

Left Bruck loops are closely related to grouplike structures called gyrogroups. The term of *gyrogroup* is introduced by A.A. Ungar in [23] to designate the noncommutative- nonassociative algebraic structure formed by relativistically admissible velocities and their Einstein composition law (see [20, 21]). Such a noncommutativity-nonassociativity is generated by the *Thomas precession* well known in Physics. The grouplike foundations of these algebraic structures are layed in [20, 21, 23, 25]. In [25] gyrogroups were split up into gyrocommutative gyrogroups and nongyrocommutative ones just as groups are classified into abelian and nonabelian. It turns out that the algebraic structure of gyrocommutative gyrogroups coincides with a known type of loops (Ungar [21] introduced the term of "K-loop" to designate this type of loop). We recall this interesting fact that the Thomas precession coincides with the left inner mapping of the loop theory. Later A. Kreuzer [10] proved that K-loops and left Bruck loops are actually the same. In [6] was observed that a nongyrocommutative gyrogroup could be identified with a homogeneous loop satisfying the *left loop property (LLP)* $l_{a,b} = l_{a.b,b}$ (the loop property is one of the characteristic properties of gyrogroups).

For the sake of conciseness we will not give the axiomatic definitions of gyrogroups (one may refer to [20, 23, 24, 25]). Instead we shall think of them as of their loop-theoretical equivalents. Thus a *gyrocommutative gyrogroup* is a left Bruck loop (or a K-loop) while a *nongyrocommutative gyrogroup* is a homogeneous loop with the left loop property. As an abelian group is a group with the commutative law, a gyrocommutative gyrogroup could be seen as a (nongyrocommutative) gyrogroup satisfying additionally the *gyrocommutative law*

$$a.b = l_{a,b}(b.a). \tag{2}$$

L.V. Sabinin [18] showed that the gyrocommutative law is actually the "gyroversion" of the left Bruck identity (1). Using properties of homogeneous loops we observed [6] that the gyrocommutative law (2) can be generalized up to an identity (that we called the *Kikkawa identity*) characterizing homogeneous loops with the automorphic inverse property ([7], Proposition

1.13 (3)). However A.A. Ungar already obtained ([24], Theorem 3.1 (v)), by another way, such a generalization. This proves again the homogeneous loop foundations of gyrogroups.

In this paper we extend the loop-theoretical interpretation of gyrogroups up to left distributive quasigroups with additional properties.

3. Isotopy of Left Distributive Quasigroups and Gyrogroups

A quasigroup $(Q, .)$ is called a *quasigroup of reflection* if it is left distributive and satisfies the *left law of keyes*

$$x.xy = y. \tag{3}$$

One notes that, as a left distributive quasigroup, Q is *idempotent* i.e. $x.x = x$ for any x in Q. The term of "quasigroup of reflection", used in [7] to designate this type of quasigroups, is motivated by its close relation with the geometry of symmetric spaces (see also Section 4 below). In [15] D.A. Robinson studied right distributive quasigroups satisfying the right law of keyes and called them right-sided quasigroups. As such, quasigroups of reflection could also be called *left-sided quasigroups*. We observe that the link between left distributive quasigroups satisfying the left law of keyes and left Bol loops was established in earlier literature (see [2]). The results below regarding quasigroups of reflection are dual to those of D.A. Robinson in [15] about the isotopic study of right-sided quasigroups.

Theorem 1. *Every quasigroup of reflection is isotopic to a gyrocommutative gyrogroup.*

Proof. Let $(Q, .)$ be a quasigroup of reflection and e a fixed element in Q. Consider the isotopy of $(Q, .)$ given by $x + y = (x/e).ey$. Then $(Q, +)$ is a quasigroup and e is its identity since $e + e = e$ (recall that (Q,.) is idempotent) so that $(Q, +, e)$ is a loop. Next, from (3) we have $(L_x)^2 = id$, $\forall x \in Q$, where id denotes the identity mapping, and therefore by [1], Theorem 9.11 (see also [2], Theorem 3) we get that $(Q, +, e)$ is a left Bol loop. If $x \in (Q, +, e)$ then the inverse x^{-1} of x is given by $x^{-1} = e.x$ since $x + (e.x) = (x/e).e(e.x) = (x/e).x = (x/e).((x/e).e) = e$ (by (3)). Therefore, for any $x, y \in (Q, +, e)$, $x^{-1} + y^{-1} = ((e.x)/e).(e.ey) = (e.(x/e)).(e.ey) = e.((x/e).ey) = e.(x + y) = (x + y)^{-1}$ (here we used the equality $(e.x)/e = e.(x/e)$ which follows from the left distributivity and (3); indeed $ex.ee = e.xe$ implies $ex.e = e.xe$ that is, replacing x by x/e, $(e.(x/e)).e = e.((x/e).e)$ and this leads to $e.(x/e) = (e.x)/e)$. Thus $(Q, +, e)$ has the automorphic inverse property. Then we conclude that $(Q, +, e)$ is a left Bruck loop, i.e. a gyrocommutative gyrogroup. □

Remark 1. Since any quasigroup of reflection is idempotent, the isotopy $x + y = (x/e).ey$ is actually the e-isotope of $(Q, .)$. Indeed from the left law of keyes we have $(L_x)^2 = id$ and $(L_x^{-1})^2 = id$ (since by the left distributive law L_x and L_x^{-1} are automorphisms of $(Q, .)$) so that $x + y = (x/e).ey = R_e^{-1}x.L_e y = R_e^{-1}x.L_e^{-1}y = (x/e).(e\backslash y)$.

Corollary 1. *The mapping $x \mapsto x + x$ is a permutation of $(Q,+,e)$.*

Proof. Since $y + y = (y/e).ey = (y/e).(e\backslash y)$, then for any z in Q we have $(z.e) + (z.e) = ((z.e)/e).(e\backslash(z.e)) = z.(e\backslash(z.e)) = z.(e.ze) = ze.(z.ze) = ze.e.$ so we get $(z.e) + (z.e) = (z.e).e$ for all z in Q. Setting $x = z.e$ we have $x + x = R_e x$. Therefore, R_e being a permutation of Q, we conclude that the mapping $x \mapsto x + x$ is also a permutation of Q. □

By [15], Theorem 4, one also deduces the following result: If two gyrocommutative gyrogroups are isotopic to the same quasigroup of reflection (the isotopy being the one described above) then they are isomorphic.

Theorem 2. *If $(Q, +, e)$ is a gyrocommutative gyrogroup such that the mapping $x \mapsto x+x$ is a permutation of Q, then $(Q, +, e)$ is isotopic to some quasigroup of reflection.*

Proof. Define on Q a binary operation

$$x.y = x^2 + y^{-1} \tag{4}$$

for any x, y in Q, where $x^2 = x + x$ and y^{-1} is the inverse of y in $(Q, +, e)$. Then $(Q, .)$ is a quasigroup since the mappings $x \mapsto x^2 = x + x$ and $x \mapsto x^{-1}$ are permutations of Q. Next, $(Q, +, e)$ being a left Bol loop with the automorphic inverse property, we have $z.(x.y) = z^2 + (x^2 + y^{-1})^{-1} = z^2 + ((x^2)^{-1} + y) = z^2 + ((x^{-1})^2 + y) = z^2 + [(x^{-1})^2 + (z^2 + ((z^2)^{-1} + y))] = [z^2 + ((x^{-1})^2 + z^2)] + ((z^2)^{-1} + y) = [z^2 + ((x^{-1})^2 + z^2)] + (z^2 + y^{-1})^{-1} =$

$[(z^2+x^{-1})+(z^2+x^{-1})]+(z^2+y^{-1})^{-1}$ (by (1)) $= (z^2+x^{-1})^2+(z^2+y^{-1})^{-1} = (z.x).(z.y)$ (by (4)), so that we get the left distributive law. On the other hand $x^2 + (x^2 + y^{-1})^{-1} = x^2 + ((x^2)^{-1} + y) = y$ which is the left law of keyes. Thus $(Q, .)$ is a quasigroup of reflection. □

Remark 2. The requirement for the mapping $x \mapsto x + x$ to be a permutation of a given gyrocommutative gyrogroup $(Q, +)$ is essential for the global algebraic theory of such gyrogroups (but as it could be seen from the proof of Proposition 2 below, this requirement can be deleted in the local smooth theory). This naturally arises from the role of the mapping $x \mapsto x^2$ in the algebraic theory of Bol loops. Indeed, from the results of R.H. Bruck ([3], Chapt. VII, Theorem 5.2) and V.D. Belousov ([1], Theorems 11.9 and 11.10) a way of constructing gyrocommutative gyrogroups is as follows. Consider a left Bol loop $(B, .)$ such that the mapping $x \mapsto x^2$ is a permutation of B. Then the *core* $(B, \oplus)$ of $(B, .)$ defined by $x \oplus y = x.y^{-1}x$ is a quasigroup that turns out to be a quasigroup of reflection. Now construct a loop-isotope $(B, +, e)$ of $(B, \oplus)$, $e \in B$, as in the proof of Theorem 1 and then $(B, +, e)$ is a gyrocommutative gyrogroup. This construction can be extended up to groups, making use of the Glauberman construction of a Bruck loop from a group with an involutory automorphism (see [4, 15]). At this point it is relevant to observe that the links between gyrocommutative gyrogroups and groups of odd order are also considered by T. Foguel and A.A. Ungar in [26], where they use involutory decompositions of groups to produce gyrogroups.

Below (Section 4) is pointed out the existence of gyrocommutative gyrogroups for which the mapping $x \mapsto x^2$ is a permutation.

We now turn our attention to nongyrocommutative gyrogroups and their relations with left distributive quasigroups satisfying some conditions. Specifically we shall consider a left distributive quasigroup $(Q, .)$ with a left identity e satisfying

$$(*) \qquad [\alpha(x.y)].(x\backslash z) = \alpha([\alpha(x.y)].y).[(\alpha(x.y))\backslash z]$$

for any x, y, z in Q, where α is a permutation of Q. We consider the following isotope of $(Q, .)$:

$$(5) \qquad x \circ y = R^{-1}(x.Ry)$$

for any x, y in Q, where R is a permutation of Q defined by $Rx = x.e$. We denote by λ_x and L_x the left translations by x in $(Q, .)$ and $(Q, \circ)$ respectively. The statements below concern left distributive quasigroups with a left identity; but similar statements hold for any left distributive quasigroup since, being a F-quasigroup (see [1]), any left distributive quasigroup is isotopic to a left distributive quasigroup with a left identity.

Lemma 1. *The operation (5) satisfies the relation*

$$(*)' \qquad \lambda_{x \circ y} \lambda_x^{-1} \lambda_{x \circ y} = \lambda_{(x \circ y) \circ y}.$$

Proof. In $(*)$ replace y by $\alpha^{-1} y$ and take $\alpha = R^{-1}$. Then, by (5), we have $\lambda_{x \circ y} \lambda_x^{-1} z = \lambda_{(x \circ y) \circ y} \lambda_{x \circ y}^{-1} z$, for any x, y, z in Q so that we get $(*)'$. $\square$

Lemma 2. *The groupoid $(Q, \circ)$, with $(\circ)$ defined as in (5), is a loop with e as its identity element. The corresponding quasigroup $(Q, .)$ is a LIP quasigroup and therefore $(Q, \circ, e)$ has also the LIP.*

Proof. The first statement directly follows from (5). Now let x^r be the unique element in Q such that $x.x^r = e$ and set $y = R^{-1} x^r$ in $(*)'$. Then, by (5), we have $\lambda_x^{-1} = \lambda_{R^{-1} x^r}$. This means that $(Q, .)$ is a LIP quasigroup. Next, again from (5), we have $L_x = R^{-1} \lambda_x R$ and then $L_x^{-1} = R^{-1} \lambda_x^{-1} R$. On the other hand we have $x^{-1} = L_x^{-1} e = R^{-1} \lambda_x^{-1} R e = R^{-1} \lambda_x^{-1} e = R^{-1} x^r$. Since $\lambda_x^{-1} = \lambda_{R^{-1} x^r}$, then for any $y \in Q$, $\lambda_x^{-1} R y = \lambda_{R^{-1} x^r} R y \Leftrightarrow R^{-1} \lambda_x^{-1} R y = R^{-1} \lambda_{R^{-1} x^r} R y$ that is $L_x^{-1} y = R^{-1} \lambda_{R^{-1} x^r} R y = R^{-1}(R^{-1} x^r . R y) = R^{-1} x^r \circ y = L_{R^{-1} x^r} y = L_{x^{-1}} y$ and thus $(Q, \circ, e)$ has the left inverse property. $\square$

Theorem 3. *Any left distributive quasigroup $(Q, .)$ with a left identity satisfying $(*)$ is isotopic to some nongyrocommutative gyrogroup $(Q, \circ)$, the isotopy being given by (5).*

Proof. If e is the left identity of $(Q, .)$ then from Lemma 2 we know that $(Q, \circ)$ is a loop with the LIP. Next, from [1] (Theorem 9.10 and Corollary, p.165-167), we get that the permutations $l_{x,y}^{-1}$ are automorphisms of $(Q, \circ)$ so that $(Q, \circ)$ is a homogeneous loop. Further, from $(*)$ with $\alpha = R^{-1}$, we have $[R^{-1}(x.y)].(x \backslash z) = R^{-1}([R^{-1}(x.y)].y).[(R^{-1}(x.y)) \backslash z]$ that is, replacing y by Ry and using (5), $\lambda_{x \circ y} \lambda_x^{-1} z = \lambda_{(x \circ y) \circ y} \lambda_{x \circ y}^{-1} z$, or $\lambda_x^{-1} z = \lambda_{x \circ y}^{-1} \lambda_{(x \circ y) \circ y} \lambda_{x \circ y}^{-1} z$, i.e., replacing z by $\lambda_{x \circ y} z$, $\lambda_x^{-1} \lambda_{x \circ y} z = \lambda_{x \circ y}^{-1} \lambda_{(x \circ y) \circ y} z$.

Thus we obtain $\lambda_x^{-1}\lambda_{x\circ y} = \lambda_{x\circ y}^{-1}\lambda_{(x\circ y)\circ y}$. Multiplying each member of this last equality by $R^{-1}\lambda_y^{-1}$ from the left and by R from the right and using the relation $L_x = R^{-1}\lambda_x R$, we get $L_y^{-1}L_x^{-1}L_{x\circ y} = L_y^{-1}L_{x\circ y}^{-1}L_{(x\circ y)\circ y}$ i.e. $l_{x,y}^{-1} = l_{x\circ y,y}^{-1}$ and one recognizes the left loop property (LLP). Therefore $(Q,\circ,e)$ is a homogeneous loop with the LLP, i.e. a nongyrocommutative gyrogroup. $\square$

THEOREM 4. *Let $(Q,\circ,e)$ be a nongyrocommutative gyrogroup, R a permutation of Q and $(.)$ a binary operation on Q defined by (5). If R is such that $R\phi = \phi R$ for any automorphism ϕ of $(Q,\circ,e)$, then $(Q,.)$ is a left distributive quasigroup with e as its left identity and $(Q,.)$ satisfies $(*)$.*

PROOF. One knows that the defined groupoid $(Q,.)$ is a quasigroup. From Theorem 9.10 in [1] (p.165) we have $x.y = x\circ(y\circ x^{-1})$ wherefrom it follows that e is a left identity for $(Q,.)$. The permutations $l_{x,y} = L_{x\circ y}^{-1}L_xL_y$ are automorphisms of $(Q,\circ,e)$ and therefore so are $l_{x,y}^{-1}$. Now from (5), $l_{x,y}^{-1} = R^{-1}\lambda_y^{-1}\lambda_x^{-1}\lambda_{x\circ y}R$ that is $Rl_{x,y}^{-1}R^{-1} = \lambda_y^{-1}\lambda_x^{-1}\lambda_{x\circ y}$. Let us denote $\theta_{y,x} = \lambda_y^{-1}\lambda_x^{-1}\lambda_{x\circ y}$. Then since $Rl_{x,y}^{-1}R^{-1} = l_{x,y}^{-1}$, we have $\theta_{y,x} = l_{x,y}^{-1}$. Since $l_{x,y}^{-1}$ are automorphisms of $(Q,\circ,e)$, from [1] (Lemma 9.1, p.164) we know that $\theta_{y,x}$ are also automorphisms of the quasigroup $(Q,.)$ that is, for any u,v in Q, $\lambda_y^{-1}\lambda_x^{-1}\lambda_{x\circ y}(u.v) = (\lambda_y^{-1}\lambda_x^{-1}\lambda_{x\circ y}u).(\lambda_y^{-1}\lambda_x^{-1}\lambda_{x\circ y}v)$. In particular for $x = R^{-1}y'$ (where y' is uniquely defined by $y.y' = Re$), we have $\lambda_y^{-1}\lambda_{R^{-1}y'}^{-1}\lambda_{(R^{-1}y')\circ y}(u.v) = (\lambda_y^{-1}\lambda_{R^{-1}y'}^{-1}\lambda_{(R^{-1}y')\circ y}u).(\lambda_y^{-1}\lambda_{R^{-1}y'}^{-1}\lambda_{(R^{-1}y')\circ y}v)$ or, by the second statement of Lemma 2, $\lambda_{(R^{-1}y')\circ y}(u.v) = \lambda_{(R^{-1}y')\circ y}u.\lambda_{(R^{-1}y')\circ y}v$, which is the left distributivity in $(Q,.)$, since $w = (R^{-1}y')\circ y$ ranges through all Q.

Next, from the LLP in $(Q,\circ,e)$ and by (5), we have $R^{-1}\lambda_{x\circ y}^{-1}\lambda_x\lambda_yR = R^{-1}\lambda_{(x\circ y)\circ y}^{-1}\lambda_{x\circ y}\lambda_yR$, that is $\lambda_{x\circ y}^{-1}\lambda_x\lambda_{x\circ y}^{-1} = \lambda_{(x\circ y)\circ y}^{-1}$. Thus $(*)'$ is fulfilled and so is $(*)$. This completes our proof. $\square$

4. Homogeneous Spaces, Exact Decompositions and Gyrogroups

In this section we make some observations about the connection between gyrogroups and some aspects of differential geometry. For this purpose we use the links between homogeneous spaces, loops and exact decompositions of groups.

Exact decompositions of groups are considered by A.A. Ungar in [22]. Let $A = SB$ be a group with a subgroup B and a subset S such that
(i) $x \in S \Rightarrow x^{-1} \in S$
(ii) $bxb^{-1} \in S$, $\forall\, x \in S$, $\forall\, b \in B$
(iii) the decomposition $a = xb$ is unique for any $a \in A$, where $x \in S$ and $b \in B$
(iv) if $b \in B$ then $bx = xb$ for any $x \in S$ only if $b = e$, where e is the identity element of A.
Then the decomposition $A = SB$ of the group A is said to be *exact*.

From (iii) it follows that if $x_1,\ x_2\ \in S$, then $x_1x_2 \in A$ and we have the unique decomposition $x_1x_2 = x_{12}b_{12}$, where $x_{12} \in S$ and $b_{12} \in B$. Therefore $x_{12} = pr_S(x_1x_2)$, where pr_S denotes the projection on S parallely to the subgroup B. Such a projection defines a binary operation, say $(+)$, on S induced from the one of A, that is $x_1 + x_2 = pr_S(x_1x_2)$. Then the groupoid $(S, +)$ is said to be *induced* by the exact decomposition $A = SB$. The notion of an exact decomposition of groups led to the notion of a *weakly associative group* (WAG) [22] i.e. a LIP left quasigroup $(Q, ., \backslash)$ with a two-sided identity element, whose left inner mappings are its automorphisms.

By a *local gyrocommutative gyrogroup* i.e. a local left Bruck loop $(U, ., e)$, we mean a neighborhood U of the identity e such that the functions $U \times U \to U$, $(x, y) \mapsto x\backslash y$ and $(x, y) \mapsto x/y$ are smooth (or differentiable) and such that $(U, ., e)$ has a structure of left Bruck loop.

As it is already shown in [9] reductive homogeneous spaces and, in particular, symmetric spaces constitute interesting examples of exact decompositions especially from the point of view of their connection with gyrocommutative gyrogroups. In fact symmetric homogeneous spaces are suitable differential geometric counterparts of such gyrogroups. A *symmetric homogeneous space* is a triple (G, H, s), where G is a connected Lie group, H a closed subgroup of G, s an involutory automorphism of G (i.e. $s \in Aut(G)$, $s^2 = id$) such that H lies between G_0 and the identity component of G_0, where G_0 is the closed subgroup of elements of G left fixed by s. Let g and h denote the Lie algebras corresponding to G and H respectively. Then there exists a canonical decomposition $g = m + h$, where m is an $Ad(H)$-invariant subspace of g and m is isomorphic to the tangent algebra of the homogeneous space G/H at the origin H (see [8]). If define $Q = \{expX,\ X \in m\}$ then $Q \subset G$ and it turns out that the decomposition

$G = QH$ is exact and the groupoid $(Q, +)$ induced by this decomposition is a gyrocommutative gyrogroup (global or local according to whether the decomposition is global or local, i.e. Q is a neighborhood of the origin H in G/H. This in turn depends on the type (noncompact or compact) of the space (G, H, s); see [9], Theorems 3.1 and 3.2). Conversely, up to some condition on (G, H, s), the space m has a structure of a gyrocommutative gyrogroup. Specifically we have the following

THEOREM 5 ([9], Theorem 3.4). *Let (G, H, s) be a symmetric space of noncompact type and $(Q, +)$ the gyrocommutative gyrogroup induced by the exact decomposition of (G, H, s). Let m be the $Ad(H)$-invariant subspace of the Lie algebra g of G and $(\circ)$ the binary operation on m defined by $X \circ Y = exp^{-1}(expX + expY)$. Then $(m, \circ)$ is a gyrocommutative gyrogroup.*

REMARK 3. The uniqueness of the vector $X \circ Y$ in m follows from that (G, H, s) is of noncompact type and, in this case, the mapping *exp* is a diffeomorphism. Besides, $(m, \circ)$ is an example of gyrocommutative gyrogroups (i.e. left Bruck loops) for which the mapping $X \mapsto X^2$ is a permutation (see Section 3). Indeed, for $(m, \circ)$ as defined above, one has $X^2 = X \circ X = exp^{-1}(expX + expX) = exp^{-1}(expX expX)$ (since $expX + expX \in exp\, m = Q \subset G = QH) = exp^{-1}(exp2X) = 2X$ and the mapping $X \mapsto 2X$ is clearly a permutation of m. We also observe that, deleting the condition of noncompact type for (G, H, s), we obtain by [5] (Chapt.II, Lemma 2.4) that $(m, \circ)$ must be a *local* gyrocommutative gyrogroup. Note that $(m, \circ)$ may be endowed with a structure of *gyrovector space* [25] as follows. The groupoid $(Q, +)$ being a smooth loop induced by a symmetric space, an multiplication operation $texpX$ by a real scalar t can be defined [19] on $(Q, +)$ for any X in m; then this operation may be shifted on $(m, \circ)$ by the help of the *exp* function by setting $t \odot X = exp^{-1}(texpX)$, for any X in m and thus $(m, \circ, \odot)$ is a gyrovector space. One notes that $t \odot X$ is uniquely defined by $texpX$ since, (G, H, s) being of noncompact type, the function *exp* is a diffeomorphism.

We find relevant here to point out the possibility of producing local gyrocommutative gyrogroups from symmetric homogeneous spaces without appealing to their corresponding symmetric Lie algebras. By such a way one can also obtain local WAGs. Local gyrocommutative gyrogroups are

precisely of interest from the point of view of their links with the geometry of locally symmetric affine connection spaces (see [19]. At this point we recall that the *geodesic loop* at a fixed point of a given locally symmetric affine connection space is always a local left Bruck loop, i.e. a local gyrocommutative gyrogroup). Beforehand, in order to match up our present subject, we recall the definition of a symmetric space in the sense of O. Loos [12]. A *symmetric space* is a manifold M with a differentiable multiplication $\mu : M \times M \to M$ with the following properties:
(S1) $\mu(x,x) = x$,
(S2) $\mu(x, \mu(x,y)) = y$,
(S3) $\mu(x, \mu(y,z)) = \mu(\mu(x,y), \mu(x,z))$,
(S4) every x has a neighborhood U such that $\mu(x,y) = y$ implies $y = x$ for all y in U.

From (S1)-(S3) it follows that the underlying algebraic structure of a symmetric space M is a quasigroup of reflection $(M, .)$ with the binary operation given by $x.y = \mu(x,y)$ for all x, y in M.

Now let (G, e) be a connected Lie group, e being its identity element. Denote by s an involutory automorphism of G and let G^s be the set of points left fixed by s. We take G^s as a closed subgroup of G which does not contain a nontrivial normal subgroup of G (this amounts to an effective action of G on the coset space G/G^s). Thus (G, G^s, s) is a symmetric space, G/G^s is an effective symmetric homogeneous space and therefore is reductive (see [8]). Define $G_s = \{xs(x^{-1}),\ x \in G\}$.

Proposition 1. *There exists an isomorphism $G_s \cong G/G^s$. The decomposition $G = G_s G^s$ is locally exact and G_s is a local weakly associative group.*

Proof. The isomorphism $G_s \cong G/G^s$ is established in [12] (Theorem 1.3, p.73). Now suppose $a \in G_s$. Then $a = xs(x^{-1})$ for some $x \in G$. Therefore $a^{-1} = s(x)x^{-1} = s(x)s^2(x^{-1}) = s(x)s(s(x)^{-1}) \in G_s$ and thus we get (i). Take again $a \in G_s$ i.e. $a = xs(x^{-1}),\ x \in G$. Then for any $h \in G^s$ we have $hah^{-1} = hxs(x^{-1})h^{-1} = hx(hs(x))^{-1} = hx(s(h)s(x))^{-1} = (hx)s((hx)^{-1}) \ \in G_s$ since $hx \in G$ and this proves (ii). Next, from [5] (Chapt.II, Lemma 2.4), we get the local uniqueness of the decomposition $x = ah$ for any $x \in G$, where $a \in G_s$ and $h \in G^s$ (we recall that $G_s \cong G/G^s$ and that the tangent space to G/G^s at its identity element is isomorphic to a vector subspace of the Lie algebra of G, complementary to the Lie algebra of G^s). The last condition (iv) follows from the hypothesis that the largest normal subgroup of G contained in G^s is $\{ e \}$. Finally, from [22] (Theorems 9.1 and 9.3), we deduce that G_s has a (local) structure of WAG. $\square$

PROPOSITION 2. *The groupoid $(G_s, .)$ with the operation product $a.b = ab^{-1}a$ for any a, b in G_s, is locally a quasigroup of reflection, isotopic to some local gyrocommutative gyrogroup.*

PROOF. First we note that the operation product is well defined since, if $a = xs(x)^{-1}$ and $b = ys(y)^{-1}$ for some $x, y \in G$ then using the fact that $s^2 = id$, we get $ab^{-1}a = [xs(x^{-1}y)]s([xs(x^{-1}y)]^{-1}) \in G_s$. The groupoid $(G_s, .)$ is locally a quasigroup since the mappings $x \mapsto x^{-1}$ and $x \mapsto x^2$ are permutations of G (the first one by the group structure of G and the second by the fact that the equation $y^2 = a$, for a sufficiently close to $e \in G$, has a unique solution in G because of the smoothness of the group operation on G and that the function *exp* is locally a diffeomorphism (see [5, 19])). A direct computation shows that $(G_s, .)$ is left distributive and satisfies the left law of keyes i.e. $(G_s, .)$ is a quasigroup of reflection. Now consider the isotopy of $(G_s, .)$ defined by $x + y = (x/e).ey$. Then by Theorem 1, $(G_s, +, e)$ is a (local) gyrocommutative gyrogroup. $\square$

REMARK 4. One observes that $(G_s, +, e)$ is actually the loop given by $x + y = x^{1/2}yx^{1/2}$, where $x^{1/2}$ is the solution of the equation $z^2 = x$ in G_s and also verifies that $(G_s, +)$ satisfies the left Bol identity and the automorphic inverse property (see [3], Chapt.VII, Theorem 5.2, where the verifications were done for Moufang loops. But it is easy to see that the same approach also accounts in our situation, where the emphasis is more on the process of producing local gyrocommutative gyrogroups). From [5] (Chapt.II, Theorem 3.2), it follows that the process described above is applicable for any homogeneous space, provided it is symmetric. Therefore, by Proposition 2, from any symmetric homogeneous space we always can get a local gyrocommutative gyrogroup (this is another way of producing such gyrogroups without referring to the geometry of locally symmetric affine connection spaces, as mentioned above).

Example. From a loop-theoretic standpoint semidirect products for some types of loops may be used to construct examples of exact decompositions and thereby obtain induced loops. As a consequence the geometry of gyrocommutative gyrogroups may be expressed in terms of reductive symmetric homogeneous spaces induced by some special semidirect products.

The general construction of a *semidirect product* of a quasigroup (loop) by its *left transassociant* is given by L.V. Sabinin [16, 17]. By definition a left transassociant of a loop Q always contains the group $L_0(Q)$ of all its left inner mappings. However we will be interested here by a narrower construction, specifically the one of a homogeneous loop $(G, ., e)$ by a subgroup K of $Aut(G)$ such that $L_0(G) \subset K \subset Aut(G)$. From [16, 17, 7] the semidirect product $G \bowtie K$ is defined to be the cartesian product $G \times K$ together with the binary operation given by

$$(x, \alpha)(y, \beta) = (x.\alpha y, l_{x,\alpha y}\alpha\beta) \tag{6}$$

for $(x, \alpha),\ (y, \beta) \in G \times K$. Since $(G, ., e)$ is a loop, $G \bowtie K$ is a group with identity (e, id) and the inverse of an element $(x, \alpha) \in G \bowtie K$ is given by

$$(x, \alpha)^{-1} = (\alpha^{-1}x^{-1}, \alpha^{-1}), \tag{7}$$

where x^{-1} is the inverse of x in $(G, ., e)$. If G is a gyrogroup, then $G \bowtie K$ is the *gyrosemidirect product* defined in [25].

Consider now in $G \bowtie K$ the subgroup $\tilde{K} = \{e\} \bowtie K$ and the subset $\tilde{G} = G \bowtie \{id\}$. Then $\tilde{K} \cong K$ and, as loops, $\tilde{G} \cong G$ ([16], Theorems 3, 4). Moreover, from [7] (Lemma 2.3), it follows that the decomposition $G \bowtie K = \tilde{G}\tilde{K}$ is exact.

Assume now that G is a connected homogeneous Lie loop (i.e. G admits a natural differentiable structure) with the automorphic inverse property (in [7] such a loop is called a *symmetric Lie loop*) and let $\tilde{K}$ $(\cong K)$ be the closure of $L_0(G)$. Then $L = G \bowtie K$ is a connected Lie group with $\tilde{K}$ as its closed subgroup so that $L/\tilde{K}$ is a reductive homogeneous space. Since G is a symmetric Lie loop, the mapping s of L defined by $s(x, \alpha) = (x, \alpha)^{-1}$ is an involutory automorphism of L ([7], Theorem 6.1). With the notations as in Propositions 1, 2, we have $L^s = \tilde{K}$ and $L_s = \{(x, \alpha)^2,\ x \in G,\ \alpha \in K\}$. Therefore if define on L_s a binary operation $(\odot)$ by

$$(x, \alpha)^2 \odot (y, \beta)^2 = (x, \alpha)^2[(y, \beta)^2]^{-1}(x, \alpha)^2,$$

$(L_s, \odot)$ turns out to be a local quasigroup of reflection, according to Proposition 2 and, since $\tilde{e} = (e, id) \in L_s$, the isotopy of $(L_s, \odot)$ given by

$$(x, \alpha)^2 + (y, \beta)^2 = ((x, \alpha)^2/\tilde{e}) \odot \tilde{e}(y, \beta)^2$$

transforms L_s into a local gyrocommutative gyrogroup $(L_s, +)$.

It is now clear that gyrocommutative gyrogroups have strong relations with the geometry of homogeneous spaces ([25, 9]). Unfortunately it seems that it is not the case for nongyrocommutative gyrogroups, one of the major difficulties turning around the left loop property. Imposing some additional condition on groups possessing an exact decomposition, one can get homogeneous loops. For instance, if $(A, .)$ is a group, B a subgroup and S a subset of A such that
(a) the decomposition $A = S.B$ is exact (in the sense of (i)-(iv))
(b) $(S.S) \cap A = \{a_0\}$ $(a_0 \in A)$,
then one can show that the induced groupoid $(S, +)$ has a homogeneous loop structure but, in general, $(S, +)$ has not the left loop property.

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