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Crystalline Fundamental Groups I — Isocrystals on Log Crystalline Site and Log Convergent Site

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Abstract. In this paper, we define the crystalline fundamental groups for fine log schemes (satisfying certain condition) over a perfect field of positive characteristic by using the category of isocrystals on log crystalline site and prove their fundamental properties, such as Hurewicz isomorphism, the bijectivity of crystalline Frobenius and the comparison theorem with (log version of) de Rham fundamental groups (log version of Berthelot-Ogus theorem for fundamental groups).

Contents

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Introduction

In arithmetic geometry, we consider various cohomologies such as Betti, etale, de Rham and crystalline ones for algebraic varieties. Their definitions are quite different, but all of them satisfy several nice properties in common, and in certain situations, we have various comparison theorems between them. Grothendieck proposed the philosophy that cohomology theory is motivic: That is, there exists a theory of motivic cohomology such that all the reasonable cohomology theories factor through it. The comparison theorems can be thought as a consequence of this philosophy. Now most mathematicians believe his philosophy, and it is one of the central subjects in arithmetic geometry.

On the other hand, various theory of (the pro-unipotent quotient of) rational fundamental groups have been studied in arithmetic geometry: The topological rational fundamental group is studied by Quillen ([Q]). The etale fundamental group is defined and studied by Grothendieck ([SGA1]). The de Rham fundamental group is studied by Sullivan ([Su]) and Chen ([Che]). The theory of mixed Hodge structure on rational fundamental groups is studied by Morgan ([Mo]), Hain ([Ha]), Navarro Aznar ([Nav]) and Wojtkowiak ([W]).

In his celebrated paper [D2], Deligne gave the construction of the topological, l-adic and de Rham fundamental groups by means of Tannakian categories and fiber functors, and he extended the philosophy of Grothendieck to the case of the pro-unipotent quotient of rational fundamental groups. If we follow his philosophy, there should be a theory of crystalline fundamental groups for smooth varieties over a perfect field of characteristic $p > 0$, and if the given variety is liftable to characteristic zero, there should be the comparison theorem between de Rham fundamental group and crystalline fundamental group. But in [D2], the crystalline fundamental group is defined only for smooth varieties which are liftable to a field of characteristic zero, and in this case, it is defined by introducing the crystalline Frobenius to the de Rham fundamental group of the lift. Since the crystalline fundamental group should be defined for any smooth variety X over a perfect field of characteristic $p > 0$ (even if X is not liftable), and since it should depend only on the reduction, the construction of the crystalline fundamental group in [D2] is not the best possible way.

In this paper, we construct the theory of crystalline fundamental groups for schemes (satisfying certain conditions) over a perfect field of characteristic $p > 0$ by using the theory of Tannakian categories and fiber functors. In fact, we work in more general setting: We work not only for schemes, but also for log schemes in the sense of Fontaine, Illusie and Kato ([Kk1]).

A log scheme is a scheme endowed with a 'log structure', which is a pair of certain sheaf of monoids and certain homomorphism of sheaves of monoids. (For precise definition, see [Kk1] or Chapter 2 in this paper.) It is a generalization of the notion of a scheme, and it is useful for the study of degeneration of varieties, for example. Recently, a generalization of the above four cohomology theories to the case of log schemes and a generalization of the comparison theorems are developed ([Kk1], [Nak], [Nak-Kk], [Kf2], [Ma], [T1] etc.). Hence it might be natural to say that the cohomology theory of log schemes should be also motivic.

Moreover, one is tempted to say that there should exist the theory of rational fundamental group for log schemes which is motivic. If we believe this philosophy, there should be various rational fundamental groups for fundamental groups (corresponding to the above four cohomology theories) which are related by various comparison theorems. In this paper, we consider de Rham fundamental groups and crystalline fundamental groups for log schemes: We extend the definition of de Rham fundamental group to the case of certain log schemes, and give the definition of crystalline fundamental group for certain log schemes, and prove the comparison theorem between them.

Now let us briefly explain the content of each chapter.

In Chapter 1, first we recall the definition and basic properties of Tannakian categories. After that, we introduce the notion of nilpotent part $\mathcal{NC}$ of an abelian tensor category $\mathcal{C}$. It is a useful notion for us because it corresponds to the pro-unipotent quotient of a pro-algebraic group via Tannaka duality of Saavedra Rivano ([Sa]) in the case where $\mathcal{C}$ is a neutral Tannakian category. In general, the category $\mathcal{NC}$ is not necessarily abelian even if $\mathcal{C}$ is so. We give a simple sufficient condition for the category $\mathcal{NC}$ to be abelian. This criterion is useful to prove that the several categories which we consider later are in fact Tannakian categories.

In Chapter 2, first we review basic definitions concerning log schemes, which are developed in [Kk1]. We extend some of the basic notions to the case of formal schemes, because we often use log formal schemes in later chapters. After that, we prove the following proposition: For a log smooth exact morphism $f : (X, M) \longrightarrow (\operatorname{Spec} k, N)$ of certain fine log schemes (where k is a field), $X_{f\text{-triv}} := \{x \in X \mid (f^*N)_{\bar{x}} \overset{\sim}{=} M_{\bar{x}}\}$ is open dense in X if X is reduced. (For precise statement, see Theorem 2.3.2.) In the case of a log smooth integral morphism of fs log schemes, this is due to Tsuji ([T2]). In later chapters, this proposition is used to provide a simple and geometric sufficient condition for several categories associated to a morphism of log (formal) schemes to be Tannakian.

In Chapter 3, we develop the theory of de Rham fundamental groups for log schemes. For a morphism of log schemes $f : (X, M) \longrightarrow (\operatorname{Spec} k, N)$ (where k is a field) and a k-rational point x in $X_{f\text{-triv}}$ satisfying certain condition, de Rham fundamental group $\pi_1^{\mathrm{dR}}((X, M)/(\operatorname{Spec} k, N), x))$ is defined to be the Tannaka dual of the category $\mathcal{N}C((X, M)/(\operatorname{Spec} k, N))$ of nilpotent integrable log connections. So it is a pro-unipotent algebraic group over k. Then we prove the following theorem: For a pair (X, D) of a proper smooth scheme X over a field k and a normal crossing divisor D in X, we can associate a log structure M on X. Then, if the characteristic of k is zero, the de Rham fundamental group of the log scheme (X, M) is isomorphic to that of $X - D$. In particular, de Rham fundamental group of the log scheme (X, M) of this type depends only on $X - D$ (and base point). After that, we investigate the relation between the category of integrable log connections and the category of isocrystals on log infinitesimal site (a log or a log formal version of infinitesimal site of Grothendieck [G]), by the method in [B-O, §2], [O1, §1].

In Chapter 4, we develop the theory of crystalline fundamental groups for log schemes. For morphisms $(X, M) \xrightarrow{f} (\operatorname{Spec} k, N) \hookrightarrow (\operatorname{Spf} W, N)$ of (formal) log schemes (where k is a perfect field of characteristic $p > 0$ and W is the Witt ring of k) and a k-rational point x in $X_{f\text{-triv}}$ satisfying certain condition, crystalline fundamental group $\pi_1^{\text{crys}}((X, M)/(\operatorname{Spf} W, N), x))$ is defined to be the Tannaka dual of the category $\mathcal{N}I_{\text{crys}}((X, M)/(\operatorname{Spf} W, N))$ of nilpotent isocrystals on log crystalline site (which is defined by Kato [Kk1]). It is a pro-unipotent algebraic group over the fraction field of W. We prove two important basic properties of crystalline fundamental groups: One is the crystalline version of Hurewicz isomorphism, which gives the isomorphism of abelianization of crystalline fundamental group and the dual of first log crystalline cohomology group. The other is the bijectivity of crystalline Frobenius. After proving them, we consider the following problem: Let (X, D) be a pair of a proper smooth scheme X over a perfect field k of characterictic $p > 0$ and a normal crossing divisor D in X, and let us denote the log structure on X associated to this pair by M. Then does the crystalline fundamental group of (X, M) over $\operatorname{Spf} W$ depend only on $X - D$ (and base point)? (Here we should remark that it is not good to compare with the crystalline fundamental group of $X - D$: it is too big in general, for so is the crystalline cohomologies.) We give the affirmative answer to this problem in the case $\dim X \leq 2$, by using resolution of singularities by Abhyankar ([A]) and the structure theorem of proper birational morphism of surfaces due to Shafarevich ([Sha]). In Part II ([Shi]) of this series of papers, we consider this problem in another method and we give the affirmative answer in general case. Finally in this chapter, we investigate a relation between the category of isocrystals on log infinitesimal site, the category of HPD-isostratifications and the category of integrable formal log connections, following the argument in [O2, (0.7)] and Kato [Kk1, §6]. We also prove the descent property of the category of isocrystals on log crystalline site.

In Chapter 5, we prove the comparison theorem between de Rham fundamental groups and crystalline fundamental groups. The statement is as follows: Let k be a perfect field of characterictic $p > 0$, W the Witt ring of

k and V a totally ramified finite extension of W. Denote the fraction fields of W, V by K_0, K, respectively. Now assume we are given the following commutative diagram of fine log schemes

$$\begin{array}{ccccc} (X_k, M) & \hookrightarrow & (X, M) & \hookleftarrow & (X_K, M) \\ \downarrow & & f\downarrow & & \downarrow \\ (\operatorname{Spec} k, N) & \hookrightarrow & (\operatorname{Spec} V, N) & \hookleftarrow & (\operatorname{Spec} K, N) \\ & \searrow & \downarrow & & \\ & & (\operatorname{Spec} W, N), & & \end{array}$$

where the two squares are Cartesian, f is proper log smooth integral and X_k is reduced. Assume moreover that $H^0_{\mathrm{dR}}((X, M)/(\operatorname{Spf} V, N)) = V$ holds, and that we are given a V-valued point x of $X_{f\text{-triv}}$. Denote the special fiber (resp. generic fiber) of x by x_k (resp. x_K). Then there exists a canonical isomorphism of pro-algebraic groups

$$\pi_1^{\mathrm{crys}}((X_k, M)/(\operatorname{Spf} W, N), x_k) \times_{K_0} K \overset{\sim}{=} \pi_1^{\mathrm{dR}}((X_K, M)/(\operatorname{Spec} K, N), x_K).$$

In the case of cohomologies (without log structures), the corresponding theorem is proved by Berthelot and Ogus ([B-O2]). So we call this theorem as the Berthelot-Ogus theorem for fundamental groups.

The method of proof is as follows: First, we introduce the notion of log convergent site, which is a log version of the convergent site of Ogus ([O1]), and define the convergent fundamental group

$$\pi_1^{\mathrm{conv}}((X_k, M)/(\operatorname{Spf} ?, N), x_k) \quad (? = W, V)$$

by using the category $\mathcal{N}I_{\mathrm{conv}}((X_k, M)/(\operatorname{Spf} ?, N))$ of nilpotent isocrystals on log convergent site. Then we prove the base change property

$$\pi_1^{\mathrm{conv}}((X_k, M)/(\operatorname{Spf} W, N), x_k) \times_{K_0} K \overset{\sim}{=} \pi_1^{\mathrm{conv}}((X_k, M)/(\operatorname{Spf} V, N), x_k).$$

Second, we investigate the relation between the category of nilpotent isocrystals on log convergent site and the category of nilpotent isocrystals on log infinitesimal site. This, together with the result in Chapter 3, allows us to prove the comparison

$$\pi_1^{\mathrm{conv}}((X_k, M)/(\operatorname{Spf} V, N), x_k) \overset{\sim}{=} \pi_1^{\mathrm{dR}}((X_K, M)/(\operatorname{Spec} K, N), x_K)$$

between de Rham fundamental group and convergent fundamental group. Finally, we investigate the relation between the category of nilpotent isocrystals on log convergent site and the category of nilpotent isocrystals on log crystalline site (here we use the results in Chapter 3 and Chapter 4), and prove the comparison

$$\pi_1^{\mathrm{conv}}((X_k, M)/(\mathrm{Spf}\, W, N), x_k) \overset{\sim}{=} \pi_1^{\mathrm{crys}}((X_k, M)/(\mathrm{Spf}\, W, N), x_k)$$

between convergent fundamental group and crystalline fundamental group. Combining the above three isomorphisms, we finish the proof.

In Part II ([Shi]) of this series of papers, we will study the general theory of log convergent cohomology systematically and relate it to rigid cohomology in certain case, and give an application to the study of crystalline fundamental groups.

After writing the first version of this paper, the author learned that Chiarellotto and Le Stum ([Chi-LS], [Chi]) also studied on closely related subjects. They also study the crystalline version (rather, one should say 'rigid version') of the rational fundamental group using rigid analytic geometry (without log structures).

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Conventions

(1) In this paper, all sheaves are considered with respect to small etale site unless otherwise stated.
(2) Let V be a complete discrete valuation ring of mixed characteristic $(0, p)$. A formal scheme T is called a formal V-scheme if T is a p-adic Noetherian formal scheme over $\mathrm{Spf}\, V$ and $\Gamma(U, \mathcal{O}_T)$ is topologically of finite type over V for any open affine $U \subset T$.
(3) Let V be as above and let K be the fraction field of V. For a scheme or a p-adic formal scheme T, we denote the category of coherent sheaves of $\mathcal{O}_T$-modules by $\mathrm{Coh}(\mathcal{O}_T)$. For a p-adic formal scheme T over $\mathrm{Spf}\, V$, we denote by $\mathrm{Coh}(K \otimes \mathcal{O}_T)$ the category of sheaves of $K \otimes_V \mathcal{O}_T$-modules on T which is isomorphic to $K \otimes_V F$ for some $F \in \mathrm{Coh}(\mathcal{O}_T)$. For elementary properties of $\mathrm{Coh}(K \otimes \mathcal{O}_T)$ (in the case that T is a formal V-scheme), see [O1, §1]. We call an object of $\mathrm{Coh}(K \otimes \mathcal{O}_T)$ an isocoherent sheaf on T.
(4) Tensor products of sheaves on formal V-schemes (V as above) and fiber products of formal V-schemes are p-adically completed unless otherwise stated. (Note that the exception occurs in the proof of Lemma 4.3.3.)

Chapter 1. Formalisms of Tannakian Categories

In this chapter, we recall the definition and basic properties on Tannakian categories which we will use later.

1.1. Tannakian categories

In this section, we recall the definition of Tannakian category and the structure theorem of a neutral Tannakian category, which is due to Saavedra ([Sa]). After that, we introduce a notion of nilpotence on categories.

In this section, we give no proofs. For the proofs, see [Sa], [D-M] and [D3].

First we recall the definition of tensor category.

DEFINITION 1.1.1. Let $\mathcal{C}$ be a category and let $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ be a functor.

(1) An associativity constraint for $(\mathcal{C}, \otimes)$ is a collection of functorial isomorphisms

$$\phi := \{\phi_{X,Y,Z} : X \otimes (Y \otimes Z) \longrightarrow (X \otimes Y) \otimes Z\}_{X,Y,Z \in \mathcal{C}}$$

such that the following diagram is commutative for all objects X, Y, Z, U:

$$\begin{array}{ccc} X \otimes (Y \otimes (Z \otimes U)) & \xrightarrow[\mathrm{id} \otimes \phi_{Y,Z,U}]{} & X \otimes ((Y \otimes Z) \otimes U) \\ {\scriptstyle \phi_{X,Y,Z \otimes U}} \downarrow & & {\scriptstyle \phi_{X,Y \otimes Z,U}} \downarrow \\ (X \otimes Y) \otimes (Z \otimes U) & & (X \otimes (Y \otimes Z)) \otimes U \\ {\scriptstyle \phi_{X \otimes Y,Z,U}} \downarrow & & \| \\ ((X \otimes Y) \otimes Z) \otimes U & \xrightarrow[\phi_{X,Y,Z} \otimes \mathrm{id}]{} & (X \otimes (Y \otimes Z)) \otimes U. \end{array}$$

(2) A commutativity constraint for $(\mathcal{C}, \otimes)$ is a collection of functorial isomorphisms

$$\psi := \{\psi_{X,Y} : X \otimes Y \longrightarrow Y \otimes X\}_{X,Y \in \mathcal{C}}$$

such that $\psi_{X,Y} \circ \psi_{Y,X} = \mathrm{id}$ holds for any objects X, Y.

(3) An associativity constraint ϕ and a commutativity constraint ψ for $(\mathcal{C}, \otimes)$ are compatible if the following diagram is commutative for any objects X, Y, Z :

$$\begin{array}{ccccc} X \otimes (Y \otimes Z) & \xrightarrow{\phi_{X,Y,Z}} & (X \otimes Y) \otimes Z & \xrightarrow{\psi_{X \otimes Y,Z}} & Z \otimes (X \otimes Y) \\ \downarrow {\scriptstyle \mathrm{id} \otimes \psi_{Y,Z}} & & & & \downarrow {\scriptstyle \phi_{Z,X,Y}} \\ X \otimes (Z \otimes Y) & \xrightarrow{\phi_{X,Z,Y}} & (X \otimes Z) \otimes Y & \xrightarrow{\psi_{X,Z} \otimes \mathrm{id}} & (Z \otimes X) \otimes Y \end{array}$$

(4) A pair $(\underline{1}, u)$, where $\underline{1}$ is an object in $\mathcal{C}$ and u is an isomorphism $\underline{1} \longrightarrow \underline{1} \otimes \underline{1}$, is called a unit object of $(\mathcal{C}, \otimes)$ if the functor

$$\mathcal{C} \longrightarrow \mathcal{C} \ ; \ X \mapsto \underline{1} \otimes X$$

is an equivalence of categories.

Definition 1.1.2. Let $\mathcal{C}$ be a category, let $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ be a functor, let ϕ be an associativity constraint for $(\mathcal{C}, \otimes)$ and let ψ be a commutativity constraint for $(\mathcal{C}, \otimes)$ such that ϕ and ψ are compatible. Then a 4-ple $(\mathcal{C}, \otimes, \phi, \psi)$ is called a tensor category if it has a unit object $\underline{1} \in \mathcal{C}$.

It is known that the unit object of a tensor category is unique up to unique isomorphism. In the following, we denote a tensor category $(\mathcal{C}, \otimes, \phi, \psi)$ simply by $(\mathcal{C}, \otimes)$ or $\mathcal{C}$, by abuse of notation.

When we are given a tensor category $(\mathcal{C}, \otimes)$, we can define the functor

$$\bigotimes_{i \in I} : \mathcal{C}^I \longrightarrow \mathcal{C}$$

for a finite set I in the natural way ([D-M, (1.5)]).

Next we recall the definition of rigidity for tensor categories. To do this, first we recall the definition of internal hom.

Definition 1.1.3. Let $(\mathcal{C}, \otimes)$ be a tensor category. If the functor

$$T \mapsto \mathrm{Hom}(T \otimes X, Y); \ \mathcal{C}^o \longrightarrow (\mathrm{Set})$$

is representable, we denote by $\mathcal{H}\mathrm{om}(X, Y)$ the representing object. We denote the object $\mathcal{H}\mathrm{om}(X, \underline{1})$ by X^*.

Let $X, Y \in \mathcal{C}$ and assume the internal hom $\mathcal{H}\mathrm{om}(X, Y)$ exists. Then, by the definition of internal hom, there is a functorial morphism

$$e_{X,Y} : \mathcal{H}\mathrm{om}(X, Y) \otimes X \longrightarrow Y$$

for $X, Y \in \mathcal{C}$, which corresponds to the identity map via the isomorphism

$$\mathrm{Hom}(\mathcal{H}\mathrm{om}(X, Y) \otimes X, Y) \overset{\sim}{=} \mathrm{Hom}(\mathcal{H}\mathrm{om}(X, Y), \mathcal{H}\mathrm{om}(X, Y)).$$

Then one can define the notion of reflexive object as follows:

DEFINITION 1.1.4. An object X in $\mathcal{C}$ is called reflexive if there exists X^*, X^{**} and the map $X \longrightarrow X^{**}$ corresponding to $e_{X,\underline{1}} \circ \psi_{X,X^*}$ via the isomorphism $\mathrm{Hom}(X \otimes X^*, \underline{1}) \overset{\sim}{=} \mathrm{Hom}(X, X^{**})$ is an isomorphism.

Let $X := \{X_i\}_{i \in I}$ and $Y := \{Y_i\}_{i \in I}$ be finite families of objects in $\mathcal{C}$ and assume there exist internal hom's $\mathcal{H}\mathrm{om}(X_i, Y_i)\,(i \in I)$. Then, one can define the morphism

$$\tilde{e}_{X,Y} : \bigotimes_{i \in I} \mathcal{H}\mathrm{om}(X_i, Y_i) \longrightarrow \mathcal{H}\mathrm{om}(\otimes_{i \in I} X_i, \otimes_{i \in I} Y_i)$$

as the morphism corresponding to the morphism

$$\begin{aligned}(\bigotimes_{i \in I} \mathcal{H}\mathrm{om}(X_i, Y_i)) \otimes (\bigotimes_{i \in I} X_i) &\overset{\sim}{=} \bigotimes_{i \in I} (\mathcal{H}\mathrm{om}(X_i, Y_i) \otimes X_i) \\ &\overset{\otimes_{i \in I} e_{X_i, Y_i}}{\longrightarrow} \bigotimes_{i \in I} Y_i\end{aligned}$$

(where the first isomorphism is defined by using the associativity constraint and the commutativity constraint) via the isomorphism

$$\begin{aligned}\mathrm{Hom}(\bigotimes_{i \in I} \mathcal{H}\mathrm{om}(X_i, Y_i), &\mathcal{H}\mathrm{om}(\otimes_{i \in I} X_i, \otimes_{i \in I} Y_i)) \overset{\sim}{=} \\ &\mathrm{Hom}((\bigotimes_{i \in I} \mathcal{H}\mathrm{om}(X_i, Y_i)) \otimes (\bigotimes_{i \in I} X_i), \bigotimes_{i \in I} Y_i).\end{aligned}$$

DEFINITION 1.1.5. A tensor category $(\mathcal{C}, \otimes)$ is called rigid if $\mathcal{H}\mathrm{om}(X, Y)$ exists for all objects X and Y, the morphisms $\tilde{e}_{X,Y}$ are isomorphisms for all finite families of objects $X := \{X_i\}_{i \in I}$ and $Y := \{Y_i\}_{i \in I}$, and all objects are reflexive.

DEFINITION 1.1.6. An abelian tensor category is a tensor category $(\mathcal{C}, \otimes)$ such that $\mathcal{C}$ is an abelian category and $\otimes$ is a bi-additive functor.

Now we recall the definition of Tannakian categories. The following definition is due to Deligne ([D3, (2.8)]).

DEFINITION 1.1.7. Let k be a field. A tensor category $(\mathcal{C}, \otimes)$ is called a Tannakian category over k if it is a rigid abelian tensor category such that $\mathrm{End}(\underline{1}) = k$ holds and there exists a fiber functor (:= exact faithful tensor functor) $\mathcal{C} \longrightarrow \mathrm{Vec}_K$ for some extension field $K \supset k$, where Vec_K denotes the category of finite-dimensional vector spaces over K. If there exists a fiber functor $\mathcal{C} \longrightarrow \mathrm{Vec}_k$, then $(\mathcal{C}, \otimes)$ is called a neutral Tannakian category.

As for the structure of a neutral Tannakian category, the following theorem is shown by Saavedra ([Sa]).

THEOREM 1.1.8 (Saavedra). *Let k be a field and $(\mathcal{C}, \otimes)$ be a neutral Tannakian category over k. Let $\omega : \mathcal{C} \longrightarrow \mathrm{Vec}_k$ be a fiber functor. Then the functor*

$$(k\text{-algebras}) \longrightarrow (\text{Groups}); \quad R \mapsto \mathrm{Aut}^{\otimes}(\mathcal{C} \xrightarrow{\omega} \mathrm{Vec}_k \longrightarrow \mathrm{Mod}_R)$$

is representable by a pro-affine algebraic group $G(\mathcal{C}, \omega)$ over k. Here, $\mathrm{Aut}^{\otimes}$ is the group of tensor automorphisms and Mod_R is the category of R-modules.

Moreover, ω induces an equivalence of categories

$$\mathcal{C} \longrightarrow \mathrm{Rep}_k(G(\mathcal{C}, \omega)),$$

where, for a pro-algebraic group G, $\mathrm{Rep}_k(G)$ denotes the category of finite-dimensional rational representations over k.

Next, we introduce the notion of nilpotence on categories.

DEFINITION 1.1.9.

(1) Let $\mathcal{C}$ be an exact category and let A be an object in $\mathcal{C}$. An object $E \in \mathcal{C}$ is called nilpotent with respect to A if there exists a filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_n = E$$

such that there exist exact sequences

$$0 \longrightarrow E_{i-1} \longrightarrow E_i \longrightarrow A \longrightarrow 0$$

for $1 \leq i \leq n$. We define the nilpotent part $\mathcal{N}_A\mathcal{C}$ of the category $\mathcal{C}$ with respect to A as the full subcategory of $\mathcal{C}$ which consists of nilpotent objects with respect to A.

(2) Let $\mathcal{C}$ be an abelian tensor category. Then we define the nilpotent part $\mathcal{N}\mathcal{C}$ as the nilpotent part with respect to the unit object.

(3) Let $\mathcal{C}$ be an abelian tensor category. Then $\mathcal{C}$ is called nilpotent if $\mathcal{C} = \mathcal{N}\mathcal{C}$ holds.

By definition, the following corollary of Theorem 1.1.8 holds:

Corollary 1.1.10. *Let $\mathcal{C}$ be a nilpotent neutral Tannakian category over a field k. Then $G(\mathcal{C}, \omega)$ of Theorem 1.1.8 is a pro-unipotent algebraic group over k. Moreover, there exists a natural isomorphism of (ind-finite dimensional) vector spaces over k :*

$$(\mathrm{Lie}\, G(\mathcal{C}, \omega)^{\mathrm{ab}})^* \stackrel{\sim}{=} \mathrm{Ext}^1_{\mathcal{C}}(\underline{1}, \underline{1}),$$

where $*$ denotes the dual.

1.2. A criterion of abelian-ness

Let $\mathcal{C}$ be an abelian tensor category and consider the nilpotent part $\mathcal{N}\mathcal{C}$. In general, $\mathcal{N}\mathcal{C}$ does not necessarily have nice properties: For example, it is easy to construct an example that $\mathcal{N}\mathcal{C}$ is not an abelian category. In this section, we give a simple sufficient condition that the nilpotent part of an abelian tensor category is again abelian. In later sections, we will use this result to show several categories are in fact Tannakian.

Proposition 1.2.1. *Let $\mathcal{C}$ be an abelian tensor category and let $\underline{1}$ be the unit object of $\mathcal{C}$. Assume that* $\mathrm{End}(\underline{1})$ *is a field. Then $\mathcal{N}\mathcal{C}$ is an abelian category.*

To prove Proposition 1.2.1, we prepare some elementary lemmas:

Lemma 1.2.2. *Let the notations be as in Proposition 1.2.1 and let E be an object in $\mathcal{C}$. Then E is nilpotent if and only if there exists a sequence of objects in $\mathcal{C}$*

$$E = E'_0 \to E'_1 \to \cdots \to E'_n = 0,$$

where all maps are surjective in the category $\mathcal{C}$ *and* $\mathrm{Ker}(E'_i \to E'_{i+1})$ *is isomorphic to* $\underline{1}$ *for each* i.

Proof. For a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

satisfying $E_i/E_{i-1} \simeq \underline{1}$, define E'_i by $\mathrm{Coker}(E_i \subset E)$. Then one can check that these E'_i's satisfy the above condition by using the snake lemma.

Conversely, suppose given E'_i's satisfying the above condition. Then if we put $E_i := \mathrm{Ker}(E \to E'_i)$, E_i's form a filtration of E satisfying $E_i/E_{i-1} \cong \underline{1}$. □

Lemma 1.2.3. *Let the notations be as in Proposition 1.2.1 and let* E *be an object in* $\mathcal{NC}$. *Then*:

(1) *If* $\phi : E \to \underline{1}$ *is not a zero map, then* ϕ *is surjective.*
(2) *If* $\phi : \underline{1} \to E$ *is not a zero map, then* ϕ *is injective.*

Proof. Here we prove the assertion (1). One can prove the assertion (2) by dualizing the proof, using Lemma 1.2.2. Let

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

be a filtration as in Definition 1.1.9(1) with $A = \underline{1}$. We prove the assertion by induction on n.

If $n = 0$, the lemma is trivial. If $n = 1$, E is isomorphic to $\underline{1}$ and the lemma is proved by the assumption that $\mathrm{End}(\underline{1})$ is a field.

Let us consider the general case. Let

$$0 \longrightarrow E_{n-1} \xrightarrow{f} E \xrightarrow{g} \underline{1} \longrightarrow 0$$

be the exact sequence defined above. The inductive hypothesis for $\phi \circ f : E_{n-1} \longrightarrow \underline{1}$ shows that $\phi \circ f$ is zero or surjective. If $\phi \circ f = 0$ holds, there exists a map $\psi : \underline{1} \longrightarrow \underline{1}$ such that $\phi = \psi \circ g$. Since ψ is either zero or surjective by assumption, ϕ is also zero or surjective. If $\phi \circ f$ is surjective, ϕ is also surjective. Hence the assertion (1) is proved. □

Lemma 1.2.4. *Let the notations be as in Proposition 1.2.1 and let E be an object in $\mathcal{NC}$. Then,*

(1) *If* $0 \longrightarrow F \xrightarrow{f} E \xrightarrow{g} \underline{1} \longrightarrow 0$ *is an exact sequence in* $\mathcal{C}$*, then F is in* $\mathcal{NC}$*.*
(2) *If* $0 \longrightarrow \underline{1} \longrightarrow E \longrightarrow F \longrightarrow 0$ *is an exact sequence in* $\mathcal{C}$*, then F is in* $\mathcal{NC}$*.*

Proof. Here we prove the assertion (1). We can prove the assertion (2) by dualizing the proof of (1), using Lemma 1.2.2.

Let

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

be a filtration as in Definition 1.1.9(1) with $A = \underline{1}$. We will show the lemma by induction on n. If $n = 1$ holds, then g is an isomorphism and therefore $F = 0$. Let us consider the general case. The composition $E_{n-1} \overset{i}{\hookrightarrow} E_n \overset{g}{\to} \underline{1}$ is zero or surjective by Lemma 1.2.3.

First let us assume that $g \circ i = 0$ holds. Then the diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \underline{1} & \longrightarrow & \underline{1} & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F & \longrightarrow & E & \xrightarrow{g} & \underline{1} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} & & \uparrow & & \\
 & & E_{n-1} & = & E_{n-1} & \longrightarrow & 0 & & \\
 & & & & \uparrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

and the snake lemma imply $F \simeq E_{n-1}$. Therefore F is nilpotent.

Next let us assume that $g \circ i$ is surjective. Then the diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \uparrow & & & & \\
 & & & & \underline{1} & \longrightarrow & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F & \longrightarrow & E & \xrightarrow{g} & \underline{1} & \longrightarrow & 0 \\
 & & \uparrow & & {\scriptstyle i}\uparrow & & \| & & \\
0 & \longrightarrow & \mathrm{Ker}(g \circ i) & \longrightarrow & E_{n-1} & \xrightarrow{g \circ i} & \underline{1} & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

and the snake lemma imply that there exists an exact sequence

$$0 \longrightarrow \mathrm{Ker}(g \circ i) \longrightarrow F \longrightarrow \underline{1} \longrightarrow 0.$$

On the other hand, the inductive hypothesis implies that $\mathrm{Ker}(g \circ i)$ is nilpotent. Therefore F is also nilpotent by definition. □

PROOF OF PROPOSITION 1.2.1. Let E, F be objects in $\mathcal{NC}$ and let ϕ be a morphism from E to F. Then, $\mathrm{Ker}\phi$ and $\mathrm{Coker}\phi$ are defined in the category $\mathcal{C}$. We have only to show that $\mathrm{Ker}\phi$ and $\mathrm{Coker}\phi$ are nilpotent.

Here we only show that $\mathrm{Ker}\phi$ is nilpotent. (By dualizing the proof, we can show the case of $\mathrm{Coker}\phi$.) Let

$$0 = F_0 \subset F_1 \subset \cdots \subset F_m = F$$

be a filtration as in Definition 1.1.9(1) with $A = \underline{1}$. We will show the assertion by induction on m. If $m = 0$, the assertion is trivial. If $m = 1$, the assertion follows from Lemma 1.2.4. Let us consider the general case. For an exact sequence $0 \longrightarrow F_{m-1} \xrightarrow{i} F \xrightarrow{f} \underline{1} \longrightarrow 0$, the composition $f \circ \phi : E \longrightarrow F \longrightarrow \underline{1}$ is zero or surjective by Lemma 1.2.3. Let us assume that $f \circ \phi = 0$ holds. Then there exists a morphism $\psi : E \longrightarrow F_{m-1}$ such that $\phi = i \circ \psi$ holds. Hence Kerϕ = Kerψ holds and the assertion follows from the inductive hypothesis.

Next let us assume that $f \circ \phi$ is surjective. Then by the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}(f \circ \phi) & \longrightarrow & E & \xrightarrow{f\circ\phi} & \underline{1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \phi & & \| & & \\ 0 & \longrightarrow & F_{m-1} & \longrightarrow & F & \xrightarrow{f} & \underline{1} & \longrightarrow & 0 \end{array}$$

and the snake lemma, Kerϕ = Ker(Ker$(f \circ \phi) \longrightarrow F_{m-1}$) holds. On the other hand, by Lemma 1.2.4, Ker$(f \circ \phi)$ is nilpotent. Then, by inductive hypothesis, Kerϕ is also nilpotent. □

Chapter 2. Preliminaries on Log Schemes

In this chapter, first we will recall basic definitions and basic properties on log schemes, which are due to K. Kato ([Kk1]). After this, we prove the following proposition (Proposition 2.3.2): Assume given a log smooth exact morphism of fine log schemes $f : (X, M) \longrightarrow (S, N)$, where S is the spectrum of a perfect field k. Let us consider the following three properties:

(1) (Assuming the characteristic of k is positive) The morphism f is of Cartier type. (For the definition of a morphism of Cartier type, see Definition 2.2.16.)
(2) X is reduced.
(3) The set $X_{f\text{-triv}} := \{x \in X \mid (f^*N)_{\bar{x}} \xrightarrow{\sim} M_{\bar{x}}\}$ is open dense in X.

Then we have the implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3).

This result, together with the result in Section 1.2, gives a simple sufficient condition for some categories which we shall define in later chapters by using log schemes to be in fact Tannakian.

Here we note some conventions and notations which are used throughout this chapter and the later chapters: In the following, the monoid means a commutative monoid with a unit element, and a homomorphism of monoids is required to preserve the unit elements. When we regard a ring (which is always commutative and unital) as a monoid, we do so by means of multiplication. For a monoid P, denote its Grothendieck group by P^{gp} and denote the subgroup of invertible elements by $P^{\times}$. For a homomorphism of monoids $h : P \longrightarrow Q$, denote the associated homomorphism $P^{\mathrm{gp}} \longrightarrow Q^{\mathrm{gp}}$ of the Grothendieck groups by h^{gp}. For a monoid P and a scheme X, denote the constant sheaf on X_{et} with fiber P by P_X.

Finally, thoughout this chapter, p denotes a prime number, unless otherwise stated.

2.1. Basic definitions

First we recall the definition of pre-log structures and log structures on schemes or p-adic formal schemes ([Kk1, (1.1),(1.2)]).

DEFINITION 2.1.1.

(1) Let X be a scheme or a p-adic formal scheme. A pre-log structure on X is a pair (M, α), where M is a sheaf of monoids on X and $\alpha : M \longrightarrow \mathcal{O}_X$ is a homomorphism of sheaves of monoids.
(2) A pre-log structure (M, α) is called a log structure if α induces the isomorphism $\alpha^{-1}(\mathcal{O}_X^\times) \xrightarrow{\sim} \mathcal{O}_X^\times$.
(3) A log scheme (resp. a p-adic log formal scheme) is a triple (X, M, α), where X is a scheme (resp. a p-adic formal scheme) and (M, α) is a log structure on X. In the following, we denote the log scheme (resp. p-adic log formal scheme) (X, M, α) by (X, M), by abuse of notation.
(4) Let V be a complete discrete valuation ring of mixed characteristic $(0, p)$. Then a p-adic log formal scheme (X, M) is called a log formal V-scheme if X is a formal V-scheme.

A morphism of log schemes (resp. p-adic log formal schemes) $(X, M, \alpha) \longrightarrow (X', M', \alpha')$ is defined as the pair (f, g), where $f : X \longrightarrow X'$ is a morphism of schemes (resp. p-adic formal schemes) and $g : f^{-1}M' \longrightarrow M$ is a homomorphism of sheaves of monoids on X which makes the following diagram commutative:

$$\begin{array}{ccc} f^{-1}M' & \xrightarrow{g} & M \\ {\scriptstyle f^{-1}\alpha'}\downarrow & & {\scriptstyle \alpha'}\downarrow \\ f^{-1}\mathcal{O}_{X'} & \xrightarrow{f^*} & \mathcal{O}_X. \end{array}$$

Next we recall the definition of the log structure associated to a pre-log structure ([Kk1, (1.3)]).

Definition 2.1.2. Let X be a scheme or a p-adic formal scheme and let (M,α) be a pre-log structure on X. Then we define the log structure (M^a,α^a) associated to (M,α) as follows: M^a is defined to be the push-out of the diagram

$$\mathcal{O}_X^\times \xleftarrow{\alpha} \alpha^{-1}(\mathcal{O}_X^\times) \longrightarrow M$$

in the category of sheaves of monoids on X and α^a is defined to be the morphism

$$M^a \longrightarrow \mathcal{O}_X\,;\ (a,b) \mapsto \alpha(a)b\ (a \in M, b \in \mathcal{O}_X).$$

Next we define the definition of the pull-back of a log structure ([Kk1, (1.4)]).

Definition 2.1.3. Let $f : X \longrightarrow Y$ be a morphism of schemes or a p-adic formal schemes and let $M := (M,\alpha)$ be a log structure on Y. Then we define the pull-back f^*M of the log structure M to X as the log structure associated to the pre-log structure

$$f^{-1}M \xrightarrow{f^{-1}\alpha} f^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X.$$

Let the notations be as above. In the following, we shall often denote the log scheme or the p-adic log formal scheme (X, f^*M) by (X,M), by abuse of notation.

The category of log schemes (resp. p-adic log formal schemes) has fiber products ([Kk1, (1.6)]).

Next, we define the notion of coherence, fineness and fs-ness of log schemes or p-adic log formal schemes ([Kk1, §2], [Kk2, §1]).

Definition 2.1.4. Let P be a monoid.

(1) P is said to be integral if the natural homomorphism $P \to P^{\mathrm{gp}}$ is injective. It is equivalent to the following condition:

$$a,b,c \in P,\ ab = ac \implies b = c.$$

P is said to be fine if P is integral and finitely generated.

(2) P is said to be fs if P is fine and the following property is satisfied:

$$a \in P^{\mathrm{gp}}, a^n \in P \text{ for some } n \implies a \in P.$$

Definition 2.1.5. Let (X, M) be a log scheme or a p-adic log formal scheme. Then X is said to be coherent (resp. fine, fs) if etale locally on X, there exists a finitely generated monoid (resp. fine monoid, fs monoid) P and a homomorphism $P_X \longrightarrow \mathcal{O}_X$ whose associated log structure is isomorphic to M.

It is known ([Kk1, (2.6)]) that the category of coherent log schemes (resp. coherent p-adic log formal schemes) is closed under fiber products in the category of log schemes (resp. p-adic log formal schemes). Next proposition is due to K. Kato ([Kk1, (2.7)]):

Proposition 2.1.6. *The inclusion functor from the category of fine log schemes* (*resp. fine p-adic log formal schemes*) *to the category of coherent log schemes* (*resp. coherent p-adic log formal schemes*) *has a right adjoint.*

Proof. We only sketch the construction of the functor in the case of log schemes.

It suffices to construct the functor for a coherent log scheme (X, M, α) such that there exists a finitely generated monoid P and a homomorphism $\beta : P_X \to M$ satisfying $(P_X)^a = M$. Let $X \longrightarrow \operatorname{Spec} \mathbb{Z}[P]$ be the morphism induced by $\alpha \circ \beta$, and put $P^{\text{int}} := \operatorname{Im}(P \longrightarrow P^{\text{gp}})$. Let X' be $X \times_{\operatorname{Spec} \mathbb{Z}[P]} \operatorname{Spec} \mathbb{Z}[P^{\text{int}}]$ and let M' be the log structure on X' associated to the natural homomorphism $P^{\text{int}}_{X'} \to \mathcal{O}_{X'}$. Then the desired functor is defined by $(X, M) \mapsto (X', M')$. $\square$

In the following, we denote the right adjoint in the above proposition by $(X, M) \mapsto (X, M)^{\text{int}}$.

As a corollary to the above proposition, one can see that the category of fine log schemes (resp. fine p-adic log formal schemes) has the fiber product: For a diagram of fine log schemes

$$(X_1, M_1) \longrightarrow (Y, N) \longleftarrow (X_2, M_2),$$

it is defined by $((X_1, M_1) \times_{(Y,N)} (X_2, M_2))^{\text{int}}$. It should be noted that it does not coincide with the fiber product in the category of log schemes (resp. log formal schemes) in general. *In the rest of this paper, the fiber products of log schemes* (*resp. p-adic log formal schemes*) *are taken in the category of log* *schemes, even for the fiber products of fine log schemes* (*resp. fine p-adic log formal schemes*)*.* When we would like to consider the fiber products in the category of fine log schemes (resp. fine p-adic log formal schemes), we do not abbreviate to write the superscript $^{\text{int}}$.

Now we recall the definition of a chart. (We also introduce the terminology 'quasi-chart', which will be used in Chapter 3.)

Definition 2.1.7.

(1) For a fine (resp. cohorent) log scheme or a fine (resp. coherent) p-adic log formal scheme (X, M), a chart (resp. quasi-chart) of (X, M) is a homomorphism $P_X \longrightarrow M$ for a fine (resp. coherent) monoid P which induces $(P_X)^a \overset{\sim}{=} M$.

(2) For a morphism $f : (X, M) \longrightarrow (Y, N)$ of fine (resp. coherent) log schemes or fine (resp. coherent) p-adic log formal log schemes, a chart (resp. a quasi-chart) of f is a triple $(P_X \to M, Q_Y \to N, Q \to P)$, where $P_X \to M$ and $Q_Y \to N$ are charts (resp. quasi-charts) of M and N respectively and $Q \to P$ is a homomorphism such that the diagram

$$\begin{array}{ccc} Q_X & \longrightarrow & P_X \\ \downarrow & & \downarrow \\ f^{-1}N & \longrightarrow & M \end{array}$$

is commutative.

We define the notion of a chart (resp. a quasi-chart) for a diagram of fine (resp. coherent) log schemes or fine (resp. coherent) p-adic log formal schemes in a similar way.

For a fine log scheme or a fine p-adic log formal scheme (X, M), there exists a chart etale locally, by definition.

To prove some results for fine log schemes, it is sometimes important to take a 'nice' chart. So here we introduce the notion of good (resp. almost good) chart. (The notion of good chart is introduced in [Kf2, (2.1.4)].)

Definition 2.1.8. Let (X, M) be a fine log scheme or a fine p-adic log formal scheme and let $\phi : P_X \longrightarrow M$ be a chart. Let $x \in X$ and let $\phi_x : P \longrightarrow M_{\bar{x}}$ be the stalk of the homomorphism ϕ at $\bar{x}$. Then, the chart $P_X \longrightarrow M$ is said to be good at x (resp. almost good at x) if $\phi_x^{-1}(\mathcal{O}_{X,\bar{x}}^\times) = \{1\}$ holds (resp. $\phi_x^{-1}(\mathcal{O}_{X,\bar{x}}^\times)$ forms a group.)

REMARK 2.1.9. With the above notation, one can easily check the following:

(1) The chart $P_X \longrightarrow \mathcal{O}_X$ is good at x if and only if the composite $P \xrightarrow{\phi_{\bar{x}}} M_{\bar{x}} \longrightarrow M_{\bar{x}}/\mathcal{O}_{X,\bar{x}}^\times$ is an isomorphism.

(2) The chart $P_X \longrightarrow \mathcal{O}_X$ is almost good at x if and only if $\phi_x^{-1}(\mathcal{O}_{X,\bar{x}}^\times) = P^\times$ holds.

As for the existence of a good chart or an almost good chart, we have the following:

PROPOSITION 2.1.10.

(1) *Let $f : (X, M) \longrightarrow (Y, N)$ be a morphism of fine log schemes (resp. p-adic fine formal log schemes). Assume (Y, N) admits a chart $\varphi : Q_Y \longrightarrow N$ and let x be a point in X. Then there exists an etale neighborhood U of $\bar{x}$(:= a geometric point with image x) and a chart $(P_U \overset{\psi}{\to} M|_U, Q_Y \overset{\varphi}{\to} N, Q \to P)$ of the composite $(U, M|_U) \longrightarrow (X, M) \overset{f}{\longrightarrow} (Y, N)$ extending φ such that the chart ψ of $(U, M|_U)$ is almost good at x. (In particular, for a fine log scheme (resp. a p-adic fine log formal scheme) (X, M) and a point x, there exists etale locally a chart which is almost good at x.)*

(2) *Let (X, M) be a fine log scheme (resp. a p-adic fine log formal scheme) and let x be a point in X. Assume one of the following:*
   (a) $X = \operatorname{Spec} k$ *for a perfect field k.*
   (b) *M is an fs log structure.*

   Then, etale locally, there exists a chart which is good at x.

The following lemma, which is shown in [Kk1, (2.10)], is important for the proof of (1) of the above proposition (we omit the proof):

LEMMA 2.1.11. *Let (X, M) be a fine log scheme or a fine p-adic log formal scheme and let $x \in X$. Assume given a finitely generated group G and a homomorphism $h : G \longrightarrow M_{\bar{x}}^{\mathrm{gp}}$ such that the homomorphism*

$G \longrightarrow M^{\rm gp}_{\bar{x}}/\mathcal{O}^{\times}_{X,\bar{x}}$ induced by h is surjective. Then, if we put $P := h^{-1}(M_{\bar{x}})$, then the homomorphism $P \longrightarrow M_{\bar{x}}$ can be extended to a chart $P_U \longrightarrow M_U$ on an etale neighborhood U of x.

Proof of Proposition 2.1.10. First we prove the assertion (1). Take a neighborhood U' of $\bar{x}$ such that there exists a chart $\psi' : P'_{U'} \longrightarrow M|_{U'}$ of $(U', M|_{U'})$. Let us consider the following diagram:

$$\begin{array}{ccc} Q & \xrightarrow{\mathrm{id}\oplus 0} & Q \oplus P' \\ {\scriptstyle \varphi_{\bar{x}}}\downarrow & & \downarrow{\scriptstyle (f^*_{\bar{x}}\circ\varphi_{\bar{x}})\oplus\psi_{\bar{x}}} \\ N_{\bar{x}} & \xrightarrow{f^*_{\bar{x}}} & M_{\bar{x}}, \end{array}$$

where, for a homomorphism ? of sheaves of monoids, $?_{\bar{x}}$ denotes the stalk of ? at $\bar{x}$. Denote the right vertical homomorphism $(f^*_{\bar{x}} \circ \varphi_{\bar{x}}) \oplus \psi_{\bar{x}}$ simply by τ. Then the composite $(Q \oplus P')^{\rm gp} \xrightarrow{\tau^{\rm gp}} M^{\rm gp}_{\bar{x}} \longrightarrow M^{\rm gp}_{\bar{x}}/\mathcal{O}^{\times}_{X,\bar{x}}$ is surjective. Put $P := (\tau^{\rm gp})^{-1}(M_{\bar{x}})$. Then, by Lemma 2.1.11, the homomorphism $P \longrightarrow M_{\bar{x}}$ extends to a chart $\psi : P_U \longrightarrow M|_U$ on a neighborhood U of x. Then one can see that the triple $(P_U \xrightarrow{\psi} M|_U, Q_Y \xrightarrow{\varphi} N, Q \xrightarrow{\mathrm{id}\oplus 0} Q \oplus P' \hookrightarrow P)$ defines the desired chart.

Let us consider the assertion (2). As for the case (a), it is true by [Kk1, (2.5)(2)] when k is algebraically closed. In general case, we can take a good chart etale locally by using Lemma 2.1.11. The case (b) is nothing but [Kk2, (1.6)]. □

Remark 2.1.12. There exists a counter-example for (2) of the above proposition if we do not assume neither (a) nor (b).

We introduce the following terminology, which we shall use in Section 2.3.

Definition 2.1.13. A fine log scheme (X, M) is called a log point if X is a spectrum of a field k and there exists a good chart $P_X \to N$ globally.

Finally in this section, we recall the definition of log differential modules.

Definition 2.1.14.

(1) Let $f : (X, M) \longrightarrow (Y, N)$ be a morphism of log schemes. Then we define the module of log differentials $\omega^1_{(X,M)/(Y,N)}$ as the quotient of

$$\Omega^1_{X/Y} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} M^{\mathrm{gp}})$$

divided by the $\mathcal{O}_X$-submodule locally generated by the elements of the following types:

(a) $(d\alpha(a), 0) - (0, \alpha(a) \otimes a)$, where $a \in M$ and $\alpha : M \longrightarrow \mathcal{O}_X$ is the structure morphism of the log structure M.

(b) $(0, 1 \otimes a)$, where $a \in \mathrm{Im}(f^*N \to M)$.

(2) Let $f : (X, M) \longrightarrow (Y, N)$ be a morphism of p-adic log formal schemes. Then we define the module of log differentials $\omega^1_{(X,M)/(Y,N)}$ by $\omega^1_{(X,M)/(Y,N)} := \varprojlim_n \omega^1_{(X_n,M_n)/(Y_n,N_n)}$, where, for a p-adic log formal scheme (Z, L), Z_n is the closed subscheme defined by p^n and L_n is the log structure on Z_n which is defined as the pull-back of L.

In the following, we denote the element $(0, 1 \otimes a) \in \omega^1_{(X,M)/(Y,N)}$ $(a \in M)$ by $\mathrm{dlog}\, a$, following the tradition. As in the usual case, we denote the q-th exterior product of $\omega^1_{(X,M)/(Y,N)}$ over $\mathcal{O}_X$ by $\omega^q_{(X,M)/(Y,N)}$. In the following, if there will be no confusion concerning log structures, we will denote $\omega^q_{(X,M)/(Y,N)}$ simply as $\omega^q_{X/Y}$, by abuse of notation.

2.2. Several types of morphisms

Throughout this section, for a p-adic log formal scheme (X, M), denote by X_n $(n \in \mathbb{N}, n \geq 1)$ the closed subscheme defined by $p^n\mathcal{O}_X$ and denote by M_n the log structure on X_n which is defined as the pull-back of M.

First, we recall the definition of a closed immersion and an exact closed immersion ([Kk1, (3.1)]).

Definition 2.2.1. A morphism $i : (X, M) \longrightarrow (Y, N)$ of log schemes or p-adic fine log formal schemes is called a closed immersion (resp. an exact closed immersion) if the underlying morphism of schemes $X \longrightarrow Y$ is a closed immersion and the induced homomorphism $i^*N \longrightarrow M$ is surjective (resp. isomorphic).

Next we recall the definition of a log smooth (resp. log etale) morphism ([Kk1, (3.3)]).

DEFINITION 2.2.2. A morphism of fine log schemes $f : (X, M) \longrightarrow (Y, N)$ is log smooth (resp. log etale) if the following conditions are satisfied:

(1) The underlying morphism of schemes $X \longrightarrow Y$ is locally of finite presentation.
(2) Suppose given the following commutative diagram of fine log schemes

$$\begin{array}{ccc} (T_0, L_0) & \xrightarrow{a} & (X, M) \\ {\scriptstyle i}\downarrow & & {\scriptstyle f}\downarrow \\ (T, L) & \xrightarrow{b} & (Y, N), \end{array}$$

where i is an exact closed immersion such that the ideal $\mathcal{I} := \mathrm{Ker}(\mathcal{O}_T \longrightarrow \mathcal{O}_{T_0})$ satisfies $\mathcal{I}^2 = 0$. Then there exists etale locally (resp. uniquely) a morphism $c : (T, L) \longrightarrow (X, M)$ such that $c \circ i = a$ and $f \circ c = b$ hold.

A morphism of p-adic log formal schemes $f : (X, M) \longrightarrow (Y, N)$ is said to be formally log smooth (resp. formally log etale) if the induced morphism $(X_n, M_n) \longrightarrow (Y_n, N_n)$ are log smooth (resp. log etale) for each n.

One can check that the log smoothness and the log etaleness (resp. the formally log smoothness and the formally log etaleness) are stable under the base change in the category of fine log schemes (resp. fine p-adic log formal schemes.) The following lemma is immediate ([Kk1, (3.8)]):

LEMMA 2.2.3. *Let $f : (X, M) \longrightarrow (Y, N)$ be a morphism of fine log schemes and let us assume that $f^*N \overset{\sim}{=} M$ holds. Then f is log smooth* (*resp. log etale*) *if and only if the underlying morphism of schemes of f is smooth* (*resp. etale*). *The same result holds for a formally log smooth* (*resp. formally log etale*) *morphism of p-adic fine formal log schemes.*

We will call a morphism $f : (X, M) \longrightarrow (Y, N)$ of fine log schemes smooth in classical sense (resp. etale in classical sense) if f is log smooth (resp. log etale) and $f^*N \overset{\sim}{=} M$ holds. The terminology 'formally smooth in classical sense' (resp. 'formally etale in classical sense') will also be used in the same way.

As in the case of usual schemes, we have the following proposition ([Kk1, (3.10)]):

PROPOSITION 2.2.4. *Let $f : (X, M) \longrightarrow (Y, N)$ be a log smooth morphism of fine log schemes (resp. a formally log smooth morphism of fine p-adic formal log schemes). Then the log differential module $\omega^1_{(X,M)/(Y,N)}$ is a locally free $\mathcal{O}_X$-module of finite type.*

Next we recall the definition of an integral morphism of log schemes or p-adic log formal schemes ([Kk1, (4.2)]).

DEFINITION 2.2.5. Let $f : (X, M) \longrightarrow (Y, N)$ be a morphism of fine log schemes (resp. p-adic log formal schemes). For $x \in X$, let P_x, Q_x be $M_{\bar{x}}/\mathcal{O}^\times_{X,\bar{x}}, (f^*N)_{\bar{x}}/\mathcal{O}^\times_{X,\bar{x}}$, respectively. Then f is said to be integral if one of the following equivalent condition is satisfied:

(1) For any morphism of fine log schemes (resp. fine p-adic log formal schemes) $(Y', N') \longrightarrow (Y, N)$ to (Y, N), the fiber product $(Y', N') \times_{(Y,N)} (X, M)$ in the category of log schemes (resp. p-adic log formal schemes) is fine.
(2) For any $x \in X$, the homomorphism $Q_x \longrightarrow P_x$ induced by f is injective and for any homomorphism of fine monoids $Q_x \longrightarrow R$, the monoid defined as the push-out of the diagram $R \longleftarrow Q_x \longrightarrow P_x$ is fine.

REMARK 2.2.6. Here we give a remark on the condition (2) of the above definition. It is known ([Kk1, (4.1)]) that the following conditions on a homomorphism $h : Q \to P$ of fine monoids are equivalent:

(1) h is injective and for any homomorphism of fine monoids $Q \longrightarrow R$, the monoid defined as the push-out of the diagram $R \longleftarrow Q \xrightarrow{h} P$ is fine.
(2) The ring homomorphism $\mathbb{Z}[Q] \longrightarrow \mathbb{Z}[P]$ induced by h is flat.

The following lemma is proved in [Kk1, (4.4)] (we omit the proof):

Lemma 2.2.7. *Let $f : (X, M) \longrightarrow (Y, N)$ be a morphism of fine log schemes or fine p-adic log formal schemes and assume $N_{\bar{y}}/\mathcal{O}^{\times}_{Y,\bar{y}}$ is generated by one element for any $y \in Y$. Then f is integral.*

Now we recall an important proposition on the existence of a nice chart of a log smooth or a formally log smooth morphism which is proven in [Kk1, (3.5), (4.5)], [Kf1, (4.1)] (We omit the proof).

Theorem 2.2.8 (K. Kato). *Let $f : (X, M) \longrightarrow (Y, N)$ be a morphism of fine log schemes (resp. fine p-adic log formal schemes), and assume given a chart $Q_Y \longrightarrow N$ of Y. Then the following two conditions are equivalent:*

(1) *f is log smooth (resp. formally log smooth).*

(2) *Etale locally on X, there exists a chart $(P_X \to M, Q_Y \to N, Q \to P)$ extending the given chart of Y satisfying the following conditions:*

  (a) *The homomorphism $Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$ induced by the chart is injective and the order of the torsion part of the cokernel is invertible on X (resp. prime to p).*

  (b) *The induced morphism*

$$X \longrightarrow Y \times_{\operatorname{Spec} \mathbb{Z}[Q]} \operatorname{Spec} \mathbb{Z}[P]$$

$$(\textit{resp. } X_n \longrightarrow Y_n \times_{\operatorname{Spec} \mathbb{Z}[Q]} \operatorname{Spec} \mathbb{Z}[P]\ )$$

  is etale (resp. is etale for any n).

  (c) *There exist elements $x_1, \cdots, x_r \in P$ such that the image of x_i's by the morphism $P_X \to M \overset{\mathrm{dlog}}{\to} \omega^1_{(X,M)/(Y,N)}$ form a basis of $\omega^1_{(X,M)/(Y,N)}$ as an $\mathcal{O}_X$-module.*

If f is integral, we can take the chart as above such that the homomorphism of monoids $Q \to P$ satisfies the conditions in Remark 2.2.6.

Moreover, if we choose a point x in X, we can take, etale locally around x, the chart $(P_X \overset{\varphi}{\to} M, Q_Y \to N, Q \to P)$ such that it satisfies the above conditions and that the chart φ is almost good at x.

Remark 2.2.9. The first conclusion of the above theorem remains true if we drop the condition (c) in (2) and replace the condition (b) in (2) to the following condition (b'):

(b') The induced morphism

$$X \longrightarrow Y \times_{\operatorname{Spec}\mathbb{Z}[Q]} \operatorname{Spec}\mathbb{Z}[P]$$

$$(\text{resp. } X_n \longrightarrow Y_n \times_{\operatorname{Spec}\mathbb{Z}[Q]} \operatorname{Spec}\mathbb{Z}[P]\ )$$

is smooth (resp. is smooth for any n).

Remark 2.2.10. There is also a result on the existence of a nice chart for a log smooth morphism of fs log schemes satisfying certain condition ([Kf2, (3.1.1)]).

The above theorem has many important consequences. First we have the following:

Corollary 2.2.11. *A log smooth, integral morphism of fine log schemes is flat.*

Proof. Obvious by Remark 2.2.6 and Theorem 2.2.8. □

Next, the following proposition (the lifting of log smooth morphism of log schemes) can be shown by reducing the case of usual schemes, by using the above theorem ([Kk1, (3.14)]). We omit the proof.

Proposition 2.2.12. *Let $f:(X,M) \longrightarrow (Y,N)$ be a log smooth (resp. log smooth integral) morphism of fine log schemes, and let $i:(Y,N) \hookrightarrow (Y',N')$ be an exact closed immersion of (Y,N) into a fine p-adic log formal scheme such that Y is a scheme of definition of Y'. Assume X is affine. Then there exists a fine p-adic log formal scheme (X',M') endowed with a formally log smooth (resp. formally log smooth integral) morphism $f':(X',M') \longrightarrow (Y',N')$ and an isomorphism*

$$(X',M') \times_{(Y',N')} (Y,N) \overset{\sim}{=} (X,M).$$

Moreover, such a set of data is unique.

As the third consequence of the above theorem, we prove a certain lifting property which is slightly stronger than Definition 2.2.2 for a formally log smooth morphism of p-adic fine formal log schemes.

PROPOSITION 2.2.13. *Let $f : (X, M) \longrightarrow (Y, N)$ be a formally log smooth morphism of p-adic fine log formal schemes and let $i : (T', L') \hookrightarrow (T, L)$ be an exact closed immersion of p-adic fine formal log schemes such that the morphism $i \otimes \mathbb{Z}/p\mathbb{Z} : T'_1 \hookrightarrow T_1$ is a nilpotent closed immersion. Now let us assume that we are given the following commutative diagram:*

$$\begin{array}{ccc} (T', L') & \xrightarrow{a} & (X, M) \\ {\scriptstyle i}\downarrow & & {\scriptstyle f}\downarrow \\ (T, L) & \xrightarrow{b} & (Y, N). \end{array}$$

Then there exists etale locally a morphism $c : (T, L) \longrightarrow (X, M)$ such that $c \circ i = a$ and $f \circ c = b$ hold.

PROOF. First we prepare some notations: For a monoid P, let us denote the fine log structure on $\mathrm{Spf}\,\mathbb{Z}_p\{P\}$ (where $\mathbb{Z}_p\{P\}$ is the p-adic completion of the monoid ring $\mathbb{Z}_p[P]$) associated to the pre-log structure $P \longrightarrow \mathbb{Z}_p\{P\};\ x \mapsto x\,(x \in P)$ by P^a. For a homomorphism of monoids $\alpha : Q \longrightarrow P$, denote the morphism $(\mathrm{Spf}\,\mathbb{Z}_p\{P\}, P^a) \longrightarrow (\mathrm{Spf}\,\mathbb{Z}_p\{Q\}, Q^a)$, which is induced by α, by $\hat{\alpha}$.

Since we can work etale locally to prove the assertion, we may assume that there exists a chart $(P_X \to M, Q_Y \to N, Q \xrightarrow{\alpha} P)$ of f which satisfies the conditions in Theorem 2.2.8 (2). Then the morphism f factors as

$$(X, M) \xrightarrow{g} (Y, N) \times_{(\mathrm{Spf}\,\mathbb{Z}_p\{Q\}, Q^a), \hat{\alpha}} (\mathrm{Spf}\,\mathbb{Z}_p\{P\}, P^a) \xrightarrow{h} (Y, N),$$

and g is formally log etale. Since g is formally log etale, we can reduce to the case $f = h$ to prove the proposition. Further we can reduce to the case $f = \hat{\alpha}$.

Since we may work etale locally, we may assume T is affine. Put $\mathcal{I} := \mathrm{Ker}(\mathcal{O}_T \longrightarrow \mathcal{O}_{T'})$. Then we have the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & 1+\mathcal{I} & \longrightarrow & L & \longrightarrow & L' & \longrightarrow & 1 \\ & & \| & & \downarrow & (*) & \downarrow & & \\ 1 & \longrightarrow & 1+\mathcal{I} & \longrightarrow & L^{\mathrm{gp}} & \longrightarrow & (L')^{\mathrm{gp}} & \longrightarrow & 1, \end{array}$$

where two horizontal lines are exact and the square $(*)$ is cartesian. The commutative diagram given in the statement of proposition gives the following diagram:

$$\begin{array}{ccc} Q^{\mathrm{gp}} & \xrightarrow{\alpha^{\mathrm{gp}}} & P^{\mathrm{gp}} \\ {\scriptstyle b^*}\downarrow & & {\scriptstyle a^*}\downarrow \\ L^{\mathrm{gp}} & \xrightarrow{i^*} & (L')^{\mathrm{gp}}. \end{array}$$

The obstruction for extending b^* to a homomorphism $\gamma : P^{\mathrm{gp}} \longrightarrow L^{\mathrm{gp}}$ compatible with a^* lies in the group $\mathrm{Ext}^1(\mathrm{Coker}(\alpha^{\mathrm{gp}}), 1+\mathcal{I})$.

We prove that the group $\mathrm{Ext}^1(\mathrm{Coker}(\alpha^{\mathrm{gp}}), 1+\mathcal{I})$ vanishes. Since the orders of the torsion elements in $\mathrm{Coker}(\alpha^{\mathrm{gp}})$ are prime to p, it suffices to prove that the group $1+\mathcal{I}$ has only p-primary torsion. Since $i \otimes \mathbb{Z}/p\mathbb{Z}$ is nilpotent, there exists an integer $N \in \mathbb{N}$ such that $\mathcal{I}^N \subset p\mathcal{O}_T$ holds. Hence we have $1+\mathcal{I} = \varprojlim_n (1+\mathcal{I})/(1+\mathcal{I}^n)$. So it suffices to show that the group $(1+\mathcal{I}^n)/(1+\mathcal{I}^{n+1}) \overset{\simeq}{=} \mathcal{I}^n/\mathcal{I}^{n+1}$ has only p-primary torsion, and it is obvious.

So there exists a homomorphism $\gamma : P^{\mathrm{gp}} \longrightarrow L^{\mathrm{gp}}$ which is compatible with the above diagram. Since the square $(*)$ is cartesian, γ induces the homomorphism $P \longrightarrow L$ and it defines the desired morphism $c : (T, L) \longrightarrow (X, M) = (\mathrm{Spf}\,\mathbb{Z}\{P\}, P^a)$. So the assertion is proved. □

Next we recall the definition of an exact morphism.

Definition 2.2.14.

(1) A homomorphism of fine monoids $h : P \longrightarrow Q$ is said to be exact if $(h^{\mathrm{gp}})^{-1}(Q) = P$, where h^{gp} is the homomorphism $P^{\mathrm{gp}} \longrightarrow Q^{\mathrm{gp}}$ induced by h.
(2) A morphism $f : (X, M) \longrightarrow (Y, N)$ of fine log schemes or fine p-adic log formal schemes is said to be exact if the homomorphism $(f^*N)_{\bar{x}}/\mathcal{O}^{\times}_{X,\bar{x}} \longrightarrow M_{\bar{x}}/\mathcal{O}^{\times}_{X,\bar{x}}$ is exact for any $x \in X$.

The following facts are known ([Kk1, §4]).

Lemma 2.2.15.

(1) *An integral morphism of fine log schemes or fine p-adic log formal schemes is exact.*

(2) *If a morphism* $f : (X, M) \longrightarrow (Y, N)$ *of fine log schemes or fine p-adic log formal schemes is exact, the homomorphism* $(f^*N)_{\bar{x}}/\mathcal{O}^{\times}_{X,\bar{x}} \longrightarrow M_{\bar{x}}/\mathcal{O}^{\times}_{X,\bar{x}}$ *is injective.*

Next we recall the definition of a morphism of Cartier type.

Definition 2.2.16.

(1) Let (X, M) be a log scheme over Spec $\mathbb{F}_p$. Then we define the absolute Frobenius $F_{(X,M)} : (X, M) \longrightarrow (X, M)$ as follows: The morphism of schemes underlying $F_{(X,M)}$ is the usual absolute Frobenius of X, and the homomorphism $F^{-1}_{(X,M)}M \longrightarrow M$ is defined by the multiplication by p via the canonical identification of $F^{-1}_{(X,M)}M$ with M.

(2) Let $f : (X, M) \longrightarrow (Y, N)$ be a morphism of fine log schemes over Spec $\mathbb{F}_p$. Then f is said to be of Cartier type if f is integral and the morphism $(f, F_{(X,M)})$ from (X, M) to the fiber product of the diagram

$$(Y, N) \xrightarrow{F_{(Y,N)}} (Y, N) \xleftarrow{f} (X, M)$$

is exact.

Finally we recall a result of Kato on the existence of Cartier inverse isomorphism for log schemes ([Kk1, (4.12)]). (We omit the proof.)

Proposition 2.2.17. *Let* $f : (X, M) \longrightarrow (Y, N)$ *be a log smooth integral morphism of Cartier type between fine log schemes over* Spec $\mathbb{F}_p$. *Define the log scheme* (X', M') *by the Cartesian diagram*

$$\begin{array}{ccc} (X', M') & \xrightarrow{g} & (X, M) \\ \big\downarrow & & \big\downarrow f \\ (Y, N) & \xrightarrow{F_{(Y,N)}} & (Y, N), \end{array}$$

and let $h : (X, M) \longrightarrow (X', M)$ *be the morphism induced by* $(F_{(X,M)}, f)$. *Then there exists a canonical isomorphism of* $\mathcal{O}_{X'}$*-modules*

$$C^{-1} : \omega^q_{X'/Y} \xrightarrow{\sim} \mathcal{H}^q(\omega^{\cdot}_{X/Y})$$

characterized by

$$C^{-1}(a\mathrm{dlog}\, g^*(b_1)\wedge\cdots\wedge\mathrm{dlog}\, g^*(b_q)) = \textit{the class of } h^*(a)\mathrm{dlog}\, b_1\wedge\cdots\wedge\mathrm{dlog}\, b_q$$

for $a \in \mathcal{O}_{X'}, b_1, ..., b_q \in M$.

2.3. A note on reducedness of log schemes

For a morphism $(X, M) \longrightarrow (Y, N)$ of fine log schemes, define $X_{f\text{-triv}} \subset X$ by

$$X_{f\text{-triv}} := \{x \in X \mid (f^*N)_{\bar{x}} \xrightarrow{\sim} M_{\bar{x}}\}.$$

(That is, $X_{f\text{-triv}}$ is the locus on which the difference of the log structures f^*N and M is trivial.)

First note the following:

Proposition 2.3.1. *Let* $f : (X, M) \longrightarrow (Y, N)$ *be a morphism of fine log schemes. Then* $X_{f\text{-triv}}$ *is open in* X.

Proof. Let $x \in X_{f\text{-triv}}$ and $y = f(x)$. Let $\phi : Q \longrightarrow N$ be a chart of (Y, N) around y. Let ϕ be the composite

$$Q^{\mathrm{gp}} \longrightarrow N_{\bar{y}}^{\mathrm{gp}} \longrightarrow (f^*N)_{\bar{x}}^{\mathrm{gp}} \xrightarrow{\sim} M_{\bar{x}}^{\mathrm{gp}}.$$

Then the map

$$Q^{\mathrm{gp}} \longrightarrow M_{\bar{x}}^{\mathrm{gp}}/\mathcal{O}_{X,\bar{x}}^{\times}$$

induced by ϕ is surjective, for x is contained in $X_{f\text{-triv}}$. So if we put $R := \phi^{-1}(M_{\bar{x}})$, then, by Lemma 2.1.11, we can extend ϕ to the chart

$$\begin{array}{ccc} R_X & = & R_X \\ {\scriptstyle\phi}\downarrow & & {\scriptstyle\phi}\downarrow \\ f^*N & \longrightarrow & M \end{array}$$

of the morphism $(X, M) \longrightarrow (X, f^*N)$ etale locally. So the morphism $f^*N \longrightarrow M$ is isomorphic on a neighborhood of x. So we have the assertion. $\square$

Now we state the main result in this section.

Proposition 2.3.2. *Let k be a field and let $f:(X,M)\longrightarrow(\mathrm{Spec}\,k,N)$ be a log smooth exact morphism of fine log schemes. Assume moreover that k is perfect or $(\mathrm{Spec}\,k,N)$ is a log point (Definition 2.1.13). Let us consider the following three properties:*

(1) *(Assuming the characteristic of k is positive) f is of Cartier type.*
(2) *X is reduced.*
(3) *$X_{f\text{-triv}}\subset X$ is dense open.*

Then we have the implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3).

Remark 2.3.3. If f is log smooth integral and the log structures M,N are fs, the above three conditions are equivalent. This is due to Tsuji ([T2]).

First we show the implication (1) $\Longrightarrow$ (2).

Proof of (1) $\Longrightarrow$ (2) in Proposition 2.3.2. The following argument is due to Tsuji. Let (X',M') be the fine log scheme obtained by pulling back f by the Frobenius morphism of $(\mathrm{Spec}\ k,N)$. Then there exists a Cartier inverse isomorphism

$$C^{-1}:\omega^q_{X'}\longrightarrow\mathcal{H}^q(\omega^{\cdot}_X).$$

Then the composite

$$\mathcal{O}_X\hookrightarrow\mathcal{O}_{X'}\xrightarrow{C^{-1}}\mathcal{H}^0(\omega^{\cdot}_X)\hookrightarrow\mathcal{O}_X$$

is injective. On the other hand, this map is nothing but the char(k)-th power map. Hence X is reduced. □

Next we prove the implication (2) $\Longrightarrow$ (3). Before the proof, let us recall some terminologies concerning commutative algebra for monoids, which is developed in [Kk2, (5.1),(5.2)].

Definition 2.3.4. Let P be a monoid.

(1) A subset $I\subset P$ is called an ideal of P if $PI\subset I$ holds.
(2) An ideal I of P is called a prime ideal if $P-I$ is a submonoid of P.
(3) For a submonoid $S\subset P$, we define the monoid $S^{-1}P$ by $S^{-1}P:=\{s^{-1}a\,|\,a\in P,s\in S\}$, where

$$s_1^{-1}a_1=s_2^{-1}a_2\Leftrightarrow ts_1a_2=ts_2a_1 \text{ for some } t\in S.$$

For a monoid P, we denote the set of invertible elements of P by $P^{\times}$. Then one can see that $P - P^{\times}$ is a prime ideal of P. We have the following proposition, which is well-known in usual commutative algebra:

PROPOSITION 2.3.5. *Let P be a monoid and let $I \subset P$ be an ideal of P. Define the radical $\sqrt{I}$ of I by*

$$\sqrt{I} := \{a \in P \mid a^n \in I \textit{ for some } n \in \mathbb{N}, n \geq 1\}.$$

Then $\sqrt{I} = \bigcap_{\substack{I \subset J \\ J:\text{prime ideal}}} J$ *holds.*

PROOF. One can check the inclusion $\sqrt{I} \subset \bigcap_{\substack{I \subset J \\ J:\text{prime ideal}}} J$ easily.

Conversely, let us assume $a \notin \sqrt{I}$. Let S be a submonoid of P generated by a and let $f : P \longrightarrow S^{-1}P$ be the natural map. Then, by definition of a, $f(I)$ does not meet with $(S^{-1}P)^{\times}$. Hence $f^{-1}(S^{-1}P - (S^{-1}P)^{\times})$ is a prime ideal of P which contains I and to which a does not belong. So the assertion is proved. □

Now we begin the proof of (2) $\Longrightarrow$ (3).

PROOF OF (2) $\Longrightarrow$ (3) IN PROPOSITION 2.3.2. Since we already know the openness of $X_{f\text{-triv}}$, we may assume X is irreducible and we only have to show $X_{f\text{-triv}}$ is non-empty. Moreover, since we may work etale locally, we may assume that $(\operatorname{Spec} k, N)$ is a log point. Let $Q_k \longrightarrow N$ be a good chart.

Let $x \in X$ and take a chart $(Q_k \to N, P_X \to N, Q \xrightarrow{\alpha} P)$ of f satisfying the conditions in Theorem 2.2.8 which is almost good at x (here we replaced X by an etale neighborhood of x). Put $I := \alpha(Q - \{1\})P \subset P$. We remark that I is not equal to P. Let us consider the following commutative diagram induced by the chart which is chosen above:

$$\begin{array}{ccc} Q & \xrightarrow{\alpha} & P \\ \downarrow & & \downarrow \pi \\ (f^*N)_{\bar{x}}/\mathcal{O}^{\times}_{X,\bar{x}} & \longrightarrow & M_{\bar{x}}/\mathcal{O}^{\times}_{X,\bar{x}}. \end{array}$$

Since $Q_k \to N$ is a good chart, the left vertical arrow is an isomorphism, and the exactness of f implies the injectivity of the lower horizontal arrow. So $\pi \circ \alpha$ is injective. On the other hand, one can easily check that the homomorphism π sends $P^\times$ to 1. Hence $\alpha(Q - \{1\})$ does not meet with $P^\times$. So I is not equal to P.

Now let us consider the following Cartesian diagram of schemes:

$$\begin{array}{ccc} X & \xrightarrow{h} & \mathrm{Spec}\; k[P]/(I) \\ \uparrow & & \uparrow \\ X' & \xrightarrow{g} & \mathrm{Spec}\; k[P]/(\sqrt{I}). \end{array}$$

Since X is reduced, we have $X = X'$. Let $J_1, J_2, \cdots, J_r$ be the minimal prime ideals of P which contain I. Since X is irreducible, we may assume $\mathrm{Im}(g)$ is contained in $Y := \mathrm{Spec}\; k[P]/(J_1) \overset{\sim}{=} \mathrm{Spec}\; k[P - J_1]$. Put $U := \mathrm{Spec}\; k[(P - J_1)^{\mathrm{gp}}]$ and regard it as a dense open subscheme of Y via the morphism induced by the ring homomorphism $k[P - J_1] \hookrightarrow k[(P - J_1)^{\mathrm{gp}}]$. Let V be $g^{-1}(\mathrm{Im}(g) \cap U)$. Then V is a non-empty set of X. It suffices to show that $V \subset X_{f\text{-triv}}$ holds.

Since the injectivity of the morphism of log structures follows from the exactness of f and Proposition 2.2.15, it suffices to show the surjectivity. To show this, we only have to show the following claim: For any element $p \in P$, there exist $p', p'' \in P - J_1$ and $q \in Q$ such that $pp' = qp''$ holds.

To show this claim, we prepare two lemmas:

LEMMA 2.3.6. *Let the notation be as above. Then we have* $\sqrt{I} = I$.

PROOF. Let $p \in \sqrt{I}$ and set $y = h(x)$. Since X is reduced and h is flat, the ring $(k[P]/(I))_y$ is reduced. Hence $p = 0$ holds in $(k[P]/(I))_y$. Let $\mathcal{P}$ be the kernel of the map $k[P] \longrightarrow k[P]/(I) \longrightarrow \kappa(\bar{y})$. Then there exists an element $q = \sum_{k \in P} c_k k \notin \mathcal{P}$, such that $qp = \sum_{k \in P} c_k (kp) \in (I)$ holds.

Let us assume $p \notin I$. Let k be an element of P which is not contained in $\mathcal{P}$. Then, by the almost goodness of the chart at x, one can check that k is contained in $P^\times$. So, for such an element k, we have $kp \notin I$ and hence we have $c_k = 0$. Therefore we have $q \in \mathcal{P}$ and this is a contradiction. So $p \in I$ holds and the assertion is proved. □

Lemma 2.3.7. *For any $p \in J_1$, there exists an element $p' \in P - J_1$ such that $pp' \in I$.*

Proof. Assume $p \in \cap_{i \le l} J_i$ and $p \notin \cup_{i>l} J_i$ holds. If we take an element $p' \in J_{l+1} - J_1$ (note that there exists such an element because of the minimality of J_i's), we have $pp' \in \cap_{i \le l+1} J_i$. Repeating this process, we can show that there exists an element $p' \in P - J_1$ such that $pp' \in \cap_{i=1}^{r} J_i = \sqrt{I} = I$. □

Proof of (2) $\Longrightarrow$ (3) in Theorem 2.3.2 (continued). We will show the following assertion: For any element $p \in P$, there exist $p', p'' \in P - J_1$ and $q \in Q$ such that $pp' = qp''$ holds.

Let $p_0 = p \in P$. If $p_0 \notin J_1$ holds, we are done. Otherwise, there exists elements $r_0 \in P - J_1, q_0 \in Q - \{1\}$ and $p_1 \in P$ such that $p_0 r_0 = q_0 p_1$ holds by Lemma 2.3.7.

If $p_1 \notin J_1$ holds, we are done. Otherwise, there exists elements $r_1 \in P - J_1, q_1 \in Q - \{1\}$ and $p_2 \in P$ such that $p_1 r_1 = q_1 p_2$ holds.

Similarly, we can define $p_{i+1} \in P$ if we have $p_i \in J_1$. If we have $p_i \notin J_1$ for some i, we are done. So let us assume the contrary, that is, we assume $p_i \in J_1$ for all i. Then we have $(p_i) = (q_i p_{i+1})$ as ideals in $(P - J_1)^{-1}P$ and since $q_i \notin ((P - J_1)^{-1}P)^{\times}$, $(p_i) \subsetneq (p_{i+1})$ holds. This is a contradiction, since we can construct an infinite ascending chain of ideals in a Noetherian ring $k[(P - J_1)^{-1}P]$. □

2.4. Examples

In this section, we give some typical examples of log schemes or p-adic log formal schemes.

Example 2.4.1. Let X be a scheme or a p-adic formal scheme. Then the pair $(\mathcal{O}_X^{\times}, \mathcal{O}_X^{\times} \hookrightarrow \mathcal{O}_X)$ is a log structure on X. This log structure is called the trivial log structure on X. The functor

$$(\text{Schemes}) \longrightarrow (\text{Log schemes}),$$

$$(\text{resp.} \quad (p\text{-adic formal schemes}) \longrightarrow (p\text{-adic log formal schemes}) \quad)$$

defined by $X \mapsto (X, \text{triv. log str.})$ is fully faithful. So, one can naturally regard a scheme (resp. a p-adic formal scheme) as a log scheme (resp. a p-adic log formal scheme) which is endowed with the trivial log structure.

The trivial log structure is fs, since it is associated to the pre-log structure $\{1\}_X \longrightarrow \mathcal{O}_X$.

Example 2.4.2. Let P be a monoid and let R be a ring (resp. a p-adically complete ring). Let $R[P]$ (resp. $R\{P\}$) be the monoid ring (resp. the p-adic completion of the monoid ring) with coefficient ring R. Put $X := \operatorname{Spec} R[P]$ (resp. $X := \operatorname{Spf} R\{P\}$.) and $S := \operatorname{Spec} R$ (resp. $S := \operatorname{Spf} R$).

Let $\varphi' : P \longrightarrow R[P]$ (resp. $\varphi' : P \longrightarrow R\{P\}$) be the natural monoid homomorphism and let $\varphi : P_X \longrightarrow \mathcal{O}_X$ be the homomorphism of sheaves of monoids induced by φ'. The log structure on X associated to the pre-log structure (P_X, φ) is called the canonical log structure on X.

The canonical log structure is fine if P is fine, and it is fs if P is fs. If P is fine and the order of the torsion part of P^{gp} is invertible on $\operatorname{Spec} R$ (resp. prime to p), the natural homomorphism $(X, \text{can.}\log\text{str.}) \longrightarrow (S, \text{triv.}\log\text{str.})$ is log smooth.

Example 2.4.3. Let X be a toric variety of dimension d over a field k. Then, locally, X can be written as $\operatorname{Spec} k[P]$ for an fs monoid P satisfying $P^{\text{gp}} \cong \mathbb{Z}^d$. Let us endow it with the canonical log structure. Then, one can glue the log schemes $(\operatorname{Spec} k[P], \text{can.}\log\text{str.})$ and it gives an fs log structure on X. This log structure is called the canonical log structure of a toric variety X. By the previous example, the natural morphism $(X, \text{can.}\log\text{str.}) \longrightarrow (\operatorname{Spec} k, \text{triv.}\log\text{str.})$ is log smooth.

Example 2.4.4. Let X be a scheme and D a closed subscheme, and denote the open immersion $X - D \hookrightarrow X$ by j. Then the inclusion

$$M := \mathcal{O}_X \cap j_*\mathcal{O}_{X-D}^{\times} \hookrightarrow \mathcal{O}_X$$

defines a log structure (which is not necessarily fine) on X. This log structure is called the log structure associated to the pair (X, D). Let us give two typical examples of fs log structures associated to certain pairs.

First, let us assume that X is regular and that D is a normal crossing divisor on X. Let x be a point in X. Then there exists a connected etale affine neighborhood U of $\bar{x}$ and a regular sequence $z_1, \cdots, z_n \in \Gamma(U, \mathcal{O}_U)$ such that $D \times_X U$ is defined by the equation $z_1 z_2 \cdots z_r = 0$ for some $r \leq n$. let $\varphi : \mathbb{N}_U^r \longrightarrow M|_U$ be the morphism defined by $\varphi(e_i) := z_i$, where

$e_i\,(1 \leq i \leq r)$ is the natural basis of $\Gamma(U, \mathbb{N}^r_U) = \mathbb{N}^r$. Then the composite $\mathbb{N}^r_U \longrightarrow M|_U \hookrightarrow \mathcal{O}_U$ defines a pre-log structure, and one can check that $M|_U$ is isomorphic to the log structure associated to the above pre-log structure via the homomorphism $(\mathbb{N}^r_U)^a \longrightarrow M|_U$ induced by φ. Hence M is an fs log structure, and if X is defined over a field k, the morphism $(X, M) \longrightarrow (\operatorname{Spec} k, \text{triv.}\log \text{str.})$ is log smooth.

Second, let X be a toric variety and let $D \subset X$ be the complement of the open torus of X. Then, it is known ([Kk2]) that the canonical log structure on X is isomorphic to the log structure associated to the pair (X, D).

Example 2.4.5. Here we give the simplest example of a log smooth integral morphism of fs log schemes which do not satisfy the properties in Proposition 2.3.2.

Let k be a field and let $n \geq 2$ be an integer prime to the characteristic of k. Let φ be the monoid homomorphism $\mathbb{N} \longrightarrow k$ defined by $\varphi(1) = 0$, and let ψ be the monoid homomorphism $\mathbb{N} \longrightarrow k[t]/(t^n)$ defined by $\psi(1) = t$. The pair $(\mathbb{N}, \varphi)$ (resp. $(\mathbb{N}, \psi)$) naturally defines a pre-log structure on $\operatorname{Spec} k$ (resp. $\operatorname{Spec} k[t]/(t^n)$). Denote the associated log structure by N (resp. M). Let $f : (\operatorname{Spec} k[t]/(t^n), M) \longrightarrow (\operatorname{Spec} k, N)$ be the morphism naturally induced by the following commutative diagram

$$\begin{array}{ccc} k[t]/(t^n) & \overset{\alpha}{\longleftarrow} & k \\ {\scriptstyle\psi}\big\uparrow & & {\scriptstyle\varphi}\big\uparrow \\ \mathbb{N} & \overset{\beta}{\longleftarrow} & \mathbb{N}, \end{array}$$

where α, β is defined by $\alpha(x) = x\,(x \in k)$, $\beta(m) := nm\,(m \in \mathbb{N})$. Then f is a log smooth integral morphism of fs log schemes, but $\operatorname{Spec} k[t]/(t^n)$ is not reduced.

Example 2.4.6. Let S be the spectrum of a discrete valuation ring, and let s be the closed point of S. Let X be a regular scheme and let $f : X \longrightarrow S$ be a flat morphism of finite type. Denote the special fiber of f by Y and let us assume that Y_{red} is a normal crossing divisor of X. Let M (resp. N) be the log structure on X (resp. S) associated to the pair (X, Y_{red}) (resp. (S, s)). Then, f induces naturally a morphism $(X, M) \longrightarrow (S, N)$ between fs log schemes, which will also be denoted by f.

Put $Y := \sum_{i=1}^{r} n_i D_i$, where each D_i is a prime divisor and each n_i is a positive integer. Let us assume moreover that each n_i is invertible on S. Then, etale locally, one can take a chart of the form $(\mathbb{N}_S \overset{\alpha}{\to} \mathcal{O}_S, \mathbb{N}_X^k \overset{\beta}{\to} \mathcal{O}_X, \mathbb{N} \overset{\gamma}{\to} \mathbb{N}^k)$ (where k is an integer satisfying $1 \leq k \leq r$) which satisfies the following conditions:

(1) γ is defined by $\gamma(1) = (m_i)_{i=1}^k$ with $\{m_i\}_{i=1}^k \subset \{n_i\}_{i=1}^r$.

(2) The morphism

$$X \longrightarrow S \times_{\alpha^*, \operatorname{Spec} \mathbb{Z}[\mathbb{N}], \beta^*} \operatorname{Spec} \mathbb{Z}[\mathbb{N}^k]$$

is etale in the classical sense. (Here $\alpha^* : \operatorname{Spec} \mathbb{Z}[\mathbb{N}] \longrightarrow S$ and $\beta^* : \operatorname{Spec} \mathbb{Z}[\mathbb{N}^k] \longrightarrow X$ are the morphisms induced by α, β, respectively.)

So, f is a log smooth integral morphism by Lemma 2.2.7 and Theorem 2.2.8.

Let $g : (Y, M|_Y) \longrightarrow (s, N|_s)$ be $f \times_S s$. Then g is also a log smooth integral morphism between fs log schemes. By Remark 2.3.3, the following conditions are equivalent:

(a) (Assuming the characteristic of s is positive) g is of Cartier type,

(b) Y is reduced.

(c) $Y_{g\text{-triv}} \subset Y$ is dense open.

and the condition (b) is obviously equivalent to the condition that all n_i's are equal to 1.

Chapter 3. Differentials on Log Schemes

In this chapter, we define de Rham fundamental groups for certain fine log schemes by using the category of nilpotent integrable log connections, and then prove a relation between integral log connections and isocrystals on log infinitesimal site. (This is a log version or a log formal version of [B-O, §2].)

3.1. Definition of de Rham fundamental groups

In this section, we give the definition of de Rham fundamental groups for certain fine log schemes. First we give the definitions of log connections and log formal connections.

Definition 3.1.1. Let $(X, M) \longrightarrow (S, N)$ be a morphism of fine log schemes (resp. a morphism of fine formal V-schemes, where V is a complete discrete valuation ring of mixed characteristic). For $E \in \mathrm{Coh}(\mathcal{O}_X)$ (resp. $E \in \mathrm{Coh}(K \otimes \mathcal{O}_X)$, where K is the fraction field of V), a log connection on E (resp. a log formal connection on E) with respect to (X, M) over (S, N) is an additive map

$$\nabla : E \longrightarrow E \otimes \omega^1_{(X,M)/(S,N)}$$

(where $\omega^1_{(X,M)/(S,N)}$ is the log differential module (resp. the formal log differential module)) which satisfies

$$\nabla(ax) = a\nabla(x) + x \otimes da$$

for $a \in \mathcal{O}_X$ and $x \in E$.

A log connection (resp. a log formal connection) ∇ on E is integrable if the composition

$$E \xrightarrow{\nabla} E \otimes \omega^1_{(X,M)/(S,N)} \xrightarrow{\nabla} E \otimes \omega^2_{(X,M)/(S,N)}$$

is zero, where we extend ∇ to

$$E \otimes \omega^1_{(X,M)/(S,N)} \longrightarrow E \otimes \omega^2_{(X,M)/(S,N)}$$

by setting

$$x \otimes \omega \longmapsto \nabla(x) \wedge \omega + x \otimes d\omega.$$

We denote the category of coherent sheaves on X with integrable log connection with respect to (X, M) over (S, N) by $C((X, M)/(S, N))$ (resp. the category of isocoherent sheaves on X with integrable log formal connection with respect to (X, M) over (S, N) by $\hat{C}((X, M)/(S, N))$).

In the following, we sometimes denote the category $C((X, M)/(S, N))$ (resp. $\hat{C}((X, M)/(S, N))$) simply by $C((X/S)^{\log})$ (resp. $\hat{C}((X/S)^{\log})$), if there will be no confusions on log structures. When the log structures M and N are trivial, we further abbreviate to write the superscript $^{\log}$.

Let the notations be as above and let (E, ∇) be an object in $C((X/S)^{\log})$ (resp. $\hat{C}((X/S)^{\log})$). Then we define the de Rham complex $\mathrm{DR}(E, \nabla)$ associated to (E, ∇) by the complex

$$0 \longrightarrow E \xrightarrow{\nabla} E \otimes \omega^1_{X/S} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} E \otimes \omega^n_{X/S} \xrightarrow{\nabla} E \otimes \omega^{n+1}_{X/S} \xrightarrow{\nabla} \cdots,$$

where where we extend ∇ to

$$E \otimes \omega^n_{(X,M)/(S,N)} \longrightarrow E \otimes \omega^{n+1}_{(X,M)/(S,N)}$$

by setting

$$x \otimes \omega \longmapsto \nabla(x) \wedge \omega + x \otimes d\omega.$$

For $(E, \nabla), (E', \nabla') \in C((X/S)^{\log})$ (resp. $(E, \nabla), (E', \nabla') \in \hat{C}((X/S)^{\log})$), we define the tensor product $(E'', \nabla'') := (E, \nabla) \otimes (E', \nabla')$ by

$$E'' := E \otimes E', \quad \nabla'' := s \circ (\nabla \otimes \mathrm{id}) + \mathrm{id} \otimes \nabla',$$

where $s : E \otimes \omega^1_{X/S} \otimes E' \longrightarrow E \otimes E' \otimes \omega^1_{X/S}$ is defined by $s(a \otimes b \otimes c) := a \otimes c \otimes b$. Then one can check easily that the category $C((X/S)^{\log})$ (resp. $\hat{C}((X/S)^{\log})$) is an abelian tensor category with this tensor structure with the unit object $(\mathcal{O}_X, d)$, where $d : \mathcal{O}_X \longrightarrow \omega^1_{X/S}$ is the composite of the usual differential $\mathcal{O}_X \longrightarrow \Omega^1_{X/S}$ and the natural morphism $\Omega^1_{X/S} \longrightarrow \omega^1_{X/S}$. Hence we can define the nilpotent part $\mathcal{N}C((X,M)/(S,N))$ (resp. $\mathcal{N}\hat{C}((X,M)/(S,N))$). By using the result in Section 1.2, we can show the following proposition:

Proposition 3.1.2. *Let k be a field. Let $f : (X,M) \longrightarrow (\operatorname{Spec} k, N)$ be a morphism of fine log schemes and assume that $H^0_{\mathrm{dR}}((X,M)/(\operatorname{Spec} k, N)) := H^0(X, \mathrm{DR}(\mathcal{O}_X, d))$ is a field. Then the category $\mathcal{N}C((X,M)/(\operatorname{Spec} k, N))$ is Tannakian.*

Proof. Since the category $C((X,M)/(\operatorname{Spec} k, N))$ is an abelian tensor category with unit object $(\mathcal{O}_X, d)$ and $\mathrm{End}(\mathcal{O}_X, d) = H^0_{\mathrm{dR}}((X,M)/(\operatorname{Spec} k, N))$ is a field, the nilpotent part $\mathcal{N}C((X,M)/(\operatorname{Spec} k, N))$ is also an abelian category by Proposition 1.2.1. Moreover, one can check that the tensor structure on $C((X,M)/(\operatorname{Spec} k, N))$ induces that on $\mathcal{N}C((X,M)/(\operatorname{Spec} k, N))$ and that it is a rigid abelian tensor category, by using the fact that E is a locally free $\mathcal{O}_X$-module for any object $(E, \nabla) \in \mathcal{N}C((X,M)/(\operatorname{Spec} k, N))$.

Hence it suffices to show the existence of a fiber functor. Let $x \in X$ and let k' be the residue field of x. Let us define the functor $\omega : \mathcal{N}C((X,M)/(\operatorname{Spec} k, N)) \longrightarrow \mathrm{Vec}_{k'}$ by $(E, \nabla) \mapsto E|_x$. Then, since E is locally free for any $(E, \nabla) \in \mathcal{N}C((X, M)/(\operatorname{Spec} k, N))$, ω is an exact tensor functor. Then, by [D2, (2.10)], the functor ω is faithful. So ω is a fiber functor. Hence the assertion is proved. □

Now we define the de Rham fundamental groups:

DEFINITION 3.1.3. Let k be a field and let $f : (X, M) \longrightarrow (\operatorname{Spec} k, N)$ be a morphism of fine log schemes. Assume that $H^0_{\mathrm{dR}}((X, M)/(\operatorname{Spec} k, N))$ is a field. Let x be a k-rational point of $X_{f\text{-triv}}$. (Assume the existence of such a point x.) Then we define the de Rham fundamental group of (X, M) over $(\operatorname{Spec} k, N)$ with base point x by

$$\pi_1^{\mathrm{dR}}((X, M)/(\operatorname{Spec} k, N), x) := G(\mathcal{N}C((X, M)/(\operatorname{Spec} k, N)), \omega_x),$$

where ω_x is the fiber functor

$$\mathcal{N}C((X, M)/(\operatorname{Spec} k, N)) \longrightarrow \mathcal{N}C((x, M)/(\operatorname{Spec} k, N)) \simeq \mathrm{Vec}_k$$

induced by the exact closed immersion $(x, M) \hookrightarrow (X, M)$ and the notation $G(\cdots)$ is as in Theorem 1.1.8.

We will sometimes denote the de Rham fundamental group $\pi_1^{\mathrm{dR}}((X, M)/(\operatorname{Spec} k, N), x)$ simply by $\pi_1^{\mathrm{dR}}((X/\operatorname{Spec} k)^{\log}, x)$, when there will be no confusions on log structures. When the log structures are trivial, we drop the superscript $^{\log}$.

REMARK 3.1.4. The existence of the point x assures that the category $\mathcal{N}C((X, M)/(\operatorname{Spec} k, N))$ in the above definition is actually a *neutral* Tannakian category.

REMARK 3.1.5. In the case without log structures, the above definition is due to Deligne ([D2]).

Here we give one useful sufficient condition for the category $\mathcal{N}C((X, M)/(\operatorname{Spec} k, N))$ to be Tannakian:

PROPOSITION 3.1.6. *Let k be a perfect field and let $f : (X, M) \longrightarrow (\operatorname{Spec} k, N)$ be a log smooth integral morphism of finite type between fine* *log schemes. Assume moreover that X is connected and reduced. Then $H^0_{\mathrm{dR}}((X,M)/(\operatorname{Spec} k, N))$ is a field. (In particular, the category $\mathcal{NC}((X,M)/(\operatorname{Spec} k, N))$ is a Tannakian category by Proposition 3.1.2.)*

PROOF. For a finite extension $k \subset k'$ of fields and a scheme Y over $X \otimes_k k'$, let us denote the ring $H^0_{\mathrm{dR}}((Y,M)/(\operatorname{Spec} k', N))$ simply by $H^0_{\text{log-dR}}(Y/k')$.

First, let us note the following elementary claim:

CLAIM 1. Let $k \subset k' \subset k''$ be finite extensions of fields and put $X' := X \otimes_k k', f' := f \otimes_k k' : (X', M) \longrightarrow (\operatorname{Spec} k', N)$. Let U be a connected component of $X'_{f'\text{-triv}}$ and let $x \in U(k'')$. Then the homomorphism

$$\alpha : H^0_{\text{log-dR}}(U/k') \longrightarrow H^0_{\text{log-dR}}(x/k') = k''$$

is injective, and it is isomorphic if $k' = k''$ holds.

PROOF OF CLAIM 1. Since U is smooth over k and connected, it is an integral scheme. Let K be a function field of U. Then, since U is contained in $X'_{f'\text{-triv}}$, we have

$$H^0_{\text{log-dR}}(U/k') = H^0_{\mathrm{dR}}(U/k') \subset \operatorname{Ker}(K \longrightarrow \Omega^1_{K/k'}),$$

and the elements in the right hand side are algebraic over k'. Hence, for each element $a \in H^0_{\text{log-dR}}(U/k')$, the composite

$$k[a] \hookrightarrow H^0_{\text{log-dR}}(U/k') \xrightarrow{\alpha} H^0_{\text{log-dR}}(x/k') = k''$$

is injective, since it is a homomorphism of fields. So α is injective. On the other hand, we have the natural inclusion $k' \subset H^0_{\text{log-dR}}(U/k')$. Hence α is surjective if $k' = k''$ holds. $\square$

Next, we prove the following claim, which is the key point of the proof:

CLAIM 2. Let k, X, f be as in the statement of the proposition, and let U be a connected component of $X_{f\text{-triv}}$. Then the natural homomorphism

$$\beta : H^0_{\text{log-dR}}(X/k) \longrightarrow H^0_{\text{log-dR}}(U/k)$$

is injective.

Let $X_{f\text{-triv}} = \coprod_{i=1}^{n} U_i$ be the decomposition into the connected components with $U = U_1$. First, to prove the claim 2, we prepare the following claim:

CLAIM 3. To prove the claim 2, one may assume the following condition $(*)$:

$(*)$ For each i, $U_i(k)$ is non-empty.

PROOF OF CLAIM 3. Let $k \subset k'$ be a finite Galois extension such that $U_i(k') \neq \emptyset$ for any i. Put $X' := X \otimes_k k'$, $U_i' := U_i \otimes_k k'$, $f' := f \otimes_k k' : (X', M) \longrightarrow (\operatorname{Spec} k', N)$ and put $U' := U_1'$. Let

$$X' = \coprod_{j=1}^{m} X_j',$$

$$U_i' \cap X_j' = \coprod_{l \in I_{ij}} O_{ijl}$$

be the decomposition of X', $U_i' \cap X_j'$ into the connected components, respectively. (Here I_{ij} is an index set depending on i and j.) Then

$$(X_j')_{f'\text{-triv}} = \coprod_{i=1}^{n} \coprod_{l \in I_{ij}} O_{ijl},$$

$$U_i' = \coprod_{j=1}^{m} \coprod_{l \in I_{ij}} O_{ijl}$$

give the decomposition of $(X_j')_{f'\text{-triv}}$, U_i' into the connected components, respectively.

Let x be a k'-valued point in $U_i \subset X$. Then, since $k \subset k'$ is Galois, each O_{ijl} $(1 \leq j \leq m,\ l \in I_{ij})$ contains a k'-valued point above x. Hence $O_{ijl}(k')$ is non-empty for all i, j, l. On the other hand, since $k \subset k'$ is Galois, X_j' contains a k'-valued point above x for each j. Hence the intersection

$X'_j \cap U'_i$ is non-empty for all i, j, that is, I_{ij} is non-empty for all i, j. Let us choose an element $l_j \in I_{1j}$ for $1 \leq j \leq m$.

For $1 \leq j \leq m$, let β_j be the natural homomorphism

$$H^0_{\text{log-dR}}(X'_j/k') \longrightarrow \prod_{l \in I_{1j}} H^0_{\text{log-dR}}(O_{1jl}/k')$$

and let γ_j be the composite

$$H^0_{\text{log-dR}}(X'_j/k') \xrightarrow{\beta_j} \prod_{l \in I_{1j}} H^0_{\text{log-dR}}(O_{1jl}/k') \xrightarrow{\text{proj.}} H^0_{\text{log-dR}}(O_{1jl_j}/k').$$

Let us consider the following diagram:

$$\begin{array}{ccccc} H^0_{\text{log-dR}}(X/k) & \xrightarrow{\delta_X} & H^0_{\text{log-dR}}(X'/k') & \xrightarrow{\sim} & \prod_{j=1}^m H^0_{\text{log-dR}}(X'_j/k') \\ \beta \downarrow & & & & \prod_j \beta_j \downarrow \\ H^0_{\text{log-dR}}(U/k) & \xrightarrow{\delta_U} & H^0_{\text{log-dR}}(U'/k') & \xrightarrow{\sim} & \prod_{j=1}^m \prod_{l \in I_{1j}} H^0_{\text{log-dR}}(O_{1jl}/k'), \end{array}$$

where δ_X, δ_U are the homomorphisms induced by the scalar extension.

Since we have

$$H^0_{\text{log-dR}}(X'/k') \cong H^0_{\text{log-dR}}(X/k) \otimes_k k',$$

$$H^0_{\text{log-dR}}(U'/k') \cong H^0_{\text{log-dR}}(U/k) \otimes_k k',$$

the homomorphisms δ_X, δ_U are injective. Therefore, if we prove that γ_j's are injective, it implies the injectivity of $\prod_j \beta_j$ and so we obtain the injectivity of β. So, to prove the claim, we may replace X, U, k by X'_j, O_{1jl_j}, k' respectively. Since $O_{ijl}(k')$ is non-empty for each i, j, l, we may assume the condition $(*)$ to prove the claim. □

PROOF OF CLAIM 2. By claim 3, we may assume the condition $(*)$ to prove the claim 2. So we assume it.

For an element $z \in H^0_{\text{log-dR}}(X/k)$, denote the closed set $\{x \in X \mid z = 0 \,\text{in}\, k(x)\}$ (where $k(x)$ is the residue field of x) by C_z and regard it as a reduced closed subscheme of X. Since X is reduced, it suffices to show that $C_a = X$ holds for $a \in \mathrm{Ker}(\beta)$.

Let $a \in \mathrm{Ker}(\beta)$ and let a_i be the image of a in $H^0_{\text{log-dR}}(U_i/k)$. By claim 1 (in the case $k = k' = k''$) for U_i and a k-rational point in U_i (whose existence is assured by the condition $(*)$), the composite

$$k \subset H^0_{\text{log-dR}}(X/k) \longrightarrow H^0_{\text{log-dR}}(U_i/k)$$

is an isomorphism. Hence there exists a unique element $b_i \in k \subset H^0_{\text{log-dR}}(X/k)$ such that $a_i = b_i$ holds in $H^0_{\text{log-dR}}(U_i/k)$. Hence we have $U_i \subset C_{a-b_i}$. Let us denote the closure of U_i in X (regarded as a reduced closed subscheme) by D_i. Then we have $D_i \subset C_{a-b_i}$. Since D_i is reduced, $a - b_i$ is equal to zero in $H^0_{\text{log-dR}}(D_i/k)$.

Now let us assume that $D_i \cap D_j \neq \emptyset$ holds. Then, $b_i = b_j$ holds in $H^0_{\text{log-dR}}((D_i \cap D_j)_{\text{red}})$ since they are both the image of a in $H^0_{\text{log-dR}}((D_i \cap D_j)_{\text{red}})$. Since $b_i, b_j \in k$, we have $b_i = b_j$ in k. Since $X_{f\text{-triv}} \subset X$ is open dense by Proposition 2.3.2, we have $X = \bigcup_i D_i$. This fact and the connectedness of X implies that $b_i = b_1$ holds for all i. On the other hand, since $a \in \mathrm{Ker}(\beta)$, we have $b_1 = 0$. So $b_i = 0$ holds for all i. Therefore we have $D_i \subset C_{a-b_i} = C_a$ for all i. So one get the equality $X = C_a$ and so the assertion is proved. □

Now we finish the proof of the proposition by using the claims. Let U be a connected component of $X_{f\text{-triv}}$ and let x be a closed point of U and let k' be the residue field of x. Let us consider the following diagram:

$$k \subset H^0_{\text{log-dR}}(X/k) \xrightarrow{\beta} H^0_{\text{log-dR}}(U/k) \xrightarrow{\alpha} H^0(x/k) = k'.$$

By claim 1, α is injective and by claim 2, β is injective. So there are inclusions of rings $k \subset H^0_{\text{log-dR}}(X/k) \subset k'$, and they imply that $H^0_{\text{log-dR}}(X/k)$ is a field. □

Remark 3.1.7. Let $f : (X, M) \longrightarrow (\mathrm{Spec}\, k, N)$ be a log smooth integral morphism of finite type between fine log schemes with k perfect and X connected. Then the above proposition says the reducedness of X is a sufficient condition for the category $\mathcal{NC}((X, M)/(\mathrm{Spec}\, k, N))$ to be Tannakian. Here we remark that it is not a necessary condition, by giving a simple counter-example.

Let us assume the characteristic of k is zero. Let X be $\mathrm{Spec}\, k[t,s]/(t^m s^n)$ $(m,n \in \mathbb{N}, (m,n)=1, n \geq 2)$ and let M be the log structure associated to the monoid homomorphism

$$\alpha : \mathbb{N}^2 \longrightarrow k[t,s]/(t^m s^n);\ (1,0) \mapsto t,\ (0,1) \mapsto s.$$

Let N be the log structure associated to the monoid homomorphism $\beta : \mathbb{N} \longrightarrow k, 1 \mapsto 0$ and let $f : (X,M) \longrightarrow (\mathrm{Spec}\, k, N)$ be the morphism of log schemes induced by the following diagram:

$$\begin{array}{ccc} k[t,s]/(t^m s^n) & \xleftarrow{\ a\ } & k \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\beta} \\ \mathbb{N}^2 & \xleftarrow{\ b\ } & \mathbb{N}, \end{array}$$

where a,b is defined by $a(x)=x\,(x \in k)$, $b(1)=(m,n)$. Then X is not reduced. On the other hand, $H^0_{\mathrm{dR}}((X,M)/(\mathrm{Spec}\, k, N))$ is nothing but the kernel of the differential

$$d : K[t,s]/(t^m s^n) \longrightarrow \frac{K[t,s]/(t^m s^n)\mathrm{dlog}\, t \oplus K[t,s]/(t^m s^n)\mathrm{dlog}\, s}{K[t,s]/(t^m s^n)(m\mathrm{dlog}\, t + n\mathrm{dlog}\, s)}.$$

Note that the module on the right hand side above is a free $K[t,s]/(t^m s^n)$-module generated by $\mathrm{dlog}\, s$. We can calculate the differential as follows:

$$d(\sum_{\substack{0 \leq i,j \\ i<m \,\mathrm{or}\, j<n}} a_{ij} t^i s^j) = \sum_{\substack{0 \leq i,j \\ i<m \,\mathrm{or}\, j<n}} a_{ij}(j - \frac{ni}{m}) t^i s^j \mathrm{dlog}\, s.$$

Therefore, we have

$$\begin{aligned} a := \sum_{\substack{0 \leq i,j \\ i<m \,\mathrm{or}\, j<n}} a_{ij} t^i s^j \in \mathrm{Ker}(d) &\iff a_{ij} = 0 \text{ unless } mj - ni = 0 \\ &\iff a_{ij} = 0 \text{ unless } i = j = 0 \\ &\iff a \in k. \end{aligned}$$

Hence $H^0_{\mathrm{dR}}((X,M)/(\mathrm{Spec}\, k, N))$ is a field. So the category $\mathcal{NC}((X,M)/(\mathrm{Spec}\, k, N))$ is Tannakian.

The basic example of a morphism $f : (X, M) \longrightarrow (\operatorname{Spec} k, N)$ for which we would like to investigate the de Rham fundamental group, is the following: k is a field of cahracteristic zero, N is the trivial log structure, X is a proper smooth variety over k, M is the log structure associated to a pair (X, D) (where D is a normal crossing divisor) and f is the morphism induced by the structure morphism of X. In this case, we have the following proposition:

Proposition 3.1.8. *Let the notations be as above and let x be a k-valued point of $U := X - D$. Let j be the open immersion $(U, \text{triv.} \log \text{str.}) \hookrightarrow (X, M)$. Then the homomorphism*

$$j_* : \pi_1^{\mathrm{dR}}(U/\operatorname{Spec} k, x) \longrightarrow \pi_1^{\mathrm{dR}}((X, M)/\operatorname{Spec} k, x)$$

is an isomorphism. In particular, the de Rham fundamental group of (X, M) with base point x depends only on U and x in this case.

Proof. We prove that the functor

$$j^* : \mathcal{N}C((X, M)/\operatorname{Spec} k) \longrightarrow \mathcal{N}C(U/\operatorname{Spec} k)$$

induced by j is an equivalence of categories.

For $(E, \nabla), (E', \nabla') \in \mathcal{N}C((X, M)/\operatorname{Spec} k)$ (resp. $\mathcal{N}C(U/\operatorname{Spec} k)$), the group of homomorphism $\operatorname{Hom}(E, E')$ in the category $\mathcal{N}C((X, M)/\operatorname{Spec} k)$ (resp. $\mathcal{N}C(U/\operatorname{Spec} k)$) is naturally isomorphic to the group $H^0(X, \mathrm{DR}(\mathcal{H}\mathrm{om}((E, \nabla), (E', \nabla'))))$ (resp. $H^0(U, \mathrm{DR}(\mathcal{H}\mathrm{om}((E, \nabla), (E', \nabla')))))$, and the set of the isomorphism class of extensions

$$0 \longrightarrow (\mathcal{O}_X, d) \longrightarrow (\tilde{E}, \tilde{\nabla}) \longrightarrow (E, \nabla) \longrightarrow 0$$

is isomorphic to the set $H^1(X, \mathrm{DR}(E, \nabla))$ (resp. $H^1(U, \mathrm{DR}(E, \nabla)))$. Since the categories $\mathcal{N}C((X, M)/\operatorname{Spec} k)$, $\mathcal{N}C(U/\operatorname{Spec} k)$ are nilpotent, it suffices to show the homomorphisms

$$j^i_{(E,\nabla)} : H^i(X, \mathrm{DR}(E, \nabla)) \longrightarrow H^i(U, \mathrm{DR}(E|_U, \nabla|_U))$$

are isomorphisms for $E \in \mathcal{N}C((X,M)/\operatorname{Spec} k)$ and $i = 0, 1$ to prove the categorical equivalence of j^*. Moreover, by using five lemma, one can see that it suffices to show the homomorphisms

$$j^i : H^i_{\mathrm{dR}}((X,M)/\operatorname{Spec} k) \longrightarrow H^i_{\mathrm{dR}}(U/\operatorname{Spec} k)$$

are isomorphisms for $i \in \mathbb{N}$.

To show j^i's are isomorphisms, one may assume $k = \mathbb{C}$. Let $X^{\mathrm{an}}, U^{\mathrm{an}}$, D^{an} be the complex analytic spaces associated to X, U, D, respectively. Let $j^m_* \Omega^{\cdot}_{U^{\mathrm{an}}}$ (resp. $\Omega^{\cdot}_{X^{\mathrm{an}}}(\operatorname{dlog} D^{\mathrm{an}})$) be the sheaf of meromophic forms on X which are holomorphic on U^{an} (resp. which are holomorphic on U^{an} and logarithmic along D^{an}.) Then, we have the isomorphism

$$H^i_{\mathrm{dR}}(U/\operatorname{Spec} k) \overset{\sim}{=} H^i(X^{\mathrm{an}}, j^m_* \Omega^{\cdot}_{U^{\mathrm{an}}})$$

by [D1, (6.4.3), (6.6.1)], and we have the GAGA isomorphism

$$H^i_{\mathrm{dR}}((X,M)/\operatorname{Spec} k) \overset{\sim}{=} H^i(X^{\mathrm{an}}, \Omega^{\cdot}_{X^{\mathrm{an}}}(\operatorname{dlog} D^{\mathrm{an}})).$$

Via these isomorphimsms, the homomorphism j^i is compatible with the homomorphism

$$j^{i,\mathrm{an}} : H^i(X^{\mathrm{an}}, \Omega^{\cdot}_{X^{\mathrm{an}}}(\operatorname{dlog} D^{\mathrm{an}})) \longrightarrow H^i(X^{\mathrm{an}}, j^m_* \Omega^{\cdot}_{U^{\mathrm{an}}})$$

defined by the inclusion of sheaves $\Omega^{\cdot}_{X^{\mathrm{an}}}(\operatorname{dlog} D^{\mathrm{an}}) \hookrightarrow j^m_* \Omega^{\cdot}_{U^{\mathrm{an}}}$. By [D1, (3.13)], the homomorphism $j^{i,\mathrm{an}}$ is an isomorphism. Hence j^i is also an isomorphism and so the assertion is proved. □

3.2. Log connections, stratifications and isocrystals on log infinitesimal site

Throughout this section, let p be a fixed prime number, let V be a complete discrete valuation ring of mixed characteristic $(0, p)$ and let K be the fraction field of V. Let $f : (X, M) \longrightarrow (S, N)$ be a log smooth morphism of fine log schemes over $\mathbb{Q}$ (resp. a formally log smooth morphism of fine log formal V-schemes). In this section, we prove the equivalence of the following three categories:

(1) The category $C((X,M)/(S,N))$ of integrable log connections (resp. The category $\hat{C}((X,M)/(S,N))$ of integrable log formal connections).

(2) The category $\mathrm{Str}((X,M)/(S,N))$ of log stratifications (resp. The category $\widehat{\mathrm{Str}}((X,M)/(S,N))$ of formal log stratifications). (For definition, see Definition 3.2.10.)

(3) The category $C_{\mathrm{inf}}((X,M)/(S,N))$ of crystals on log infinitesimal site $((X,M)/(S,N))_{\mathrm{inf}}$ (resp. The category $I_{\mathrm{inf}}((X,M)/(S,N))$ of isocrystals on log infinitesimal site $((X,M)/(S,N))_{\mathrm{inf}}$). (For definition, see Definitions 3.2.12, 3.2.13).

We shall use this equivalence in the proof of Berthelot-Ogus theorem for fundamental groups in Chapter 5.

Our method of proof is a log version of that in [B-O, §2] and [O1, §1]: First, we prove that the category in (1) is equivalent to the following auxiliary category.

(4) The category $D((X,M)/(S,N))$ (resp. $\hat{D}((X,M)/(S,N)))$ of pairs (E,φ), where E is a coherent sheaf (resp. an isocohorent sheaf) on X and φ is an order-preserving, $\varprojlim_n \mathcal{O}_{X^n}$-linear ring homomorphism $\mathcal{D}\mathrm{iff}(\mathcal{O}_X,\mathcal{O}_X) \longrightarrow \mathcal{D}\mathrm{iff}(E,E)$ (resp. $\mathcal{D}\mathrm{iff}(K\otimes_V \mathcal{O}_X, K\otimes_V \mathcal{O}_X) \longrightarrow \mathcal{D}\mathrm{iff}(E,E)$), where $\mathcal{D}\mathrm{iff}$ means the sheaf of log differential operators. (For precise definitions, see Definitions 3.2.6, 3.2.8.)

Then, we prove the equivalence of the category in (4) and that in (2), and finally we prove the equivalence of the categories in (2) and (3).

First, we recall the notion of log infinitesimal neighborhood which is due to Kato ([Kk1, (5.8)]).

Proposition-Definition 3.2.1 (Kato). *Let $\mathcal{C}$ be the category of closed immersions $(X,M) \longrightarrow (Y,N)$ of log schemes such that M is fine and N is coherent and let $\mathcal{C}_n$ be the category of exact closed immersions $(X,M) \longrightarrow (Y,N)$ of fine log schemes such that X is defined in Y by an ideal J with the property $J^{n+1}=0$. Then the canonical functor $\mathcal{C}_n \longrightarrow \mathcal{C}$ has a right adjoint $\phi : \mathcal{C} \longrightarrow \mathcal{C}_n$. For $(X,M) \longrightarrow (Y,N)$ in $\mathcal{C}$, the object obtained by applying ϕ has the form $(X,M) \longrightarrow (Z,M_Z)$. We call this (Z,M_Z) the n-th log infinitesimal neighborhood of (X,M) in (Y,N).*

Let $f:(X,M) \longrightarrow (Y,N)$ be a closed immersion of log schemes such that M is fine and N is coherent and assume that there exists a quasi-chart $(P_X \to M, Q_Y \to N, Q \overset{\alpha}{\to} P)$ of f such that P is fine and α^{gp} is surjective. Then the n-th infinitesimal neighborhood (Z,M_Z) of (X,M) in

(Y, N) is constructed as follows ([Kk1, (5.8)]): Put $R := (\alpha^{\mathrm{gp}})^{-1}(P) \subset Q^{\mathrm{gp}}$. Then the natural homomorphism $Q \to Q^{\mathrm{gp}}$ factors as $Q \overset{\beta}{\to} R \to Q^{\mathrm{gp}}$. Let $\beta^* : \operatorname{Spec} \mathbb{Z}[R] \longrightarrow \operatorname{Spec} \mathbb{Z}[Q]$ be the morphism induced by β. Put

$$Y' = Y \times_{\operatorname{Spec} \mathbb{Z}[Q], \beta^*} \operatorname{Spec} \mathbb{Z}[R]$$

and endow Y' with the inverse image N' of the canonical log structure of $\operatorname{Spec} \mathbb{Z}[R]$. Then Z is the (classical) n-th infinitesimal neighborhood of X in Y' and M_Z is the inverse image of N'.

REMARK 3.2.2. We can define the notion of n-th log infinitesimal neighborhood for a locally closed immersion $(X, M) \longrightarrow (Y, N)$ of log schemes such that M is fine and N is coherent, because it is well-defined locally and we can glue it.

For a morphism of fine log schemes $(X, M) \longrightarrow (S, N)$, we can define the n-th log infinitesimal neighborhood of (X, M) in $(X, M) \times_{(S,N)} (X, M)$ (note that the fiber product is taken in the category of log schemes) because the diagonal map $(X, M) \longrightarrow (X, M) \times_{(S,N)} (X, M)$ is locally closed and $(X, M) \times_{(S,N)} (X, M)$ is a coherent log scheme. We denote it by (X^n, M^n). We can regard $\mathcal{O}_{X^n}$ as $\mathcal{O}_X$-modules in two ways through the morphisms

$$f_i : (X^n, M^n) \longrightarrow (X, M) \times_{(S,N)} (X, M) \overset{i\text{-th proj.}}{\longrightarrow} (X, M) \ (i = 1, 2).$$

We call the $\mathcal{O}_X$-module structure of $\mathcal{O}_{X^n}$ defined by f_1 (resp. f_2) the left (resp. right) $\mathcal{O}_X$-module structure.

Now we introduce two kinds of morphisms between log infinitesimal neighborhoods. Before this, let us note the following lemma:

LEMMA 3.2.3. *Let $(X, M) \longrightarrow (S, N)$ be a morphism of fine log schemes and let (X^n, M^n) be the n-th log infinitesimal neighborhood of (X, M) in $(X, M) \times_{(S,N)} (X, M)$. Let $p_i \, (i = 1, 2)$ be the composition $(X^n, M^n) \longrightarrow (X, M) \times_{(S,N)} (X, M) \overset{i\text{-th proj.}}{\longrightarrow} (X, M)$. Then*:

1. *The log structure of $(X^n, M^n) \underset{p_2}{\searrow} \times_{(X,M)} \underset{p_1}{\swarrow} (X^m, M^m)$ is fine.*
2. *The map $(X, M) \longrightarrow (X^n, M^n) \times_{(X,M)} (X^m, M^m)$ is an exact closed immersion.*

Proof. Since the assertion (1) is local, we may assume there exists a chart of $(X, M) \to (S, N)$. Choose a chart $(P_X \to M, Q_S \to N, Q \xrightarrow{\alpha} P)$. Then the diagonal map $(X, M) \longrightarrow (X, M) \times_{(S,N)} (X, M)$ has a quasi-chart

$$(P_X \to M, (P \oplus_Q P)_{(X \times_S X)} \to M \oplus_N M, P \oplus_Q P \xrightarrow{\beta} P),$$

where $P \oplus_Q P$ is the monoid defined by the push-out of the diagram $P \xleftarrow{\alpha} Q \xrightarrow{\alpha} P$ and β is defined by $\beta(p, p') = pp'$ $(p, p' \in P)$. Let R be $\beta^{\mathrm{gp},-1}(P) \subset (P \oplus_Q P)^{\mathrm{gp}}$ and let γ_i $(i = 1, 2)$ be maps $P \xrightarrow{i\text{-th incl.}} P \oplus_Q P \to R$. Then the log structure of $(X^n, M^n) \times_{(X,M)} (X^m, M^m)$ is associated to $R \underset{\gamma_2}{\nwarrow} \oplus_P \underset{\gamma_1}{\nearrow} R$, and it suffices to show this monoid is integral. Let δ_i $(i = 1, 2)$ be maps

$$(P^{\mathrm{gp}} \times P^{\mathrm{gp}})/Q^{\mathrm{gp}} \longrightarrow P^{\mathrm{gp}} \times ((P^{\mathrm{gp}} \times P^{\mathrm{gp}})/Q^{\mathrm{gp}}) \xrightarrow{\mathrm{id} \times i\text{-th proj.}} P^{\mathrm{gp}} \times (P^{\mathrm{gp}}/Q^{\mathrm{gp}}),$$

where the first map is defined by $(p, p') \mapsto (pp', (p, p'))$ for $(p, p') \in P^{\mathrm{gp}}$. Then the maps δ_i are isomorphisms and via δ_i, R is isomorphic to $P \times (P^{\mathrm{gp}}/Q^{\mathrm{gp}})$. So $R \oplus_P R$ is isomorphic to

$$\{P \times (P^{\mathrm{gp}}/Q^{\mathrm{gp}})\} \oplus_P \{P \times (P^{\mathrm{gp}}/Q^{\mathrm{gp}})\} \overset{\sim}{=} P \oplus (P^{\mathrm{gp}}/Q^{\mathrm{gp}})^2.$$

So $R \oplus_P R$ is integral. This proves the assertion (1).

With the above notations, the morphism in (2) admits locally a chart of the form $(P_X \to M, (R \oplus_P R)_{X^n \times_X X^m} \to M_{X^n \times_X X^m}, R \oplus_P R \xrightarrow{\epsilon} P)$ (where we denoted the log structure of the log scheme $(X^n, M^n) \times_{(X,M)} (X^m, M^m)$ by $M_{X^n \times_X X^m}$) such that, via the isomorphism $R \oplus_P R \overset{\sim}{=} P \oplus (P^{\mathrm{gp}}/Q^{\mathrm{gp}})^2$, the homomorphism ϵ coincides with the first projection $P \oplus (P^{\mathrm{gp}}/Q^{\mathrm{gp}})^2 \longrightarrow P$. From this fact, one can check easily that the morphism $(X, M) \longrightarrow (X^n, M^n) \times_{(X,M)} (X^m, M^m)$ is an exact closed immersion. □

Let q be the composition

$$\begin{aligned}(X^n, M^n) \times_{(X,M)} (X^m, M^m) &\to (X, M) \times_{(S,N)} (X, M) \times_{(S,N)} (X, M) \\ &\xrightarrow{\mathrm{pr}_{13}} (X, M) \times_{(S,N)} (X, M)\end{aligned}$$

and consider the following diagram:

$$\begin{array}{ccc}
(X, M) & \longrightarrow & (X^n, M^n) \times_{(X,M)} (X^m, M^m) \\
\big\downarrow & & \big\downarrow{\scriptstyle q} \\
(X^{n+m}, M^{n+m}) & \longrightarrow & (X, M) \times_{(S,N)} (X, M).
\end{array}$$

(Here the morphisms other than q are the ones which are induced by the definition of log infinitesimal neighborhoods.) Then, by the above lemma and the universality of log infinitesimal neighborhood, there exists uniquely a morphism

$$(X^n, M^n) \times_{(X,M)} (X^m, M^m) \longrightarrow (X^{n+m}, M^{n+m}),$$

which is compatible with the above diagram. We denote this map by $\delta_{n,m}$.

In terms of charts, the morphism $\delta_{n,m}$ can be described as follows: Assume $(X, M) \longrightarrow (S, N)$ has a chart $(P_X \to M, Q_S \to N, Q \to P)$ and $X = \operatorname{Spec} A$ and $S = \operatorname{Spec} B$ are affine. Let R be as in the proof of Lemma 3.2.3 and let Y be $\operatorname{Spec}((A \otimes_B A) \otimes_{\mathbb{Z}[P \oplus_Q P]} \mathbb{Z}[R])$. Then there exists a natural morphism $Y \longrightarrow \operatorname{Spec} \mathbb{Z}[R]$. Let L be the log structure on Y defined as the pull-back of the canonical log structure on $\operatorname{Spec} \mathbb{Z}[R]$. Then (Y, L) admits a chart

$$\varphi : R \longrightarrow (A \otimes_B A) \otimes_{\mathbb{Z}[P \oplus_Q P]} \mathbb{Z}[R], \quad r \mapsto (1 \otimes 1) \otimes r \ \ (r \in R).$$

We define a map $\delta : (Y, L) \times_{(X,M)} (Y, L) \longrightarrow (Y, L)$ as the map which is induced by the following diagram of rings and monoids:

$$\begin{array}{ccc}
\{(A \otimes_B A) \otimes_{\mathbb{Z}[P \oplus_Q P]} \mathbb{Z}[R]\} \otimes_A \{(A \otimes_B A) \otimes_{\mathbb{Z}[P \oplus_Q P]} \mathbb{Z}[R]\} & \xleftarrow{\varphi \otimes \varphi} & R \oplus_P R \\
{\scriptstyle f}\big\uparrow & & \big\uparrow{\scriptstyle g} \\
(A \otimes_B A) \otimes_{\mathbb{Z}[P \oplus_Q P]} \mathbb{Z}[R] & \xleftarrow{\varphi} & R,
\end{array}$$

where maps f, g are defined by

$$f((a \otimes b) \otimes 1) = \{(a \otimes 1) \otimes 1\} \otimes \{(1 \otimes b) \otimes 1\},$$

$$f((1 \otimes 1) \otimes (x, y)) = \{(1 \otimes 1) \otimes (x, x^{-1})\} \otimes \{(1 \otimes 1) \otimes (x, y)\},$$

$$g((x, y)) = (x, x^{-1}) \oplus (x, y),$$

for $a, b \in A$ and $(x, y) \in R$ $(x, y \in P^{\mathrm{gp}})$.

Note that (X^n, M^n) is the (classical) n-th infinitesimal neighborhood of (X, M) in (Y, L). One can check that the morphism δ induces a morphism $(X^n, M^n) \times_{(X,M)} (X^m, M^m) \longrightarrow (X^{n+m}, M^{n+m})$ and this map coincides with $\delta_{n,m}$ above.

Now we define another important morphism between log infinitesimal neighborhoods. Let t be the composition $(X^n, M^n) \to (X, M) \times_{(S,N)} (X, M) \overset{\mathrm{pr}_2 \times \mathrm{pr}_1}{\to} (X, M) \times_{(S,N)} (X, M)$ and consider the following diagram:

$$\begin{array}{ccc} (X, M) & \longrightarrow & (X^n, M^n) \\ \Big\downarrow & & \Big\downarrow{\scriptstyle t} \\ (X^n, M^n) & \longrightarrow & (X, M) \times_{(S,N)} (X, M). \end{array}$$

(Here the morphisms other than t are the ones which are induced by the definition of log infinitesimal neighborhoods.) Then, by the universality of log infinitesimal neighborhood, there exists uniquely a map $(X^n, M^n) \longrightarrow (X^n, M^n)$ which is compatible with the above diagram. We denote this map by τ_n.

In terms of charts, the morphism τ_n can be described as follows: Assume $(X, M) \longrightarrow (S, N)$ has a chart $(P_X \to M, Q_S \to N, Q \to P)$ and $X = \operatorname{Spec} A$ and $S = \operatorname{Spec} B$ are affine. Define R, φ and (Y, L) as in the case of $\delta_{n,m}$. We define a map $\tau : (Y, L) \longrightarrow (Y, L)$ as the map which is induced by the following diagram of rings and monoids:

$$\begin{array}{ccc} (A \otimes_B A) \otimes_{\mathbb{Z}[P \oplus_Q P]} \mathbb{Z}[R] & \overset{\varphi}{\longleftarrow} & R \\ {\scriptstyle f}\Big\uparrow & & \Big\uparrow{\scriptstyle g} \\ (A \otimes_B A) \otimes_{\mathbb{Z}[P \oplus_Q P]} \mathbb{Z}[R] & \overset{\varphi}{\longleftarrow} & R, \end{array}$$

where maps f, g are defined by

$$f((a \otimes b) \otimes 1) = (b \otimes a) \otimes 1,$$

$$f((1 \otimes 1) \otimes (x, y)) = (1 \otimes 1) \otimes (y, x),$$
$$g((x, y)) = (y, x),$$

for $a, b \in A$ and $(x, y) \in R$ $(x, y \in P^{\mathrm{gp}})$.

Then one can check that τ induces a map $(X^n, M^n) \longrightarrow (X^n, M^n)$ and this map coincides with τ_n above.

Remark 3.2.4. What we described about log infinitesimal neighborhood holds also in the case of log formal V-schemes (with appropriate modifications). We use the same notations also in the case of fine log formal V-schemes.

Next we prove the relation between log differential module and the log infinitesimal neighborhood, which is also due to Kato.

Before stating the proposition, we prepare some notations. Let $f : (X, M) \longrightarrow (S, N)$ be a morphism of fine log schemes. Let $\Delta_n : (X, M) \hookrightarrow (X^n, M^n)$ be the n-th log infinitesimal neighborhood of the diagonal morphism $(X, M) \longrightarrow (X, M) \times_{(S,N)} (X, M)$. Let p_i $(i = 1, 2)$ be the composite of the canonical map $(X^n, M^n) \longrightarrow (X, M) \times_{(S,N)} (X, M)$ and the i-th projection. Let $U \longrightarrow X$ be an etale morphism of schemes and let m be an element of $\Gamma(U, M)$. Then we have $p_i^*(m) \in \Gamma(U, M^n)$ $(i = 1, 2)$ and

$$\Delta_n^* \circ p_1^*(m) = m = \Delta_n^* \circ p_2^*(m). \tag{3.2.1}$$

(Note that the etale site of X^n and that of X is equivalent via Δ_n^*. So we can regard $M^n, \mathcal{O}_{X^n}$ as sheaves on X_{et}.) Let us consider the following homomorphisms:

$$\Gamma(U, M^n) \xrightarrow{a} \Gamma(U, M^n/\mathcal{O}_{X^n}^\times) \xrightarrow{b} \Gamma(U, M/\mathcal{O}_X^\times),$$

where a is the natural projection and b is the homomorphism induced by Δ_n. Then, by equation (3.2.1), we have $b \circ a(p_1^*(m)) = b \circ a(p_2^*(m))$. Now, since Δ_n is an exact closed immersion, b is an isomorphism. Hence we have $a(p_1^*(m)) = a(p_2^*(m))$. Hence, locally on U, there exists an element u of $\mathcal{O}_{X^n}^\times$ such that $p_2^*(m) = p_1^*(m)u$ holds. One can check, by using the fineness of M^n, that one can glue this element u and it defines uniquely an element in $\Gamma(U, \mathcal{O}_{X^n}^\times) \subset \Gamma(U, M^n)$. We denote this element by $(m^{-1}, m)_n$

in this section. By the uniqueness of the element $(m^{-1},m)_n$, the image of $(m^{-1},m)_n$ by the homomorphism $\Gamma(U,M^n) \longrightarrow \Gamma(U,M^{n'})$ $(n'<n)$ coincides with $(m^{-1},m)_{n'}$. On the other hand, by using the equation $p_2^*(m) = p_1^*(m)(m^{-1},m)_n$ and the fineness of M, we have $\Delta_n^*((m^{-1},m)_n) = 1$. In particular, we have $(m^{-1},m)_n - 1 \in \mathrm{Ker}(\Gamma(U,\mathcal{O}_{X^n}) \longrightarrow \Gamma(U,\mathcal{O}_X))$.

The above element is compatible with a chart in the following sense: With the above notation, let us assume that we have a chart $(Q_S \to N, P_X \overset{\varphi}{\to} M, Q \overset{c}{\to} P)$. Let $d : P \oplus_Q P \longrightarrow P$ be the homomorphism defined by $d(p,p') = pp'$ and put $R := d^{\mathrm{gp},-1}(P)$. Then, as we have seen, (X^n, M^n) has naturally a chart $R \to M^n$, which we denote by ψ. Let $p \in P$. Then one can check that the element $(\varphi(p)^{-1}, \varphi(p))_n$ coincides with the image of $(p^{-1},p) \in R(\subset (P \oplus_Q P)^{\mathrm{gp}})$ by ψ.

Now the relation between log differential module and the log infinitesimal neighborhood is described as follows ([Kk1, (5.8.1)]):

PROPOSITION 3.2.5. *Let* $(X,M) \overset{f}{\longrightarrow} (S,N)$ *be a morphism of fine log schemes and let* $\mathcal{I}$ *be* $\mathrm{Ker}(\mathcal{O}_{X^1} \longrightarrow \mathcal{O}_X)$. *Then there exists a canonical isomorphism of* $\mathcal{O}_X$*-modules*

$$\Phi : \mathcal{I} \overset{\sim}{\longrightarrow} \omega^1_{(X,M)/(S,N)},$$

which satisfies the following condition: For $m \in M$, *the image of the element* $\alpha^1((m^{-1},m)_1) - 1 \in \mathcal{I}$ (*where* $\alpha^1 : M^1 \longrightarrow \mathcal{O}_{X^1}$ *is the homomorphism defining the log structure*) *by* Φ *is equal to* $\mathrm{dlog}\,(m)$.

Henceforth we sometimes identify these modules via Φ.

PROOF. Denote the structure homomorphism $M \longrightarrow \mathcal{O}_X$ of the log scheme (X,M) by α. For an $\mathcal{O}_X$-module E, let us define the module $\mathrm{Der}_{(S,N)}((X,M),E)$ of log derivations of (X,M) to E over (S,N) to be the module consists of pairs (D, Dlog) of $D \in \mathrm{Der}_S(X,E)$(:= the module of derivations of X to E over S in the classical sense) and $\mathrm{Dlog} : M \to E$ satisfying

(1) $\mathrm{Dlog}(ab) = \mathrm{Dlog}(a) + \mathrm{Dlog}(b) \qquad (a,b \in M)$,
(2) $\alpha(a)\mathrm{Dlog}(a) = D(\alpha(a)) \qquad (a \in M)$,
(3) $\mathrm{Dlog}(f^*(c)) = 0 \qquad (c \in f^{-1}N,\ \ f^* : f^{-1}N \to M)$.

Then it follows from [Kf1, (5.2)] that the homomorphism

$$\Psi : \mathrm{Hom}_{\mathcal{O}_X}(\omega^1_{(X,M)/(S,N)}, E) \longrightarrow \mathrm{Der}_{(S,N)}((X,M),E);\ u \mapsto (u\circ d, u\circ \mathrm{dlog}\,)$$

is an isomorphism.

Next we define a homomorphism

$$\Lambda : \mathrm{Der}_{(S,N)}((X,M),E) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I},E).$$

Let (D, Dlog) be an element in $\mathrm{Der}_{(S,N)}((X,M),E)$. Let us introduce a structure of a ring on $\mathcal{O}_X \oplus E$ by

$$(x,e)\cdot(x',e') = (xx', xe' + x'e).$$

Then it is an $\mathcal{O}_X$-algebra. Let $\pi : \underline{\mathrm{Spec}}\,(\mathcal{O}_X \oplus E) \longrightarrow X$ be the corresponding morphism of schemes, and let $\theta : X \longrightarrow \underline{\mathrm{Spec}}\,(\mathcal{O}_X \oplus E)$ be the morphism induced by the projection $\mathcal{O}_X \oplus E \longrightarrow \mathcal{O}_X$. Then these morphisms induce the morphisms of log schemes

$$\begin{aligned}\pi &: (\underline{\mathrm{Spec}}\,(\mathcal{O}_X \oplus E), \pi^*M) \longrightarrow (X,M),\\ \theta &: (X,M) \longrightarrow (\underline{\mathrm{Spec}}\,(\mathcal{O}_X \oplus E), \pi^*M),\end{aligned}$$

respectively. Denote the structure morphism $\pi^*M \longrightarrow \mathcal{O}_X \oplus E$ of the log scheme $(\underline{\mathrm{Spec}}\,(\mathcal{O}_X \oplus E), \pi^*M)$ by β. Let us consider the following commutative diagrams:

$$\begin{array}{ccc} \pi^{-1}\mathcal{O}_X & \xleftarrow{\pi^{-1}\alpha} & \pi^{-1}M \\ {\scriptstyle\gamma_1}\big\downarrow & & {\scriptstyle\delta_1}\big\downarrow \\ \mathcal{O}_X \oplus E & \xleftarrow{\ \beta\ } & \pi^*M, \end{array} \qquad \begin{array}{ccc} \pi^{-1}\mathcal{O}_X & \xleftarrow{\pi^{-1}\alpha} & \pi^{-1}M \\ {\scriptstyle\gamma_2}\big\downarrow & & {\scriptstyle\delta_2}\big\downarrow \\ \mathcal{O}_X \oplus E & \xleftarrow{\ \beta\ } & \pi^*M, \end{array}$$

where the homomorphisms $\gamma_i, \delta_i\ (i = 1,2)$ are defined by

$$\gamma_1(x) = (x, 0), \qquad \gamma_2(x) = (x, D(x)),\ (x \in \pi^{-1}\mathcal{O}_X),$$

$$\delta_1(m) = m, \qquad \delta_2(m) = m\cdot(1, \mathrm{Dlog}(m)),\ (m \in \pi^{-1}M).$$

(Note that $(1, \mathrm{Dlog}(m)) \in (\mathcal{O}_X \oplus E)^\times \subset \pi^*M$ holds.) Each of the above diagram defines a morphism $(\underline{\mathrm{Spec}}\,(\mathcal{O}_X \oplus E), \pi^*M) \longrightarrow (X, M)$. So they defines a morphism $g : (\underline{\mathrm{Spec}}\,(\mathcal{O}_X \oplus E), \pi^*M) \longrightarrow (X, M) \times_{(S,M)} (X, M)$, and one can check the commutativity of the following diagrams:

$$\begin{array}{ccc} (X, M) & \xrightarrow{\Delta} & (X, M) \times_{(S,N)} (X, M) \\ \| & & g \uparrow \\ (X, M) & \xrightarrow{\theta} & (\underline{\mathrm{Spec}}\,(\mathcal{O}_X \oplus E), \pi^*M), \end{array}$$

$$\begin{array}{ccc} (X, M) \times_{(S,N)} (X, M) & \xrightarrow{\mathrm{pr}_1} & (X, M) \\ g \uparrow & & \| \\ (\underline{\mathrm{Spec}}\,(\mathcal{O}_X \oplus E), \pi^*M) & \xrightarrow{\pi} & (X, M), \end{array}$$

By the first diagram and the universal property of log infinitesimal neighborhood, there exists uniquely a morphism $g^1 : (\underline{\mathrm{Spec}}\,(\mathcal{O}_X \oplus E), \pi^*M) \longrightarrow (X^1, M^1)$ such that the composite of g^1 with the natural morphism $(X^1, M^1) \longrightarrow (X, M) \times_{(S,N)} (X, M)$ is equal to g. By the above two diagrams, one can see that there exists the following commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{X^1} & \longrightarrow & \mathcal{O}_X \\ g^{1,*} \downarrow & & \| \\ \mathcal{O}_X \oplus E & \xrightarrow{\theta^*} & \mathcal{O}_X, \end{array}$$

which is $\mathcal{O}_X$-linear with respect to left $\mathcal{O}_X$-module structure on $\mathcal{O}_{X^1}$ and the structure of $\mathcal{O}_X$-module on $\mathcal{O}_X \oplus E$ via π^*. Hence $g^{1,*}$ induces an $\mathcal{O}_X$-linear homomorphism $\overline{g}^{1,*} : \mathcal{I} \longrightarrow E$. We define $\Lambda((D, \mathrm{Dlog})) \in \mathrm{Hom}(\mathcal{I}, E)$ by $\Lambda((D, \mathrm{Dlog})) := \overline{g}^{1,*}$.

Next we construct a homomorphism

$$\Lambda' : \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}, E) \longrightarrow \mathrm{Der}_{(S,N)}((X, M), E).$$

Suppose given an element φ in $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}, E)$. Let us introduce a structure of an $\mathcal{O}_X$-algebra on $\mathcal{O}_X \oplus \mathcal{I}$ as in the case of $\mathcal{O}_X \oplus E$ in the previous paragraph. Then the $\mathcal{O}_X$-linear homomorphism of rings $i : \mathcal{O}_X \oplus \mathcal{I} \longrightarrow \mathcal{O}_{X^1}$

defined by $i(x,y) = x+y$ (where we consider the left $\mathcal{O}_X$-module structure on $\mathcal{O}_{X^1}$) is an isomorphism. Let $\tilde{\varphi} : \mathcal{O}_X \oplus \mathcal{I} \longrightarrow \mathcal{O}_X \oplus E$ be the $\mathcal{O}$-algebra homomorphism defined by $\tilde{\varphi}((x,y)) := (x, \varphi(y))$, and let $\tilde{D}$ and $\widetilde{\mathrm{Dlog}}$ be the following homomorphisms:

$$\tilde{D} : \mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_X \xrightarrow{\Delta_1^*} \mathcal{O}_{X^1} \xrightarrow{i^{-1}} \mathcal{O}_X \oplus \mathcal{I} \xrightarrow{\tilde{\varphi}} \mathcal{O}_X \oplus E \xrightarrow{\text{proj.}} E,$$

$$\widetilde{\mathrm{Dlog}} : M^1 \xrightarrow{\alpha^1} \mathcal{O}_{X^1} \xrightarrow{i^{-1}} \mathcal{O}_X \oplus \mathcal{I} \xrightarrow{\tilde{\varphi}} \mathcal{O}_X \oplus E \xrightarrow{\text{proj.}} E,$$

where $\alpha^1 : M^1 \longrightarrow \mathcal{O}_{X^1}$ is the structure homomorphism of (X^1, M^1). One define (D, Dlog) by $D(x) := \tilde{D}(1 \otimes x), \mathrm{Dlog}(m) := \widetilde{\mathrm{Dlog}}((m^{-1}, m)_1)$. Then one can check that the pair (D, Dlog) defines an element in $\mathrm{Der}_{(S,N)}((X,M),E)$. We define $\Lambda'(\varphi)$ by $\Lambda'(\varphi) := (D, \mathrm{Dlog})$. Hence we have defined the homomorphism Λ'.

One can check that Λ and Λ' are mutually inverse. (Details are left to the reader.) Hence Λ is an isomorphism. Define the functorial isomorphism

$$\Phi_E : \mathrm{Hom}_{\mathcal{O}_X}(\omega^1_{X/S}, E) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}, E)$$

by $\Phi_E := \Lambda \circ \Psi$, and define $\Phi : \mathcal{I} \longrightarrow \omega^1_{X/S}$ by $\Phi := \Phi_{\omega^1_{X/S}}(\mathrm{id}_{\omega^1_{X/S}})$. Then Φ is an isomorphism.

Finally we show that the element $\alpha^1((m^{-1}, m)_1) - 1 \in \mathcal{I}$ is sent to $\mathrm{dlog}\,(m)$ by Φ. Let us define the homomorphism $\tilde{\Phi} : \mathcal{O}_X \oplus \mathcal{I} \longrightarrow \mathcal{O}_X \oplus \omega^1_{X/S}$ by $\tilde{\Phi}(x,y) := (x, \Phi(y))$. Then, by using the descriptions of $\Psi, \Lambda, \Lambda^{-1} = \Lambda'$ above, one can see that the image of the element $(m^{-1}, m)_1 \in M^1$ by the composite

$$M^1 \xrightarrow{\alpha^1} \mathcal{O}_{X^1} \xrightarrow{i^{-1}} \mathcal{O}_X \oplus \mathcal{I} \xrightarrow{\tilde{\Phi}} \mathcal{O}_X \oplus \omega^1_{X/S} \xrightarrow{\text{proj.}} \omega^1_{X/S}$$

is equal to $\mathrm{dlog}\,(m)$. On the other hand, we have $z := \alpha^1((m^{-1}, m)_1) - 1 \in \mathcal{I}$. So we have $i^{-1} \circ \alpha^1((m^{-1}, m)_1) = (1, z)$. Then one can see $\mathrm{dlog}\,(m)$ is equal to $\Phi(z)$. So the assertion is proved and the proof is finished. □

Now we define the notion of the sheaf of log differential operators.

Definition 3.2.6. Let $(X, M) \longrightarrow (S, N)$ be a morphism of fine log schemes (resp. fine log formal V-schemes). For $E, F \in \mathrm{Coh}(\mathcal{O}_X)$ (resp.

Coh($K \otimes \mathcal{O}_X$)), we define the sheaf of log differential operators (resp. formal log differential operators) of order $\leq n$ $\mathcal{D}\mathrm{iff}^n(E,F)$ by $\mathcal{D}\mathrm{iff}^n(E,F) = \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{O}_{X^n} \otimes E, F)$ and the sheaf of log differential operators $\mathcal{D}\mathrm{iff}(E,F)$ by $\mathcal{D}\mathrm{iff}(E,F) = \varinjlim_n \mathcal{D}\mathrm{iff}^n(E,F)$.

We define an $\mathcal{O}_{X^n}$-module structure of $\mathcal{D}\mathrm{iff}^n(E,F)$ by $(af)(x) = f(ax)$ for $a \in \mathcal{O}_{X^n}, f \in \mathcal{D}\mathrm{iff}^n(E,F)$, $x \in \mathcal{O}_{X^n} \otimes E$. This induces a $\varprojlim_n \mathcal{O}_{X^n}$-module structure of $\mathcal{D}\mathrm{iff}(E,F)$.

For $f \in \mathcal{D}\mathrm{iff}^n(E,E)$ and $g \in \mathcal{D}\mathrm{iff}^m(E,E)$, we define the product $f * g \in \mathcal{D}\mathrm{iff}^{n+m}(E,E)$ by the composition

$$\mathcal{O}_{X^{n+m}} \otimes E \overset{\delta^*_{n,m}\otimes\mathrm{id}}{\longrightarrow} \mathcal{O}_{X^n} \otimes \mathcal{O}_{X^m} \otimes E \overset{\mathrm{id}\otimes g}{\longrightarrow} \mathcal{O}_{X^n} \otimes E \overset{f}{\longrightarrow} E.$$

This operation $*$ defines an $\mathcal{O}_X$-algebra structure of $\mathcal{D}\mathrm{iff}(E,E)$.

In the rest of this section, we put $\mathcal{O} := \mathcal{O}_X$ for a scheme X and $\mathcal{O} := K \otimes_V \mathcal{O}_X$ for a p-adic formal scheme X.

In certain case, the local structure of the sheaf of log differential operators $\mathcal{D}\mathrm{iff}(\mathcal{O},\mathcal{O})$ of $\mathcal{O}$ can be described as follows:

LEMMA 3.2.7. *Let $(X,M) \longrightarrow (S,N)$ be a log smooth morphism of fine log schemes over $\mathbb{Q}$ or a formally log smooth morphism of fine log formal V-schemes. Assume that X,S are affine and that there exists a chart $(P_X \overset{\varphi}{\to} M, Q_S \to N, Q \to P)$ satisfying the conditions (a), (b), (c) of (2) in Theorem 2.2.8. Let $x_1,\cdots,x_m \in P$ be elements such that* $\mathrm{dlog}\, x_i$ $(1 \leq i \leq m)$ *forms a basis of $\omega^1_{X/S}$, and put $\xi_{i,n} := (\varphi(x_i)^{-1}, \varphi(x_i))_n - 1 \in \Gamma(X^n, \mathcal{O}_{X^n})$. (Note that $\xi_{i,n}$'s are compatible with respect to n.) Then:*

(1) *$\mathcal{O}_{X^n}$ is a free left $\mathcal{O}_X$-module with basis*

$$\{\xi_n^a := \prod_{i=1}^m \xi_{i,n}^{a_i} \,|\, 0 \leq |a| := \sum_{i=1}^m a_i \leq n\},$$

where $a := (a_1, a_2, \cdots, a_m)$ is a multi-index of length m.

(2) *Let $\{D_a\}_{0\leq|a|\leq n}$ be the dual base of $\{\xi_n^a\}_{0\leq|a|\leq n}$ in $\mathcal{D}\mathrm{iff}^n(\mathcal{O},\mathcal{O})$, and denote their image in $\mathcal{D}\mathrm{iff}(\mathcal{O},\mathcal{O})$ by the same letter. (Since $\xi_{i,n}$'s are compatible with respect to n, the definition of $D_a \in \mathcal{D}\mathrm{iff}(\mathcal{O},\mathcal{O})$ is independent of n.) Then there exists a ring isomorphism*

$$\Phi : \mathcal{O}[T_1, T_2, \cdots, T_m] \longrightarrow \mathcal{D}\mathrm{iff}(\mathcal{O},\mathcal{O}),$$

which sends T_i *to* $D_{(i)} := D_{(0,\cdots,\overset{i}{\check{1}},\cdots,0)}$.

Proof. Here we prove the assertions only in the case of fine log schemes over $\mathbb{Q}$. (The similar method can be applied to the case of fine log formal V-schemes. Details are left to the reader.)

Let $\alpha : P \oplus_Q P \longrightarrow P$ be the homomorphism induced by the summation. Put $R := \alpha^{\mathrm{gp},-1}(P)$ and let Y be $(X \times_S X) \times_{\mathrm{Spec}\,\mathbb{Z}[P \oplus_Q P]} \mathrm{Spec}\,\mathbb{Z}[R]$. Let L be the log structure on Y induced by the canonical log structure on $\mathrm{Spec}\,\mathbb{Z}[R]$.

Now we show the assertion (1) by induction on n. The assertion is trivial if $n = 0$. Let $\Delta : (X, M) \hookrightarrow (Y, L)$ be the exact closed immersion which is naturally induced from the diagonal morphism $(X, M) \longrightarrow (X, M) \times_{(S,N)} (X, M)$, and put $\mathcal{I} := \mathrm{Ker}(\mathcal{O}_Y \overset{\Delta^*}{\longrightarrow} \mathcal{O}_X)$. Then, by the paragraph after Proposition-Definition 3.2.1 and Proposition 3.2.5, we have the canonical ismorphism $\mathcal{I}/\mathcal{I}^2 \overset{\sim}{\longrightarrow} \omega^1_{X/S}$ which sends $\xi_{i,1} \in \mathcal{I}/\mathcal{I}^2$ to $\mathrm{dlog}\, x_i$. Hence $\mathcal{I}/\mathcal{I}^2$ is a free $\mathcal{O}_X$-module with basis $\{\xi_{i,1}\}_{i=1}^m$. By using the exact sequence

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{O}_{X^1} \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

we see the assertion for $n = 1$.

Let us consider the general case. Let f be the composite

$$(Y, L) \longrightarrow (X, M) \times_{(S,N)} (X, M) \overset{\text{1st proj.}}{\longrightarrow} (X, M).$$

Then, one can see that the chart $(P_X \to M, R_Y \overset{\gamma}{\to} L, P \overset{\beta}{\to} R)$ (here β is the composite $P \overset{\text{1st incl.}}{\longrightarrow} P \oplus_Q P \longrightarrow R$) satisfies the conditions (a), (b) of (2) in Theorem 2.2.8. Hence f is log smooth. On the other hand, $\Delta : (X, M) \hookrightarrow (Y, N)$ is an exact closed immersion and $f \circ \Delta = \mathrm{id}$ holds. Hence we have $Y_{f\text{-triv}} \supset \Delta(X)$. So f is smooth in classical sense on a neighborhood of $\Delta(X)$. So we have $\mathcal{I}^n/\mathcal{I}^{n+1} \overset{\sim}{=} \mathrm{Sym}^n(\mathcal{I}/\mathcal{I}^2)$ and it is a free $\mathcal{O}_X$-module with basis $\{\xi_n^a \,|\, |a| = n\}$. Then the assertion follows from the inductive hypothesis and the exact sequence

$$0 \longrightarrow \mathcal{I}^n/\mathcal{I}^{n+1} \longrightarrow \mathcal{O}_{X^n} \longrightarrow \mathcal{O}_{X^{n-1}} \longrightarrow 0.$$

Next we prove the assertion (2). Let ψ be the composite $R_Y \overset{\gamma}{\longrightarrow} L \longrightarrow \mathcal{O}_Y$ and put $\xi_i := \psi((x_i^{-1}, x_i))$. Then we have $\xi_i \bmod \mathcal{I}^{n+1} = \xi_{i,n}$. Put $\xi^a := \prod_{i=1}^m \xi_i^{a_i}$ for a multi-index $a = (a_1, \cdots, a_m)$ of length m.

By definition, $\mathcal{D}\mathrm{iff}(\mathcal{O}_X, \mathcal{O}_X) = \bigoplus_a \mathcal{O}_X \cdot D_a$ holds. We will calculate $D_a * D_b$ for multi-indices a, b of length m. Recall we have defined the morphism $\delta : (Y, L) \times_{(X,M)} (Y, L) \longrightarrow (Y, L)$ which is compatible with $\delta_{k,l} : (X^k, M^k) \times_{(X,M)} (X^l, M^l)$. First the image of ξ_i by the map $\delta^* : \mathcal{O}_Y \longrightarrow \mathcal{O}_Y \otimes \mathcal{O}_Y$ is calculated as follows :

$$\begin{aligned}\delta^*(\xi_i) &= \delta^*(\beta((x_i^{-1}, x_i)) - 1) \\ &= \beta((x_i^{-1}, x_i)) \otimes \beta((x_i^{-1}, x_i)) - 1 \\ &= (\beta((x_i^{-1}, x_i)) - 1) \otimes (\beta((x_i^{-1}, x_i)) - 1) \\ &\qquad + (\beta((x_i^{-1}, x_i)) - 1) \otimes 1 + 1 \otimes (\beta((x_i^{-1}, x_i)) - 1) \\ &= \xi_i \otimes \xi_i + \xi_i \otimes 1 + 1 \otimes \xi_i.\end{aligned}$$

Hence for a multi-index $c = (c_1, c_2, ..., c_m)$ of length m, we have

$$\begin{aligned}\delta^*(\xi^c) &= \prod_{i=1}^m (\xi_i \otimes \xi_i + \xi_i \otimes 1 + 1 \otimes \xi_i)^{c_i} \\ &= \prod_{i=1}^m \Big(\sum_{\substack{0 \leq p_i, q_i, r_i \\ p_i + q_i + r_i = c_i}} \frac{c_i!}{p_i! q_i! r_i!} \xi_i^{p_i + q_i} \otimes \xi_i^{q_i + r_i} \Big).\end{aligned}$$

So we have

$$\delta_{k,l}^*(\xi_{k+l}^c) = \prod_{i=1}^m \Big(\sum_{\substack{0 \leq p_i, q_i, r_i \\ p_i + q_i + r_i = c_i}} \frac{c_i!}{p_i! q_i! r_i!} \xi_{i,k}^{p_i + q_i} \otimes \xi_{i,l}^{q_i + r_i} \Big).$$

Therefore,

$$D_a * D_b(\xi_{|a|+|b|}^c) = \prod_{i=1}^m \Big(\sum_{(*)} \frac{c_i!}{p_i! q_i! r_i!} \Big), \tag{3.2.2}$$

holds, where the sum (*) is taken over (p_i, q_i, r_i) satisfying

$$p_i, q_i, r_i \geq 0, \quad p_i + q_i + r_i = c_i, \quad p_i + q_i = a_i, \quad q_i + r_i = b_i.$$

The symmetry of the right hand side of the equation (3.2.2) implies $D_b * D_a = D_a * D_b$. Hence we can define a ring homomorphism Φ :

$\mathcal{O}[T_1, T_2, ..., T_m] \to \mathcal{D}\mathrm{iff}(\mathcal{O}, \mathcal{O})$ which sends T_i to $D_{(i)}$. Moreover, by the equation (3.2.2), we can write $D_a * D_b$ by the linear combination of D_c's $(c \leq a+b)$

$$D_a * D_b = \sum_{c \leq a+b} k_c D_c, \tag{3.2.3}$$

where k_c are integers and k_{a+b} is non zero.

Now let $a := (a_1, \cdots, a_m)$ be a multi-index which satisfies $|a| \geq 2$. Choose a multi-index $b := (b_1, \cdots, b_m)$ satisfying $b \neq 0, a$ and $b_i \leq a_i \, (1 \leq i \leq m)$. Then the equation (3.2.3) implies that D_a can be expressed by a linear combination of $D_b * D_{a-b}$ and D_c $(c < a)$ over $\mathbb{Q}$. (Note that X is defined over $\mathbb{Q}$.) Hence D_a is in a sub-$\mathcal{O}_X$-algebra of $\mathcal{D}\mathrm{iff}(\mathcal{O}_X, \mathcal{O}_X)$ which is generated by D_c $(c < a)$. Using this fact inductively, one can show that Φ is surjective.

Finally we prove Φ is injective. Let us put $D^{*a} := D_{(1)}^{*a_1} * \cdots D_{(m)}^{*a_m}$ for a multi-index a of length m. Assume that $\Phi(\sum_{|a| \leq n} f_a T^a) = 0$ for some $n \in \mathbb{N}$ and $f_a \in \mathcal{O}_X$ $(|a| \leq n)$. Then

$$0 = \sum_{|a| \leq n} f_a D^{*a} \tag{3.2.4}$$

holds. By the equation (3.2.3), for any multi index a,

$$D^{*a} = \sum_{c \leq a} k_{a,c} D_c$$

holds for some $k_{a,c} \in \mathbb{Z}$ such that $k_{a,a} \neq 0$. Using this, the right hand side of the equation (3.2.4) can be written as

$$\sum_{|a| < n} g_a D_a + \sum_{|a| = n} k_{a,a} f_a D_a$$

for some $g_a \in \mathcal{O}_X$. Since D_a's form a basis of $\mathcal{D}\mathrm{iff}(\mathcal{O}, \mathcal{O})$, $k_{a,a} f_a (|a| = n)$ equals to zero in the above expression, hence $f_a = 0$ if $|a| = n$. One can show that $f_a = 0$ holds for all a by using this argument inductively. Hence Φ is injective. □

We define the category $D((X,M)/(S,N))$ (resp. $\hat{D}((X,M)/(S,N)))$ by using the sheaf of log differential operators (resp. the sheaf of formal log differential operators) as follows:

DEFINITION 3.2.8. Let $(X,M) \longrightarrow (S,N)$ be a morphism of fine log schemes (resp. fine log formal V-schemes). Then we define the category $D((X,M)/(S,N))$ (resp. $\hat{D}((X,M)/(S,N)))$ as follows: An object is a pair (E,φ), where E is a coherent sheaf on X (resp. an isocoherent sheaf on X) and φ is an order-preserving $\varprojlim_n \mathcal{O}_{X^n}$-linear ring homomorphism $\mathcal{D}\mathrm{iff}(\mathcal{O},\mathcal{O}) \longrightarrow \mathcal{D}\mathrm{iff}(E,E)$. A morphism from (E,φ) to (E',φ') is an $\mathcal{O}$-linear homomorphism $\alpha : E \longrightarrow E'$ which makes the following diagram commutative:

$$\begin{array}{ccc} \mathcal{D}\mathrm{iff}(\mathcal{O},\mathcal{O}) & \xrightarrow{\varphi} & \mathcal{D}\mathrm{iff}(E,E) \\ {\scriptstyle\varphi'}\big\downarrow & & {\scriptstyle\alpha\circ-}\big\downarrow \\ \mathcal{D}\mathrm{iff}(E',E') & \xrightarrow{-\circ(\mathrm{id}\otimes\alpha)} & \mathcal{D}\mathrm{iff}(E,E'). \end{array}$$

Then, we have the following equivalence:

PROPOSITION 3.2.9. *Let $f : (X,M) \longrightarrow (S,N)$ be a log smooth morphism of fine log schemes over $\mathbb{Q}$ (resp. a formally log smooth morphism of fine log formal V-schemes). Then we have an equivalence of categories*

$$C((X,M)/(S,N)) \simeq D((X,M)/(S,N)).$$

$$(\textit{resp.}\quad \hat{C}((X,M)/(S,N)) \simeq \hat{D}((X,M)/(S,N)).\quad)$$

PROOF. We give a proof only in the case where f is a log smooth morphism of fine log schemes over $\mathbb{Q}$. (The case of fine log formal V-schemes can be proved in a similar way. Details are left to the reader.)

First we introduce two auxiliary categories. Let $C^1((X,M)/(S,N))$ be the category of log connections (which are not necessarily integrable), and let $D^1((X,M)/(S,N))$ be the category of pairs (E,φ), where E is a coherent sheaf on X and $\varphi : \mathcal{D}\mathrm{iff}^1(\mathcal{O}_X,\mathcal{O}_X) \longrightarrow \mathcal{D}\mathrm{iff}^1(E,E)$ is an $\mathcal{O}_{X^1}$-linear homomorphism which sends the natural projection $\mathcal{O}_{X^1} \longrightarrow \mathcal{O}_X$ to the natural projection $\mathcal{O}_{X^1} \otimes E \longrightarrow E$. (A morphism from (E,φ) to (E',φ')

is defined as an $\mathcal{O}_X$-linear homomorphism $\alpha : E \longrightarrow E'$ which makes the following diagram commutative:

$$\begin{array}{ccc} \mathcal{D}\mathrm{iff}^1(\mathcal{O}_X, \mathcal{O}_X) & \xrightarrow{\varphi} & \mathcal{D}\mathrm{iff}^1(E, E) \\ {\scriptstyle \varphi'}\Big\downarrow & & {\scriptstyle \alpha\circ -}\Big\downarrow \\ \mathcal{D}\mathrm{iff}^1(E', E') & \xrightarrow{-\circ(\mathrm{id}\otimes\alpha)} & \mathcal{D}\mathrm{iff}^1(E, E').) \end{array} \tag{3.2.5}$$

Then there exists a canonical fully-faithful funtor $C((X, M)/(S, N)) \longrightarrow C^1((X, M)/(S, N))$ and a canonical functor $D((X, M)/(S, N)) \longrightarrow D^1((X, M)/(S, N))$.

To prove the proposition, it suffices to show the following three claims.

CLAIM 1. There exists an equivalence of categories $\Sigma : C^1((X, M)/(S, N)) \xrightarrow{\sim} D^1((X, M)/(S, N))$.

CLAIM 2. The functor $D((X, M)/(S, N)) \longrightarrow D^1((X, M)/(S, N))$ is fully faithful.

CLAIM 3. $(E, \nabla) \in C^1((X, M)/(S, N))$ is an object in $C((X, M)/(S, N))$ if and only if $\Sigma((E, \nabla))$ is an object in $D((X, M)/(S, N))$.

Now we prove the above three claims.

PROOF OF CLAIM 1. Let (E, ∇) be an object in $C^1((X, M)/(S, N))$. Let us identify $\mathcal{I} := \mathrm{Ker}(\mathcal{O}_{X^1} \longrightarrow \mathcal{O}_X)$ with $\omega_{(X,M)/(S,N)}$ and define $\theta : E \longrightarrow E \otimes \mathcal{O}_{X^1}$ by $\theta(e) = \nabla(e) + e \otimes 1$ for $e \in E$. Then we can calculate $\theta(ae)$ for $a \in \mathcal{O}_X$ as follows:

$$\begin{aligned} \theta(ae) &= \nabla(ae) + ae \otimes 1 \\ &= (a \otimes 1) \cdot \nabla(e) + (1 \otimes a - a \otimes 1) \cdot (e \otimes 1) + (a \otimes 1) \cdot (e \otimes 1) \\ &= (1 \otimes a) \cdot (\nabla(e) + e \otimes 1) \\ &= (1 \otimes a)\theta(e), \end{aligned}$$

where the third equality follows from the equation $(a \otimes 1) \cdot \nabla(e) = (1 \otimes a) \cdot \nabla(e)$. The above calculation show that θ is right $\mathcal{O}_X$-linear. Now let us define $\epsilon : \mathcal{O}_{X^1} \otimes E \longrightarrow E \otimes \mathcal{O}_{X^1}$ by extending θ $\mathcal{O}_{X^1}$-linearly, and let $\varphi : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_{X^1}, \mathcal{O}_X) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_{X^1} \otimes E, E)$ be the homomorphism $\alpha \mapsto \alpha \circ \epsilon \, (\alpha \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_{X^1}, \mathcal{O}_X))$. Then one can check that the pair (E, φ) is in $D^1((X, M)/(S, N))$. We define the functor Σ by $\Sigma((E, \nabla)) := (E, \varphi)$.

Let us define the quasi-inverse Σ' of the functor Σ. Let (E, φ) be an object in $D^1((X, M)/(S, N))$. For an element α in $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}, \mathcal{O}_X)$, let us define $\tilde{\alpha} \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_{X^1}, \mathcal{O}_X)$ as the unique element satisfying $\tilde{\alpha}(1) = 0, \tilde{\alpha}|_{\mathcal{I}} = \alpha$. Then define the homomorphism

$$\nabla : E \longrightarrow \mathcal{H}om(\mathcal{H}om(\mathcal{I}, \mathcal{O}_X), E) \overset{\sim}{=} E \otimes \mathcal{I} = E \otimes \omega^1_{(X,M)/(S,N)}$$

by $e \mapsto (\alpha \mapsto \varphi(\tilde{\alpha})(1 \otimes e))$, and let us define $\Sigma'((E, \varphi))$ by $\Sigma'((E, \varphi)) := (E, \nabla)$. One can check that this functor Σ' gives the quasi-inverse of Σ. $\square$

Proof of claim 2. Let $(E, \varphi), (E', \varphi') \in D((X, M)/(S, N))$ and assume given a homomorphism $\alpha : E \longrightarrow E'$ which makes the diagram (3.2.5) commutative. It suffices to show the commutativity of the following diagram for any n:

$$\begin{array}{ccc} \mathcal{D}\mathrm{iff}^n(\mathcal{O}, \mathcal{O}) & \xrightarrow{\varphi} & \mathcal{D}\mathrm{iff}^n(E, E) \\ {\scriptstyle \varphi'}\downarrow & & \downarrow{\scriptstyle \alpha\circ -} \\ \mathcal{D}\mathrm{iff}^n(E', E') & \xrightarrow{-\circ(\mathrm{id}\otimes\alpha)} & \mathcal{D}\mathrm{iff}^n(E, E'). \end{array}$$

We prove this assertion by induction on n. To show this, we may consider etale locally. Hence we may assume that we are in the situation in Lemma 3.2.7. Then, by Lemma 3.2.7 (2), it suffices to show the equality

$$\alpha \circ \varphi(D * D') = \varphi'(D * D') \circ (\mathrm{id} \otimes \alpha)$$

for log differential operators D, D' of order $k, l \leq n-1$, respectively. This is shown as follows:

$$\begin{aligned} \alpha \circ \varphi(D * D') &= \alpha \circ (\varphi(D) * \varphi(D')) \quad (\varphi \text{ is a ring hom}) \\ &= \alpha \circ \varphi(D') \circ (\mathrm{id} \circ \varphi(D)) \circ (\delta^*_{k,l} \otimes \mathrm{id}) \\ &= \varphi'(D') \circ (\mathrm{id} \otimes \varphi'(D)) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \alpha) \\ &\quad \circ (\delta^*_{k,l} \otimes \mathrm{id}) \quad (\text{induction}) \end{aligned}$$

$$= \varphi'(D') \circ (\mathrm{id} \otimes \varphi'(D)) \circ (\delta_{k,l}^* \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \alpha)$$
$$= (\varphi'(D) * \varphi'(D')) \circ (\mathrm{id} \otimes \alpha)$$
$$= \varphi'(D * D') \circ (\mathrm{id} \otimes \alpha) \quad (\varphi' \text{ is a ring hom}).$$

Hence the claim is proved. □

PROOF OF CLAIM 3. Let $(E, \nabla) \in C^1((X, M)/(S, N))$ and put $(E, \varphi) := \Sigma((E, \nabla))$. First, note that (E, ∇) (resp. (E, φ)) is in the category $C((X, M)/(S, N))$ (resp. $D((X, M)/(S, N))$) if and only if it is in the category $C((X, M)/(S, N))$ (resp. $D((X, M)/(S, N))$) etale locally. Hence we may consider etale locally, that is, we can assume that the morphism $(X, M) \longrightarrow (S, N)$ is as in the situation of Lemma 3.2.7. Let $\xi_{i,n}\,(1 \le i \le m, n \in \mathbb{N})$ be as in Lemma 3.2.7. (Then the elements $\{\xi_{i,1}\}_{i=1}^m$ form a basis of $\mathrm{Ker}(\mathcal{O}_{X^1} \longrightarrow \mathcal{O}_X) \cong \omega^1_{(X,M)/(S,N)}$.) For a local section $e \in E$, put

$$\nabla(e) = \sum_{i=1}^m e_i \otimes \xi_{i,1} \in E \otimes \omega^1_{(X,M)/(S,N)},$$
$$\nabla(e_i) = \sum_{j=1}^m e_{ij} \otimes \xi_{j,1} \in E \otimes \omega^1_{(X,M)/(S,N)}.$$

Then,

$$\nabla \circ \nabla(e) = \sum_{\substack{1 \le i,j \le m \\ i<j}} (e_{ji} - e_{ij}) \xi_{i,1} \wedge \xi_{j,1} \in E \otimes \omega^2_{(X,M)/(S,N)} \tag{3.2.6}$$

holds. Then, by definition of the functor Σ, one can see that φ is calculated as follows:

$$\begin{cases} \varphi(D_{(i)})(1 \otimes e) = e_i, \\ \varphi(D_{(i)})(\xi_{k,1} \otimes e) = \delta_{ik} e, \end{cases}$$

where δ_{ik} is Kronecker's delta. Using the above equations, definition of the product $*$ and the equations in the proof of Lemma 3.2.7, we can deduce the following equations:

$$\begin{cases} \varphi(D_{(i)}) * \varphi(D_{(j)})(1 \otimes e) = e_{ji}, \\ \varphi(D_{(i)}) * \varphi(D_{(j)})(\xi_{k,2} \otimes e) = \delta_{ik}\delta_{jk} e + \delta_{ik} e_j + \delta_{jk} e_i, \\ \varphi(D_{(i)}) * \varphi(D_{(j)})(\xi_{k,2}\xi_{l,2} \otimes e) = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) e. \end{cases}$$

From these equations and the equation (3.2.6), we have the following:

$$\begin{aligned}(E, \nabla) \quad : \text{integrable} &\underset{\text{def}}{\Longleftrightarrow} \forall e \in E, \nabla \circ \nabla(e) = 0 \\ &\Longleftrightarrow \forall e \in E, e_{ij} = e_{ji} \\ &\Longleftrightarrow \varphi(D_{(i)}) * \varphi(D_{(j)}) = \varphi(D_{(j)}) * \varphi(D_{(i)}).\end{aligned}$$

By this equivalence and (2) of Lemma 3.2.7, we can deduce the following: (E, ∇) is integrable if and only if there exists an $\mathcal{O}_X$-linear order-preserving ring homomorphism $\varphi : \mathcal{D}\mathrm{iff}(\mathcal{O}_X, \mathcal{O}_X) \longrightarrow \mathcal{D}\mathrm{iff}(E, E)$ extending φ.

Hence, to prove the assertion of the claim, it suffices to show that any $\mathcal{O}_X$-linear order preserving ring homomorphism $\varphi : \mathcal{D}\mathrm{iff}(\mathcal{O}_X, \mathcal{O}_X) \longrightarrow \mathcal{D}\mathrm{iff}(E, E)$ whose restriction to $\mathcal{D}\mathrm{iff}^1(\mathcal{O}_X, \mathcal{O}_X)$ is $\mathcal{O}_{X^1}$-linear is necessarily $\varprojlim_n \mathcal{O}_{X^n}$-linear. Let $x \in \varprojlim_n \mathcal{O}_{X^n}$ and $D \in \mathcal{D}\mathrm{iff}(\mathcal{O}_X, \mathcal{O}_X)$. We prove the equality $x\varphi(D) = \varphi(xD)$ by induction on the order n of D. When the order of D is ≤ 1, this equality follows from the assumption that $\varphi|_{\mathcal{D}\mathrm{iff}^1(\mathcal{O}_X, \mathcal{O}_X)}$ is $\mathcal{O}_{X^1}$-linear. To prove the equation in the case $n \geq 2$, we may assume $D := D' * D''$ for some D', D'' whose orders n', n'' are less than that of D. Let x_n be the image of x in $\mathcal{O}_{X^n}$, and put $\delta^*_{n',n''}(x_n) = \sum_k y_k \otimes z_k$($y_k \in \mathcal{O}_{X^{n'}}, z_k \in \mathcal{O}_{X^{n''}}$). Then we have the following equations:

$$\begin{aligned}\varphi(xD) &= \varphi(x_n(D' * D'')) \\ &= \varphi(\sum_k y_k D' * z_k D'') \\ &= \sum_k \varphi(y_k D' * z_k D'') \\ &= \sum_k \varphi(y_k D') * \varphi(z_k D'') \quad (\varphi \text{ is a ring hom.}) \\ &= \sum_k y_k \varphi(D') * z_k \varphi(D'') \quad (\text{inductive hypothesis}) \\ &= x_n(\varphi(D') * \varphi(D'')) = x\varphi(D).\end{aligned}$$

Hence the proof of the claim is completed. □

Since we have proved the three claims above, the proof of the proposition is now finished. □

Next we define log stratifications and formal log stratifications.

DEFINITION 3.2.10. Let $(X, M) \longrightarrow (S, N)$ be a morphism of fine log schemes (resp. fine log formal V-schemes) and let E be a coherent (resp. an isocoherent) sheaf on X. Then, a log stratification (resp. a formal log stratification) on E is a family of morphisms $\epsilon_n : \mathcal{O}_{X^n} \otimes E \longrightarrow E \otimes \mathcal{O}_{X^n} (n = 0, 1, 2...)$ satisfying the following conditions:

(1) Each ϵ_n is an $\mathcal{O}_{X^n}$-linear isomorphism and $\epsilon_0 = \mathrm{id}$.
(2) For any $n' > n$, $\epsilon_{n'}$ modulo $\mathrm{Ker}(\mathcal{O}_{X^{n'}} \longrightarrow \mathcal{O}_{X^n})$ coincides with ϵ_n.
(3) (Cocycle condition) For any n and n',

$$(\mathrm{id} \otimes \delta^*_{n,n'}) \circ \epsilon_{n+n'} = (\epsilon_n \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \epsilon_{n'}) \circ (\delta^*_{n,n'} \otimes \mathrm{id}) : \mathcal{O}_{X^{n+n'}} \otimes E \longrightarrow E \otimes \mathcal{O}_{X^n} \otimes \mathcal{O}_{X^{n'}}$$

holds.

We denote the category of coherent sheaves with log stratifications (resp. isocoherent sheaves with formal log stratifications) on (X, M) over (S, N) by $\mathrm{Str}((X, M)/(S, N))$. (resp. $\widehat{\mathrm{Str}}((X, M)/(S, N))$.)

Then we can show the following proposition:

PROPOSITION 3.2.11. *Let $(X, M) \longrightarrow (S, N)$ be a log smooth morphism of fine log schemes (resp. a formally log smooth morphism of fine log formal V-schemes) and let E be a coherent (resp. an isocoherent) sheaf on X. Then there exists a canonical equivalence of categories*

$$D((X, M)/(S, N)) \simeq \mathrm{Str}((X, M)/(S, N)).$$

$$(\textit{resp.} \quad \hat{D}((X, M)/(S, N)) \simeq \widehat{\mathrm{Str}}((X, M)/(S, N)). \quad)$$

PROOF. We treat only the case of fine log schemes. The case of fine log formal V-schemes can be shown similarly.

Since $\mathcal{O}_{X^n}$ is locally free by Lemma 3.2.7, $E \otimes \mathcal{O}_{X^n} \overset{\sim}{=} \mathcal{H}om(\mathcal{H}om(\mathcal{O}_{X^n}, \mathcal{O}), E)$ holds. Given an object (E, φ) in the category

$D((X,M)/(S,N))$, we define the homomorphism $\epsilon_n : \mathcal{O}_{X^n} \otimes E \longrightarrow E \otimes \mathcal{O}_{X^n}$ by the composite

$$\begin{aligned}\mathcal{O}_{X^n} \otimes E &\longrightarrow \mathcal{H}om(\mathcal{H}om(\mathcal{O}_{X^n} \otimes E, E), E) \\ &\xrightarrow{\varphi^*} \mathcal{H}om(\mathcal{H}om(\mathcal{O}_{X^n}, \mathcal{O}), E) \overset{\sim}{=} E \otimes \mathcal{O}_{X^n}.\end{aligned}$$

It is obvious that these ϵ_n's satisfy the condition (2) of Definition 3.2.10. In the conditions in (1), the condition $\epsilon_0 = \mathrm{id}$ is clear and $\mathcal{O}_{X^n}$-linearity of ϵ_n follows from the assumption that φ is $\varprojlim_n \mathcal{O}_{X^n}$-linear. To prove that ϵ_n is isomorphic, let us introduce a homomorphism $\eta_n : E \otimes \mathcal{O}_{X^n} \longrightarrow \mathcal{O}_{X^n} \otimes E$ which is defined by $\eta_n(e \otimes x) = \tau_n^*(x) \otimes e$, where τ_n^* is the homomorphism $\mathcal{O}_{X^n} \longrightarrow \mathcal{O}_{X^n}$ induced by the morphism $\tau_n : (X^n, M^n) \longrightarrow (X^n, M^n)$. Then η_n is a τ_n^*-linear, bijective and it reduces to identity modulo $\mathrm{Ker}(\mathcal{O}_{X^n} \longrightarrow \mathcal{O}_X)$. Hence $(\eta_n \circ \epsilon_n)^2$ is $\mathcal{O}_{X^n}$-linear and reduces to identity modulo $\mathrm{Ker}(\mathcal{O}_{X^n} \longrightarrow \mathcal{O}_X)$. So it is bijective. Hence ϵ_n is also bijective. Finally we can prove the condition (3) by using the assumption that Φ is a ring homomorphism. (We leave the detail to the reader.) So $(E, \{\epsilon_n\}_n)$ defines an object in the category $\mathrm{Str}((X,M)/(S,N))$.

Next suppose we are given a stratification ϵ_n ($n \in \mathbb{N}$) on E. Then for $\alpha \in \mathcal{D}\mathrm{iff}^n(\mathcal{O}, \mathcal{O})$, we define $\varphi(\alpha) \in \mathcal{D}\mathrm{iff}^n(E,E)$ by the composition

$$\mathcal{O}_{X^n} \otimes E \xrightarrow{\epsilon_n} E \otimes \mathcal{O}_{X^n} \xrightarrow{\mathrm{id}\otimes\alpha} E.$$

Then obviously φ is an order-preserving, $\varprojlim_n \mathcal{O}_{X^n}$-linear homomorphism. Using the condition (3) of Definition 3.2.10, one can show that φ is a ring homomorphism. Hence the pair (E, φ) is in the category $D((X,M)/(S,N))$.

One can check easily that the above two funtors are the quasi-inverse of each other. Hence the assertion is proved. □

Now we define the infinitesimal site for a morphism of fine log schemes or fine log formal V-schemes. It is a log version or a log formal version of the infinitesimal site defined by Grothendieck ([G]).

DEFINITION 3.2.12. Let $(X,M) \longrightarrow (S,N)$ be a morphism of fine log schemes (resp. fine log formal V-schemes). Then we define the infinitesimal site $((X,M)/(S,N))_{\mathrm{inf}}$ of (X,M) over (S,N) as follows. An object is a 4-ple (U, T, M_T, ϕ), where U is a scheme (resp. a formal V-scheme) which is etale (resp. formally etale) over X, (T, M_T) is a fine log scheme (resp. fine log formal V-scheme) over (S, N), ϕ is a nilpotent exact closed immersion $(U, M) \longrightarrow (T, M_T)$ over (S, N). A morphism from (U, T, M_T, ϕ) to $(U', T', M_{T'}, \phi')$ is defined as a pair of morphisms $f : (U, M) \longrightarrow (U', M), g : (T, M_T) \longrightarrow (T', M_{T'})$ over (S, N) satisfying $g \circ \phi = \phi' \circ f$. A collection of morphisms $\{(U_i, T_i, M_{T_i}, \phi_i) \longrightarrow (U, T, M_T, \phi)\}_i$ is a covering if the induced collection of morphisms $\{T_i \longrightarrow T\}_i$ is an etale covering and $U_i = T_i \times_T U$ holds. We sometimes denote (U, T, M_T, ϕ) simply by T.

Next we define the notion of a crystal or an isocrystal on infinitesimal site as follows:

Definition 3.2.13. Let $(X, M) \longrightarrow (S, N)$ be a morphism of fine log schemes (resp. fine log formal V-schemes). Then a crystal (resp. an isocrystal) on the infinitesimal site $((X, M)/(S, N))_{\mathrm{inf}}$ is a sheaf $\mathcal{E}$ on $((X, M)/(S, N))_{\mathrm{inf}}$ satisfying the following conditions:

(1) For $T \in ((X, M)/(S, N))_{\mathrm{inf}}$, the sheaf $\mathcal{E}_T$ on T induced by $\mathcal{E}$ is a coherent sheaf (resp. an isocoherent sheaf) on T.
(2) For a morphism $f : T \longrightarrow T'$ in $((X, M)/(S, N))_{\mathrm{inf}}$, the canonical homomorphism $f^*\mathcal{E}_{T'} \longrightarrow \mathcal{E}_T$ is an isomorphism.

We denote the category of crystals (resp. isocrystals) on the infinitesimal site of (X, M) over (S, N) by $C_{\mathrm{inf}}((X, M)/(S, N))$ (resp. $I_{\mathrm{inf}}((X, M)/(S, N))$).

We often denote the category $C_{\mathrm{inf}}((X, M)/(S, N))$ (resp. $I_{\mathrm{inf}}((X, M)/(S, N))$) simply by $C_{\mathrm{inf}}((X/S)^{\log})$ (resp. $I_{\mathrm{inf}}((X/S)^{\log})$) when there will be no confusion on log structures. When the log structures are trivial, we abbreviate to write the superscript $^{\log}$.

Now we prove the equivalence of the category of log stratifications (resp. formal log stratifications) and the category of crystals (resp. isocrystals) on the infinitesimal site:

Proposition 3.2.14. *Let $(X, M) \longrightarrow (S, N)$ be a log smooth morphism of fine log schemes (resp. a formally log smooth morphism of fine log formal V-schemes). Then there exists a canonical equivalence of categories*

$$\mathrm{Str}((X, M)/(S, N)) \simeq C_{\mathrm{inf}}((X, M)/(S, N)).$$

$$(\textit{resp.}\quad \widehat{\mathrm{Str}}((X,M)/(S,N)) \simeq I_{\mathrm{inf}}((X,M)/(S,N)).\quad)$$

Proof. We only sketch how to construct the functors giving the equivalence of categories.

First suppose given an object $\mathcal{E}$ in $I_{\mathrm{inf}}((X,M)/(S,N))$. Let p_i $(i=1,2)$ be the morphisms $(X^n, M^n) \to (X,M) \times_{(S,N)} (X,M) \overset{i\text{-th proj.}}{\to} (X,M)$. Then, by the definition of crystals (resp. isocrystals), there exist isomorphisms $p_2^*\mathcal{E}_X \overset{\sim}{\to} \mathcal{E}_{X^n} \overset{\sim}{\leftarrow} p_1^*\mathcal{E}_X$, that is, the isomorphism $\epsilon_n : \mathcal{O}_{X^n} \otimes \mathcal{E}_X \to \mathcal{E}_X \otimes \mathcal{O}_{X^n}$. Then one can check that the pair $(\mathcal{E}_X, \{\epsilon_n\}_n)$ defines an object in the category $\mathrm{Str}((X,M)/(S,N))$ (resp. $\widehat{\mathrm{Str}}((X,M)/(S,N))$).

Next suppose we are given an object $(E, \{\epsilon_n\})$ in the category $\mathrm{Str}((X,M)/(S,N))$ (resp. $\widehat{\mathrm{Str}}((X,M)/(S,N))$). Let (U,T,L,ϕ) be an object of $((X,M)/(S,N))_{\mathrm{inf}}$. Since $(U,M) \longrightarrow (S,N)$ is log smooth (resp. formally log smooth), there exists a section of $\phi : (U,M) \longrightarrow (T,L)$ etale locally on T. (Definition 2.2.2, Proposition 2.2.13.) Let $s : (T',L) \longrightarrow (X,M)$ be the composite of this section (T' is etale over T) with the morphism $(U,M) \longrightarrow (X,M)$. Then E determines a sheaf s^*E on T'. Suppose there exist two sections $s,t : (T',L) \longrightarrow (X,M)$. Then, by the universality of log infinitesimal neighborhood, there exists an integer n such that $s \times t : (T',L) \longrightarrow (X,M) \times_{(S,N)} (X,M)$ factors as

$$(T',L) \xrightarrow{u} (X^n, M^n) \longrightarrow (X,M) \times_{(S,N)} (X,M).$$

Then pullback of $\epsilon_n : \mathcal{O}_{X^n} \otimes E \longrightarrow E \otimes \mathcal{O}_{X^n}$ by u determines an isomorphism $t^*E \xrightarrow{\sim} s^*E$. This fact and the cocycle condition of the log stratification (resp. the log formal stratification) $\{\epsilon_n\}$ show that the above sheaf s^*E descents to T. One can check also that the resulting sheaf on T is independent of the choice of the sections, and that it is coherent (resp. isocoherent). (In the case of formal log schemes, we use the rigid analytic faithfully flat descent of Gabber ([O1, (1.9)]).) The fact that the above E_T's form an object of $C_{\mathrm{inf}}((X,M)/(S,N))$ (resp. $I_{\mathrm{inf}}((X,M)/(S,N))$) can be proved also by using the cocycle condition of $\{\epsilon_n\}$.

It can be checked easily that the above two functors are the inverse of each other. □

Combining Propositions 3.2.9, 3.2.11 and 3.2.14, we obtain the following result, which is the main result in this section:

Theorem 3.2.15. *Let $f : (X, M) \longrightarrow (S, N)$ be a log smooth morphism of fine log schemes over $\mathbb{Q}$ (resp. formally log smooth morphism of fine log formal V-schemes). Then the following three categories are equivalent*:

(1) *The category $C((X, M)/(S, N))$ of integrable log connections (resp. The category $\hat{C}((X, M)/(S, N))$ of integrable log formal connections).*
(2) *The category $\mathrm{Str}((X, M)/(S, N))$ of log stratifications (resp. The category $\widehat{\mathrm{Str}}((X, M)/(S, N))$ of formal log stratifications).*
(3) *The category $C_{\mathrm{inf}}((X, M)/(S, N))$ of crystals (resp. The category $I_{\mathrm{inf}}((X, M)/(S, N))$ of isocrystals) on log infinitesimal site $((X, M)/(S, N))_{\mathrm{inf}}$.*

Corollary 3.2.16. *Let $f : (X, M) \longrightarrow (\mathrm{Spec}\, V, N)$ be a proper log smooth morphism of fine log schemes. Let us denote the generic fiber of f by $(X_K, M) \longrightarrow (\mathrm{Spec}\, K, N)$ and the p-adic completion by $(\hat{X}, M) \longrightarrow (\mathrm{Spf}\, V, N)$. Then there exists a canonical equivalence of categories*

$$C((X_K, M)/(\mathrm{Spec}\, K, N)) \longrightarrow I_{\mathrm{inf}}((\hat{X}, M)/(\mathrm{Spf}\, V, N)).$$

Proof. First, by Theorem 3.2.15, the category $C((X_K, M)/(\mathrm{Spec}\, K, N))$ is equivalent to the category $\mathrm{Str}((X_K, M)/(\mathrm{Spec}\, K, N))$. Next, by [O1, (1.4)], there exists canonically an equivalence of categories $\mathrm{Coh}(\mathcal{O}_{X_K^n}) \xrightarrow{\sim} \mathrm{Coh}(K \otimes \mathcal{O}_{\hat{X}^n})$ (where $X_K^n, \hat{X}^n$ are the n-th log infinitesimal neighborhood of the diagonal morphism), and this functor induces an equivalence of categories $\mathrm{Str}((X_K, M)/(\mathrm{Spec}\, K, M)) \xrightarrow{\sim} \widehat{\mathrm{Str}}((\hat{X}, M)/(\mathrm{Spf}\, V, N))$. Moreover, again by Theorem 3.2.15, $\widehat{\mathrm{Str}}((\hat{X}, M)/(\mathrm{Spf}\, V, N))$ is categorically equivalent to $I_{\mathrm{inf}}((\hat{X}, M)/(\mathrm{Spf}\, V, N))$. By combining them, we obtain the assertion. □

Finally, we give some remarks on the category of isocrystals on log infinitesimal site which we need later.

Remark 3.2.17. Let $(X, M) \longrightarrow (S, N)$ be a formally log smooth morphism of fine log formal V-schemes. We remark that the category $I_{\mathrm{inf}}((X, M)/(S, N))$ is an abelian tensor category.

Indeed, by Theorem 3.2.15 (Proposition 3.2.14), it suffices to show that the category $\widehat{\mathrm{Str}}((X,M)/(S,N))$ is an abelian tensor category. Since $\mathcal{O}_{X^n}$ is a locally free $\mathcal{O}_X$-module with respect to both right and left $\mathcal{O}_X$-module structures, formal log stratifications are inherited to kernels and cokernels of sheaves naturally. Hence kernels and cokernels exist in the category $\widehat{\mathrm{Str}}((X,M)/(S,N))$, and it is easy to check that the image and the coimage coincide. Therefore $\widehat{\mathrm{Str}}((X,M)/(S,N))$ is an abelian category. The tensor structure is given by $(E,\{\epsilon_n\})\otimes(E',\{\epsilon'_n\}) := (E\otimes E', \{\mathrm{id}\otimes\epsilon'_n\circ\epsilon_n\otimes\mathrm{id}\})$.

Remark 3.2.18. Assume that we are given the following commutative diagram of fine log schemes (resp. fine log formal V-schemes)

$$\begin{array}{ccc} (X,M) & \xleftarrow{\ g\ } & (X',M') \\ {\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow \\ (S,N) & \longleftarrow & (S',N'), \end{array}$$

where f and f' are log smooth (resp. formally log smooth). Let (X^n,M^n), $((X')^n,(M')^n)$ $(n\in\mathbb{N})$ be the n-th log infinitesimal neighborhood of $(X,M),(X',M')$ in $(X,M)\times_{(S,N)}(X,M),(X',M')\times_{(S',N')}(X',M')$, respectively. Then, by the above diagram, we have the compatible family of morphisms $((X')^n,(M')^n)\longrightarrow(X^n,M^n)$, and it induces the functor

$$g^*:\mathrm{Str}((X,M)/(S,N))\longrightarrow\mathrm{Str}((X',M')/(S',N')).$$

$$(\text{resp.}\quad g^*:\widehat{\mathrm{Str}}((X,M)/(S,N))\longrightarrow\widehat{\mathrm{Str}}((X',M')/(S',N')).\quad)$$

Hence, by the equivalence of categories in Proposition 3.2.14, The 'pull-back' functor

$$g^*:C_{\mathrm{inf}}((X,M)/(S,N))\longrightarrow C_{\mathrm{inf}}((X',M')/(S',N'))$$

$$(\text{resp.}\quad g^*:I_{\mathrm{inf}}((X,M)/(S,N))\longrightarrow I_{\mathrm{inf}}((X',M')/(S',N'))\quad)$$

is defined.

It is possible to define the pull-back functor of the topos associated to the infinitesimal site in general, but we do not pursue this subject here.

REMARK 3.2.19. Assume that we are given the following cartesian square of fine log formal V-schemes

$$\begin{array}{ccc} (X,M) & \xleftarrow{g} & (X',M) \\ {\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow \\ (S,N) & \xleftarrow{h} & (S',N), \end{array}$$

where f and f' are formally log smooth and S' is defined by $S' := \underline{\mathrm{Spec}}\,\mathcal{O}_S/(p\text{-torsion})$. Define $X^n, (X')^n$ as in Remark 3.2.18. Then one can see easily (by [O1, (1.2)]) that the functor $\mathrm{Coh}(K \otimes \mathcal{O}_{X^n}) \longrightarrow \mathrm{Coh}(K \otimes \mathcal{O}_{(X')^n})$ gives an equivalence of categories. Hence so does the functor

$$g^* : \widehat{\mathrm{Str}}((X,M)/(S,N)) \longrightarrow \widehat{\mathrm{Str}}((X',M')/(S',N')).$$

So, the pull-back functor

$$g^* : I_{\mathrm{inf}}((X,M)/(S,N)) \longrightarrow I_{\mathrm{inf}}((X',M')/(S',N'))$$

gives an equivalence of categories.

REMARK 3.2.20. Let $(X,M) \longrightarrow (S,N)$ be a formally log smooth morphism of fine log formal V-schemes, and assume we are given the following diagram of formal V-schemes

$$\begin{array}{ccccccc} X & \leftarrow & X^{(0)} & \begin{smallmatrix}\leftarrow\\ \leftarrow\end{smallmatrix} & X^{(1)} & \begin{smallmatrix}\leftarrow\\ \leftarrow\\ \leftarrow\end{smallmatrix} & X^{(2)} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S & \leftarrow & S^{(0)} & \begin{smallmatrix}\leftarrow\\ \leftarrow\end{smallmatrix} & S^{(1)} & \begin{smallmatrix}\leftarrow\\ \leftarrow\\ \leftarrow\end{smallmatrix} & S^{(2)}, \end{array}$$

where the horizontal lines are etale hypercoverings (up to level 2). Then one can define functors

$$p : I_{\mathrm{inf}}((X,M)/(S,N)) \longrightarrow I_{\mathrm{inf}}((X^{(0)},M)/(S^{(0)},N)),$$

$$p_i : I_{\mathrm{inf}}((X^{(0)},M)/(S^{(0)},N)) \longrightarrow I_{\mathrm{inf}}((X^{(1)},M)/(S^{(1)},N)) \quad (i = 1,2),$$

$$p_{ij} : I_{\mathrm{inf}}((X^{(1)},M)/(S^{(1)},N)) \longrightarrow I_{\mathrm{inf}}((X^{(2)},M)/(S^{(2)},N)) \qquad (1 \le i < j \le 3),$$

which are induced by the above diagram. Let Δ_{inf} be the category of the descent data of isocrystals with respect to these functors, that is, the category of pairs $(\mathcal{E}, \varphi)$, where $\mathcal{E} \in I_{\mathrm{inf}}((X^{(0)}, M)/(S^{(0)}, N))$ and φ is an isomorphism $p_2(\mathcal{E}) \xrightarrow{\sim} p_1(\mathcal{E})$ in $I_{\mathrm{inf}}((X^{(1)}, M)/(S^{(1)}, N))$ satisfying the condition $p_{12}(\varphi) \circ p_{23}(\varphi) = p_{13}(\varphi)$ in $I_{\mathrm{inf}}((X^{(2)}, M)/(S^{(2)}, N))$. Then there exists a canonical functor

$$I_{\mathrm{inf}}((X, M)/(S, N)) \longrightarrow \Delta_{\mathrm{inf}}$$

induced by p. By using rigid analytic faithfully flat descent of Gabber ([O1, (1.9)]), one can check that this functor gives an equivalence of categories. That is, the category of isocrystals on log infinitesimal site admits the descent for etale coverings.

Remark 3.2.21. One can find the calculations similar to those in this section in a paper of Ogus [O3, 1.1]. He defines the rings of HPD-differential operators for log smooth morphisms of fine log schemes of positive characteristic. On the other hand, our rings of differential operators are defined without using PD-structures and hence behave well for log smooth morphisms of fine log schemes over $\mathbb{Q}$ or formally log smooth morphisms of fine log formal V-schemes.

Chapter 4. Crystalline Fundamental Groups

Throughout this chapter, let p be a prime and let k be a perfect field of characteritic p. Denote the Witt ring of k by W and denote the fraction field of W by K_0. Let (X, M) be a fine log scheme over k, let $(X, M) \xrightarrow{f} (\mathrm{Spec}\, k, N) \overset{i}{\hookrightarrow} (\mathrm{Spf}\, W, N)$ be morphisms of fine log formal W-schemes and let x be a k-valued point of $X_{f\text{-triv}}$. In this chapter, we give a definition of crystalline fundamental group $\pi_1^{\mathrm{crys}}((X, M)/(\mathrm{Spf}\, W, N), x)$ of (X, M) over $(\mathrm{Spf}\, W, N)$ with base point x, and prove some basic properties.

4.1. Definition of crystalline fundamental groups

Let (X, M) be a fine log scheme over k and let $(X, M) \xrightarrow{f} (\mathrm{Spec}\, k, N) \overset{i}{\hookrightarrow} (\mathrm{Spf}\, W, N)$ be morphisms of fine log formal W-schemes. Denote the canonical PD-structure on W by γ. Put $W_n := W \otimes \mathbb{Z}/p^n\mathbb{Z}$ for $n \in \mathbb{N}$ and denote the PD-structure on W_n induced by γ also by γ.

K. Kato defined in his article [Kk1] the notion of the log crystalline site $((X, M)/(\operatorname{Spec} W_n, N, \gamma))_{\text{crys}}$ and crystals on it. In this section, first we recall the basic definitions on log crystalline site. We also consider the log crystalline site $((X, M)/(\operatorname{Spf} W, N, \gamma))_{\text{crys}}$ over a formal base. Then, using the category of isocrystals on the log crystalline site over a formal base, we give the definition of crystalline fundamental groups. Finally, we give some basic properties which they should satisfy: The first one is the crystalline version of Hurewicz isomorphism and the second one is the bijectivity of crystalline Frobenius.

First let us recall the definition of log crystalline site:

Definition 4.1.1. Let (X, M) be a fine log scheme over k and let $(X, M) \xrightarrow{f} (\operatorname{Spec} k, N) \overset{i}{\hookrightarrow} (\operatorname{Spf} W, N)$ be morphisms of fine log formal W-schemes. Let W_n, γ be as above. Then we define the log crystalline site $((X, M)/(\operatorname{Spec} W_n, N))_{\text{crys}}$ (resp. $((X, M)/(\operatorname{Spf} W, N))_{\text{crys}}$) as follows: An object is a 5-ple (U, T, L, i, δ), where U is a scheme etale over X, (T, L) is a fine log scheme over $(\operatorname{Spec} W_n, N)$ (resp. a fine log scheme over $(\operatorname{Spec} W_n, N)$ for some n), $i : (U, M) \hookrightarrow (T, L)$ is an exact closed immersion over $(\operatorname{Spec} W_n, N)$ (resp. over $(\operatorname{Spf} W, N)$), and δ is a PD-structure on the defining ideal of U in $\mathcal{O}_T$ which is compatible with γ. A morphism from (U, T, L, i, δ) to $(U', T', L', i', \delta')$ is a pair of morphisms $f : (U, M) \longrightarrow (U', M')$, $g : (T, L) \longrightarrow (T', L')$ over $(\operatorname{Spec} W_n, N)$ (resp. over $(\operatorname{Spf} W, N)$) satisfying $i' \circ f = g \circ i$ which is compatible with PD-strutures. A covering is the one which is induced by the (classical) etale topology of T.

We often denote the 5-ple (U, T, L, i, δ) simply by T, and we often denote the site $((X, M)/(\operatorname{Spec} W_n, N))_{\text{crys}}$ (resp. $((X, M)/(\operatorname{Spec} W, N))_{\text{crys}}$) simply by $(X/W_n)^{\log}_{\text{crys}}$ (resp. $(X/W)^{\log}_{\text{crys}}$), when there will be no confusion on log structures.

We define the structure sheaf $\mathcal{O}_{X/W_n}$ (resp. $\mathcal{O}_{X/W}$) on $(X/W_n)^{\log}_{\text{crys}}$ (resp. $(X/W)^{\log}_{\text{crys}}$) by

$$\mathcal{O}_{X/W_n}(T) := \Gamma(T, \mathcal{O}_T).$$

$$(\text{resp.} \quad \mathcal{O}_{X/W}(T) := \Gamma(T, \mathcal{O}_T). \quad)$$

Next we recall the notion of crystals and isocrystals on log crystalline site:

Definition 4.1.2. Let the notations be as above. A crystal on the site $(X/W_n)^{\log}_{\mathrm{crys}}$ is a sheaf of $\mathcal{O}_{X/W_n}$-modules $\mathcal{F}$ such that, for any morphism $g : T' \longrightarrow T$ in $(X/W_n)^{\log}_{\mathrm{crys}}$, the induced map $g^*\mathcal{F}_T \longrightarrow \mathcal{F}_{T'}$ is isomorphic, where $\mathcal{F}_T$ is a sheaf on T induced by $\mathcal{F}$. A crystal $\mathcal{F}$ on $(X/W_n)^{\log}_{\mathrm{crys}}$ is called coherent (resp. locally free) if each sheaf $\mathcal{F}_T$ is a coherent (resp. locally free) $\mathcal{O}_T$-module for any $T \in (X/W_n)^{\log}_{\mathrm{crys}}$. Similarly we define the notion of crystals on $(X/W)^{\log}_{\mathrm{crys}}$ and coherent (resp. locally free) crystals on $(X/W)^{\log}_{\mathrm{crys}}$. We denote the category of coherent crystals on $(X/W_n)^{\log}_{\mathrm{crys}}$ (resp. $(X/W)^{\log}_{\mathrm{crys}}$) by $C_{\mathrm{crys}}((X/W_n)^{\log})$ (resp. $C_{\mathrm{crys}}((X/W)^{\log})$).

Definition 4.1.3. Let the notations be as above. We define the category of isocrystals $I_{\mathrm{crys}}((X,M)/(\mathrm{Spf}\, W,N))$ (we often denote simply by $I_{\mathrm{crys}}((X/W)^{\log})$) on the log crystalline site $(X/W)^{\log}_{\mathrm{crys}}$ as follows. The objects are coherent crystals on $(X/W)^{\log}_{\mathrm{crys}}$. For coherent crystals $\mathcal{E}$ and $\mathcal{F}$, morphisms are defined by

$$\mathrm{Hom}_{I_{\mathrm{crys}}((X/W)^{\log})}(\mathcal{E},\mathcal{F}) := K_0 \otimes_W \mathrm{Hom}_{C_{\mathrm{crys}}((X/W)^{\log})}(\mathcal{E},\mathcal{F}).$$

When we regard a crystal $\mathcal{F}$ as an isocrystal, we denote this object by $K_0 \otimes \mathcal{F}$.

Let (X,M) be a fine log scheme over k and let $(X,M) \xrightarrow{f} (\mathrm{Spec}\, k,N) \overset{i}{\hookrightarrow} (\mathrm{Spf}\, W,N)$ be morphisms of fine log formal W-schemes. Then we denote the nilpotent part $\mathcal{N}_{K_0\otimes\mathcal{O}_{X/W}} I_{\mathrm{crys}}((X,M)/(\mathrm{Spf}\, W,N))$ of the category $I_{\mathrm{crys}}((X,M)/(\mathrm{Spf}\, W,N))$ with respect to $K_0 \otimes \mathcal{O}_{X/W}$ by $\mathcal{N}I_{\mathrm{crys}}((X,M)/(\mathrm{Spf}\, W,N))$. (We often denote it simply by $\mathcal{N}I_{\mathrm{crys}}((X/W)^{\log})$, when there will be no confusion on log structures.)

Then we have the following proposition:

Proposition 4.1.4. *Let (X,M) be a fine log scheme over k and let $(X,M) \xrightarrow{f} (\mathrm{Spec}\, k,N) \overset{i}{\hookrightarrow} (\mathrm{Spf}\, W,N)$ be morphisms of fine log formal W-schemes. Assume that $K_0 \otimes H^0((X/W)^{\log}_{\mathrm{crys}}, \mathcal{O}_{X/W})$ is isomorphic to a field. Then the category $\mathcal{N}I_{\mathrm{crys}}((X/W)^{\log})$ is a Tannakian category.*

Proof. Let $\mathcal{C}'$ be the category of sheaves of $\mathcal{O}_{X/W}$-modules on $(X/W)^{\log}_{\mathrm{crys}}$ and let $\mathcal{C}$ be the abelian category obtained from $\mathcal{C}'$ by inverting the morphism 'multiplication by p'. Then $\mathcal{C}$ is an abelian tensor category and $I_{\rm crys}((X/W)^{\log})$ is a full subcategory of $\mathcal{C}$. Hence $\mathcal{N}I_{\rm crys}((X/W)^{\log})$ is a full subcategory of $\mathcal{NC}$.

First we claim that $\mathcal{N}I_{\rm crys}((X/W)^{\log})$ is equivalent to $\mathcal{NC}$. That is, we claim that each object in $\mathcal{NC}$ is an isocrystal.

Let E be an object in $\mathcal{NC}$. To show the assertion, we may assume there exists an exact sequence in $\mathcal{C}$

$$0 \longrightarrow F \longrightarrow E \xrightarrow{f} K_0 \otimes \mathcal{O}_{X/W} \longrightarrow 0,$$

where F is an object in $\mathcal{N}I_{\rm crys}((X/W)^{\log})$. Let F_0 be a crystal representing F and E_0 be an object in $\mathcal{C}'$ representing E. Then there exists a morphism $f_0 : E_0 \longrightarrow \mathcal{O}_{X/W}$ which represents $p^n f$ for some n. Then $\mathrm{Ker}(f_0)$ is isomorphic to F in the category $\mathcal{C}$. So there exists morphisms in $\mathcal{C}'$

$$g_1 : \mathrm{Ker}(f_0) \longrightarrow F_0,$$

$$h_1 : F_0 \longrightarrow \mathrm{Ker}(f_0),$$

such that $h_1 \circ g_1 = p^m$ and $g_1 \circ h_1 = p^m$ holds for some $m \in \mathbb{N}$. Also, $\mathrm{Im}(f_0)$ is isomorphic to $K_0 \otimes \mathcal{O}_{X/W}$ in the category $\mathcal{C}$. So there exists morphisms in $\mathcal{C}'$

$$g_2 : \mathrm{Im}(f_0) \longrightarrow \mathcal{O}_{X/W},$$

$$h_2 : \mathcal{O}_{X/W} \longrightarrow \mathrm{Im}(f_0),$$

such that $h_2 \circ g_2 = p^l$ and $g_2 \circ h_2 = p^l$ holds for some $l \in \mathbb{N}$. If we pull back the exact sequence

$$0 \longrightarrow \mathrm{Ker}(f_0) \longrightarrow E_0 \longrightarrow \mathrm{Im}(f_0) \longrightarrow 0$$

by h_2 and then push it out by g_1, we get the exact sequence

$$0 \longrightarrow F_0 \longrightarrow E_0' \longrightarrow \mathcal{O}_{X/W} \longrightarrow 0,$$

where E_0' is an object in $\mathcal{C}'$ which also represents E. By the above exact sequence, one can check that E_0' is a crystal. Therefore, E is an isocrystal. So the category $\mathcal{N}I_{\rm crys}((X/W)^{\log})$ is equivalent to $\mathcal{NC}$.

Then, by Proposition 1.2.1, the category $\mathcal{N}I_{\mathrm{crys}}((X/W)^{\log})$ is an abelian category. Moreover it is easy to check that $\mathcal{N}I_{\mathrm{crys}}((X/W)^{\log})$ has a canonical tensor structure which makes this category a rigid abelian tensor category.

To show the category $\mathcal{N}I_{\mathrm{crys}}((X/W)^{\log})$ is Tannakian, it suffices to prove that there exists a fiber functor $\xi : \mathcal{N}I_{\mathrm{crys}}((X/W)^{\log}) \to \mathrm{Vec}_L$, where L is a field.

We prove the existence of a fiber functor as above. Since $K_0 \otimes_W H^0((X/W)^{\log}_{\mathrm{crys}}, \mathcal{O}_{X/W})$ is a field, $p^n \neq 0$ holds in $H^0((X/W)^{\log}_{\mathrm{crys}}, \mathcal{O}_{X/W})$. Hence there exists an object $T_n := (U_n, T_n, L_n, i_n, \delta_n)$ in $(X/W)^{\log}_{\mathrm{crys}}$ such that T_n is affine and $p^n \neq 0$ holds in $\Gamma(T_n, \mathcal{O}_{T_n})$. Put $A := \prod_{n \in \mathbb{N}} \Gamma(T_n, \mathcal{O}_{T_n})$. Then A is a W-algebra and the canonical homomorphism $W \longrightarrow A$ is injective. So we have $K_0 \otimes_W A \neq 0$. Let I be a maximal ideal of $K_0 \otimes_W A$ and put $L := (K_0 \otimes_W A)/I$. Then we can define the functor

$$\begin{aligned} \mathcal{N}I_{\mathrm{crys}}((X,M)/(\mathrm{Spf}\, W, N)) &\xrightarrow{\mathrm{ev}} (K_0 \otimes_W A\text{-modules}) \\ &\longrightarrow \mathrm{Vec}_L, \end{aligned}$$

where ev is defined by $\mathcal{E} \mapsto K_0 \otimes_W (\prod_{n \in \mathbb{N}} \mathcal{E}(T_n))$ and the second arrow is induced by the ring homomorphism $K_0 \otimes_W A \longrightarrow L$. We denote this functor by ξ. One can check that ξ is an exact tensor functor, noting that the essential image of ev consists of free modules. Moreover, by [D2, (2.10)], ξ is faithful. Hence ξ is the desired fiber functor. □

Now we define crystalline fundamental groups:

DEFINITION 4.1.5 (Definition of π_1^{crys}). Let (X, M) be a fine log scheme over k and let $(X, M) \xrightarrow{f} (\mathrm{Spec}\ k, N) \hookrightarrow (\mathrm{Spf}\, W, N)$ be morphisms of fine log formal W-schemes such that the underlying morphism of schemes of f is of finite type and that $K_0 \otimes_W H^0((X/W)^{\log}_{\mathrm{crys}}, \mathcal{O}_{X/W})$ is a field. Let x be a k-valued point of $X_{f\text{-triv}}$ over $\mathrm{Spf}\, W$. Then we define the crystalline fundamental group of (X, M) over $(\mathrm{Spf}\ W, N)$ with base point x by

$$\pi_1^{\mathrm{crys}}((X, M)/(\mathrm{Spf}\ W, N), x) := G(\mathcal{N}I_{\mathrm{crys}}((X/W)^{\log}), \omega_x),$$

where ω_x is the fiber functor

$$\mathcal{N}I_{\mathrm{crys}}((X/W)^{\log}) \longrightarrow \mathcal{N}I_{\mathrm{crys}}((x/W)^{\log}) \simeq \mathrm{Vec}_K$$

and the notation $G(\cdots)$ on the right hand side is as in Theorem 1.1.8. When there will be no confusion on log structures, we will denote the crystalline fundamental group $\pi_1^{\mathrm{crys}}((X,M)/(\mathrm{Spf}\,W,N),x)$ simply by $\pi_1^{\mathrm{crys}}((X/W)^{\log},x)$. When the log structures are trivial, we abbreviate to write the superscript $^{\log}$.

REMARK 4.1.6. The existence of the point x assures that the category $\mathcal{N}I_{\mathrm{crys}}((X/W)^{\log})$ in the above definition is actually a *neutral* Tannakian category.

Here we give a useful sufficient condition for the category $\mathcal{N}I_{\mathrm{crys}}((X/W)^{\log})$ to be Tannakian, which is analogous to Proposition 3.1.6:

PROPOSITION 4.1.7. *Let (X,M) be a fine k-scheme over k and let $(X,M)\xrightarrow{f}(\mathrm{Spec}\,k,N)\overset{i}{\hookrightarrow}(\mathrm{Spf}\,W,N)$ be morphisms of fine log formal W-schemes. Assume that f is log smooth integral of finite type and that X is connected and reduced. Then $K_0\otimes_W H^0((X/W)^{\log}_{\mathrm{crys}},\mathcal{O}_{X/W})$ is a field. In particular, the category $\mathcal{N}I_{\mathrm{crys}}((X/W)^{\log})$ is a Tannakian category by Proposition 4.1.4.*

PROOF. For a finite extension $k\subset k'$ of fields and a scheme Y over $X\otimes_k k'$, let us denote the ring $H^0(((Y,M)/(\mathrm{Spec}\,W(k'),N))_{\mathrm{crys}},\mathcal{O}_{Y/W(k')})$ simply by $H^0_{\text{log-crys}}(Y/W(k'))$.

First, let us note the following claim:

CLAIM 1. Let $k\subset k'\subset k''$ be finite extensions of fields and put $X':=X\otimes_k k', f':=f\otimes_k k':(X',M)\longrightarrow(\mathrm{Spec}\,k',N)$. Let U be a connected component of $X'_{f'\text{-triv}}$ and let $x\in U(k'')$. Then the homomorphism

$$\alpha:\mathbb{Q}\otimes_{\mathbb{Z}}H^0_{\text{log-crys}}(U/W(k'))\longrightarrow\mathbb{Q}\otimes_{\mathbb{Z}}H^0_{\text{log-crys}}(x/W(k'))=\mathbb{Q}\otimes_{\mathbb{Z}}W(k'')$$

is injective, and it is isomorphic if $k'=k''$ holds.

PROOF OF CLAIM 1. Let us take a diagram

$$U\overset{q}{\longleftarrow}\coprod_{i\in I}U_i\overset{\iota}{\hookrightarrow}\coprod_{i\in I}\tilde{U}_i,$$

where U_i's and $\tilde{U}_i$'s are affine and connected, q is an open covering of U and ι is a closed immersion such that each $\tilde{U}_i$ is formally smooth over $\mathrm{Spf}\, W(k')$ and that $\tilde{U}_i \times_{\mathrm{Spf}\, W} \mathrm{Spec}\, k = U_i$ holds. Let i_0 be an index such that $x \in U_{i_0}$.

First we prove that the restriction $\alpha_1 : H^0_{\text{log-crys}}(U/W(k')) \longrightarrow H^0_{\text{log-crys}}(U_{i_0}/W(k'))$ is injective. Let a be an element in $\mathrm{Ker}(\alpha_1)$. Since the restriction $H^0_{\text{log-crys}}(U/W(k')) \longrightarrow \oplus_{i\in I} H^0_{\text{log-crys}}(U_i/W(k'))$ is injective, it suffices to show that the restriction of a to $H^0_{\text{log-crys}}(U_i/W(k'))$ is zero for any $i \in I$. Let $U_{=0}$ (resp. $U_{\neq 0}$) be the union of U_i's such that the restriction of a to $H^0_{\text{log-crys}}(U_i/W(k'))$ is zero (resp. is not zero). It suffices to prove that $U_{\neq 0}$ is empty. Assume the contrary. Then, since U is connected, there exist $i_1, i_2 \in I$ such that the restriction of a to $H^0_{\text{log-crys}}(U_{i_1}/W(k'))$ (resp. $H^0_{\text{log-crys}}(U_{i_2}/W(k'))$) is zero (resp. not zero) and that $U_{i_1} \cap U_{i_2}$ is non-empty. Let $\tilde{U}'$ be the open sub formal scheme of $\tilde{U}_{i_2}$ such that $\tilde{U}' \times_{\tilde{U}_{i_2}} U_{i_2} = U_{i_1} \cap U_{i_2}$ holds. Let a' is the image of a by the homomorphism

$$H^0_{\text{log-crys}}(U/W(k')) \longrightarrow H^0_{\text{log-crys}}(U_{i_2}/W(k')) \hookrightarrow \Gamma(\tilde{U}_{i_2}, \mathcal{O}_{\tilde{U}_{i_2}}).$$

(The injectivity of the second arrow follows easily from [Kk1, (6.2),(6.4)].) Then the restriction of a' to $\Gamma(\tilde{U}', \mathcal{O}_{\tilde{U}'})$ is zero, since it is the image of a by the homomorphism

$$H^0_{\text{log-crys}}(U/W(k')) \longrightarrow H^0_{\text{log-crys}}(U_{i_1} \cap U_{i_2}/W(k')) \hookrightarrow \Gamma(\tilde{U}', \mathcal{O}_{\tilde{U}'}).$$

Now, note that the restriction $\Gamma(\tilde{U}_{i_2}, \mathcal{O}_{\tilde{U}_{i_2}}) \longrightarrow \Gamma(\tilde{U}', \mathcal{O}_{\tilde{U}'})$ is injective. Hence $a' = 0$ holds. So the restriction of a to $H^0_{\text{log-crys}}(U_{i_2}/W(k'))$ is zero, and it is a contradiction. Hence we have $U_{\neq 0} = \emptyset$. Therefore α_1 is injective.

Next we prove the injectivity of the restriction

$$\alpha_2 : H^0_{\text{log-crys}}(U_{i_0}/W(k')) \longrightarrow H^0_{\text{log-crys}}(x/W(k')) = W(k'').$$

Let D be the p-adically complete PD-envelope of the closed immersion $x \hookrightarrow U_{i_0} \hookrightarrow \tilde{U}_{i_0}$. Then we have the following commutative diagram:

$$\begin{array}{ccc} H^0_{\text{log-crys}}(U_{i_0}/W(k')) & \xrightarrow{\alpha_2} & H^0_{\text{log-crys}}(x/W(k')) \\ \big\downarrow & & \big\downarrow \\ \Gamma(\tilde{U}_{i_0}, \mathcal{O}_{\tilde{U}_{i_0}}) & \xrightarrow{b} & \Gamma(D, \mathcal{O}_D), \end{array}$$

where the vertical arrows are injective. Hence it suffices to prove the injectivity of b. It follows from the fact that b is isomorphic to the inclusion of rings

$$W(k'')\{t_1, \cdots, t_r\} \hookrightarrow W(k'')\langle t_1, \cdots, t_r\rangle$$

etale locally (for some r), where $W(k'')\langle t_1, \cdots, t_r\rangle$ is the p-adic completion of PD-polynomial ring.

Since α_1, α_2 are injective, so is α. On the other hand, we have the natural inclusion $W(k') \subset H^0_{\text{log-crys}}(U/W(k'))$. Hence α is surjective if $k' = k''$ holds. Hence we obtain the assertion. □

Next, we prove the following claim:

CLAIM 2. Let k, X, f be as in the statement of the proposition, and let U be a connected component of $X_{f\text{-triv}}$. Then the natural homomorphism

$$\beta : \mathbb{Q} \otimes_{\mathbb{Z}} H^0_{\text{log-crys}}(X/W(k)) \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} H^0_{\text{log-crys}}(U/W(k))$$

is injective.

Let $X_{f\text{-triv}} = \coprod_{i=1}^n U_i$ be the decomposition into the connected components with $U = U_1$. First, to prove the claim 2, we prepare the following claim:

CLAIM 3. To prove the claim 2, one may assume the following condition $(*)$:

$(*)$: For each i, $U_i(k)$ is non-empty.

PROOF OF CLAIM 3. The proof of this claim is the same as that in claim 3 in Proposition 3.1.6, noting that we have the base change isomorphism

$$W(k') \otimes_{W(k)} H^0_{\text{log-crys}}(X/W) \overset{\simeq}{=} H^0_{\text{log-crys}}(X \times_k k'/W(k')),$$

$$W(k') \otimes_{W(k)} H^0_{\text{log-crys}}(U/W) \overset{\simeq}{=} H^0_{\text{log-crys}}(U \times_k k'/W(k')),$$

for a finite extension $k \subset k'$, which can be deduced from [Hy-Kk, (2.23)] in the same way as the case of usual crystalline cohomology. □

PROOF OF CLAIM 2. By claim 3, we may assume the condition $(*)$ to prove the claim 2. So we assume it. Let a be an element in $\mathrm{Ker}(\beta)$.

By claim 1 (in the case $k = k' = k''$) for U_i and a k-rational point in U_i (whose existence is assured by the condition $(*)$), the composite

$$K_0 \subset \mathbb{Q} \otimes_{\mathbb{Z}} H^0_{\text{log-crys}}(X/W) \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} H^0_{\text{log-crys}}(U_i/W)$$

is an isomorphism. Hence there exists a unique element $b_i \in K_0 \subset \mathbb{Q} \otimes_{\mathbb{Z}} H^0_{\text{log-crys}}(X/W)$ $(1 \leq i \leq n)$ such that the restrictions of a and b_i coincide in $\mathbb{Q} \otimes_{\mathbb{Z}} H^0_{\text{log-crys}}(U_i/W)$.

Let us take a diagram

$$(X, M) \overset{q}{\longleftarrow} \coprod_{j \in J} (X_i, M) \overset{\iota}{\hookrightarrow} \coprod_{j \in J} (\tilde{X}_j, M_{\tilde{X}_j}),$$

where X_j's and $\tilde{X}_j$'s are affine and connected, q is an open covering of X and ι is an exact closed immersion such that each $(\tilde{X}_j, M_{\tilde{X}_j})$ is formally log smooth integral over $(\operatorname{Spf} W, N)$ and that $(\tilde{X}_j, M_{\tilde{X}_j}) \times_{(\operatorname{Spf} W, N)} (\operatorname{Spec} k, N) = (X_j, M)$ holds. Let $a \in \operatorname{Ker}(\beta)$ and let a_j $(j \in J)$ be the image of a via the composite

$$\begin{aligned} \mathbb{Q} \otimes_{\mathbb{Z}} H^0_{\text{log-crys}}(X/W) \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} H^0_{\text{log-crys}}(X_j/W) &\hookrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma(\tilde{X}_j, \mathcal{O}_{\tilde{X}_j}) \\ &=: \mathbb{Q} \otimes_{\mathbb{Z}} A_j. \end{aligned}$$

It suffices to show that $a_j = 0$ holds for any $i \in I$.

By the connectedness of X, it suffices to show the following two assertions:

(1) $a_j = 0$ holds when $X_j \cap U$ is not empty.

(2) When $X_j \cap X_{j'}$ is not empty and $a_j = 0$ holds, then $a_{j'} = 0$ holds.

First let us prove the assertion (1). Assume a_j is not equal to zero. Then, by multiplying by p^m for some $m \in \mathbb{Z}$, we may assume that $a_j \in A_j, a_j \notin pA_j$. (Note that A_j is flat over V.) For $1 \leq i \leq n$, let $\tilde{X}_{ji}$ be the open sub formal scheme of $\tilde{X}_j$ satisfying $\tilde{X}_{ji} \times_{\tilde{X}_j} X_j = X_j \cap U_i$. Then the image of a_j and b_i in $\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma(\tilde{X}_{ji}, \mathcal{O}_{\tilde{X}_{ji}})$ coincide. Hence we have $b_i \in W$. Denote $a_j \bmod p \in \Gamma(X_j, \mathcal{O}_{X_j}) =: B_j$, $b_i \bmod p \in k$ simply by $\overline{a}_j, \overline{b}_i$, respectively. Then $\overline{a}_j \neq 0$ holds.

For an element $z \in B_j$, denote the closed set $\{x \in X_j \mid z = 0 \operatorname{in} k(x)\}$ (where $k(x)$ is the residue field of x) by C_z and regard it as a reduced closed subscheme of X_j. Then, we have $U_i \cap X_j \subset C_{\overline{a}_j - \overline{b}_i}$. Let us denote the closure of $U_i \cap X_j$ in X_j (regarded as a reduced closed subscheme) by D_i. Then we have $D_i \subset C_{\overline{a}_j - \overline{b}_i}$. Since D_i is reduced, $\overline{a}_j - \overline{b}_i$ is equal to zero in $\Gamma(D_i, \mathcal{O}_{D_j})$. Now let us assume that $D_i \cap D_{i'} \neq \emptyset$ holds. Then, $\overline{b}_i = \overline{b}_{i'}$ holds in $\Gamma((D_i \cap D_{i'})_{\mathrm{red}}, \mathcal{O}_{(D_i \cap D_{i'})_{\mathrm{red}}})$ since they both coincides with the image of $\overline{a}_j$. Since $\overline{b}_i, \overline{b}_{i'} \in k$, we have $\overline{b}_i = \overline{b}_{i'}$ in k. Since $X_{j,f\text{-triv}} \subset X_j$ is open dense by Proposition 2.3.2, we have $X_j = \bigcup_i D_i$. This fact, connectedness of X_j and the assumption $X_j \cap U_1 \neq \emptyset$ imply the equation $\overline{b}_i = \overline{b}_1$ for any i satisfying $X_j \cap U_i \neq \emptyset$. On the other hand, since $a \in \mathrm{Ker}(\beta)$, we have $\overline{b}_1 = 0$. So $\overline{b}_i = 0$ holds for all i satisfying $X_j \cap U_i \neq \emptyset$. Therefore we have $D_i \subset C_{\overline{a}_j - \overline{b}_i} = C_{\overline{a}_j}$ for all such i's. So one get the equality $X_j = C_{\overline{a}_j}$. Since X_j is reduced, we have $\overline{a}_j = 0$ in B_j. This is a contradiction. Hence $a_j = 0$ holds.

Next let us prove the assertion (2). Since $X_j \cap X_{j'}$ is not empty, there exists an index $1 \leq i_0 \leq n$ such that $X_j \cap X_{j'} \cap U_{i_0} \neq \emptyset$ holds. Since $a_j = 0$ holds, the restriction of a to $\mathbb{Q} \otimes_{\mathbb{Z}} H^0_{\text{log-crys}}(X_j \cap X_{j'} \cap U_{i_0}/W)$ is equal to zero. Since the map $\mathbb{Q} \otimes_{\mathbb{Z}} H^0_{\text{log-crys}}(U_{i_0}/W) \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} H^0_{\text{log-crys}}(X_j \cap X_{j'} \cap U_{i_0}/W)$ is injective (which can be proved as in the proof of injectivity of the homomorphism α_1 in claim 1), the restriction of a to $\mathbb{Q} \otimes_{\mathbb{Z}} H^0_{\text{log-crys}}(U_{i_0}/W)$ is zero. Then, by the argument in the proof of the assertion (1) (replacing $\tilde{X}_j, X_j, U_1$ by $\tilde{X}_{j'}, X_{j'}, U_{i_0}$, respectively), one can show the equality $a_{j'} = 0$. $\square$

Now we finish the proof of the proposition by using the claims. Let U be a connected component of $X_{f\text{-triv}}$ and let x be a closed point of U and let k' be the residue field of x. Let us consider the following diagram:

$$
\begin{aligned}
K_0 \subset \mathbb{Q} \otimes_{\mathbb{Z}} H^0_{\text{log-crys}}(X/W) &\xrightarrow{\beta} \mathbb{Q} \otimes_{\mathbb{Z}} H^0_{\text{log-crys}}(U/W) \\
&\xrightarrow{\alpha} \mathbb{Q} \otimes_{\mathbb{Z}} H^0(x/W) = \mathbb{Q} \otimes_{\mathbb{Z}} W(k').
\end{aligned}
$$

By claim 1, α is injective and by claim 2, β is injective. So there are inclusions of rings $K_0 \subset H^0_{\text{log-crys}}(X/W) \subset \mathbb{Q} \otimes_{\mathbb{Z}} W(k')$, and they imply that $H^0_{\text{log-dR}}(X/k)$ is a field. $\square$

In the rest of this section, we give two most basic properties of crystalline fundamental groups. The first one is Hurewicz isomorphism.

Theorem 4.1.8 (Hurewicz Isomorphism). *Let* $(X,M) \xrightarrow{f} (\operatorname{Spec} k, N) \hookrightarrow (\operatorname{Spf} W, N)$ *as in Definition 4.1.5. Then, for any* k*-rational point* x *of* $X_{f\text{-triv}}$ *over* $\operatorname{Spf} W$*, there is a canonical isomorphism*

$$(\operatorname{Lie} \pi_1^{\mathrm{crys}}((X/W)^{\log}, x))^* \cong K_0 \otimes_W H^1((X/W)^{\log}_{\mathrm{crys}}, \mathcal{O}_{X/W}).$$

Proof. Obvious by Corollary 1.1.10. □

Next we define the crystalline Frobenius and show that it is isomorphic under certain condition. Let (X,M) be a fine log scheme over k and $(X,M) \xrightarrow{f} (\operatorname{Spec} k, N) \hookrightarrow (\operatorname{Spf} W, N)$ be as in Definition 4.1.5. Suppose there exists a lifting

$$\sigma : (\operatorname{Spf} W, N) \longrightarrow (\operatorname{Spf} W, N)$$

of the absolute Frobenius of $(\operatorname{Spec} k, N)$ and let

$$F^{\mathrm{abs}} : (X,M) \longrightarrow (X,M)$$

be the absolute Frobenius. Then there exists the functor

$$(F^{\mathrm{abs}})^* : \mathcal{N}I_{\mathrm{crys}}((X,M)/(\operatorname{Spf} W,N)) \longrightarrow \mathcal{N}I_{\mathrm{crys}}((X,M)/(\operatorname{Spf} W,N))$$

and it induces the morphism of crystalline fundamental groups

$$\begin{aligned}(F^{\mathrm{abs}})_* : \pi_1^{\mathrm{crys}}&((X,M)/(\operatorname{Spf} W,N),x) \otimes_{K_0,\sigma^*} K_0 \\ &\longrightarrow \pi_1^{\mathrm{crys}}((X,M)/(\operatorname{Spf} W,N),x)\end{aligned}$$

for any k-rational point x of $X_{f\text{-triv}}$, where $\sigma^* : K_0 \longrightarrow K_0$ is the homomorphism induced by σ. We call this map (absolute) crystalline Frobenius.

Let $(X^{(p)}, M^{(p)})$ be $(X,M) \times_{(\operatorname{Spf} W,N),\sigma} (\operatorname{Spf} W,N)$. Then F^{abs} induces the morphism

$$F : (X,M) \longrightarrow (X^{(p)}, M^{(p)})$$

called relative Frobenius. This map induces the morphism of crystalline fundamental groups

$$F_* : \pi_1^{\mathrm{crys}}((X, M)/(\mathrm{Spf}\, W, N), x) \longrightarrow \pi_1^{\mathrm{crys}}((X^{(p)}, M^{(p)})/(\mathrm{Spf}\, W, N), F(x)).$$

we call this map relative crystalline Frobenius.

Theorem 4.1.9. *Let the notations be as above and moreover assume that f is log smooth integral and of Cartier type. Then the absolute crystalline Frobenius*

$$\begin{aligned}(F^{\mathrm{abs}})_* : \pi_1^{\mathrm{crys}}((X, M)/(\mathrm{Spf}\, W, N), x) \otimes_{K_0,\sigma} K_0 \\ \longrightarrow \pi_1^{\mathrm{crys}}((X, M)/(\mathrm{Spf}\, W, N), x)\end{aligned}$$

and the relative crystalline Frobenius

$$F_* : \pi_1^{\mathrm{crys}}((X, M)/(\mathrm{Spf}\, W, N), x) \longrightarrow \pi_1^{\mathrm{crys}}((X^{(p)}, M^{(p)})/(\mathrm{Spf}\, W, N), F(x))$$

are isomorphisms.

Proof. Let s be the natural morphism $(X^{(p)}, M^{(p)}) \longrightarrow (X, M)$. It suffices to prove that the functors

$$s^* : \mathcal{N}I_{\mathrm{crys}}((X/W)^{\log}) \longrightarrow \mathcal{N}I_{\mathrm{crys}}((X^{(p)}/W)^{\log}),$$

$$F^* : \mathcal{N}I_{\mathrm{crys}}((X^{(p)}/W)^{\log}) \longrightarrow \mathcal{N}I_{\mathrm{crys}}((X/W)^{\log}),$$

give equivalences of categories. For $\mathcal{E} := K_0 \otimes \mathcal{E}_0, \mathcal{E} := K_0 \otimes \mathcal{E}_0' \in \mathcal{N}I_{\mathrm{crys}}((X/W)^{\log})$ (resp. $\mathcal{N}I_{\mathrm{crys}}((X^{(p)}/W)^{\log})$, the group of homomorphism $\mathrm{Hom}(\mathcal{E}, \mathcal{E}')$ in the category $\mathcal{N}I_{\mathrm{crys}}((X/W)^{\log})$ (resp. $\mathcal{N}I_{\mathrm{crys}}((X^{(p)}/W)^{\log})$) is naturally isomorphic to the group

$$\mathbb{Q} \otimes_{\mathbb{Z}} H^0((X/W)^{\log}_{\mathrm{crys}}, \mathcal{H}\mathrm{om}(\mathcal{E}_0, \mathcal{E}'_0)) =: H^0((X/W)^{\log}_{\mathrm{crys}}, \mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{E}'))$$

$$\begin{aligned}(\text{resp.} \quad \mathbb{Q} \otimes_{\mathbb{Z}} H^0((X^{(p)}/W)^{\log}_{\mathrm{crys}}, \mathcal{H}\mathrm{om}(\mathcal{E}_0, \mathcal{E}'_0)) \\ =: H^0((X^{(p)}/W)^{\log}_{\mathrm{crys}}, \mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{E}')) \quad),\end{aligned}$$

and the set of the isomorphism classes of extensions

$$0 \longrightarrow K_0 \otimes \mathcal{O}_{X/W} \longrightarrow \tilde{\mathcal{E}} \longrightarrow \mathcal{E} \longrightarrow 0$$

$$(\text{resp.} \quad 0 \longrightarrow K_0 \otimes \mathcal{O}_{X^{(p)}/W} \longrightarrow \tilde{\mathcal{E}} \longrightarrow \mathcal{E} \longrightarrow 0 \quad)$$

is isomorphic to

$$\mathbb{Q} \otimes_{\mathbb{Z}} H^1((X/W)^{\log}_{\text{crys}}, \mathcal{E}_0) =: H^1((X/W)^{\log}_{\text{crys}}, \mathcal{E})$$

$$(\text{resp.} \quad \mathbb{Q} \otimes_{\mathbb{Z}} H^1((X^{(p)}/W)^{\log}_{\text{crys}}, \mathcal{E}_0) =: H^1((X/W)^{\log}_{\text{crys}}, \mathcal{E}) \quad).$$

Since the categories $\mathcal{N}I_{\text{crys}}((X/W)^{\log})$, $\mathcal{N}I_{\text{crys}}((X^{(p)}/W)^{\log})$ are nilpotent, it suffices to show that the homomorphisms

$$s^* : H^i((X/W)^{\log}_{\text{crys}}, \mathcal{E}) \otimes_{K_0,\sigma} K_0 \longrightarrow H^i((X^{(p)}/W)^{\log}_{\text{crys}}, s^*\mathcal{E}),$$

$$F^* : H^i((X^{(p)}/W)^{\log}_{\text{crys}}, \mathcal{E}') \longrightarrow H^i((X^{(p)}/W)^{\log}_{\text{crys}}, F^*\mathcal{E}'),$$

are isomorphisms for any $\mathcal{E} \in \mathcal{N}I_{\text{crys}}((X/W)^{\log}), \mathcal{E}' \in \mathcal{N}I_{\text{crys}}((X^{(p)}/W)^{\log})$ and $i = 0, 1$ to prove the desired equivalences of categories. Moreover, by using five lemma, one knows that it suffices to show the homomorphisms

$$\begin{aligned} s^* : K_0 \otimes_{\sigma,K_0} (K_0 \otimes_W H^i((X/W)^{\log}_{\text{crys}}, \mathcal{O}_{X/W})) \\ \longrightarrow K_0 \otimes_W H^i((X^{(p)}/W)^{\log}_{\text{crys}}, \mathcal{O}_{X^{(p)}/W}), \end{aligned}$$

$$\begin{aligned} F^* : K_0 \otimes_W H^i((X^{(p)}/W)^{\log}_{\text{crys}}, \mathcal{O}_{X^{(p)}/W}) \\ \longrightarrow K_0 \otimes_W H^i((X/W)^{\log}_{\text{crys}}, \mathcal{O}_{X/W}), \end{aligned}$$

are isomorphisms for $i \in \mathbb{N}$. But they follow from [Hy-Kk, (2.23), (2.24)]. Hence we obtain the assertion. □

4.2. On independence of compactification

Let k be a perfect field of characteristic $p > 0$ and let W be the Witt ring of k. Let $f : X \longrightarrow \operatorname{Spec} k$ be a proper smooth morphism of schemes and let D be a normal crossing divisor in X. Then, as we saw in Section 2.4, we can define the log structure M on X associated to the pair (X, D) such that $(X, M)_{f\text{-triv}} = X - D =: U$. Therefore, for $x \in (X - D)(k)$, we can define the crystalline fundamental group $\pi_1^{\text{crys}}((X, M)/\operatorname{Spf} W, x)$. It is expected that this fundamental group is the crystalline realization of the motivic fundamental group of U. So it is natural to consider the following problem:

Problem 4.2.1. Let the notations be as above. Then, is the crystalline fundamental group $\pi_1^{\text{crys}}((X, M)/\operatorname{Spf} W, x)$ independent of the choice of the compactification (X, D) of U as above?

In this section, we will give an affirmative answer to this problem under the condition $\dim X \leq 2$. We need this condition here because we use the resolution of singularities (see [A] or [L]) and the structure of proper birational morphisms of surfaces ([Sha]).

THEOREM 4.2.2. *Let X_i be proper smooth varieties over k and let D_i be a normal crossing divisor on X_i $(i = 1, 2)$. Denote the log structure on X_i associated to the pair (X_i, D_i) by M_i. Assume that we are given an isomorphism $X_1 - D_1 \overset{\sim}{=} X_2 - D_2$ and we denote this scheme by U. Assume moreover that* $\dim U \leq 2$ *holds. Then, for any k-rational point x in U, there exists an isomorphism*

$$\pi_1^{\text{crys}}((X_1, M_1)/\operatorname{Spf} W, x) \overset{\sim}{=} \pi_1^{\text{crys}}((X_2, M_2)/\operatorname{Spf} W, x).$$

PROOF. Let X' be the closure of the image of diagonal map $U \longrightarrow X_1 \times X_2$. Let D' be $X' - U$. Then by the embedded resolution for $D' \subset X'$, there exists a proper birational morphism $X_3 \overset{g}{\longrightarrow} X'$ such that $D_3 := f^{-1}(D')_{\text{red}}$ is a normal crossing divisor on X_3. (For this embedded resolution, see [A].) Since $\text{pr}_i \circ g : X_3 \longrightarrow X_i$ $(i = 1, 2)$ is proper birational, we may assume that there exists a proper birational morphism $X_2 \overset{f}{\longrightarrow} X_1$ which is identity on U. If $\dim U = 1$ holds, this implies $X_2 \overset{\sim}{=} X_1$ and the theorem is obvious.

Hence we may assume $\dim U = 2$. By [Sha, p.55], f can be written as a composition of blow ups at closed points which are not in U. Hence we may assume that f is a blow up at a closed point which is not in U. We prove that the functor $f^* : \mathcal{N}I_{\mathrm{crys}}((X_1, M_1)/\mathrm{Spf}\, W) \longrightarrow \mathcal{N}I_{\mathrm{crys}}((X_2, M_2)/\mathrm{Spf}\, W)$ gives an equivalence of categories. By the argument in the proof of Theorem 4.1.9, it suffices to show that the homomorphisms

$$f^* : \mathbb{Q} \otimes_{\mathbb{Z}} H^j(((X_1, M_1)/\mathrm{Spf}\, W)_{\mathrm{crys}}, \mathcal{E}) \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} H^j((X_2, M_2)/\mathrm{Spf}\, W, f^*\mathcal{E})$$

are isomorphisms for any $\mathcal{E} \in \mathcal{N}I_{\mathrm{crys}}((X_1, M_1)/\mathrm{Spf}\, W)$ and $j = 0, 1$. Hence the theorem is reduced to the following lemma:

Lemma 4.2.3. *Let (X_1, M_1), (X_2, M_2) and f be as above and let $\mathcal{E}$ be a locally free, coherent crystal on the site $(X_1/W)^{\log}_{\mathrm{crys}}$. Then the homomorphism*

$$H^j((X_1/W)^{\log}_{\mathrm{crys}}, \mathcal{E}) \longrightarrow H^j((X_2/W)^{\log}_{\mathrm{crys}}, f^*\mathcal{E})$$

is an isomorphism for $j = 0, 1$.

Proof. First, by base change theorem for log crystalline cohomology ([Hy-Kk, (2.23)]), we can enlarge the field k to a finite extension. Hence we may assume that f is a blow-up at a point $z \in D_1(k)$.

For a site $\mathcal{S}$, we denote the topos of sheaves on the site $\mathcal{S}$ by $\mathcal{S}^{\sim}$. Then there exists a canonical morphism of topoi

$$u_{X_i/W} : (X_i/W)^{\log\sim}_{\mathrm{crys}} \longrightarrow X^{\sim}_{i,\mathrm{et}}.$$

Then there exists a map of complexes

$$Ru_{X_1/W,*}\mathcal{E} \longrightarrow Rf_* Ru_{X_2/W,*} f^*\mathcal{E} \tag{4.2.1}$$

which induces the map between the log crystalline cohomologies above. Then it suffices to show that the map (4.2.1) is a quasi-isomorphism. To prove it, we may work etale locally. Hence we may assume (X_1, D_1, z) lifts to (Y_1, C_1, y), where Y_1 is a formal scheme which is formally smooth over $\mathrm{Spf}\, W$, C_1 is a relative simple normal crossing divisor on Y_1, and y is a W-valued point of C_1. let us denote the blow up of Y_1 with center y by Y_2 and the inverse image of C_1 by C_2. Then (Y_2, C_2) is a lifting of (X_2, D_2).

We denote the map $(Y_2, C_2) \longrightarrow (Y_1, C_1)$ by h. By [Kk1, (6.2)], the formal log connection (E, ∇) on (Y_1, C_1) corresponding the crystal $\mathcal{E}$ is defined, and by [Kk1, (6.4)], the map (4.2.1) is expressed by the map

$$E \otimes \omega_1^{\cdot} \longrightarrow Rh_*(h^* E \otimes \omega_2^{\cdot}),$$

where $\omega_i^{\cdot} := \omega^{\cdot}_{(Y_i, C_i)/W}$, the right hand side is the de Rham complex of (E, ∇) and $h^* E \otimes \omega_2^{\cdot}$ is the de Rham complex of $h^*(E, \nabla)$. If we can show the equality $Rh_* \omega_2^q = \omega_1^q$, then the right hand side can be written as

$$h_*(h^* E \otimes \omega_2^{\cdot}) \simeq E \otimes h_* \omega_2^{\cdot} \simeq E \otimes \omega_1^{\cdot}$$

and the lemma is proved. Hence it suffices to prove that $Rh_* \omega_2^q = \omega_1^q$.

First we consider the case that there are two components of C_1 which contain y. In this case, the map h is formally log etale. Hence, by [Kk1, (3.12)], $h^* \omega_1^q \cong \omega_2^q$ holds. So we have

$$Rh_* \omega_2^q = Rh_* h^* \omega_1^q = \omega_1^q.$$

Second we consider the case that there is only one component of C_1 which contains y. Denote this component by C_{11}. In this case, we have the following exact sequences

$$0 \longrightarrow h^* \omega_1^0 \longrightarrow \omega_2^0 \longrightarrow 0, \tag{4.2.2}$$

$$0 \longrightarrow h^* \omega_1^1 \longrightarrow \omega_2^1 \longrightarrow \omega_B^1 \longrightarrow 0, \tag{4.2.3}$$

$$0 \longrightarrow h^* \omega_1^2 \longrightarrow \omega_2^2 \longrightarrow \omega_2^2|_B \longrightarrow 0, \tag{4.2.4}$$

where B is the exceptional curve, and if we denote the intersection point of B and C_{11} by w, ω_B^1 is defined by $\Omega^1_{B/W}(\log w)$. Hence ω_B^1 and $\omega_2^2|_B$ is calculated as follows:

$$\omega_B^1 = \Omega^1_{B/W}(\log w) \cong \mathcal{O}_B(-1),$$

$$\omega_2^2|_B = h^* \omega_1^2(B)|_B \cong \mathcal{O}(B)|_B \cong \mathcal{O}_B(-1).$$

Therefore $Rh_*\omega_B^1 = 0$ and $Rh_*(\omega_2^2|_B) = 0$ hold. So, by the exact sequences (4.2.2), (4.2.3) and (4.2.4), we have

$$Rh_*\omega_2^q \overset{\sim}{=} Rh_*h^*\omega_1^q = \omega_1^q.$$

Hence the proof of the lemma is finished. □

Since the proof of Lemma 4.2.3 is finished, the proof of Theorem 4.2.2 is completed.

Remark 4.2.4. In the forthcoming paper [Shi], we consider Problem 4.2.1 in another method and we give the affirmative answer in general case.

4.3. Formal log connections, HPD-isostratifications and isocrystals on log crystalline site

Throughout this section, let k be a perfect field of characteristic $p > 0$, let W be the Witt ring of k and put $K_0 := \operatorname{Frac} W$, $W_n := W \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$. In this section, we define the log version of HPD-isostratifications defined in [O2, §7], and study the relation between HPD-isostratifications and isocrystals on log crystalline site. As a consequence, we prove the descent property of the category of isocrystals on log crystalline site. Then we recall the relation between isocrystals on log crystalline site and integrable formal log connections, which is essentially due to Kato ([Kk1]). We use these results in the proof of Berthelot-Ogus theorem for fundamental groups in Chapter 5.

First let us define the notion of HPD-stratifications and HPD-isostratifications on log schemes:

Definition 4.3.1. Let S be either $\operatorname{Spec} W_n$ or $\operatorname{Spf} W$. Let (X, M) be a fine log scheme over k and let $(X, M) \longrightarrow (\operatorname{Spec} k, N) \hookrightarrow (S, N)$ be morphisms of fine log formal W-schemes. Let $(X, M) \hookrightarrow (Y, J)$ be a closed immersion of (X, M) into a fine log formal W-scheme (Y, J) over (S, N). Let $D := D_{(X,M)}((Y, J))^{\log}$ be the p-adically complete log PD-envelope. Let $D(1)$ be $D_{(X,M)}((Y, J) \times_{(S,N)} (Y, J))^{\log}$ and let $D(2)$ be $D_{(X,M)}((Y, J) \times_{(S,N)} (Y, J) \times_{(S,N)} (Y, J))^{\log}$. Then we have the projections $p_i : D(1) \longrightarrow D$ $(i = 1, 2)$, $p_{ij} : D(2) \longrightarrow D(1)$ $(1 \le i < j \le 3)$ and the diagonal map $\Delta : D \longrightarrow D(1)$.

(1) We define an HPD-stratification on an $\mathcal{O}_D$-module E as an isomorphism $\epsilon : p_2^*E \longrightarrow p_1^*E$ which satisfies the following two conditions:

(a) $\Delta^*(\epsilon) = \mathrm{id} \in \mathrm{End}_D(E)$.
(b) (Cocycle condition)

$$p_{12}^*(\epsilon) \circ p_{23}^*(\epsilon) = p_{13}^*(\epsilon) \in \mathrm{Hom}_{D(2)}(q_3^*E, q_1^*E),$$

where $q_3 = p_2 \circ p_{23}$ and $q_1 = p_1 \circ p_{12}$.

We denote the category of coherent sheaves of $\mathcal{O}_D$-modules with HPD-stratifications by $\mathrm{HPD}((X,M) \hookrightarrow (Y,J))$.

(2) Let us assume $S = \mathrm{Spf}\, W$. Then we define an HPD-isostratification on a $K_0 \otimes_W \mathcal{O}_D$-module E as an isomorphism $\epsilon : p_2^*E \longrightarrow p_1^*E$ satisfying the two conditions in (1). We denote the category of isocoherent sheaves on D with HPD-isostratifications by $\mathrm{HPDI}((X,M) \hookrightarrow (Y,J))$.

Let (X,M) be a fine log scheme over k, let $(X,M) \longrightarrow (\mathrm{Spec}\, k, N) \hookrightarrow (\mathrm{Spf}\, W, N)$ be morphisms fine log formal W-schemes and let $(X,M) \hookrightarrow (Y,J)$ be a closed immersion of (X,M) into a fine log formal W-scheme (Y,J). Denote the fine log scheme $(Y,J) \times_{(\mathrm{Spf}\, W,N)} (\mathrm{Spec}\, W_n, N)$ by (Y_n, J_n). Then we have the canonical functor (see [Kk1, §6])

$$C_{\mathrm{crys}}((X,M)/(\mathrm{Spec}\, W_n, N)) \longrightarrow \mathrm{HPD}((X,M) \hookrightarrow (Y_n, J_n)), \tag{4.3.1}$$

which is compatible with respect to n. Hence it induces the functor

$$C_{\mathrm{crys}}((X,M)/(\mathrm{Spf}\, W, N)) \longrightarrow \mathrm{HPD}((X,M) \hookrightarrow (Y, J)), \tag{4.3.2}$$

and by inverting 'multiplication by p', it induces the functor

$$I_{\mathrm{crys}}((X,M)/(\mathrm{Spf}\, W, N)) \longrightarrow \mathrm{HPDI}((X,M) \hookrightarrow (Y, J)). \tag{4.3.3}$$

Let us consider the case that (Y,J) is formally log smooth over $(\mathrm{Spf}\, W, N)$. Then, the proof of [Kk1, (6.2)] shows that the functor (4.3.1) is an equivalence of categories. Hence so is the functor (4.3.2). From this, one can conclude that the functor (4.3.3) is fully-faithful.

Moreover, one can prove the following:

Proposition 4.3.2. *Let the notations be as above and assume moreover that (Y,J) is formally log smooth over $(\mathrm{Spf}\, W, N)$ and that we have*

$(Y, J) \times_{(\mathrm{Spf}\, W, N)} (\mathrm{Spec}\, k, N) = (X, M)$. *Then the functor* (*4.3.3*) *is an equpivalence of categories.*

Proof. We only have to show the above functor is essentially surjective. Since the functor (4.3.2) is an equivalence of categories, it suffices to show the following lemma (for $U = \emptyset$).

Lemma 4.3.3. *Let the situation be as above.* (*Note that in this situation, the p-adically complete log PD-envelope D in Definition 4.3.1 coincides with Y.*) *Let L be a p-torsion free coherent sheaf of $\mathcal{O}_Y$-modules and let ϵ be an HPD-isostratification on $K_0 \otimes L$. Let U be an open subset of Y* (*possibly empty*), *let L' be a p-torsion free coherent sheaf of $\mathcal{O}_Y|_U$-modules and let α be an isomorphism $K_0 \otimes L' \longrightarrow K_0 \otimes L|_U$. Assume that L' is $\epsilon|_U$-stable when viewed via α. Then, there exists a coherent extension L'' of L' to Y and an isomorphism $\alpha' : K_0 \otimes L'' \longrightarrow K_0 \otimes L$ extending α such that L'' is ϵ-stable when viewed via α'.*

Proof. The proof is the log version of [O2, (0.7.4)].

Replacing α by $p^n\alpha$ for a suitable n, we may assume that α comes from a morphism $L' \longrightarrow L|_U$. Replacing L' with its image in $L|_U$, we may assume that $L' \subset L|_U$. Choose m so that p^m annihilates $L|_U/L'$, and let L_m be the restriction of L to the subscheme of definition Y_m defined by p^m and $L'_m \subset L_m|_U$ be the image of L' in $L_m|_U$. Choose a coherent extension M_m of L'_m to Y_m and let M be the inverse image of M_m. Then $K_0 \otimes M$ is isomorphic to $K_0 \otimes L$ and $M|_U$ is isomorphic to L'. Hence we may assume that $L' \cong L|_U$ holds.

The rest of the argument is based on the proof of rigid analytic faithfully flat descent due to O.Gabber (cf. [O2, (0.7.2)], [O1, (1.9)]). Let $D(1), D(2), p_i : D(1) \longrightarrow Y\,(i = 1, 2), p_{ij} : D(2) \longrightarrow D(1)\,(1 \leq i < j \leq 3)$ be as in Definition 4.3.1. Put $E := K_0 \otimes L$ and we define the map $\theta : E \longrightarrow p_1^*E$ by $\theta(x) = \epsilon(p_2^*x)$.

By [Kk1, (6.5)], the maps $p_i : D(1) \longrightarrow Y$ are flat. Hence $p_1^*L \subset p_1^*E$ holds. Put $\phi := \theta|_L$ and let L'' be $\theta^{-1}(p_1^*L)$.

First check that $L'' \subset L$ holds. Let $\Delta_E : p_1^*E \longrightarrow E$ be a map $x \otimes \gamma \mapsto x\Delta^*(\gamma)$ for $x \in E$ and $\gamma \in \mathcal{O}_{D(1)}$. Then $\Delta_E \circ \theta = \mathrm{id}$ holds and Δ_E sends p_1^*L into L. If x is a local section of L'', $\theta(x) \in p_1^*L$ holds and hence $x = \Delta_E \circ \theta(x) \in L$ holds. Hence $L'' \subset L$ holds and so L'' is coherent.

Hence we have only to show that L'' is compatible with θ. (Since $L|_U$ is compatible with θ, $L''|_U = L|_U$ holds and $K_0 \otimes L'' = K_0 \otimes \theta^{-1}p_1^* L \overset{\sim}{=} \theta^{-1}p_1^*(K_0 \otimes L) = E$ holds.) Since p_1 is flat, $L'' \subset L$ implies $p_1^* L'' \subset p_1^* L$. Hence we have to show that the map $\phi' = \theta|''_L : L'' \longrightarrow p_1^* L$ factors through $p_1^* L''$. Since the assertion is etale local, we may assume that $Y = \operatorname{Spf} A$ holds and that there exists a chart $(P_Y \to L, Q_W \to N, Q \to P)$ of $(Y, L) \longrightarrow (\operatorname{Spf} W, N)$.

Until the end of this proof, we denote the p-adically completed tensor product by $\hat{\otimes}$ and use the symbol $\otimes$ only for usual tensor products.

Let f_1 (resp. f_2) be the map $P \oplus_Q P \longrightarrow P$ (resp. $P \oplus_Q P \oplus_Q P \longrightarrow P$) induced by the summation and let $R(i)$ be $(f_i^{\mathrm{gp}})^{-1}(P)$ for $i = 1, 2$. Let $A(1)$ (resp. $A(2)$) be the rings

$$(A \hat{\otimes} A) \hat{\otimes}_{\mathbb{Z}_p\{P \oplus_Q P\}} \mathbb{Z}_p\{R(1)\} \quad (\text{resp. } (A \hat{\otimes} A \hat{\otimes} A) \hat{\otimes}_{\mathbb{Z}_p\{P \oplus_Q P \oplus_Q P\}} \mathbb{Z}_p\{R(2)\}).$$

Then the natural map $A(1) \hat{\otimes} A(1) \longrightarrow A(2)$ is isomorphic. If we set $I(1) := \operatorname{Ker}(A(1) \longrightarrow A)$ and $I(2) := \operatorname{Ker}(A(2) \longrightarrow A)$, we can identify $I(1) \hat{\otimes} A(1) + A(1) \hat{\otimes} I(1)$ with $I(2)$ via the above isomorphism. By [Kk1, (5.6)], $D(i)$ is the usual PD-envelope of Y in $\operatorname{Spf} A(i)$. So it is affine. Put $D(i) = \operatorname{Spf} B(i)$ for $i = 1, 2$. By [B-O, (3.7)], the ideal $I(1)B(1) \hat{\otimes} B(1) + B(1) \hat{\otimes} I(1)B(1)$ of $B(1) \hat{\otimes} B(1)$ has the induced PD-structure. Hence there exists a map $\beta : B(2) \longrightarrow B(1) \hat{\otimes} B(1)$ which makes the following diagram commutative:

$$\begin{array}{ccccc} A(1) & \xrightarrow{p_{13}^*} & A(2) & \xrightarrow{\sim} & A(1) \hat{\otimes} A(1) \\ \downarrow & & \downarrow & & \downarrow \\ B(1) & \xrightarrow{p_{13}^*} & B(2) & \xrightarrow{\beta} & B(1) \hat{\otimes} B(1). \end{array}$$

Let δ be the composition $B(1) \longrightarrow B(2) \xrightarrow{\beta} B(1) \hat{\otimes} B(1)$. We denote the group of global sections of E also by E. Then, by the cocycle condition, the following diagram is commutative:

$$(4.3.4) \qquad \begin{array}{ccc} E & \xrightarrow{\theta} & E \hat{\otimes} B(1) \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \otimes \delta} \\ E \hat{\otimes} B(1) & \xrightarrow[\theta \otimes \mathrm{id}]{} & E \hat{\otimes} B(1) \hat{\otimes} B(1). \end{array}$$

Consider the following diagram:

$$\begin{array}{ccccc}
 & & L'' \otimes B(1) & \xrightarrow{\phi' \otimes \mathrm{id}} & L \hat{\otimes} B(1) \otimes B(1) \\
 & & \downarrow & & \downarrow \\
L'' & \xrightarrow{\phi'} & L \hat{\otimes} B(1) & \xrightarrow{\phi \otimes \mathrm{id}} & E \hat{\otimes} B(1) \otimes B(1) \\
\downarrow & & \downarrow & & \| \\
E & \xrightarrow{\theta} & E \hat{\otimes} B(1) & \xrightarrow{\theta \otimes \mathrm{id}} & E \hat{\otimes} B(1) \otimes B(1).
\end{array}$$

By definition, the square on the bottom left is Cartesian. Since p_1 is flat, the large rectangle on the right is also Cartesian. Thus it suffices to prove that the composition

$$L'' \to E \xrightarrow{\theta} E \hat{\otimes} B(1) \xrightarrow{\theta \otimes \mathrm{id}} E \hat{\otimes} B(1) \otimes B(1) \xrightarrow{\mathrm{pr} \otimes \mathrm{id}} E/L \hat{\otimes} B(1) \otimes B(1)$$

is the zero map. Since E/L is p-torsion, the natural map

$$E/L \hat{\otimes} B(1) \otimes B(1) \longrightarrow E/L \hat{\otimes} B(1) \hat{\otimes} B(1)$$

is isomorphic, so it suffices to show that our map becomes zero after we follow it with this isomorphism. If $x \in L''$, then $\theta(x) \in L \hat{\otimes} B(1)$ holds, and so $(\theta \otimes \mathrm{id})\theta(x) = \mathrm{id} \otimes \delta(x) \in L \hat{\otimes} B(1) \hat{\otimes} B(1)$ holds by the diagram (4.3.4). This proves the assertion. □

Since the proof of Lemma 4.3.3 is finished, the proof of Proposition 4.3.2 is also finished.

As a consequence of Lemma 4.3.3, we can prove the following descent property (cf. [O2, (0.7.5)]):

Corollary 4.3.4. *Let (X, M) be a fine log scheme over k and let $(X, M) \xrightarrow{f} (\mathrm{Spec}\, k, N) \hookrightarrow (\mathrm{Spf}\, W, N)$ be a morphism of fine log formal W-schemes such that f is formally log smooth. Then the descent for finite Zariski open coverings holds for the category $I_{\mathrm{crys}}((X, M)/(\mathrm{Spf}\, W, N))$.*

Proof. The proof is the same as that of [O2, (0.7.5)].

We have only to show the descent for the open covering $\{X_i \longrightarrow X\}_{i=1,2}$ such that (X_2, M) lifts to a fine log formal W-scheme (Y, L) which is formally log smooth over $(\mathrm{Spf}\, W, N)$. Let U be $X_1 \cap X_2$. Now we are given crystals $\mathcal{E}_i$ over (X_i, M) for $i = 1, 2$ and an isomorphism of isocrystals $\alpha : K_0 \otimes \mathcal{E}_1 \longrightarrow K_0 \otimes \mathcal{E}_2$ on (U, M). If we regard these objects via the categorical equivalence of Proposition 4.3.2, we are given a coherent sheaf L_2 on Y, an HPD-isostratification ϵ_2 on $K_0 \otimes L_2$ such that L_2 is ϵ_2-stable, a coherent sheaf L_1 on $Y|_U$, an HPD-isostratification ϵ_1 on $K_0 \otimes L_1$ such that L_1 is ϵ_1-stable and an isomorphism $\alpha : K_0 \otimes L_1 \longrightarrow K_0 \otimes L_2$ on $Y|_U$ which is compatible with α. (L_i's correspond to $\mathcal{E}_i$'s.) By the above lemma, there exists a coherent sheaf L'_2 on Y and an HPD-isostratification ϵ'_2 on $K_0 \otimes L'_2$ such that L'_2 is ϵ'_2-stable, $K_0 \otimes L_2$ is isomorphic to $K_0 \otimes L'_2$ with HPD-isostratification structures and the isomorphism $\alpha' : K_0 \otimes L_1 \longrightarrow K_0 \otimes L_2 \cong K_0 \otimes L'_2$ on U is induced by $L_1 \xrightarrow{\sim} L'_2$. If we denote the crystal corresponds to (L'_2, ϵ'_2) by $\mathcal{E}'_2$, the isomorphism α' gives the descent data of crystals $\mathcal{E}_1$ and $\mathcal{E}'_2$ and there exists a crystal $\mathcal{E}$ on (X, M) such that $\mathcal{E}|_{X_1} \cong \mathcal{E}_1$ and $\mathcal{E}|_{X_2} \cong \mathcal{E}'_2$ hold. Since $K_0 \otimes \mathcal{E}'_2$ is isomorphic to $K_0 \otimes \mathcal{E}_2$, the isocrystal $K_0 \otimes \mathcal{E}$ is the object corresponding to the descent data. □

More generally, we can prove the descent property of the category of isocrystals on log crystalline site for etale coverings. To state this, let us introduce the following notation:

Notation 4.3.5. Let $\mathcal{S}_k$ be the category of fine log formal W-schemes of the form $\coprod_{i \in I} (\mathrm{Spf}\, W(k_i), N_i)$, where $|I|$ is finite and each k_i is a finite extension of k. Let $(S, N) := \coprod_{i \in I} (\mathrm{Spf}\, W(k_i), N_i)$ be an object in $\mathcal{S}_k$ and let $(X, M) \xrightarrow{f} \coprod_{i \in I} (\mathrm{Spec}\, k_i, N_i) \overset{i}{\hookrightarrow} (S, N)$ be morphisms of fine log formal W-schemes. Let X_i be the scheme $(i \circ f)^{-1}(\mathrm{Spf}\, W(k_i))$. Then we denote the category $\prod_{i \in I} I_{\mathrm{crys}}((X_i, M)/(\mathrm{Spf}\, W(k_i), N_i))$ simply by $I_{\mathrm{crys}}((X, M)/(S, N))$.

Assume moreover that we are given a closed immersion $(X, M) \hookrightarrow (Y, J)$ of (X, M) into a fine log formal W-scheme (Y, J) over (S, N). Denote the structure morphism $(Y, J) \longrightarrow (S, N)$ by g and put $Y_i := g^{-1}(\mathrm{Spf}\, W(k_i))$. Then we denote the category $\prod_{i \in I} \mathrm{HPDI}((X_i, M) \hookrightarrow (Y_i, J))$ simply by $\mathrm{HPDI}((X, M) \hookrightarrow (Y, J))$.

Then the descent property is described as follows:

Proposition 4.3.6. *Put $S := \mathrm{Spf}\, W$. Let (X, M) be a fine log scheme over k and let $(X, M) \xrightarrow{f} (\mathrm{Spec}\, k, N) \overset{i}{\hookrightarrow} (S, N)$ be morphisms of fine log formal W-schemes such that f is formally log smooth of finite type. Assume we are given a commutative diagram of formal W-schemes*

$$\begin{array}{ccccccccc} & X & \leftarrow & X^{(0)} & \leftleftarrows & X^{(1)} & \overset{\leftarrow}{\leftleftarrows} & X^{(2)} \\ i \circ f & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & S & \leftarrow & S^{(0)} & \leftleftarrows & S^{(1)} & \overset{\leftarrow}{\leftleftarrows} & S^{(2)}, \end{array}$$

where the horizontal lines are etale hypercoverings (up to level 2). Let

$$p : I_{\mathrm{crys}}((X, M)/(S, N)) \longrightarrow I_{\mathrm{crys}}((X^{(0)}, M)/(S^{(0)}, N)),$$

$$p_i : I_{\mathrm{crys}}((X^{(0)}, M)/(S^{(0)}, N)) \longrightarrow I_{\mathrm{crys}}((X^{(1)}, M)/(S^{(1)}, N)) \quad (i = 1, 2),$$

$$p_{ij} : I_{\mathrm{crys}}((X^{(1)}, M)/(S^{(1)}, N)) \longrightarrow I_{\mathrm{crys}}((X^{(2)}, M)/(S^{(2)}, N)) \qquad (1 \leq i < j \leq 3),$$

be the functors induced by the above diagram. Let Δ_{crys} be the category of pairs $(\mathcal{E}, \varphi)$, where $\mathcal{E} \in I_{\mathrm{crys}}((X^{(0)}, M)/(S^{(0)}, N))$ and φ is an isomorphism $p_2(\mathcal{E}) \xrightarrow{\sim} p_1(\mathcal{E})$ in $I_{\mathrm{crys}}((X^{(1)}, M)/(S^{(1)}, N))$ satisfying the condition $p_{12}(\varphi) \circ p_{23}(\varphi) = p_{13}(\varphi)$ in $I_{\mathrm{crys}}((X^{(2)}, M)/(S^{(2)}, N))$. Then the functor

$$\Phi : I_{\mathrm{crys}}((X, M)/(S, N)) \longrightarrow \Delta_{\mathrm{crys}}$$

induced by p gives an equivalence of categories.

Proof. Let $X = \bigcup_{\lambda \in \Lambda} U_\lambda$ be an open covering by finite number of affine open subschemes. Put $X_{(0)} := \coprod_{\lambda \in \Lambda} U_\lambda$, $X_{(1)} := X_{(0)} \times_X X_{(0)}$ and $X_{(2)} := X_{(0)} \times_X X_{(0)} \times_X X_{(0)}$. Put $X_{(j)}^{(i)} := X_{(j)} \times_X X^{(i)}$. Then we have the following diagram:

$$\begin{array}{ccccccc} X_{(2)} & \leftarrow & X_{(2)}^{(0)} & \leftleftarrows & X_{(2)}^{(1)} & \overset{\leftarrow}{\leftleftarrows} & X_{(2)}^{(2)} \\ \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow \\ X_{(1)} & \leftarrow & X_{(1)}^{(0)} & \leftleftarrows & X_{(1)}^{(1)} & \overset{\leftarrow}{\leftleftarrows} & X_{(1)}^{(2)} \\ \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\ X_{(0)} & \leftarrow & X_{(0)}^{(0)} & \leftleftarrows & X_{(0)}^{(1)} & \overset{\leftarrow}{\leftleftarrows} & X_{(0)}^{(2)} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X & \leftarrow & X^{(0)} & \leftleftarrows & X^{(1)} & \overset{\leftarrow}{\leftleftarrows} & X^{(2)} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S & \leftarrow & S^{(0)} & \leftleftarrows & S^{(1)} & \overset{\leftarrow}{\leftleftarrows} & S^{(2)}. \end{array} \tag{4.3.5}$$

Let $\Delta_{\mathrm{crys},(i)}$ $(i = 0, 1, 2)$ (resp. Δ'_{crys}) be the category defined as Δ_{crys} by using the diagram

$$\begin{array}{ccccccc} X_{(i)} & \leftarrow & X_{(i)}^{(0)} & \overleftarrow{\leftarrow} & X_{(i)}^{(1)} & \overset{\leftarrow}{\overleftarrow{\leftarrow}} & X_{(i)}^{(2)} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S & \leftarrow & S^{(0)} & \overleftarrow{\leftarrow} & S^{(1)} & \overset{\leftarrow}{\overleftarrow{\leftarrow}} & S^{(2)} \end{array}$$

$$\left(\text{resp.} \begin{array}{ccccccc} X & \leftarrow & X_{(0)} & \overleftarrow{\leftarrow} & X_{(1)} & \overset{\leftarrow}{\overleftarrow{\leftarrow}} & X_{(2)} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S & \leftarrow & S & \overleftarrow{\leftarrow} & S & \overset{\leftarrow}{\overleftarrow{\leftarrow}} & S \end{array}\right).$$

Then there exist functors

$$q_i : \Delta_{\mathrm{crys},(0)} \longrightarrow \Delta_{\mathrm{crys},(1)} \quad (i = 1, 2),$$

$$q_{ij} : \Delta_{\mathrm{crys},(1)} \longrightarrow \Delta_{\mathrm{crys},(2)} \quad (1 \leq i < j \leq 3),$$

which are induced by the diagram (4.3.5). We define the category $\tilde{\Delta}_{\mathrm{crys}}$ as the category of pairs $(\mathcal{E}, \varphi)$, where $\mathcal{E} \in \Delta_{\mathrm{crys},(0)}$ and φ is an isomorphism $q_2(\mathcal{E}) \xrightarrow{\sim} q_1(\mathcal{E})$ in $\Delta_{\mathrm{crys},(1)}$ satisfying the condition $p_{12}(\varphi) \circ p_{23}(\varphi) = p_{13}(\varphi)$ in $\Delta_{\mathrm{crys},(2)}$. Then we have the functors

$$I_{\mathrm{crys}}((X^{(i)}, M)/(S, N)) \longrightarrow \Delta_{\mathrm{crys},(i)} \quad (i = 0, 1, 2), \tag{4.3.6}$$

and a diagram of functors

$$\begin{array}{ccc} \Delta'_{\mathrm{crys}} & \longrightarrow & \tilde{\Delta}_{\mathrm{crys}} \\ \uparrow & & \uparrow \\ I_{\mathrm{crys}}((X, M)/(S, N)) & \xrightarrow{\Phi} & \Delta_{\mathrm{crys}}, \end{array} \tag{4.3.7}$$

which are induced by the diagram (4.3.5).

Assume that the functors (4.3.6) are equivalences. Then the upper horizontal arrow in the diagram (4.3.7) is an equivalence. On the other hand, the vertical arrows in the diagram (4.3.7) are equivalences by Corollary 4.3.4, because the vertical hypercoverings in the diagram (4.3.5) are Čech coverings by finite Zariski open subschemes. Therefore Φ is also an equivalence. So, to prove the proposition, it suffices to prove the equivalence of the functors (4.3.6). So we may replace X by $X_{(i)}$ $(i = 0, 1, 2)$.

Note that a connected component X_0 of $X_{(i)}$ is an open subscheme of an affine open subscheme U of X. Since the morphism $f|_U : (U, M) \longrightarrow (\operatorname{Spec} k, N)$ is log smooth, there exists an exact closed immersion $(X, M) \hookrightarrow (Y', J)$ into a fine log formal W-scheme (Y', J) over (S, N) such that (Y', J) is formally log smooth over (S, N) and that $(Y', J) \times_{(S,N)} (\operatorname{Spec} k, N) = (U, M)$ holds. Then the morphism $U \hookrightarrow Y'$ is a homeomorphism. Let $Y \hookrightarrow Y'$ be the open subscheme of Y' which is homeomorphic to X_0 via the homeomorphism $U \hookrightarrow Y'$. Then we have $(Y, J) \times_{(S,N)} (\operatorname{Spec} k, N) = (X_0, M)$. Therefore, to prove the proposition, we may assume the existence of an exact closed immersion $(X, M) \hookrightarrow (Y, J)$ into a fine log formal W-scheme (Y, J) over (S, N) such that (Y, J) is formally log smooth over (S, N) and that $(Y, J) \times_{(S,N)} (\operatorname{Spec} k, N) = (X, M)$ holds.

Now we prove the proposition under the assumption of the previous paragraph. Since we have an equivalence of sites $X_{\text{et}} \simeq Y_{\text{et}}$, we can form the commutative diagram

$$
\begin{array}{ccccccc}
X & \leftarrow & X^{(0)} & \underset{\leftarrow}{\leftarrow} & X^{(1)} & \underset{\leftarrow}{\overset{\leftarrow}{\rightarrow}} & X^{(2)} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Y & \leftarrow & Y^{(0)} & \underset{\leftarrow}{\leftarrow} & Y^{(1)} & \underset{\leftarrow}{\overset{\leftarrow}{\rightarrow}} & Y^{(2)}
\end{array}
\tag{4.3.8}
$$

saisfying the following conditions:

(1) The horizontal lines are etale hypercoverings, and the upper horizontal line is the one given in the diagram (4.3.5).
(2) The morphisms $(X^{(i)}, M) \longrightarrow (Y^{(i)}, J)$ $(i = 0, 1, 2)$ induced by the vertical arrows are exact closed immersions over $(S^{(i)}, N)$,
(3) $(Y^{(i)}, J)$ is formally log smooth over $(S^{(i)}, N)$ and $(Y^{(i)}, J) \times_{(S^{(i)},N)} (S_1^{(i)}, N) = (X^{(i)}, M)$ holds, where $S_1^{(i)} := \underline{\operatorname{Spec}}\, \mathcal{O}_{S^{(i)}}/p\mathcal{O}_{S^{(i)}}$.

Then we have the equivalences of categories

$$
I_{\text{crys}}((X, M)/(S, N)) \xrightarrow{\sim} \operatorname{HPDI}((X, M) \hookrightarrow (Y, J)),
\tag{4.3.9}
$$

$$
I_{\text{crys}}((X^{(i)}, M)/(S^{(i)}, N)) \xrightarrow{\sim} \operatorname{HPDI}((X^{(i)}, M) \hookrightarrow (Y^{(i)}, J)).
\tag{4.3.10}
$$

Let

$$
p' : \operatorname{HPDI}((X, M) \hookrightarrow (Y, J)) \longrightarrow \operatorname{HPDI}((X^{(0)}, M) \hookrightarrow (Y^{(0)}, J)),
$$

$$p'_i : \mathrm{HPDI}((X^{(0)}, M) \hookrightarrow (Y^{(0)}, J)) \longrightarrow \mathrm{HPDI}((X^{(1)}, M) \hookrightarrow (Y^{(1)}, J)) \quad (i = 1, 2),$$

$$p'_{ij} : \mathrm{HPDI}((X^{(1)}, M) \hookrightarrow (Y^{(1)}, J)) \longrightarrow \mathrm{HPDI}((X^{(2)}, M) \hookrightarrow (Y^{(2)}, J)) \quad (1 \leq i < j \leq 3),$$

be the functors induced by the diagram (4.3.8). Then the functor p' (resp. p'_i, p'_{ij}) is compatible with the functor p (resp. p_i, p_{ij}) via the equivalence of categories (4.3.9) (resp. (4.3.10)). Let Δ_{HPDI} be the category of pairs $((E, \epsilon), \varphi)$, where $(E, \epsilon) \in \mathrm{HPDI}((X^{(0)}, M) \hookrightarrow (Y^{(0)}, J))$ and φ is an isomorphism $p'_2((E, \epsilon)) \xrightarrow{\sim} p'_1((E, \epsilon))$ in $\mathrm{HPDI}((X^{(1)}, M) \hookrightarrow (Y^{(1)}, J))$ satisfying the condition $p'_{12}(\varphi) \circ p'_{23}(\varphi) = p'_{13}(\varphi)$ in $\mathrm{HPDI}((X^{(2)}, M) \hookrightarrow (Y^{(2)}, J))$. Then the equivalences (4.3.10) induces an equivalence of categories

$$\Delta_{\mathrm{crys}} \xrightarrow{\sim} \Delta_{\mathrm{HPDI}}, \tag{4.3.11}$$

and one can check that the functor

$$\Psi : \mathrm{HPDI}((X, M) \hookrightarrow (Y, J)) \longrightarrow \Delta_{\mathrm{HPDI}}$$

induced by p' is compatible with Φ via the equivalences of categories (4.3.9) and (4.3.11). Hence it suffices to prove that the functor Ψ gives an equivalence of categories.

Let $(D(i), M_{D(i)})$ (resp. $(D(i)^{(j)}, M_{D(i)^{(j)}})$) be the p-adically complete log PD-envelope of (X, M) (resp. $(X^{(j)}, M)$) in the $(i+1)$-fold fiber product of (Y, J) (resp. $(Y^{(j)}, J)$) over (S, N) (resp. $(S^{(j)}, N)$). Let

$$r : Y^{(0)} \longrightarrow Y, \quad r_k : Y^{(1)} \longrightarrow Y^{(0)} \ (k = 1, 2),$$

$$r_{kl} : Y^{(2)} \longrightarrow Y^{(1)} \ (1 \leq k < l \leq 3),$$

$$r(i) : D(i)^{(0)} \longrightarrow D(i), \quad r(i)_k : D(i)^{(1)} \longrightarrow D(i)^{(0)} \ (k = 1, 2),$$

$$r(i)_{kl} : D(i)^{(2)} \longrightarrow D(i)^{(1)} \ (1 \leq k < l \leq 3),$$

be the projections and let $\Delta(Y)$ (resp. $\Delta(D(i))$) be the category of pairs (E, φ), where $E \in \mathrm{Coh}(K_0 \otimes \mathcal{O}_{Y^{(0)}})$ (resp. $E \in \mathrm{Coh}(K_0 \otimes \mathcal{O}_{D(i)^{(0)}})$) and

φ is an isomorphism $r_2^* E \xrightarrow{\sim} r_1^* E$ (resp. $r(i)_2^* E \xrightarrow{\sim} r(i)_1^* E$) such that $r_{12}^*(\varphi) \circ r_{23}^*(\varphi) = r_{13}^*(\varphi)$ (resp. $r(i)_{12}^*(\varphi) \circ r(i)_{23}^*(\varphi) = r(i)_{13}^*(\varphi)$) holds. Then we have the canonical functor

$$r^* : \mathrm{Coh}(K_0 \otimes \mathcal{O}_Y) \longrightarrow \Delta(Y),$$

$$r(i)^* : \mathrm{Coh}(K_0 \otimes \mathcal{O}_{D(i)}) \longrightarrow \Delta(D(i)),$$

which are induced by $r, r(i)$, respectively. To prove the the categorical equivalence of Ψ, it suffices to prove the following assertions:

(1) The functor r^* is an equivalence of categories.

(2) The functors $r(i)^*$ $(i = 1, 2)$ are fully-faithful.

Since the diagram

$$Y \leftarrow Y^{(0)} \underset{\leftarrow}{\leftarrow} Y^{(1)} \underset{\leftarrow}{\overset{\leftarrow}{\leftarrow}} Y^{(2)}$$

is an etale hypercovering, the assertion (1) follows from the rigid analytic faithfully flat descent of Gabber ([O1, (1.9)]). Moreover, one can check, as in the proof of [O1, (1.9)], that the descent of morphisms in the category $\mathrm{Coh}(K_0 \otimes \mathcal{O}_{D(i)})$ holds for etale hypercoverings. So we have the assertion (2) if we have the following claim: The diagram

$$D(i) \leftarrow D(i)^{(0)} \underset{\leftarrow}{\leftarrow} D(i)^{(1)} \underset{\leftarrow}{\overset{\leftarrow}{\leftarrow}} D(i)^{(2)}$$

is an etale hypercovering.

We prove the above claim. Since the morphisms $X^{(j)} \longrightarrow D(i)^{(j)}$ $(i = 1, 2, j = 0, 1, 2)$ are homeomorphisms, the diagram is a hypercovering. Hence it suffices to show that the transition morphisms are formally etale. Here we only prove the formal etaleness of the morphism $D(1)^{(0)} \longrightarrow D(1)$, which we denote by g. (The other cases are left to the reader.) Let $x \in D(1)^{(0)}$ and put $y := g(x)$. Then, by [Kk1, (6.5)], we have the isomorphisms

$$\mathcal{O}_{D(1)^{(0)},\bar{x}} \overset{\sim}{=} \mathcal{O}_{Y^{(0)},\bar{x}} \langle t_1, \cdots, t_r \rangle,$$

$$\mathcal{O}_{D(1),\bar{y}} \overset{\sim}{=} \mathcal{O}_{Y,\bar{y}} \langle t_1, \cdots, t_r \rangle,$$

for some $r \in \mathbb{N}$, and the homomorphism $g^* : \mathcal{O}_{D(1),\bar{y}} \longrightarrow \mathcal{O}_{D(1)^{(0)},\bar{x}}$ is identified with the homomorphism

$$\mathcal{O}_{Y^{(0)},\bar{x}} \langle t_1, \cdots, t_r \rangle \longrightarrow \mathcal{O}_{Y,\bar{y}} \langle t_1, \cdots, t_r \rangle$$

induced by the morphism $Y^{(0)} \longrightarrow Y$ via the above isomorphisms. Hence g is formally etale. So the claim is proved and the proof of the proposition is now finished. □

Finally let us recall the relation between HPD-isostratifications and integrable formal log connections, which is essentially due to K. Kato ([Kk1, (6.2)]).

Let (X, M) be a fine log scheme over k, let $(X, M) \xrightarrow{f} (\operatorname{Spec} k, N) \hookrightarrow (\operatorname{Spf} W, N)$ be morphisms fine log formal W-schemes such that f is log smooth, and let $(X, M) \hookrightarrow (Y, J)$ be a closed immersion of (X, M) into a fine log formal W-scheme (Y, J) which is formally log smooth over $(\operatorname{Spf} W, N)$ and satisfies $(Y, J) \times_{(\operatorname{Spf} W, N)} (\operatorname{Spec} k, N) = (X, M)$. Denote the fine log scheme $(Y, J) \times_{(\operatorname{Spf} W, N)} (\operatorname{Spec} W_n, N)$ by (Y_n, J_n). Then, by [Kk1, (6.2)], there exists a canonical exact fully-faithful functor

$$\Phi_n : C_{\text{crys}}((X, M)/(\operatorname{Spec} W_n, N)) \longrightarrow C((Y_n, J_n)/(\operatorname{Spec} W_n, N)),$$

which is compatible with n. Hence we have the following proposition:

PROPOSITION 4.3.7. *Let the notations be as above. Then we have a canonical exact fully-faithful functor*

$$I_{\text{crys}}((X, M)/(\operatorname{Spf} W, N)) \longrightarrow \hat{C}((Y, J)/(\operatorname{Spf} W, N)).$$

PROOF. For $K_0 \otimes E \in I_{\text{crys}}((X, M)/(\operatorname{Spf} W, N))$, the functor is defined by $K_0 \otimes E \mapsto K_0 \otimes_W \varprojlim_n \Phi_n(E_n)$, where E_n is the restriction of E to $C_{\text{crys}}((X, M)/(\operatorname{Spec} W_n, N))$. The exactness and the fully-faithfulness follow from those of Φ_n's. □

Chapter 5. Berthelot-Ogus Theorem for Fundamental Groups

Throughout this chapter, let k be a perfect field of characteristic $p > 0$, let W be a Witt ring of k and let K_0 be the fraction field of W. Let V be a finite totally ramified extension of W and let K be the fraction field of V.

The purpose of this chapter is to give a proof of the following theorem:

Theorem (=Theorem 5.3.2). *Assume we are given the following commutative diagram of fine log schemes*

$$\begin{array}{ccccc} (X_k, M) & \hookrightarrow & (X, M) & \hookleftarrow & (X_K, M) \\ \downarrow & & f\downarrow & & \downarrow \\ (\operatorname{Spec} k, N) & \hookrightarrow & (\operatorname{Spec} V, N) & \hookleftarrow & (\operatorname{Spec} K, N) \\ & \searrow & \downarrow & & \\ & & (\operatorname{Spec} W, N), & & \end{array}$$

where the two squares are Cartesian, f is proper log smooth integral and X_k is reduced. Assume that $H^0_{\mathrm{dR}}((X,M)/(\operatorname{Spf} V, N)) = V$ holds, and that we are given a V-valued point x of $X_{f\text{-triv}}$. Denote the special fiber (*resp. generic fiber*) *of x by x_k* (*resp. x_K*)*. Then there exists a canonical isomorphism of pro-algebraic groups*

$$\pi_1^{\mathrm{crys}}((X_k, M)/(\operatorname{Spf} W, N), x_k) \times_{K_0} K \overset{\simeq}{=} \pi_1^{\mathrm{dR}}((X_K, M)/(\operatorname{Spec} K, N), x_K).$$

The above theorem gives the comparison between crystalline fudamental groups and de Rham fundamental groups. In the case of cohomologies, this type of theorem (in the case without log structures) has been proved by Berthelot and Ogus ([B-O2]). So we call the above theorem as the Berthelot-Ogus theorem for fundamental groups.

The idea of proof of the above theorem is as follows: First, in Section 5.1, we define the convergent fundamental group

$$\pi_1^{\mathrm{conv}}((X_k, M)/(\operatorname{Spf} ?, N), x_k) \quad (? = W, V)$$

by using the category of nilpotent isocrystals on log convergent site and we prove the base change property

(5.0.1)

$$\pi_1^{\mathrm{conv}}((X_k, M)/(\operatorname{Spf} W, N), x_k) \times_{K_0} K \overset{\simeq}{=} \pi_1^{\mathrm{conv}}((X_k, M)/(\operatorname{Spf} V, N), x_k).$$

Second, in Section 5.2, we prove the comparison

$$(5.0.2) \quad \pi_1^{\mathrm{conv}}((X_k, M)/(\operatorname{Spf} V, N), x_k) \overset{\simeq}{=} \pi_1^{\mathrm{dR}}((X_K, M)/(\operatorname{Spec} K, N), x_K)$$

between de Rham fundamental group and convergent fundamental group. Finally, in Section 5.3, we prove the comparison

$$(5.0.3) \quad \pi_1^{\mathrm{conv}}((X_k, M)/(\operatorname{Spf} W, N), x_k) \overset{\simeq}{=} \pi_1^{\mathrm{crys}}((X_k, M)/(\operatorname{Spf} W, N), x_k)$$

between convergent fundamental group and crystalline fundamental group. Combining (5.0.1), (5.0.2) and (5.0.3), we finish the proof.

5.1. Convergent fundamental groups

In this section, we give a definition of log convergent site for certain log schemes, and using the category of nilpotent isocrystals on log convergent site, we give a definition of convergent fundamental groups. Convergent fundamental groups is useful because it can be defined even over a ramified base. (Note that one cannot define crystalline fundamental groups over a very ramified base because there exists no PD-structure on it.) After defining it, we prove the base change property of convergent fundamental groups.

Throughout this section, for a formal V-scheme X, we denote the closed subscheme defined by $p \in \mathcal{O}_X$ by X_1 and $(X_1)_{\text{red}}$ by X_0, unless otherwise stated.

First we define the notion of enlargement and log convergent site.

Definition 5.1.1. Let $(X, M) \longrightarrow (S, N)$ be a morphism of fine log formal V-schemes. Then we define an enlargement of (X, M) over (S, N) as a triple (T, L, z), where (T, L) is a fine log formal V-scheme over (S, N) and z is a morphism $(T_0, L) \longrightarrow (X, M)$ over (S, N). We define a morphism of enlargements $g : (T, L, z) \longrightarrow (T', L', z')$ as a morphism $g : (T, L) \longrightarrow (T', L')$ over (S, N) which satisfies $z' \circ g_0 = z$, where $g_0 : (T_0, L) \longrightarrow (T'_0, L')$ is a morphism induced by g. We often denote (T, L, z) simply by T.

Definition 5.1.2. Let $(X, M) \longrightarrow (S, N)$ be as above. Then we define the log convergent site $((X, M)/(S, N))_{\text{conv}}$ as follows. The objects of this category are enlargements of (X, M) over (S, N). the morphisms are the morphisms of enlargements defined above, and the coverings are the ones which are induced by the etale topology on T.

We denote the sheaf on $((X, M)/(S, N))_{\text{conv}}$ defined by $T \mapsto \Gamma(T, \mathcal{O}_T)$ (resp. $T \mapsto \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma(T, \mathcal{O}_T)$) by $\mathcal{O}_{X/S}$ (resp. $\mathcal{K}_{X/S}$).

We often denote the log convergent site $((X, M)/(S, N))_{\text{conv}}$ shortly by $(X/S)^{\log}_{\text{conv}}$, when there will be no confusion on log structures. In the case $S = \operatorname{Spf} V$, we denote it by $(X/V)^{\log}_{\text{conv}}$, by abuse of notation.

Remark 5.1.3. Let $(X, M) \longrightarrow (S, N)$ be as above, and let $(X', M') \hookrightarrow (X, M)$ be an exact closed immersion of fine log formal V-schemes over (S, N) such that the underlying morphism between topological spaces is a homeomorphism. Then, one can easily check that the canonical functor

$$((X', M')/(S, N))_{\mathrm{conv}} \longrightarrow ((X, M)/(S, N))_{\mathrm{conv}}$$

is an equivalence of categories. In particular, there exists a canonical equivalence of categories

$$((X_0, M)/(S, N))_{\mathrm{conv}} \xrightarrow{\sim} ((X, M)/(S, N))_{\mathrm{conv}}.$$

Definition 5.1.4. Let $(X, M) \longrightarrow (S, N)$ be as above. An isocrystal on the log convergent site $((X, M)/(S, N))_{\mathrm{conv}}$ is a sheaf $\mathcal{E}$ on $((X, M)/(S, N))_{\mathrm{conv}}$ satisfying the following conditions:

(1) For $T \in (X/V)^{\log}_{\mathrm{conv}}$, the sheaf $\mathcal{E}_T$ on T induced by $\mathcal{E}$ is an isocoherent sheaf on T.

(2) For a morphism $f : T' \longrightarrow T$ in $(X/V)^{\log}_{\mathrm{conv}}$, the homomorphism $f^*\mathcal{E}_T \longrightarrow \mathcal{E}_{T'}$ induced by $\mathcal{E}$ is an isomorphism.

We denote the category of isocrystals on the site $((X, M)/(S, N))_{\mathrm{conv}}$ by $I_{\mathrm{conv}}((X, M)/(S, N))$. We often denote it simply by $I_{\mathrm{conv}}((X/S)^{\log})$ when there will be no confusion on log structures. In the case $S = \mathrm{Spf}\, V$, we denote it by $I_{\mathrm{conv}}((X/V)^{\log}$, by abuse of notation.

One can check immediately that $\mathcal{K}_{X/S}$ is an isocrystal.

We denote the nilpotent part $\mathcal{N}_{\mathcal{K}_{X/S}} I_{\mathrm{conv}}((X, M)/(S, N))$ of $I_{\mathrm{conv}}((X, M)/(S, N))$ with respect to the object $\mathcal{K}_{X/S}$ simply by $\mathcal{N} I_{\mathrm{conv}}((X, M)/(S, N))$ (or $\mathcal{N} I_{\mathrm{conv}}((X/V)^{\log})$, when there will be no confusion on log structures).

Remark 5.1.5. Assume that we are given the following commutative diagram of fine log formal V-schemes

$$\begin{array}{ccc} (X, M) & \xleftarrow{\ g\ } & (X', M') \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle f'}\big\downarrow \\ (S, N) & \longleftarrow & (S', N'). \end{array}$$

Then, for an enlargement $T := (T, L, z)$ of (X', M') over (S', N'), the triple $g_*T := (T, L, g \circ z)$ is an enlargement of (X, M) over (S', N'). We define the pull-back functor

$$g^* : I_{\mathrm{conv}}((X, M)/(S, N)) \longrightarrow I_{\mathrm{conv}}((X', M')/(S', N'))$$

by $g^*\mathcal{E}(T) := \mathcal{E}(g_*T)$.

REMARK 5.1.6. Assume that we are given the following cartesian square of fine log formal V-schemes

$$\begin{array}{ccc} (X,M) & \xleftarrow{g} & (X',M) \\ {\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow \\ (S,N) & \xleftarrow{h} & (S',N), \end{array}$$

where S' is defined by $S' := \underline{\operatorname{Spec}}\,\mathcal{O}_S/(p\text{-torsion})$. We remark that the pull-back functor

$$g^* : I_{\text{conv}}((X,M)/(S,N)) \longrightarrow I_{\text{conv}}((X',M)/(S',N))$$

gives an equivalence of categories.

We define the quasi-inverse Φ of the functor g^* as follows: Let $T := (T,L,z)$ be an enlargement of (X,M) over (S,N). By taking the base change by h, we can define canonically an enlargement $T' := (T \times_S S', L, ((T\times_S S')_0, L) \xrightarrow{z'} (X',M))$ of (X',M) over (S',N). Let $q : T\times_S S' \longrightarrow T$ be the first projection. Then the pull-back functor

$$q^* : \operatorname{Coh}(K\otimes\mathcal{O}_T) \longrightarrow \operatorname{Coh}(K\otimes\mathcal{O}_{T\times_S S'})$$

is an equivalence of categories. For an object $\mathcal{E}$ in the category $I_{\text{conv}}((X',M)/(S',N))$, we define $\Phi(\mathcal{E})$ by $\Phi(\mathcal{E})(T) := (q^*)^{-1}(\mathcal{E}_{T'})(T)$, where $\mathcal{E}_{T'}$ is the isocoherent sheaf on $T\times_S S'$ induced by $\mathcal{E}$. Then one can check easily that the functor Φ gives the quasi-inverse of g^*.

REMARK 5.1.7. Let $(X,M) \longrightarrow (S,N)$ be as above, and assume we are given the following diagram of fine log formal V-schemes

$$\begin{array}{ccccccc} X & \leftarrow & X^{(0)} & \leftleftarrows & X^{(1)} & \overset{\leftarrow}{\leftleftarrows} & X^{(2)} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S & \leftarrow & S^{(0)} & \leftleftarrows & S^{(1)} & \overset{\leftarrow}{\leftleftarrows} & S^{(2)}, \end{array}$$

where the horizontal lines are etale hypercoverings (up to level 2). Then one can define functors

$$p : I_{\mathrm{conv}}((X,M)/(S,N)) \longrightarrow I_{\mathrm{conv}}((X^{(0)},M)/(S^{(0)},N)),$$

$$p_i : I_{\mathrm{conv}}((X^{(0)},M)/(S^{(0)},N)) \longrightarrow I_{\mathrm{conv}}((X^{(1)},M)/(S^{(1)},N)) \quad (i=1,2),$$

$$p_{ij} : I_{\mathrm{conv}}((X^{(1)},M)/(S^{(1)},N)) \longrightarrow I_{\mathrm{conv}}((X^{(2)},M)/(S^{(2)},N)) \quad (1 \le i < j \le 3),$$

which are induced by the above diagram. Let Δ_{conv} be the category of the descent data of isocrystals with respect to these functors, that is, the category of pairs $(\mathcal{E},\varphi)$, where $\mathcal{E} \in I_{\mathrm{conv}}((X^{(0)},M)/(S^{(0)},N))$ and φ is an isomorphism $p_2(\mathcal{E}) \xrightarrow{\sim} p_1(\mathcal{E})$ in $I_{\mathrm{conv}}((X^{(1)},M)/(S^{(1)},N))$ satisfying the condition $p_{12}(\varphi) \circ p_{23}(\varphi) = p_{13}(\varphi)$ in $I_{\mathrm{conv}}((X^{(2)},M)/(S^{(2)},N))$. Then there exists the canonical functor

$$I_{\mathrm{conv}}((X,M)/(S,N)) \longrightarrow \Delta_{\mathrm{conv}}$$

induced by p. By using rigid analytic faithfully flat descent of Gabber ([O1, (1.9)]), one can check that this functor gives an equivalence of categories. That is, the category of isocrystals on log convergent site admits the descent for etale coverings.

We prove that the category of nilpotent convergent log isocrystals of (X,M) over $(\mathrm{Spf}\, V, N)$ is Tannakian under certain condition:

Proposition 5.1.8. *Let $(X,M) \longrightarrow (\mathrm{Spf}\, V, N)$ be a morphism of fine log formal V-schemes. Assume moreover that the 0-th cohomology $H^0((X/V)^{\log}_{\mathrm{conv}}, \mathcal{K}_{X/S})$ of $\mathcal{K}_{X/S}$ is isomorphic to a field. Then the category $\mathcal{N}I_{\mathrm{conv}}((X,M)/(\mathrm{Spf}\, V,N))$ is a Tannakian category.*

Proof. Let $\mathcal{C}$ be the category of sheaves of $\mathcal{K}_{X/V}$-modules on $(X/V)^{\log}_{\mathrm{conv}}$. (Note that $\mathcal{C}$ is an abelian tensor category.) Then $\mathcal{N}\mathcal{C} = \mathcal{N}I_{\mathrm{conv}}((X,M)/(\mathrm{Spf}\, V,N))$ holds. So, by Proposition 1.2.1, $\mathcal{N}I_{\mathrm{conv}}((X,M)/(\mathrm{Spf}\, V,N))$ is an abelian category. Moreover, one can check easily that the tensor structure of $\mathcal{C}$ induces the tensor structure of

$\mathcal{N}I_{\mathrm{conv}}((X,M)/(\mathrm{Spf}\,V,N))$ and that this category is a rigid abelian tensor category with this tensor structure.

Hence, to show the category $\mathcal{N}I_{\mathrm{conv}}((X,M)/(\mathrm{Spf}\,V,N))$ is Tannakian, it suffices to prove the existence of a fiber functor $\mathcal{N}I_{\mathrm{conv}}((X,M)/(\mathrm{Spf}\,V,N)) \to \mathrm{Vec}_L$, where L is a field. Let us consider the homomorphism of evaluation

$$H^0((X/V)^{\log}_{\mathrm{conv}}, \mathcal{K}_{X/S}) \longrightarrow \mathbb{Q}\otimes_{\mathbb{Z}}\Gamma(T,\mathcal{O}_T)$$

for enlargements T. Since $H^0((X/V)^{\log}_{\mathrm{conv}}, \mathcal{K}_{X/S})$ is not zero, there exists an enlargement T such that $\mathbb{Q}\otimes_{\mathbb{Z}}\Gamma(T,\mathcal{O}_T)\neq 0$ holds. We may assume that T is affine. Let I be a maximal ideal of $\mathbb{Q}\otimes_{\mathbb{Z}}\Gamma(T,\mathcal{O}_T)$ and put $L := (\mathbb{Q}\otimes_{\mathbb{Z}}\Gamma(T,\mathcal{O}_T))/I$. Let us consider the functor

$$\begin{aligned}\mathcal{N}I_{\mathrm{conv}}((X,M)/(\mathrm{Spf}\,V,N)) &\xrightarrow{\mathrm{ev}} (\mathbb{Q}\otimes_{\mathbb{Z}}\Gamma(T,\mathcal{O}_T)\text{-modules})\\ &\longrightarrow \mathrm{Vec}_L,\end{aligned}$$

which is induced by the evaluation at T and the ring homomorphism $\mathbb{Q}\otimes_{\mathbb{Z}}\Gamma(T,\mathcal{O}_T) \longrightarrow L$. We denote this functor by ξ. Then one can check easily that ξ is an exact tensor functor, noting that the essential image of ev consists of free modules. Moreover, by [D2, (2.10)], ξ is faithful. Hence ξ is the desired fiber functor. □

Now we give the definition of convergent fundamental groups:

Definition 5.1.9 (Definition of π_1^{conv}). Let $(X,M)\longrightarrow(\mathrm{Spf}\,V,N)$ be a morphism of fine log formal V-schemes such that $H^0((X/V)^{\log}_{\mathrm{conv}}, \mathcal{K}_{X/S})$ is a field, and let x be a k-rational point of $X_{f\text{-triv}}$. Then the convergent fundamental group of (X,M) over $(\mathrm{Spf}\,V,N)$ with base point x is defined by

$$\pi_1^{\mathrm{conv}}((X,M)/(\mathrm{Spf}\,V,N),x) := G(\mathcal{N}I_{\mathrm{conv}}((X/V)^{\log}),\omega_x),$$

where ω_x is the fiber functor

$$I_{\mathrm{conv}}((X/V)^{\log}) \longrightarrow I_{\mathrm{conv}}((x/V)^{\log}) \simeq \mathrm{Vec}_K,$$

and the notation $G(\cdots)$ is as in Theorem 1.1.8.

Remark 5.1.10. The existence of the point x assures that the category $\mathcal{N}I_{\mathrm{conv}}((X/V)^{\log})$ in the above definition is actually a *neutral* Tannakian category.

We have the following useful criterion for the category $\mathcal{N}I_{\mathrm{conv}}((X/V)^{\log})$ to be Tannakian.

Proposition 5.1.11. *Let $f : (X, M) \longrightarrow (\mathrm{Spf}\, V, N)$ be a morphism of fine log formal V-scheme and assume that the morphism $f_0 : (X_0, M) \longrightarrow (\mathrm{Spec}\, k, N)$ induced by f is log smooth integral of finite type and X is connected. Then $H^0((X/V)^{\log}_{\mathrm{conv}}, \mathcal{K}_{X/S})$ is a field. In particular, the category $\mathcal{N}I_{\mathrm{conv}}((X/V)^{\log})$ is a Tannakian category.*

Before the proof, we prepare a lemma.

Lemma 5.1.12. *Let $(X, M) \longrightarrow (\mathrm{Spf}\, V, N)$ be a morphism of fine log formal V-schemes and let $h : (X, M) \hookrightarrow (Y, J)$ be an exact closed immersion over $(\mathrm{Spf}\, V, N)$ such that (Y, J) is formally log smooth over $(\mathrm{Spf}\, V, N)$ and that the underlying morphism of topological spaces of h is a homeomorphism. Then the homomorphism*

$$\alpha : H^0((X/V)^{\log}_{\mathrm{conv}}, \mathcal{K}_{X/S}) \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma(Y, \mathcal{O}_Y)$$

defined by the evaluation at the enlargement $Y := (Y, J, (Y_0, J) \overset{\simeq}{=} (X_0, M) \hookrightarrow (X, M))$ is injective.

Proof. Let x be an element of $\mathrm{Ker}(\alpha)$. Let $T := (T, L, z)$ be an enlargement and let $x_T \in \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma(T, \mathcal{O}_T)$ be the value of x at (T, L, z). Since (Y, J) is formally log smooth over $(\mathrm{Spf}\, V, N)$, there exists etale locally a morphism $\tilde{z} : (T, L) \longrightarrow (Y, J)$ over $(\mathrm{Spf}\, V, N)$ such that the following diagram is commutative:

$$\begin{array}{ccc} (T_0, L) & \longrightarrow & (T, L) \\ {\scriptstyle z}\big\downarrow & & {\scriptstyle \tilde{z}}\big\downarrow \\ (X, M) & \xrightarrow{h} & (Y, J). \end{array}$$

So the morphism $\tilde{z}$ defines a morphism of enlargements $T \longrightarrow Y$. So, by the assumption $\alpha(x) = 0$, we conclude that x_T is equal to zero etale locally.

Then, by rigid analytic faithfully flat descent, x_T is equal to zero for any T. Hence we have $x = 0$, as is desired. □

PROOF OF PROPOSITION 5.1.11. First, by Remark 5.1.3, we may assume that $(X, M) = (X_0, M)$. Hence we may assume that (X, M) is reduced, connected and log smooth integral over $(\operatorname{Spec} k, N)$.

For a finite extension $k \subset k'$ of fields and a scheme Y over $X \otimes_k k'$, let us denote the ring $H^0(((Y, M)/(\operatorname{Spec} V(k'), N))_{\mathrm{conv}}, \mathcal{K}_{Y/V(k)})$ simply by $H^0_{\text{log-conv}}(Y/V(k'))$, where $V(k')$ is the unramified extension of V with residue field k'.

First, let us note the following claim:

CLAIM 1. Let $k \subset k' \subset k''$ be finite extensions of fields and put $X' := X \otimes_k k', f' := f \otimes_k k' : (X', M) \longrightarrow (\operatorname{Spec} k', N)$. Let U be a connected component of $X'_{f'\text{-triv}}$ and let $x \in U(k'')$. Then the homomorphism

$$\alpha : H^0_{\text{log-conv}}(U/V(k')) \longrightarrow H^0_{\text{log-conv}}(x/V(k')) = \mathbb{Q} \otimes_{\mathbb{Z}} V(k'')$$

is injective, and it is isomorphic if $k' = k''$ holds.

PROOF OF CLAIM 1. Note that, for any open subscheme Y of $X_{f\text{-triv}}$, there exists the canonical isomorphism

$$H^0(((Y, M)/(\operatorname{Spf} V, N))_{\mathrm{conv}}, \mathcal{K}_{Y/\operatorname{Spf} V}) \overset{\sim}{=} H^0((Y/\operatorname{Spf} V)_{\mathrm{conv}}, \mathcal{K}_{Y/\operatorname{Spf} V}),$$

where $(Y/\operatorname{Spf} V)_{\mathrm{conv}}$ is the convergent site for schemes without log structures defined by Ogus ([O1]). Then the injectivity of α follows from [O1, (4.1)].

On the other hand, we have the natural inclusion $\mathbb{Q} \otimes_{\mathbb{Z}} V(k') \subset H^0_{\text{log-conv}}(U/V(k'))$. Hence α is surjective if $k' = k''$ holds. Hence we obtain the assertion. □

Next, we prove the following claim:

CLAIM 2. Let k, X, f be as in the statement of the proposition (we assume $X = X_0$), and let U be a connected component of $X_{f\text{-triv}}$. Then the natural homomorphism

$$\beta : H^0_{\text{log-conv}}(X/V) \longrightarrow H^0_{\text{log-conv}}(U/V)$$

is injective.

Let $X_{f\text{-triv}} = \coprod_{i=1}^{n} U_i$ be the decomposition into the connected components with $U = U_1$. First, to prove the claim 2, we prepare the following claim:

CLAIM 3. To prove the claim 2, one may assume the following condition $(*)$:

$(*)$: For each i, $U_i(k)$ is non-empty.

PROOF OF CLAIM 3. If we prove the injectivity of the homomorphisms

$$\delta_X : H^0_{\text{log-conv}}(X/V) \longrightarrow H^0_{\text{log-conv}}(X \times_k k'/V(k')),$$

$$\delta_U : H^0_{\text{log-conv}}(U/W) \longrightarrow H^0_{\text{log-conv}}(U \times_k k'/V(k')),$$

the proof goes like that of claim 3 in Proposition 3.1.6. We prove the injectivity of δ_X. (The injectivity of δ_U can be shown in the same way.) Let x be an element in $\mathrm{Ker}(\delta_X)$. Let $X = \cup_{i=1}^{l} X_i$ be an affine open covering of X. By using the descent property of $I_{\mathrm{conv}}((X/V)^{\log})$, one can see that we can replace X by X_i, that is, we may assume that X is affine. Let (Y, J) be a fine log formal V-scheme which is formally log smooth integral over $(\mathrm{Spf}\, V, N)$ such that $(Y, J) \times_{(\mathrm{Spf}\, V, N)} (\mathrm{Spec}\, k, N) = (X, M)$ holds. (There exists such a fine log formal V-scheme by Corollary 2.2.12.) Then, we can form the following commutative diagram

$$\begin{array}{ccc} H^0_{\text{log-conv}}(X/V) & \xrightarrow{\delta_X} & H^0_{\text{log-conv}}(X \times_k k'/V(k')) \\ \big\downarrow & & \big\downarrow \\ \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma(Y, \mathcal{O}_Y) & \longrightarrow & \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma(Y \times_{\mathrm{Spf}\, V} \mathrm{Spf}\, V(k'), \mathcal{O}_{Y \times_{\mathrm{Spf}\, V} \mathrm{Spf}\, V(k')}), \end{array}$$

where the vertical arrows are the evaluation at Y, $Y \times_{\mathrm{Spf}\, V} \mathrm{Spf}\, V(k')$, respectively. The vertical arrows are injective by Lemma 5.1.12, and the lower horizontal arrow is obviously injective. Hence δ_X is injective, as is desired. $\square$

PROOF OF CLAIM 2. Using claim 3, we can prove the claim in almost the same way as the proof of claim 2 in Proposition 4.1.7. So we only mention the difference.

First, we replace $H^0_{\text{log-crys}}$ by $H^0_{\text{log-conv}}$, and replace p by a uniformizer of V in the proof. Second, we have to prove the injectivity of the restriction

$$H^0_{\text{log-conv}}(U_{i_0}/V) \longrightarrow H^0_{\text{log-conv}}(X_j \cap X_{j'} \cap U_{i_0}/V)$$

when $X_j \cap X_{j'} \cap U_{i_0}$ is not empty. (For the notation, see the proof of claim 2 in Proposition 4.1.7.) By using the descent property of $I_{\text{conv}}((U_{i_0}/V)^{\log})$, we may assume U_{i_0} is affine, and in this case, we can prove the injectivity by taking a formally log smooth integral lift of U_{i_0} and by using Lemma 5.1.12. Details are left to the reader. □

Now we finish the proof of the proposition by using the claims. Let U be a connected component of $X_{f\text{-triv}}$ and let x be a closed point of U and let k' be the residue field of x. Let us consider the following diagram:

$$K \subset H^0_{\text{log-conv}}(X/V) \xrightarrow{\beta} H^0_{\text{log-conv}}(U/V) \xrightarrow{\alpha} H^0_{\text{log-conv}}(x/V) = \mathbb{Q} \otimes_{\mathbb{Z}} V(k').$$

By claim 1, α is injective and by claim 2, β is injective. So there are inclusions of rings $K \subset H^0_{\text{log-crys}}(X/W) \subset \mathbb{Q} \otimes_{\mathbb{Z}} V(k')$, and they imply that $H^0_{\text{log-dR}}(X/k)$ is a field. □

Now we prove the base change property of convergent fundamental groups.

PROPOSITION 5.1.13. *Let $f : (X, M) \longrightarrow (\operatorname{Spf} V, N)$ be a morphism of fine log formal V-schemes such that $H^0((X/V)^{\log}_{\text{conv}}, \mathcal{K}_{X/V}) = K$ holds and let x be a k-valued point of $X_{f\text{-triv}}$ over* $\operatorname{Spf} V$*. Let V' be a complete discrete valuation ring with residue field k' which is finite flat over V and let K' be the fraction field of K. Let $f' : (X', M) \longrightarrow (\operatorname{Spf} V', N)$ be the base change of f by the morphism $(\operatorname{Spf} V', N) \longrightarrow (\operatorname{Spf} V, N)$ and let x' be the k'-valued point of X' which is sent to x by the morphism $X' \longrightarrow X$. Then $H^0((X'/V')^{\log}_{\text{conv}}, \mathcal{K}_{X'/V'}) = K'$ holds and there exists a canonical isomorphism of pro-algebraic groups*

$$\pi_1^{\text{conv}}((X, M)/(\operatorname{Spf} V, N), x) \times_K K' \xrightarrow{\sim} \pi_1^{\text{conv}}((X', M)/(\operatorname{Spf} V', N), x').$$

Proof. Denote the morphism $(X', M) \longrightarrow (X, M)$ by π, and put $\mathcal{C} := \mathcal{N}I_{\mathrm{conv}}((X, M)/(\mathrm{Spf}\, V, N))$ and $\mathcal{C}' := \mathcal{N}I_{\mathrm{conv}}((X, M)/(\mathrm{Spf}\, V', N))$. Then $\mathcal{C}$ is equivalent to the category $\mathrm{Rep}_K(\pi_1^{\mathrm{conv}}((X, M)/(\mathrm{Spf}\, V, N), x))$ of finite-dimensional rational representations of $\pi_1^{\mathrm{conv}}((X, M)/(\mathrm{Spf}\, V, N), x)$ over K. Let $\mathcal{C}_{K'}$ be the category of objects in $\mathcal{C}$ with a structure of K'-module which is compatible with K-linear structure. Then, by [D2, (4.6)(ii)], the category $\mathcal{C}_{K'}$ is equivalent to the category $\mathrm{Rep}_K(\pi_1^{\mathrm{conv}}((X, M)/(\mathrm{Spf}\, V, N), x) \times_K K')$. So it suffices to prove that there exists a canonical equivalence between the categories $\mathcal{C}'$ and $\mathcal{C}_{K'}$.

First we define the functor $\Phi : \mathcal{C}' \longrightarrow \mathcal{C}_{K'}$. Let $\mathcal{E} \in \mathcal{C}'$ and let $T := (T, L, z)$ be an enlargement of (X, M) over $(\mathrm{Spf}\, V, N)$. Let T' be $T \times_{\mathrm{Spf}\, V} \mathrm{Spf}\, V'$ and let $z' : (T'_0, L) \longrightarrow (X', M)$ be the composite

$$(T'_0, L) = ((T \times_X X')_0, L) \hookrightarrow (T_0, L) \times_{(X,M)} (X', M) \overset{\text{2nd proj.}}{\longrightarrow} (X', M).$$

Then the triple $T' := (T', L, z')$ defines an enlargement of (X', M) over $(\mathrm{Spf}\, V', N)$. Then we define the value $\Phi(\mathcal{E})_T$ of $\Phi(\mathcal{E})$ at T by $\Phi(\mathcal{E})_T := \alpha_* \mathcal{E}_{T'}$, where α is the projection $T' \longrightarrow T$. Since $\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma(T', \mathcal{O}_{T'})$ has a canonical structure of K'-module, $\Phi(\mathcal{E})$ has a structure of K'-module, and one can check that $\Phi(\mathcal{E})$ defines an object in $\mathcal{C}_{K'}$. Hence the functor $\Phi : \mathcal{C}' \longrightarrow \mathcal{C}_{K'}$ is defined.

Next we define the functor $\Psi : \mathcal{C}_{K'} \longrightarrow \mathcal{C}'$. Let $\mathcal{E}$ be an object in $\mathcal{C}_{K'}$ and let (T, L, z) be an enlargement of (X', M) over $(\mathrm{Spf}\, V', N)$. Then the triple $(T, L, \pi \circ z)$ defines an enlargement of (X, M) over $(\mathrm{Spf}\, V, N)$. Put $T' := T \times_{\mathrm{Spf}\, V} \mathrm{Spf}\, V'$. Denote the projection $(T', L) \longrightarrow (T, L)$ by α and denote the morphism $(T'_0, L) \longrightarrow (T_0, L)$ induced by α by α_0. Then the triple $(T', L, \pi \circ z \circ \alpha_0)$ defines an enlargement of (X, M) over $(\mathrm{Spf}\, V, N)$ and the morphism α defines a morphism of enlargements $T' := (T', L, \pi \circ z \circ \alpha_0) \longrightarrow T := (T, L, z)$, which is also denoted by α. Since $\mathcal{E}$ is an isocrystal, there exists an isomorphism $\alpha^* \mathcal{E}_T \overset{\sim}{\longrightarrow} \mathcal{E}_{T'}$. Let us denote it by h. We define the morphism $h_1 : \mathcal{E}_T \longrightarrow \alpha_* \mathcal{E}_{T'}$ by the composite

$$\mathcal{E}_T \longrightarrow \alpha_* \alpha^* \mathcal{E}_T \overset{\alpha_* h}{\longrightarrow} \alpha_* \mathcal{E}_{T'},$$

and the morphism $h_2 : \mathcal{E}_T \longrightarrow \alpha_* \mathcal{E}_{T'}$ by the composite

$$\mathcal{E}_T \longrightarrow \alpha_* \alpha^* \mathcal{E}_T = \alpha_* (\alpha^{-1} \mathcal{E}_T \otimes_K K')$$

$$
\begin{aligned}
&= \alpha_*(\alpha^{-1}\mathcal{E}_T \otimes_{K'} (K' \otimes_K K')) \\
&\xrightarrow{s} \alpha_*(\alpha^{-1}\mathcal{E}_T \otimes_{K'} (K' \otimes_K K')) \\
&= \alpha_*(\alpha^{-1}\mathcal{E}_T \otimes_K K') = \alpha_*\alpha^*\mathcal{E}_T \\
&\xrightarrow{h} \alpha_*\mathcal{E}_{T'},
\end{aligned}
$$

where the equality on the second and fourth lines are defined by using the K'-module structure of $\mathcal{E}_T$ and the homomorphism s is defined by $e \otimes (x \otimes y) \mapsto e \otimes (y \otimes x)$ $(e \in \alpha^{-1}\mathcal{E}_T, x, y \in K')$. Then we define the value $\Psi(\mathcal{E})_T$ of $\Psi(\mathcal{E})$ at T by $\Psi(\mathcal{E})_T := \mathrm{Ker}(h_1 - h_2)$. Note that $\mathcal{E}_T$ is a free $K' \otimes_V \mathcal{O}_T$-module when T is affine. By using this, one can check that $\Psi(\mathcal{E})$ is in the category $\mathcal{C}'$.

One can check that the functors Φ, Ψ are the quasi-inverses of each other. Hence the categories $\mathcal{C}'$ and $\mathcal{C}_{K'}$ are equivalent and the proof is finished. □

In particular, we have the following:

Corollary 5.1.14. *Let (X, M) be a fine log scheme over k and let $(X, M) \xrightarrow{f} (\mathrm{Spec}\, k, N) \hookrightarrow (\mathrm{Spf}\, V, N)$ be morphisms of fine log formal V-schemes such that $H^0((X/V)^{\log}_{\mathrm{conv}}, \mathcal{K}_{X/V}) = K$ holds. Let x be a k-valued point in $X_{f\text{-triv}}$. Let V' be a totally ramified finite extension of V and let K' be the fraction field of V'. Then we have the canonical isomorphism*

$$
\pi_1^{\mathrm{conv}}((X, M)/(\mathrm{Spf}\, V, N), x) \times_K K' \overset{\sim}{=} \pi_1^{\mathrm{conv}}((X, M)/(\mathrm{Spf}\, V', N), x).
$$

5.2. Comparison between log infinitesimal site and log convergent site

Throughout this section, for a formal V-scheme X, we denote the closed subscheme defined by $p \in \mathcal{O}_X$ by X_1 and $(X_1)_{\mathrm{red}}$ by X_0, unless otherwise stated.

In this section, we prove that the nilpotent part of the category of isocrystals on log infinitesimal site is canonically equivalent to the nilpotent part of the category of isocrystals on log convergent site for certain log schemes. Since we have already shown (in Chapter 3) the comparison between the category of isocrystals on log infinitesimal site and the category of integrable log connections, we can deduce the comparison theorem between de Rham fundamental groups and convergent fundamental groups for certain log schemes.

Let $(X, M) \longrightarrow (S, N)$ be a morphism of fine log formal V-schemes. Then we define a functor

$$\iota : I_{\mathrm{conv}}((X, M)/(S, N)) \longrightarrow I_{\mathrm{inf}}((X, M)/(S, N))$$

as follows. For $T := (U, T, L, \phi) \in ((X, M)/(S, N))_{\mathrm{inf}}$, (T_0, L) is isomorphic to (U_0, M). Hence the triple $\iota^*(T) := (T, L, (T_0, L) \xleftarrow{\sim} (U_0, M) \to (X, M))$ is an enlargement of (X, M) over (S, N). Then, for $\mathcal{E} \in I_{\mathrm{conv}}((X, M)/(S, N))$, we define $\iota(\mathcal{E}) \in I_{\mathrm{inf}}((X, M)/(S, N))$ by $\iota(\mathcal{E})_T = \mathcal{E}_{\iota^*(T)}$.

Since $I_{\mathrm{inf}}((X, M)/(S, N))$ is an abelian tensor category by Remark 3.2.17, we can define the nilpotent part $\mathcal{N} I_{\mathrm{inf}}((X, M)/(S, N))$. Since ι is an exact functor, ι induces the functor

$$\bar{\iota} : \mathcal{N} I_{\mathrm{conv}}((X, M)/(S, N)) \longrightarrow \mathcal{N} I_{\mathrm{inf}}((X, M)/(S, N)).$$

The main result of this section is the following:

Theorem 5.2.1. *Let $(X, M) \xrightarrow{f} (S, N)$ be a formally log smooth integral morphism of fine log formal V-schemes and assume that $X_{f\text{-triv}}$ is dense open in X. Then the above functor $\bar{\iota}$ gives an equivalence of categories.*

First, by Remarks 3.2.19 and 5.1.6, we may assume S is flat over $\mathrm{Spf}\, V$. (Then X is also flat over $\mathrm{Spf}\, V$ by Corollary 2.2.11.) Next, let us note the following lemma:

Lemma 5.2.2. *To prove Theorem 5.2.1, we may assume that the morphism $f : (X, M) \longrightarrow (S, N)$ admits a chart.*

Proof. We prove Theorem 5.2.1 under the assumption that the theorem is true if f admits a chart.

One can take a diagram

$$\begin{array}{ccccccc} X & \leftarrow & X^{(0)} & \overleftarrow{\leftarrow} & X^{(1)} & \overleftarrow{\overleftarrow{\leftarrow}} & X^{(2)} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S & \leftarrow & S^{(0)} & \overleftarrow{\leftarrow} & S^{(1)} & \overleftarrow{\overleftarrow{\leftarrow}} & S^{(2)}, \end{array}$$

where the horizontal lines are etale hypercoverings (up to level 2), and the morphisms $(X^{(i)}, M) \longrightarrow (S^{(i)}, N)\,(i = 0, 1, 2)$ admit a chart. Let us define Δ_{inf} (resp. Δ_{conv}) as in Remark 3.2.20 (resp. Remark 5.1.7) and let Δ'_{inf}, (resp. Δ'_{conv}) be the full subcategory of Δ_{inf} (resp. Δ_{conv}) which consists of the pairs $(\mathcal{E}, \varphi)$ such that $\mathcal{E}$ is nilpotent.

By Remark 3.2.20 (resp. Remark 5.1.7), there exists the canonical equivalence of categories

$$\jmath_{\mathrm{inf}} : I_{\mathrm{inf}}((X/S)^{\log}) \longrightarrow \Delta_{\mathrm{inf}}.$$

$$(\text{resp.} \quad \jmath_{\mathrm{conv}} : I_{\mathrm{conv}}((X/S)^{\log}) \longrightarrow \Delta_{\mathrm{conv}}. \quad)$$

One can see easily that the functor $\jmath_{\mathrm{inf}}$ (resp. $\jmath_{\mathrm{conv}}$) induces the fully faithful functor

$$\overline{\jmath}_{\mathrm{inf}} : \mathcal{N}I_{\mathrm{inf}}((X/S)^{\log}) \longrightarrow \Delta'_{\mathrm{inf}}.$$

$$(\text{resp.} \quad \overline{\jmath}_{\mathrm{conv}} : \mathcal{N}I_{\mathrm{conv}}((X/S)^{\log}) \longrightarrow \Delta'_{\mathrm{conv}}. \quad)$$

Let $\overline{\iota}_\Delta : \Delta'_{\mathrm{conv}} \longrightarrow \Delta'_{\mathrm{inf}}$ be the functor induced by $\overline{\iota}$'s for $(X^{(i)}, M) \longrightarrow (S^{(i)}, N)\,(i = 0, 1, 2)$. Then, since the morphisms $(X^{(i)}, M) \longrightarrow (S^{(i)}, N)$ $(i = 0, 1, 2)$ admits a chart, $\overline{\iota}_\Delta$ gives an equivalence of categories. Let us consider the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{N}I_{\mathrm{conv}}((X/S)^{\log}) & \xrightarrow{\overline{\iota}} & \mathcal{N}I_{\mathrm{inf}}((X/S)^{\log}) \\
{\scriptstyle \overline{\jmath}_{\mathrm{conv}}}\big\downarrow & & {\scriptstyle \overline{\jmath}_{\mathrm{inf}}}\big\downarrow \\
\Delta'_{\mathrm{conv}} & \xrightarrow{\overline{\iota}_\Delta} & \Delta'_{\mathrm{inf}}.
\end{array}$$

Since the functors $\overline{\jmath}_{\mathrm{conv}}, \overline{\jmath}_{\mathrm{inf}}$ and $\overline{\iota}_\Delta$ are fully-faithful, $\overline{\iota}$ is also fully-faithful.

Now let us prove the essential surjectivity of the functor $\overline{\iota}$ by induction of the rank of $\mathcal{E} \in \mathcal{N}I_{\mathrm{inf}}((X/S)^{\log})$. By inductive hypothesis, we may assume that there exists an object $\mathcal{F} \in \mathcal{N}I_{\mathrm{conv}}((X/S)^{\log})$ and an exact sequence

$$0 \longrightarrow \overline{\iota}(\mathcal{K}_{X/S}) \longrightarrow \mathcal{E} \longrightarrow \overline{\iota}(\mathcal{F}) \longrightarrow 0.$$

Applying $\overline{\jmath}_{\mathrm{inf}}$ and using the commutative diagram in the previous paragraph, we get the following exact sequence:

$$(5.2.1) \qquad 0 \longrightarrow \overline{\iota}_\Delta \circ \overline{\jmath}_{\mathrm{conv}}(\mathcal{K}_{X/S}) \longrightarrow \overline{\jmath}_{\mathrm{inf}}(\mathcal{E}) \longrightarrow \overline{\iota}_\Delta \circ \overline{\jmath}_{\mathrm{conv}}(\mathcal{F}) \longrightarrow 0.$$

Since the functor $\overline{\iota}_\Delta$ gives an equivalence of categories, there exists an object $\mathcal{G}$ in Δ'_{conv} and an exact sequence

$$0 \longrightarrow \overline{j}_{\mathrm{conv}}(\mathcal{K}_{X/S}) \longrightarrow \mathcal{G} \longrightarrow \overline{j}_{\mathrm{conv}}(\mathcal{F}) \longrightarrow 0, \tag{5.2.2}$$

such that we get the exact sequence (5.2.1) when we apply the functor $\overline{\iota}_\Delta$ to the exact sequence (5.2.2). Moreover, since the functor j_{conv} is an equivalence of categories, there exists an object $\mathcal{H}$ in $I_{\mathrm{conv}}((X/S)^{\log})$ and an exact sequence

$$0 \longrightarrow \mathcal{K}_{X/S} \longrightarrow \mathcal{H} \longrightarrow \mathcal{F} \longrightarrow 0, \tag{5.2.3}$$

such that we get the exact sequence (5.2.2) when we apply the functor j_{conv} to the exact sequence (5.2.3). (Note that $\overline{j}_{\mathrm{conv}}$ is the restriction of j_{conv} to the category $\mathcal{N}I_{\mathrm{conv}}((X/S)^{\log})$.) By the exact sequence (5.2.3), $\mathcal{H}$ is an object in $\mathcal{N}I_{\mathrm{conv}}((X/S)^{\log})$ and we have

$$\overline{j}_{\mathrm{inf}} \circ \overline{\iota}(\mathcal{H}) = \overline{\iota}_\Delta \circ \overline{j}_{\mathrm{conv}}(\mathcal{H}) = \overline{\iota}_\Delta(\mathcal{G}) = \overline{j}_{\mathrm{inf}}(\mathcal{E}).$$

Since $\overline{j}_{\mathrm{inf}}$ is fully-faithful, we have $\overline{\iota}(\mathcal{H}) = \mathcal{E}$. Hence $\mathcal{E}$ is contained in the essential image of $\overline{\iota}$ and the proof of lemma is finished. □

To prove Theorem 5.2.1, we introduce the notion of the system of universal enlargements. First we define an enlargement in a fine log formal V-scheme.

Definition 5.2.3. Let $(X, M) \longrightarrow (S, N)$ be a morphism of fine log formal V-schemes and $i : (X, M) \longrightarrow (Y, M')$ be an exact closed immersion of fine log formal V-schemes over (S, N). Then an enlargement of (X, M) in (Y, M') over (S, N) is a 4-ple (T, L, z, g), where (T, L, z) is an enlargement of (X, M) over (S, N) and g is a morphism $(T, L) \longrightarrow (Y, M')$ such that $\mathrm{incl.} \circ g_0 : (T_0, L) \longrightarrow (Y_0, M') \longrightarrow (Y, M')$ coincides with $i \circ z : (T_0, L) \longrightarrow (X, M) \longrightarrow (Y, M')$. A morphism from (T, L, z, g) to (T', L', z', g') is defined as a morphism $h : (T, L, z) \longrightarrow (T', L', z)$ of enlargements of (X, M) over (S, N) satisfying $g' \circ h = g$.

Let $(X, M) \longrightarrow (Y, M')$ be an exact closed immersion of fine log formal V-schemes. Let $\mathcal{I}$ be the defining ideal of X in Y. Then we denote the formal blow up of Y with respect to $p\mathcal{O}_Y + \mathcal{I}^{n+1}$ by $B_{X,n}(Y)$. Let $T_{X,n}(Y)$ be the open sub formal scheme of $B_{X,n}(Y)$ defined by

$$T_{X,n}(Y) := \{x \in B_{X,n}(Y) \,|\, (p\mathcal{O}_Y + \mathcal{I}^{n+1}) \cdot \mathcal{O}_{B_{X,n}(Y),x} = p\mathcal{O}_{B_{X,n}(Y),x}\},$$

and let $L_{X,n}(Y)$ be the fine log structure on $T_{X,n}(Y)$ defined as the pullback of M'. Since $(T_{X,n}(Y))_1 \longrightarrow Y_1$ factors through the closed subscheme of Y defined by $p\mathcal{O}_Y + \mathcal{I}^{n+1}$, $(T_{X,n}(Y))_0 \longrightarrow Y_0$ factors through X_0. Hence $(T_{X,n}(Y), L_{X,n}(Y))$ defines naturally an enlargement of (X, M) in (Y, M') over (S, N). We sometimes denote this enlargement by $(T_{X,n}(Y), L_{X,n}(Y), z_n, t_n)$. For $n \geq n'$, there exists a morphism $T_{X,n'}(Y) \longrightarrow T_{X,n}(Y)$ which is induced by the inclusion of ideals $p\mathcal{O}_Y + \mathcal{I}^n \subset p\mathcal{O}_Y + \mathcal{I}^{n'}$ and this map is a morphism of enlargements of (X, M) in (Y, M') over (S, N).

The following proposition is the log version of [O1, (2.3)]:

Proposition 5.2.4. *Let $(X, M) \longrightarrow (S, N)$, $i : (X, M) \longrightarrow (Y, M')$, and $(T_{X,n}(Y), L_{X,n}(Y), z_n, t_n)$ be as above. Then, for any enlargement (T', L', z', g) of (X, M) in (Y, M') over (S, N), there exists some n and a morphism of enlargements $f : (T', L', z', g) \longrightarrow (T_{X,n}(Y), L_{X,n}(Y), z_n, t_n)$ of (X, M) in (Y, M') over (S, N) and such a morphism f is unique as a morphism to the inductive system $\{(T_{X,n}(Y), L_{X,n}(Y), z_n, t_n)\}_{n\in\mathbb{N}}$.*

The inductive system $\{(T_{X,n}(Y), L_{X,n}(Y), z_n, t_n)\}_{n\in\mathbb{N}}$ is called the system of universal enlargements of (X, M) in (Y, M') over (S, N).

Proof. Let (T', L', z', g) be any enlargement of (X, M) in (Y, M') over (S, N). Then there exists some n such that $g^*(\mathcal{I}^{n+1})\mathcal{O}_{T'} \subset p\mathcal{O}_{T'}$ holds. The universality of blow up gives the map $f : T' \longrightarrow T_{X,n}(Y)$ satisfying $t_n \circ f = g$. Since $L_{X,n}(Y)$ is the pullback of M', f defines the morphism of fine log formal V-schemes $(T', L') \longrightarrow (T_{X,n}(Y), L_{X,n}(Y))$ satisfying $t_n \circ f = g$. Then one can check easily that f gives a morphism of enlargements $(T', L', z', g) \longrightarrow (T_{X,n}(Y), L_{X,n}(Y), z_n, t_n)$ of (X, M) in (Y, M') over (S, N).

The assertion that f is unique as a morphism to the inductive system $\{(T_{X,n}(Y), L_{X,n}(Y), z_n, t_n)\}$ also follows from the universality of blow ups. $\square$

Next we define the notion of convergent log stratification, which is the 'convergent' version of log stratification, for certain fine log formal V-schemes.

Let us prepare some notations. Let $f : (Y, M') \longrightarrow (S, N)$ be a morphism of fine log formal V-schemes and assume there exists a chart $(P_Y \to M', Q_S \to N, Q \to P)$ of f. Let $\alpha(1)$ (resp.$\alpha(2)$) be the homomorphism $P \oplus_Q P \to P$ (resp. $P \oplus_Q P \oplus_Q P \to P$) induced by the summation and let $R(1)$ (resp. $R(2)$) be $(\alpha(1)^{\mathrm{gp}})^{-1}(P)$ (resp. $(\alpha(2)^{\mathrm{gp}})^{-1}(P)$). Let $Y(1)$ (resp. $Y(2)$) be $(Y \times_S Y) \times_{\mathrm{Spf}\,\mathbb{Z}_p\{P \oplus_Q P\}} \mathrm{Spf}\,\mathbb{Z}_p\{R(1)\}$ (resp. $(Y \times_S Y \times_S Y) \times_{\mathrm{Spf}\,\mathbb{Z}_p\{P \oplus_Q P \oplus_Q P\}} \mathrm{Spf}\,\mathbb{Z}_p\{R(2)\}$), and let $M'(1)$ (resp. $M'(2)$) be the log structure on $Y(1)$ (resp. $Y(2)$) induced by the canonical log structure on $\mathrm{Spf}\,\mathbb{Z}_p\{R(1)\}$ (resp. $\mathrm{Spf}\,\mathbb{Z}_p\{R(2)\}$). Then $(Y(1), M'(1))$ (resp. $(Y(2), M'(2))$) is formally log etale over $((Y, M') \times_{(S,N)} (Y, M'))^{\mathrm{int}}$ (resp. $((Y, M') \times_{(S,N)} (Y, M') \times_{(S,N)} (Y, M'))^{\mathrm{int}}$) ([Kk1, (4.10)]). Moreover, there exist morphisms

$$p'_i : (Y(1), M'(1)) \longrightarrow (Y, M') \quad (i = 1, 2),$$

$$p'_{ij} : (Y(2), M'(2)) \longrightarrow (Y(1), M'(1)) \quad (1 \leq i < j \leq 3),$$
$$\Delta' : (Y, M') \longrightarrow (Y(1), M'(1))$$

induced by the projections

$$(Y, M') \times_{(S,N)} (Y, M') \longrightarrow (Y, M'),$$

$$(Y, M') \times_{(S,N)} (Y, M') \times_{(S,N)} (Y, M) \longrightarrow (Y, M') \times_{(S,N)} (Y, M'),$$

and the diagonal morphism

$$(Y, M') \longrightarrow (Y', M') \times_{(S,N)} (Y, M'),$$

respectively. Note that Δ' is an exact closed immersion and that the composition

$$(Y, M') \xrightarrow{\Delta'} (Y(1), M'(1)) \longrightarrow (Y, M') \times_{(S,N)} (Y, M')$$

coincides with the usual diagonal morphism.

Let $(X, M) \longrightarrow (S, N)$ be a morphism of fine log formal V-schemes and let $(X, M) \hookrightarrow (Y, M')$ be an exact closed immersion over (S, N) with (Y, M') as above. We denote the log scheme $(T_{X,n}(Y), L_{X,n}(Y))$ simply by (T_n, L_n) and $(T_{X,n}(Y(i)), L_{X,n}(Y(i)))$ by $(T(i)_n, L(i)_n)$. The morphisms $p'_i\,(i = 1, 2)$, $p'_{ij}\,(1 \le i < j \le 3)$ and Δ' induce compatible morphisms of enlargements $p_{i,n} : T(1)_n \to T_n$, $p_{ij,n} : T(2)_n \to T(1)_n$ and $\Delta_n : T_n \to T(1)_n$.

Now we define the notion of a convergent log stratification as follows:

Definition 5.2.5. Let the notations be as above. (In particular, we assume that the morphism $(Y, M') \longrightarrow (S, N)$ admits a chart.) For a compatible family of isocoherent sheaves E_n on T_n, a convergent log stratification on $\{E_n\}$ is a compatible family of isomorphisms

$$\epsilon_n : p_{2,n}^* E_n \longrightarrow p_{1,n}^* E_n \;(n \in \mathbb{N})$$

which satisfies the following conditions:

(1) $\Delta_n^*(\epsilon_n) = \mathrm{id}$.

(2) $p_{12,n}^*(\epsilon_n) \circ p_{23,n}^*(\epsilon_n) = p_{13,n}^*(\epsilon_n)$.

We denote the category of compatible families of isocoherent sheaves on T_n with convergent log stratifications by $\mathrm{Str}'((X, M) \hookrightarrow (Y, M')/(S, N))$.

The next proposition establishes the relation between the categories $I_{\mathrm{conv}}((X, M)/(S, N))$ and $\mathrm{Str}'((X, M) \hookrightarrow (Y, M')/(S, N))$.

Proposition 5.2.6. *Let the notations be as in Definition 5.2.5 and assume that (Y, M') is formaly log smooth over (S, N). Then the category $I_{\mathrm{conv}}((X, M)/(S, N))$ is equivalent to the category* $\mathrm{Str}'((X, M) \hookrightarrow (Y, M')/(S, N))$.

Proof. First, suppose given an object $\mathcal{E}$ in $I_{\mathrm{conv}}((X, M)/(S, N))$. Let E_n be $\mathcal{E}_{T_n}$. Then the morphisms $p_{i,n} : (T(1)_n, L(1)_n) \longrightarrow (T_n, L_n)$ induce the isomorphisms $\epsilon_n : p_{2,n}^* E_n \xrightarrow{\sim} \mathcal{E}_{T(1)_n} \xrightarrow{\sim} p_{1,n}^* E_n$ $(n \in \mathbb{N})$. One can check easily that the pair $(\{E_n\}, \{\epsilon_n\})$ gives an object of $\mathrm{Str}'((X, M) \hookrightarrow (Y, M')/(S, N))$.

Conversely, suppose given an object $(\{E_n\}, \{\epsilon_n\})$ in $\mathrm{Str}'((X, M) \hookrightarrow (Y, M')/(S, N))$. We will show how to associate an object $\mathcal{E}$ in

$I_{\mathrm{conv}}((X,M)/(S,N))$. Denote the morphism $(X,M) \hookrightarrow (Y,M')$ by i. Let (T,L,z) be an enlargement of (X,M) over (S,N). Then there exists a commutative diagram

$$\begin{array}{ccc} (T_0,L) & \xrightarrow{i\circ z} & (Y,M') \\ \downarrow & & \downarrow \\ (T,L) & \longrightarrow & (S,N). \end{array}$$

Since $(Y,M') \longrightarrow (S,N)$ is formally log smooth, etale locally there exists a map $s:(T,L) \longrightarrow (Y,M')$ compatible with the above diagram (Proposition 2.2.13). Hence there exists a morphism $(T,L) \longrightarrow (T_n,L_n)$ for some integer n and $\mathcal{E}_T$ is defined as the pullback of E_n.

Suppose there exist two morphisms $s,t:(T,L) \longrightarrow (X,M)$ which satisfies the same condition as s. Then there exists a commutative diagram

$$\begin{array}{ccc} (T_0,L) & \xrightarrow{\Delta\circ z} & (Y(1),M'(1)) \\ \downarrow & & \downarrow \\ (T,L) & \xrightarrow[(s\times t)^{\mathrm{int}}]{} & ((Y,M')\times_{(S,N)}(Y,M'))^{\mathrm{int}}, \end{array}$$

where $(s\times t)^{\mathrm{int}}$ is the morphism induced by $s\times t:(T,L) \longrightarrow (Y,M')\times_{(S,N)}(Y,M')$. Since the morphism $(Y(1),M'(1)) \longrightarrow ((Y,M')\times_{(S,N)}(Y,M'))^{\mathrm{int}}$ is formally log etale, there exists uniquely a morphism $f:(T,L) \longrightarrow (Y(1),M'(1))$ compatible with the above diagram. Then the 4-ple (T,L,z,f) forms an enlargement of (X,M) in $(Y(1),M'(1))$ over (S,N). Hence there exists a morphism $(T,L) \longrightarrow (T(1)_n,L(1)_n)$ for some integer n. Then the isomorphism $\epsilon_n : p_{2,n}^*E \longrightarrow p_{1,n}^*E$ on $T(1)_n$ induces the isomorphism $t^*E \xrightarrow{\sim} s^*E$. By this fact, the cocycle condition of $\{\epsilon_n\}$ and rigid analytic faithfully flat descent, one can check that the above sheaf $\mathcal{E}_T := s^*E$ descents to T and determines an isocoherent sheaf on T, and that this isocoherent sheaf is independent of the choice of the morphism s. Again by the cocycle condition of $\{\epsilon_n\}$, one can see that the $\mathcal{E}_T$'s form an object $\mathcal{E}$ of $I_{\mathrm{conv}}((X,M)/(S,N))$.

It can be checked easily that the above two functors are the inverse of each other. □

We prepare two propositions which are essentially due to Ogus ([O1, (2.6.4), (2.13), (2.14)]).

Proposition 5.2.7. *Let $(X, M) \hookrightarrow (Y, M')$ be an exact closed immersion of fine log formal V-schemes which are formally log smooth, integral over a fine log formal V-scheme (S, N) such that S is flat over* $\mathrm{Spf}\, V$. *Let (T_n, L_n) be as above and let z_n be the map $(T_n)_0 \longrightarrow X_0$. Let $\mathcal{O}_{Y^{\mathrm{an}}}$ (resp. $\mathcal{O}_{T_n^{\mathrm{an}}}$) be the sheaf $\varprojlim_m K \otimes_V (\mathcal{O}_Y/J^m)$ (resp. $\varprojlim_m K \otimes_V (\mathcal{O}_{T_n}/J^m)$), where J is the defining ideal of X in Y (resp. T_n). Then:*

1. *The map $\mathcal{O}_{Y^{\mathrm{an}}} \longrightarrow \mathcal{O}_{T_n^{\mathrm{an}}}$ induced by the morphism $T_n \longrightarrow Y$ is an isomorphism for each n.*
2. *The composition*

$$z_{n*}(K \otimes_V \mathcal{O}_{T_n}) \longrightarrow \mathcal{O}_{T_n^{\mathrm{an}}} \overset{\sim}{\longleftarrow} \mathcal{O}_{Y^{\mathrm{an}}}$$

is injective.

Proof. (1) is nothing but [O1, (2.6.4)]. (Note that X and Y are flat over $\mathrm{Spf}\, V$.)

If we prove the isomorphism

$$\mathcal{O}_{Y^{an}} \overset{\sim}{\longrightarrow} (K \otimes \mathcal{O}_X)[[t_1, t_2, ..., t_r]] \tag{5.2.4}$$

(where t_i's are indeterminates) locally, one can prove (2) in the same way as that in [O1, (2.13)]. So here we only prove (5.2.4). There exists etale locally a chart $(P \to \mathcal{O}_Y, Q \to \mathcal{O}_S, Q \to P)$ of $(Y, M') \longrightarrow (S, N)$ which satisfying the conditions in Theorem 2.2.8. Let S'_n be $\mathrm{Spec}\, \mathcal{O}_S/p^n \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P]$ and S' be $\varinjlim_n S'_n$. Then X and Y are formally smooth over S'. Then the isomorphism (5.2.4) is easily deduced from this. □

Proposition 5.2.8. *With the notations of the previous proposition, let $j : U \hookrightarrow X$ be a dense open subset. Then the obvious sequence of sheaves on X*

$$0 \longrightarrow z_{n*}(K \otimes_V \mathcal{O}_{T_n}) \longrightarrow \mathcal{O}_{Y^{\mathrm{an}}} \oplus j_*j^*z_{n*}(K \otimes_V \mathcal{O}_{T_n}) \longrightarrow j_*j^*(\mathcal{O}_{Y^{\mathrm{an}}})$$

is exact.

Proof. Since the proof is same as that of [O1, (2.14)], we omit it. □

Using Propositions 5.2.6, 5.2.7 and 5.2.8, we prove the fully faithfulness of the functor $\bar{\iota}$ (cf. [O1, (2.15)]):

Proposition 5.2.9. *Let $(X, M) \longrightarrow (S, N)$ be a formally log smooth integral morphism of fine log formal V-schemes which admits a chart. Then the functor*

$$\bar{\iota} : \mathcal{N}I_{\mathrm{conv}}((X, M)/(S, N)) \longrightarrow \mathcal{N}I_{\mathrm{inf}}((X, M)/(S, N))$$

is fully faithful.

Proof. The proof is similar to that of [O1, (2.15)]. First let us note that we may assume that S is flat over $\mathrm{Spf}\, V$, by Remarks 3.2.19 and 5.1.6.

Let $\mathcal{E}, \mathcal{F}$ be objects in $\mathcal{N}I_{\mathrm{conv}}((X, M)/(S, N))$. Then $\mathrm{Hom}(\mathcal{E}, \mathcal{F})$ is isomorphic to the global section of $\mathcal{H}om(\mathcal{E}, \mathcal{F})$, where $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ is defined by

$$\mathcal{H}om(\mathcal{E}, \mathcal{F})_T := \mathcal{H}om(\mathcal{E}_T, \mathcal{F}_T)$$

for each enlargement T. The similar statement is true for the category $\mathcal{N}I_{\mathrm{inf}}((X, M)/(S, N))$. Hence it suffices to show that the homomorphism

$$\bar{\iota}_{\mathcal{E}} : H^0((X/S)^{\log}_{\mathrm{conv}}, \mathcal{E}) \longrightarrow H^0((X/S)^{\log}_{\mathrm{inf}}, \bar{\iota}(\mathcal{E}))$$

induced by $\bar{\iota}$ is isomorphic for any object $\mathcal{E}$ in $\mathcal{N}I_{\mathrm{conv}}((X/S)^{\log})$. Hence the assertion is reduced to the following lemma.

Lemma 5.2.10. *Let $(X, M) \longrightarrow (S, N)$ be as in Proposition 5.2.9 (with S flat over $\mathrm{Spf}\, V$) and let $(P_X \to M, Q_S \to N, Q \to P)$ be a chart. Let $\mathcal{E}$ be an object in $I_{\mathrm{conv}}((X/S)^{\log})$ such that the isocoherent sheaf $\mathcal{E}_X$ obtained by evaluating $\mathcal{E}$ at the enlargement $(X, M, (X_0, M) \hookrightarrow (X, M))$ is a locally free $K \otimes_V \mathcal{O}_X$-module. Then the homomorphism*

$$\iota_{\mathcal{E}} : H^0((X/S)^{\log}_{\mathrm{conv}}, \mathcal{E}) \longrightarrow H^0((X/S)^{\log}_{\mathrm{inf}}, \iota(\mathcal{E}))$$

induced by ι is an isomorphism.

PROOF. Let α be the map $P \oplus_Q P \longrightarrow P$ induced by the summation and define R to be $(\alpha^{\mathrm{gp}})^{-1}(P)$. Let

$$X(1) := (X \times_S X) \times_{\mathrm{Spf}\,\mathbb{Z}_p\{P \oplus_Q P\}} \mathrm{Spf}\,\mathbb{Z}_p\{R\}$$

and let $M(1)$ be the log structure on $X(1)$ induced by the canonical log structure on $\mathrm{Spec}\,\mathbb{Z}/p^n\mathbb{Z}[R]$. Then there exists an exact closed immersion $\Delta : (X, M) \hookrightarrow (X(1), M(1))$ induced by the diagonal morphism $(X, M) \hookrightarrow (X, M) \times_{(S,N)} (X, M)$. So the morphism $(X(1), M(1)) \longrightarrow (X, M)$ induced by the projections $(X, M) \times_{(S,N)} (X, M) \longrightarrow (X, M)$ are formally smooth in the classical sense on a neighborhood of $\Delta(X)$. Hence $(X(1), M(1))$ is formally log smooth integral over (S, N) on a neighborhood of $\Delta(X)$. Hence the results in Proposition 5.2.7 and 5.2.8 are true for $(Y, M') = (X(1), M(1))$. Put $T_n := T_{n,X}(X(1))$.

Now let $\mathcal{E}$ be an object of $\mathcal{N}I_{\mathrm{conv}}((X, M)/(S, N))$. Since we have an equivalence of categories

$$I_{\mathrm{conv}}((X, M)/(S, N)) \simeq \mathrm{Str}'((X, M) \hookrightarrow (X, M)/(S, N)),$$

$\mathcal{E}$ defines an isomorphism

$$\epsilon_n : (K \otimes \mathcal{O}_{T_n}) \otimes \mathcal{E}_X \longrightarrow \mathcal{E}_X \otimes (K \otimes \mathcal{O}_{T_n})$$

for each n. If we tensor it with the map $K \otimes \mathcal{O}_{T_n} \longrightarrow \mathcal{O}_{X(1)^{\mathrm{an}}}$ of Proposition 5.2.7(2) (for $Y = X(1)$), we obtain the isomorphism

$$\epsilon' : \mathcal{O}_{X(1)^{\mathrm{an}}} \otimes \mathcal{E}_X \longrightarrow \mathcal{E}_X \otimes \mathcal{O}_{X(1)^{\mathrm{an}}}.$$

One can check easily that this isomorphism is induced by the object $\bar{\iota}(\mathcal{E}) \in I_{\mathrm{inf}}((X, M)/(S, N))$ via the equivalence of categories

$$I_{\mathrm{inf}}((X, M)/(S, N)) \simeq \widehat{\mathrm{Str}}((X, M)/(S, N)).$$

(Note that we have $\mathcal{O}_{X(1)^{\mathrm{an}}} \overset{\simeq}{=} K \otimes_V \varprojlim_n \mathcal{O}_{X^n}$, where X^n is as in Section 3.2.) So we get the following:

$$H^0((X/S)^{\log}_{\mathrm{conv}}, \mathcal{E}) = \{e \in \Gamma(X, \mathcal{E}_X) \,|\, \epsilon_n(1 \otimes e) = e \otimes 1 \text{ for all } n\}.$$

$$H^0((X/S)^{\log}_{\inf}, \iota(\mathcal{E})) = \{e \in \Gamma(X, \mathcal{E}_X) \,|\, \epsilon'(1 \otimes e) = e \otimes 1\}.$$

Moreover, the map $\iota_{\mathcal{E}}$ is identical with the natural inclusion. Then, since the map $K \otimes \mathcal{O}_{T_n} \longrightarrow \mathcal{O}_{X(1)^{\mathrm{an}}}$ is injective and $\mathcal{E}_X$ is locally free, $\iota_{\mathcal{E}}$ is an isomorphism. Hence the assertion is proved. □

By Lemma 5.2.2 and Proposition 5.2.9, it suffices to show the essential surjectivity of $\bar{\iota}$ in the case when the morphism $(X, M) \longrightarrow (S, N)$ admits a chart to prove the Theorem 5.2.1. Let us call an object E in $\mathcal{N}I_{\inf}((X, M)/(S, N))$ is convergent if there exists a convergent isocrystal $\mathcal{E} \in \mathcal{N}I_{\mathrm{conv}}((X, M)/(S, N))$ such that $\bar{\iota}(\mathcal{E}) = E$ holds. Then the following proposition holds (cf. [O1, (2.16)]):

Proposition 5.2.11. *Let $(X, M) \longrightarrow (S, N)$ be a formally log smooth integral morphism of fine log formal V-schemes which admits a chart and let U be a dense open subset of X. If $E \in \mathcal{N}I_{\mathrm{conv}}((X, M)/(S, N))$ is convergent when restricted to (U, M), then E is convergent on (X, M).*

Proof. We may assume that S is flat over $\mathrm{Spf}\, V$. Let $\mathcal{E}$ be the object in $I_{\mathrm{conv}}((U/S)^{\log})$ such that $\bar{\iota}(\mathcal{E}) = E$ holds. Let F be the sheaf $z_{n*}\mathcal{H}\mathrm{om}((K \otimes \mathcal{O}_{T_n}) \otimes E_X, E_X \otimes (K \otimes \mathcal{O}_{T_n}))$ on X. Since F is a locally free $z_{n*}(K \otimes \mathcal{O}_{T_n})$-module, we have an exact sequence

$$0 \longrightarrow F \xrightarrow{a} (F \otimes \mathcal{O}_{X(1)^{an}}) \oplus j_*j^*F \xrightarrow{b} j_*j^*(F \otimes \mathcal{O}_{X(1)^{an}}),$$

where $X(1)$ is as in Lemma 5.2.10. Let

$$\epsilon'_n : j_*j^*z_{n*}((K \otimes \mathcal{O}_{T_n}) \otimes E_X) \longrightarrow j_*j^*z_{n*}(E_X \otimes (K \otimes \mathcal{O}_{T_n}))$$

be the map defined by the structure of an isocrystal $\mathcal{E} \in I_{\mathrm{conv}}((U/S)^{\log})$ via the equivalence of categories

$$I_{\mathrm{conv}}((U/S)^{\log}) \simeq \mathrm{Str}'((U, M) \hookrightarrow (U, M)/(S, N)),$$

and let

$$\epsilon' : \mathcal{O}_{X(1)^{\mathrm{an}}} \otimes E_X \longrightarrow E_X \otimes \mathcal{O}_{X(1)^{\mathrm{an}}}$$

be the map defined by the structure of an isocrystal $E \in I_{\mathrm{inf}}((X/S)^{\log})$ via the equivalence of categories

$$I_{\mathrm{inf}}((X/S)^{\log}) \simeq \widehat{\mathrm{Str}}((X, M)/(S, N)).$$

Then (ϵ'_n, ϵ') defines an element of $\mathrm{Ker}(b)$. So there exists an element $\epsilon_n \in F$ such that $a(\epsilon_n) = (\epsilon'_n, \epsilon')$ holds. Then one can check that these ϵ_n's define, via the equivalence of categories

$$I_{\mathrm{conv}}((X/S)^{\log}) \simeq \mathrm{Str}'((X, M) \hookrightarrow (X, M)/(S, N)),$$

an isocrystal $\tilde{\mathcal{E}} \in I_{\mathrm{conv}}((X/S)^{\log})$ satisfying $\iota(\tilde{\mathcal{E}}) = E$.

It suffices to prove that isocrystal $\tilde{\mathcal{E}}$ is nilpotent. Let us prove it by induction on the rank of E. First let us consider the case $E = K \otimes \mathcal{O}_{X/S}$. Since $E_X = K \otimes \mathcal{O}_X$ is locally free, the isocrystal $\tilde{\mathcal{E}}$ satisfying $\iota(\tilde{\mathcal{E}}) = K \otimes \mathcal{O}_{X/S}$ is unique up to isomorphism by Lemma 5.2.10. Hence $\tilde{\mathcal{E}} = \mathcal{K}_{X/S}$ holds. So $\tilde{\mathcal{E}}$ is nilpotent in this case. Let us consider the general case. By inductive hypothesis, there exists an exact sequence

$$0 \longrightarrow K \otimes_V \mathcal{O}_{X/S} \xrightarrow{h} E \longrightarrow E' \longrightarrow 0$$

in $I_{\mathrm{inf}}((X/S)^{\log})$ such that E' is convergent. Since $\tilde{\mathcal{E}}_X = E_X$ is a locally free $K \otimes_V \mathcal{O}_X$-module, the homomorphism

$$\iota_{\tilde{\mathcal{E}}} : H^0((X/S)^{\log}_{\mathrm{conv}}, \tilde{\mathcal{E}}) \longrightarrow H^0((X/S)^{\log}_{\mathrm{inf}}, \iota(\tilde{\mathcal{E}}))$$

induced by ι is an isomorphism by Lemma 5.2.10. Hence there exists a morphism $h' : \mathcal{K}_{X/S} \longrightarrow \tilde{\mathcal{E}}$ which is sent to h by the functor ι. Put $\tilde{\mathcal{E}}' := \mathrm{Coker}(h')$. Then it is easy to see that $\iota(\tilde{\mathcal{E}}') = E'$ holds. Then, since E' is convergent, $\tilde{\mathcal{E}}'$ is nilpotent. (Note that, since $\mathcal{H}om_{K \otimes_V \mathcal{O}_X}(E'_X, E'_X)$ is locally free, the isocrystal $\tilde{\mathcal{E}}'$ satisfying $\iota(\tilde{\mathcal{E}}') = E'$ is unique up to isomorphism.) Hence $\tilde{\mathcal{E}}$ is also nilpotent. So the assertion is proved. □

By Proposition 5.2.11, it suffices to prove the essential surjectivity of $\bar{\iota}$ in the case that $f^*N = M$ holds for the proof of Theorem 5.2.1. Note that we have the equivalence of categories

$$I_{\mathrm{conv}}((X, \mathrm{triv.\,log\,str.})/(S, \mathrm{triv.\,log\,str.})) \simeq I_{\mathrm{conv}}((X, M)/(S, N)),$$

$$I_{\inf}((X, \text{triv. log str.})/(S, \text{triv. log str.})) \simeq I_{\inf}((X, M)/(S, N)),$$

in this case. Hence we may assume that the log structures are trivial. So, until the end of the proof of Theorem 5.2.1, we assume the log structures are trivial and abbreviate to write the log structures. Moreover, we may assume that S is flat over $\mathrm{Spf}\, V$ by Remarks 3.2.19 and 5.1.6.

Let the notations be as above and let X^n be the n-th infinitesimal neighborhood of X in $X \times_S X$. Denote the category $\mathrm{Str}'(X \hookrightarrow X/S)$ simply by $\mathrm{Str}'(X/S)$. Let X^n be the n-th infinitesimal neighborhood of X in $X \times_S X$ and let $\{T_n\}_n$ be the system of enlargements of X in $X \times_S X$. The universality of blow up induces a family of morphisms $\psi_n : X^n \longrightarrow T_n$. We can define the functor

$$\alpha : \mathrm{Str}'(X/S) \longrightarrow \widehat{\mathrm{Str}}(X/S).$$

as the one induced by the pull-back by ψ_n's. One can check easily that this functor α coincides with the functor

$$\iota : I_{\mathrm{conv}}(X/S) \longrightarrow I_{\inf}(X/S)$$

via the equivalences of categories

$$I_{\mathrm{conv}}((X/S)^{\log}) \simeq \mathrm{Str}'(X/S),$$

$$I_{\inf}((X/S)^{\log}) \simeq \widehat{\mathrm{Str}}(X/S)$$

of Propositions 3.2.14 and 5.2.6.

For 'special' objects of $\widehat{\mathrm{Str}}(X/S)$, we construct the quasi-inverse of α.

Definition 5.2.12. Let the notations be as above. Let $(E, \{\epsilon_n\})$ be an object of the category $\widehat{\mathrm{Str}}(X/S)$ and let $\tilde{E}$ be a p-torsion free coherent sheaf on X such that $K \otimes_V \tilde{E} = E$ holds. Then we call $(E, \{\epsilon_n\})$ is special if and only if there exists a sequence of integers $a(n)$ which satisfies the following conditions:

(1) $a(n) = O(\log n) \quad (n \to \infty)$ holds.

(2) The map

$$p^{a(n)}\epsilon_n : p_2^*E \longrightarrow p_1^*E$$

sends $p_2^*\tilde{E}$ into $p_1^*\tilde{E}$ and the map

$$p^{a(n)}\epsilon_n^{-1} : p_1^*E \longrightarrow p_2^*E$$

sends $p_1^*\tilde{E}$ into $p_2^*\tilde{E}$.

This definition is independent of the choice of $\tilde{E}$. We denote the full subcategory of $\widehat{\mathrm{Str}}(X/S)$ which consists of special objects by $\mathrm{S}\widehat{\mathrm{Str}}(X/S)$.

PROPOSITION 5.2.13. *Let the notations be as above. Then there exists a functor* $\beta : \mathrm{S}\widehat{\mathrm{Str}}(X/S) \to \mathrm{Str}'(X/S)$ *such that the composition* $\alpha \circ \beta$ *is equivalent to the inclusion functor* $\mathrm{S}\widehat{\mathrm{Str}}(X/S) \to \widehat{\mathrm{Str}}(X/S)$.

PROOF. In this proof, for a formal V-scheme Z, we denote the closed subscheme defined by p^n by Z/p^n. By using the explicit description of T_n's in the proof of [O1, (2.3)], one can check the following: For integers $n, m \geq 0$, we can define a compatible family of morphisms $T_{m-1}/p^n \xrightarrow{f} X^{nm}/p^n$ which makes the following diagram commutative:

$$
\begin{array}{ccc}
X^{m(n+1)} & \xrightarrow{\psi_{m(n+1)}} & T_{m(n+1)} \\
\uparrow & & \uparrow \\
X^{nm} & & T_{m-1} \\
{\scriptstyle g}\uparrow & & \uparrow \\
X^{nm}/p^n & \underset{f}{\longleftarrow} & T_{m-1}/p^n \\
\uparrow & & \Vert \\
X^{m-1}/p^n & \underset{\psi_{m-1}/p^n}{\longrightarrow} & T_{m-1}/p^n
\end{array}
\tag{5.2.5}
$$

Let $(E, \{\epsilon_n\})$ be an object in $\mathrm{S}\widehat{\mathrm{Str}}(X/S)$, let $\tilde{E}$ be a p-torsion free coherent sheaf on X satisfying $K \otimes \tilde{E} = E$ and let $\{a(n)\}_{n \in \mathbb{N}}$ be a sequence of integers which satisfies the condition in Definition 5.2.12. Then $\{\epsilon_n\}$ induces maps

$$p^{a(nm)}\epsilon_{nm} : p_2^*\tilde{E} \longrightarrow p_1^*\tilde{E},$$

$$p^{a(nm)}\epsilon_{nm}^{-1} : p_1^*\tilde{E} \longrightarrow p_2^*\tilde{E},$$

where $p_i\,(i = 1, 2)$ is the morphism $X^{nm} \to X \times X \overset{i\text{-th proj.}}{\to} X$. Let $q_i\,(i = 1, 2)$ be the morphism

$$T_{m-1}/p^n \to X^{m-1}/p^n \to (X \times X)/p^n \overset{i\text{-th proj.}}{\to} X/p^n,$$

and denote the pull-back of the morphisms $p^{a(nm)}\epsilon_{nm}$, $p^{a(nm)}\epsilon_{nm}^{-1}$ by $f \circ g$ by

$$\sigma_{m,n} : q_2^* \tilde{E} \longrightarrow q_1^* \tilde{E}$$

$$\rho_{m,n} : q_1^* \tilde{E} \longrightarrow q_2^* \tilde{E},$$

respectively. Let r_i $(i = 1, 2)$ be the morphism

$$T_{m-1} \to X^{m-1} \to X \times X \overset{i\text{-th proj.}}{\to} X.$$

We define the map $\epsilon'_m : r_2^* E \longrightarrow r_1^* E$ to be the map such that for all n, $p^{a(nm)}\epsilon'_m$ sends $r_2^* \tilde{E}$ to $r_1^* \tilde{E}$ and that $p^{a(nm)}\epsilon'_m \mod p^n r_2^* \tilde{E}$ coincides with $\sigma_{m,n}$. From the data $\sigma_{m,n}$, ϵ'_m is determined modulo $p^{n-a(nm)}(r_1^* \tilde{E})$, and the condition $a(n) = O(\log n)$ guarantees the well-definedness of ϵ'_m. By the same method, we can define ϵ''_m from $\rho_{m,n}$. Then ϵ'_m and ϵ''_m are the inverse maps of each other. One can check easily that $(E, \{\epsilon'_m\})$ defines an object of $\mathrm{Str}'((X, M)/(S, N))$. We define the functor β by $(E, \{\epsilon_n\}) \mapsto (E, \{\epsilon'_m\})$. By using the diagram (5.2.5), one can check that the composition $\alpha \circ \beta$ is equivalent to the inclusion functor $\mathrm{S}\widehat{\mathrm{Str}}(X/S) \to \widehat{\mathrm{Str}}(X/S)$. □

By Proposition 5.2.13, Theorem 5.2.1 is reduced to the following proposition.

Proposition 5.2.14. *Let the notations be as above. Then* $\mathcal{N}\widehat{\mathrm{Str}}(X/S) \subset \mathrm{S}\widehat{\mathrm{Str}}(X/S)$ *holds, that is, nilpotent objects are special.*

Proof. It is obvious that $(K \otimes_V \mathcal{O}_X, \{\mathrm{id}\})$ is special.

Let $(E', \{\epsilon'_n\}), (E'', \{\epsilon''_n\})$ be objects in $\mathrm{S}\widehat{\mathrm{Str}}(X/S)$ and let $(E, \{\epsilon_n\})$ be an object in $\widehat{\mathrm{Str}}(X/S)$. Assume further that E, E', E'' are locally free $K \otimes \mathcal{O}_X$-modules and there exists an exact sequence

$$0 \longrightarrow (E', \{\epsilon'_n\}) \longrightarrow (E, \{\epsilon_n\}) \longrightarrow (E'', \{\epsilon''_n\}) \longrightarrow 0. \tag{5.2.6}$$

If we prove that $(E, \{\epsilon_n\})$ is also special, we are done.

Note that if $(E, \{\epsilon_n\})$ is an object of $\widehat{\mathrm{Str}}(X/S)$, $(E, \{\tau_n^* \epsilon_n^{-1}\})$ is also an object of $\widehat{\mathrm{Str}}(X/S)$, where τ_n is as in Section 3.2. If we take this fact into consideration, we only have to show that, in the above situation, there exists a sequence of integers $\{a(n)\}$ such that $a(n) = O(\log n)$ holds and that $p^{a(n)}\epsilon_n : p_2^*E \longrightarrow p_1^*E$ sends $p_2^*\tilde{E}$ into $p_1^*\tilde{E}$, where $\tilde{E}$ is any fixed p-torsion free coherent sheaf satisfying $E = K \otimes \tilde{E}$ and p_i $(i = 1, 2)$ is the morphism $X^n \longrightarrow X \times_S X \overset{i\text{-th proj.}}{\longrightarrow} X$. Moreover, note that we may work locally to prove this claim.

To show this claim, we prepare a lemma:

LEMMA 5.2.15. *Let the situation be as above and assume $\Omega^1_{X/S}$ is free with basis dx_i $(1 \le i \le m)$ with $x_i \in \mathcal{O}_X$. Define $\xi_i \in \mathcal{O}_{X\times X}$ by $\xi_i := 1\otimes x_i - x_i \otimes 1$. Let (E, ∇) be an object in $\hat{C}(X/S)$. If we identify $\widehat{\mathrm{Str}}(X/S)$ with $\hat{C}(X/S)$ by Theorem 3.2.15, the object $(E, \{\epsilon_n\})$ which corresponds to (E, ∇) is given by the following formula:*

$$\epsilon_n(1 \otimes e) = \sum_{|a|\le n} \frac{1}{a!}\nabla_a(e) \otimes \xi^a,$$

where $a := (a_1, \cdots, a_m)$ is a multi-index of length m,

$$\nabla_a := (\mathrm{id} \otimes D_{(1)} \circ \nabla)^{a_1} \circ \cdots \circ (\mathrm{id} \otimes D_{(m)} \circ \nabla)^{a_m}$$

($D_{(i)}$'s are defined in Lemma 3.2.7) and $\xi^a := \prod_{i=1}^m \xi_i^{a_i}$.

PROOF. By [B-O, (2.7)], we have $D^{*a} = a! D_a$. The lemma follows from this fact and easy calculation. □

PROOF OF PROPOSITION 5.2.14 (continued). Let

$$0 \longrightarrow (E', \nabla') \longrightarrow (E, \nabla) \longrightarrow (E'', \nabla'') \longrightarrow 0$$

be an exact sequence in $\hat{C}(X/S)$ corresponding to the exact sequence (5.2.6) via the equivalence of categories $\widehat{\mathrm{Str}}(X/S) \simeq \hat{C}(X/S)$. Since the notion 'special' is a local property, we may assume that E',E'' are free $K \otimes_V \mathcal{O}_X$-modules and $E \overset{\sim}{=} E' \oplus E''$ holds as sheaves. Let $\tilde{E}', \tilde{E}''$ be free $\mathcal{O}_X$-modules such that $K \otimes \tilde{E}' = E'$ and $K \otimes \tilde{E}'' = E''$ hold, and put $\tilde{E} := \tilde{E}' \oplus \tilde{E}''$.

∇ is expressed as

$$\nabla = \begin{pmatrix} \nabla' & \\ & \nabla'' \end{pmatrix} + \begin{pmatrix} & \Gamma \\ & \end{pmatrix},$$

where Γ is a matrix with entries in $K \otimes_V \Omega^1_{X/S}$. Let e_i be the multi-index $(0, 0, \cdots, 0, \overset{i}{\hat{1}}, 0, \cdots, 0)$ of length m. Then

$$\nabla_{e_i} = \begin{pmatrix} \nabla'_{e_i} & \\ & \nabla''_{e_i} \end{pmatrix} + \begin{pmatrix} & \Gamma_i \\ & \end{pmatrix}$$

holds for some matrix Γ_i with entries in $K \otimes_V \mathcal{O}_X$. Then, for a general multi-index a of length m,

$$\begin{aligned} \frac{1}{a!}\nabla_a = & \frac{1}{a!}\begin{pmatrix} \nabla'_a & \\ & \nabla''_a \end{pmatrix} \\ & + \sum_{i=1}^{m} \sum_{\substack{b+c=a-e_i \\ b,c \geq 0}} \frac{b!c!}{a!} \cdot \frac{1}{b!}\begin{pmatrix} \nabla'_b & \\ & \nabla''_b \end{pmatrix} \begin{pmatrix} & \Gamma_i \\ & \end{pmatrix} \frac{1}{c!}\begin{pmatrix} \nabla'_c & \\ & \nabla''_c \end{pmatrix} \end{aligned}$$

holds. Since E' and E'' are special, there exists a sequence of integers $k(n)$ such that $k(n) = O(\log n)$ and

$$\frac{p^{k(n)}}{a!}\nabla'_a(e') \in \tilde{E}'$$

$$\frac{p^{k(n)}}{a!}\nabla''_a(e'') \in \tilde{E}''$$

hold for any $e' \in \tilde{E}'$, $e'' \in \tilde{E}''$ and any multi-index a of length m such that $|a| \leq n$. Hence it suffices to show that there exists a constant C such that

$$v_p(\frac{a!}{b!c!}) \leq C \log n \quad \text{for } n \gg 0$$

holds for any multi-indices a, b, c of length m such that $|a| \leq n$ and $b + c = a - e_i$ hold, where v_p is the p-adic valuation on $\mathbb{Z}$ satisfying $v_p(p) = 1$.

Let k, l, r be integers such that $0 \leq k, l, r$ and $r - 1 \leq k + l \leq r$ hold. Let $S : \mathbb{N} \longrightarrow \mathbb{N}$ be a map defined by $S(r) = a_t + a_{t-1} + \cdots + a_0$, where $r = a_t p^t + a_{t-1}p^{t-1} + \cdots a_1 p + a_0$ with $0 \leq a_i \leq p - 1$. Then $v_p(r!) = \frac{r-S(r)}{p-1}$ holds. So we have

$$v_p(\frac{r!}{k!l!}) \leq \frac{1 + S(k) + S(l) - S(r)}{p - 1}$$

$$\leq \frac{1 + (\log_p r)(p-1)}{p-1}$$
$$\leq 1 + \log_p r.$$

Let a, b, c as above and write $a = (a_1, a_2, ..., a_m)$ etc. Then we have

$$\begin{aligned} v_p(\frac{a!}{b!c!}) &= \sum_{i=1}^{m} v_p(\frac{a_i!}{b_i!c_i!}) \\ &\leq \sum_{a_i \neq 0} (1 + \log_p a_i) \\ &\leq m + \log_p(\prod_{a_i \neq 0} a_i) \\ &\leq m + m \log_p n. \end{aligned}$$

Hence there exists a required constant C. □

By Proposition 5.2.14, the essential surjectivity of $\bar{\iota}$ is deduced and the proof of Theorem 5.2.1 is now completed.

Now we give an important corollary of Theorem 5.2.1.

Corollary 5.2.16. *Let $(X, M) \xrightarrow{f} (\mathrm{Spec}\, V, N)$ be a proper log smooth integral morphism of fine log schemes and let x be a V-valued point of $X_{f\text{-triv}}$. Assume moreover that $H^0_{\mathrm{dR}}((X, M)/(\mathrm{Spec}\, V, N)) = V$ holds and that the special fiber is reduced. Let $(X_K, M) \xrightarrow{f_K} (\mathrm{Spec}\, K, N)$ be the generic fiber and $(X_k, M) \xrightarrow{f_k} (\mathrm{Spf}\, V, N)$ be the map induced by the special fiber. Denote the generic fiber and the special fiber of x by x_K, x_k, respectively. Then we have $H^0((X_k/V)^{\log}_{\mathrm{conv}}, \mathcal{K}_{X_k/V}) = K$ (hence the convergent fundamental group of (X_k, M) over $(\mathrm{Spf}\, V, N)$ with base point x_k is well-defined) and there exists a canonical isomorphism*

$$\pi_1^{\mathrm{conv}}((X_k, M)/(\mathrm{Spf}\, V, N), x_k) \cong \pi_1^{\mathrm{dR}}((X_K, M)/(\mathrm{Spec}\, K, N), x_K).$$

Proof. Let us denote the p-adic completion of f by $\hat{f} : (\hat{X}, M) \longrightarrow (\mathrm{Spf}\, V, N)$. Then the canonical functor

$$(5.2.7) \qquad \mathcal{N}I_{\mathrm{conv}}((\hat{X}, M)/(\mathrm{Spf}\, V, N)) \longrightarrow \mathcal{N}I_{\mathrm{conv}}((X_k, M)/(\mathrm{Spf}\, V, N))$$

gives an equivalence of categories by Remark 5.1.3.

Note that $\hat{X}_{\hat{f}\text{-triv}} \subset X$ is homeomorphic to $X_{k,f_k\text{-triv}} \subset X_k$ via the homeomorphism $X_k \longrightarrow \hat{X}$. Since $X_{k,f_k\text{-triv}}$ is open dense in X_k by Proposition 2.3.2, $\hat{X}_{\hat{f}\text{-triv}}$ is open dense in $\hat{X}$. So, by Theorem 5.2.1, we have the canonical equivalence of categories

$$(5.2.8) \qquad \mathcal{N}I_{\mathrm{conv}}((\hat{X},M)/(\mathrm{Spf}\, V,N)) \simeq \mathcal{N}I_{\mathrm{inf}}((\hat{X},M)/(\mathrm{Spf}\, V,N)).$$

Moreover, by Corollary 3.2.16, we have the canonical equivalence of categories

$$(5.2.9) \qquad C((X_K,M)/(\mathrm{Spec}\, K,N)) \simeq I_{\mathrm{inf}}((\hat{X},M)/(\mathrm{Spf}\, V,N)).$$

Combining the equivalences (5.2.7), (5.2.8) and (5.2.9), we get the functorial equivalence

$$\mathcal{N}I_{\mathrm{conv}}((X_k,M)/(\mathrm{Spf}\, V,N)) \simeq \mathcal{N}C((X_K,M)/(\mathrm{Spec}\, K,N)).$$

From this equivalence, we obtain the desired assertion. (Compatibility of fiber functors follows from the functoriality of the above equivalence of categories with respect to the morphism $x : \mathrm{Spec}\, V \longrightarrow X$.) □

5.3. Comparison between log convergent site and log crystalline site

Throughout this section, for a formal W-scheme X, we denote the closed subscheme defined by $p \in \mathcal{O}_X$ by X_1 and $(X_1)_{\mathrm{red}}$ by X_0, unless otherwise stated. In this section, we prove the comparison theorem between crystalline fundamental groups and convergent fundamental groups and we finish the proof of Berthelot-Ogus theorem as a consequence.

THEOREM 5.3.1. *Let (X,M) be a fine log scheme over k and let $(X,M) \xrightarrow{f} (\mathrm{Spec}\, k,N) \hookrightarrow (\mathrm{Spf}\, W,N)$ be a morphism of fine log formal W-schemes such that f is log smooth and integral. Assume moreover that $X_{f\text{-triv}}$ is dense open in X. Then there exists a canonical functor*

$$\Phi : I_{\mathrm{conv}}((X,M)/(\mathrm{Spf}\, W,N)) \longrightarrow I_{\mathrm{crys}}((X,M)/(\mathrm{Spf}\, W,N))$$

which induces an equivalence of categories

$$\overline{\Phi} : \mathcal{N}I_{\rm conv}((X, M)/(\mathrm{Spf}\, W, N)) \longrightarrow \mathcal{N}I_{\rm crys}((X, M)/(\mathrm{Spf}\, W, N)).$$

Proof. In the proof, we use the notation in Notation 4.3.5. Since the proof is long, we divide it into five steps.

Step 1. First, let us consider the following situation: Let (S, N) be an object in the category $\mathcal{S}_k$ in Notation 4.3.5. Put $S_1 := \underline{\mathrm{Spec}}\, \mathcal{O}_S/p\mathcal{O}_S$, and let (X, M) be a fine log scheme over S_1. Let $(X, M) \xrightarrow{f} (S_1, N) \overset{i}{\hookrightarrow} (S, N)$ be morphisms of fine log formal W-schemes such that f is log smooth and integral. Assume that we are given a closed immersion $(X, M) \hookrightarrow (Y, J)$ into a fine log formal W-scheme (Y, J) which is formally log smooth and integral over (S, N), and assume moreover that the diagram

$$(X, M) \hookrightarrow (Y, J) \longrightarrow (S, N)$$

admits a chart $\mathcal{C} := (P_S \to N, Q_Y \to J, R_X \to M, P \to Q \to R)$ such that $Q^{\rm gp} \to R^{\rm gp}$ is surjective. Under this assumption, we define a functor

$$\Psi_{(Y,J),\mathcal{C}} : I_{\rm conv}((X, M)/(S, N)) \longrightarrow \mathrm{HPDI}((X, M) \hookrightarrow (Y, J)).$$

Before defining the functor $\Psi_{(Y,J),\mathcal{C}}$, we prepare some notations. Let $\alpha : Q \to R, \alpha(1) : Q \oplus_P Q \to R$ and $\alpha(2) : Q \oplus_P Q \oplus_P Q \to R$ be the homomorphisms of monoids which are induced by the summation and the chart $\mathcal{C}$. Put $Q' := (\alpha^{\rm gp})^{-1}(R)$ and $Q'(i) := (\alpha(i)^{\rm gp})^{-1}(R)\, (i = 1, 2)$. Define the formal W-schemes $Y', Y'(1)$ and $Y'(2)$ by

$$Y' := Y \times_{\mathrm{Spf}\, \mathbb{Z}_p\{Q\}} \mathrm{Spf}\, \mathbb{Z}_p\{Q'\},$$

$$Y'(1) := (Y \times_S Y) \times_{\mathrm{Spf}\, \mathbb{Z}_p\{Q \oplus_P Q\}} \mathrm{Spf}\, \mathbb{Z}_p\{Q'(1)\},$$

$$Y'(2) := (Y \times_S Y \times_S Y) \times_{\mathrm{Spf}\, \mathbb{Z}_p\{Q \oplus_P Q \oplus_P Q\}} \mathrm{Spf}\, \mathbb{Z}_p\{Q'(2)\},$$

respectively. Let J' (resp. $J'(i)\, (i = 1, 2)$) be the fine log structure on Y' (resp. $Y'(i)$) induced by the canonical log structure on $\mathrm{Spf}\, \mathbb{Z}_p\{Q'\}$ (resp. $\mathrm{Spf}\, \mathbb{Z}_p\{Q'(i)\}$). Let $\{(T_n, L_n)\}_n$ (resp. $\{(T(i)_n, L(i)_n)\}_n$) be the system of universal enlargements of (X,M) in (Y',J') (resp. $(Y'(i),J'(i))$). (Note that we have the exact closed immersions $(X,M)\hookrightarrow(Y',J'),(X,M)\hookrightarrow(Y'(i),J'(i))$ which admit charts.) Let (D,M_D) (resp. $(D(i),M_{D(i)})\,(i=1,2)$) be the p-adically complete log PD-envelope of (X,M) in (Y,J) (resp. the $(i+1)$-fold fiber product of (Y,J) over (S,N).) By [Kk1, (5.6)], D (resp. $D(i)$) is the usual p-adically complete PD-envelope of X in Y' (resp. $Y'(i)$), and M_D (resp. $M_{D(i)}$) is the pull-back of J' (resp. $J'(i)$) by the morphism $D\longrightarrow Y'$ (resp. $D(i)\longrightarrow Y'(i)$), which we denote by h (resp. $h(i)$).

Put $\mathcal{I}:=\mathrm{Ker}(\mathcal{O}_{Y'}\longrightarrow\mathcal{O}_X)$ and $\mathcal{I}(i):=\mathrm{Ker}(\mathcal{O}_{Y'(i)}\longrightarrow\mathcal{O}_X)\,(i=1,2)$. Assume that $\mathcal{I},\mathcal{I}(i)\,(i=1,2)$ are all generated by m elements as $\mathcal{O}_{Y'}$-module, $\mathcal{O}_{Y'(i)}$-module, respectively. Put $n:=(p-1)m$. Then one can check the inclusions $h^*(\mathcal{I}^{n+1})\subset p\mathcal{O}_D$, $h(i)^*(\mathcal{I}(i)^{n+1})\subset p\mathcal{O}_{D(i)}$. Hence, by the universality of blowing up, the morphisms h, $h(i)$ factors as

$$D\xrightarrow{\beta}T_n\longrightarrow Y',$$

$$D(i)\xrightarrow{\beta(i)}T(i)_n\longrightarrow Y'(i),$$

respectively. The morphisms $\beta,\beta(i)$ naturally induce the morphisms of fine log formal W-schemes

$$(D,M_D)\longrightarrow(T_n,L_n),$$

$$(D(i),M_{D(i)})\longrightarrow(T(i)_n,L(i)_n),$$

which are also denoted by $\beta,\beta(i)$, respectively. Then we have the following commutative diagrams

$$\begin{array}{ccccc}(D,M_D) & \overleftarrow{\leftarrow} & (D(1),M_{D(1)}) & \overleftarrow{\overleftarrow{\leftarrow}} & (D(2),M_{D(2)})\\ \beta\downarrow & & \beta(1)\downarrow & & \beta(2)\downarrow\\ (T_n,L_n) & \overleftarrow{\leftarrow} & (T(1)_n,L(1)_n) & \overleftarrow{\overleftarrow{\leftarrow}} & (T(2)_n,L(2)_n),\end{array}\tag{5.3.1}$$

$$\begin{array}{ccc}(D,M_D) & \longrightarrow & (D(1),M_{D(1)})\\ \beta\downarrow & & \beta(1)\downarrow\\ (T_n,L_n) & \longrightarrow & (T(1)_n,L(1)_n),\end{array}\tag{5.3.2}$$

where the horizontal lines in the first diagram are induced by the projections and the horizontal lines in the second diagram is induced by the diagonal map.

Now let $\mathcal{E}$ be an object in the category $I_{\text{conv}}((X,M)/(S,N))$ and let $(E, \{\epsilon_n\})$ be the object corresponding to $\mathcal{E}$ by the equivalence of categories

$$I_{\text{conv}}((X,M)/(S,N)) \simeq \text{Str}'((X,M) \hookrightarrow (Y',J')/(S,N)).$$

Then, by the commutative diagrams (5.3.1) and (5.3.2), the pair $(\beta^*E, \beta(1)^*\epsilon_n)$ defines an object in $\text{HPDI}((X,M) \hookrightarrow (Y,J))$. We define the functor $\Psi_{(Y,J),\mathcal{C}}$ by $\mathcal{E} \mapsto (\beta^*E, \beta(1)^*\epsilon_n)$. It is easy to see that the above definition is independent of the choice of m.

Step 2. Let us consider the following situation: Let $S := \text{Spf}\, W$, let (X,M) be a fine log scheme over k and let $(X,M) \xrightarrow{f} (S_1,N) \hookrightarrow (S,N)$ be as in Step 1. Let $\iota : (X,M) \hookrightarrow (Y,J)$ be a closed immersion into a fine log formal W-scheme (Y,J) over (S,N) such that (Y,J) is formally log smooth and integral over (S,N). Denote the structure morphism $(Y,J) \longrightarrow (S,N)$ by g. Assume moreover that we are given a commutative diagram

$$\mathcal{D} := \begin{bmatrix} X & \leftarrow & X^{(0)} & \leftleftarrows & X^{(1)} & \overset{\leftarrow}{\leftleftarrows} & X^{(2)} \\ \iota\downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y & \leftarrow & Y^{(0)} & \leftleftarrows & Y^{(1)} & \overset{\leftarrow}{\leftleftarrows} & Y^{(2)} \\ g\downarrow & & \downarrow & & \downarrow & & \downarrow \\ S & \leftarrow & S^{(0)} & \leftleftarrows & S^{(1)} & \overset{\leftarrow}{\leftleftarrows} & S^{(2)} \end{bmatrix}$$

satisfying the following conditions:

(1) The horizontal arrows are etale hypercoverings.
(2) The morphisms $X^{(i)} \longrightarrow Y^{(i)}$ are closed immersions.
(3) The diagram

$$\mathcal{D}' := \begin{bmatrix} (X^{(0)},M) & \leftleftarrows & (X^{(1)},M) & \overset{\leftarrow}{\leftleftarrows} & (X^{(2)},M) \\ \downarrow & & \downarrow & & \downarrow \\ (Y^{(0)},J) & \leftleftarrows & (Y^{(1)},J) & \overset{\leftarrow}{\leftleftarrows} & (Y^{(2)},J) \\ \downarrow & & \downarrow & & \downarrow \\ (S^{(0)},N) & \leftleftarrows & (S^{(1)},N) & \overset{\leftarrow}{\leftleftarrows} & (S^{(2)},N) \end{bmatrix},$$

which is naturally induced by $\mathcal{D}$, admits a chart $\mathcal{C}$ such that, if we denote the restiction of $\mathcal{C}$ to $(X^{(i)}, M) \longrightarrow (Y^{(i)}, J)$ $(i = 0, 1, 2)$ by $Q_i \to R_i$, then $Q_i^{\mathrm{gp}} \to R_i^{\mathrm{gp}}$ is surjective.

Let $p_i : Y^{(1)} \longrightarrow Y^{(0)}$ $(i = 1, 2)$ and $p_{ij} : Y^{(2)} \longrightarrow Y^{(1)}$ $(1 \leq i < j \leq 3)$ are projections, and let Δ_{HPDI} be the category of pairs $((E, \epsilon), \varphi)$, where (E, ϵ) is an object in $\mathrm{HPDI}((X^{(0)}, M) \hookrightarrow (Y^{(0)}, J))$ and φ is an isomorphism $p_2^*(E, \epsilon) \xrightarrow{\sim} p_1^*(E, \epsilon)$ in $\mathrm{HPDI}((X^{(1)}, M) \hookrightarrow (Y^{(1)}, J))$ satisfying $p_{12}^*(\varphi) \circ p_{23}^*(\varphi) = p_{13}^*(\varphi)$ in $\mathrm{HPDI}((X^{(2)}, M) \hookrightarrow (Y^{(2)}, J))$. Under this assumption, we define a functor

$$\Psi_{(Y,J),\mathcal{D},\mathcal{C}} : I_{\mathrm{conv}}((X, M)/(S, N)) \longrightarrow \Delta_{\mathrm{HPDI}}$$

and a fully-faithful functor

$$\Lambda_{(Y,J),\mathcal{D}} : I_{\mathrm{crys}}((X, M)/(S, N)) \longrightarrow \Delta_{\mathrm{HPDI}}.$$

Let Δ_{conv} (resp. Δ_{crys}) be the category of pairs $(\mathcal{E}, \varphi)$, where $\mathcal{E}$ is an object in $I_{\mathrm{conv}}((X^{(0)}, M)/(S^{(0)}, N))$ (resp. $I_{\mathrm{crys}}((X^{(0)}, M)/(S^{(0)}, N))$) and φ is an isomorphism $p_2^*\mathcal{E} \xrightarrow{\sim} p_1^*\mathcal{E}$ in $I_{\mathrm{conv}}((X^{(1)}, M)/(S^{(1)}, N))$ (resp. $I_{\mathrm{crys}}((X^{(1)}, M)/(S^{(1)}, N))$) satisfying $p_{12}^*(\varphi) \circ p_{23}^*(\varphi) = p_{13}^*(\varphi)$ in $I_{\mathrm{conv}}((X^{(2)}, M)/(S^{(2)}, N))$ (resp. $I_{\mathrm{crys}}((X^{(2)}, M)/(S^{(2)}, N))$). Then, by Remark 5.1.7 and Proposition 4.3.6, we have the equivalences of categories

$$e_{\mathrm{conv}} : I_{\mathrm{conv}}((X, M)/(S, N)) \xrightarrow{\sim} \Delta_{\mathrm{conv}},$$

$$e_{\mathrm{crys}} : I_{\mathrm{crys}}((X, M)/(S, N)) \xrightarrow{\sim} \Delta_{\mathrm{crys}}.$$

Let $\mathcal{C}^{(i)}$ $(i = 0, 1, 2)$ be the restriction of the chart $\mathcal{C}$ to the diagram

$$(X^{(i)}, M) \hookrightarrow (Y^{(i)}, J) \longrightarrow (S^{(i)}, N).$$

Then we have the functors

$$\Psi_{(Y^{(i)},J),\mathcal{C}^{(i)}} : I_{\mathrm{conv}}((X^{(i)}, M)/(S^{(i)}, N)) \longrightarrow \mathrm{HPDI}((X^{(i)}, M) \hookrightarrow (Y^{(i)}, J)) \qquad (i = 0, 1, 2).$$

One can check easily that the functors $\Psi_{(Y^{(i)},J),\mathcal{C}^{(i)}}$ $(i = 0, 1, 2)$ induce the functor

$$\Psi' : \Delta_{\mathrm{conv}} \longrightarrow \Delta_{\mathrm{HPDI}}.$$

The desired functor $\Psi_{(Y,J),\mathcal{D},\mathcal{C}}$ is defined as the composite $\Psi' \circ e_{\mathrm{conv}}$.

On the other hand, the fully-faithful functors

(5.3.3)
$$I_{\mathrm{crys}}((X^{(i)},M)/(S^{(i)},N)) \longrightarrow \mathrm{HPDI}((X^{(i)},M) \hookrightarrow (Y^{(i)},J)) \quad (i=0,1,2)$$

in Section 4.3 (4.3.3) induce the functor

$$\Lambda' : \Delta_{\mathrm{crys}} \longrightarrow \Delta_{\mathrm{HPDI}},$$

which is also fully-faithful. The desired fully-faithful functor $\Lambda_{(Y,J),\mathcal{D}}$ is defined as the composite $\Lambda' \circ e_{\mathrm{crys}}$.

Now let us assume moreover the following conditions:

(4) ι is an exact closed immersion and $(Y,J) \times_{(S,N)} (S_1,N) = (X,M)$ holds.

(5) The squares in the diagram

$$\begin{array}{ccccccc} X & \leftarrow & X^{(0)} & \overleftarrow{\leftarrow} & X^{(1)} & \overleftarrow{\overleftarrow{\leftarrow}} & X^{(2)} \\ \iota\downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y & \leftarrow & Y^{(0)} & \overleftarrow{\leftarrow} & Y^{(1)} & \overleftarrow{\overleftarrow{\leftarrow}} & Y^{(2)} \end{array}$$

are cartesian.

Then, by Proposition 4.3.2, the functors in (5.3.3) are eqivalences. Hence so is Λ'. So the functor $\Lambda_{(Y,J),\mathcal{D}}$ gives an equivalence of categories in this case.

Step 3. Let us consider the following situation: Let $S := \mathrm{Spf}\, W$, and let (X,M) be a fine log scheme over S. Let $(X,M) \xrightarrow{f} (S_1,N) \hookrightarrow (S,N)$ be as in Step 1 and assume that X is affine. Under this assumption, we define a functor

$$\Phi_{(X,M)} : I_{\mathrm{conv}}((X,M)/(S,N)) \longrightarrow I_{\mathrm{crys}}((X,M)/(S,N)).$$

Since X is affine, there exists an exact closed immersion $\iota : (X,M) \hookrightarrow (Y,J)$ into a fine log formal W-scheme (Y,J) over (S,N) such that (Y,J) is formally log smooth integral over (S,N) and that $(Y,J) \times_{(S,N)} (S_1,N) = (X,M)$ holds. Then one can construct a diagram $\mathcal{D}$ (in Step 2) satisfying the conditions (1) to (5) in Step 2. Then we define the functor $\Phi_{(X,M)}$ by the composite $\Lambda^{-1}_{(Y,J),\mathcal{D}} \circ \Psi_{(Y,J),\mathcal{D},\mathcal{C}}$.

We should prove that the above definition of the functor $\Phi_{(X,M)}$ is independent of the choice of ι, $\mathcal{D}$ and $\mathcal{C}$. For $i = 1, 2$, let $\iota_i : (X, M) \hookrightarrow (Y_i, J_i)$,

$$\mathcal{D}_i := \begin{bmatrix} X & \leftarrow & X_i^{(0)} & \leftleftarrows & X_i^{(1)} & \overset{\leftarrow}{\leftleftarrows} & X_i^{(2)} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y_i & \leftarrow & Y_i^{(0)} & \leftleftarrows & Y_i^{(1)} & \overset{\leftarrow}{\leftleftarrows} & Y_i^{(2)} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S & \leftarrow & S_i^{(0)} & \leftleftarrows & S_i^{(1)} & \overset{\leftarrow}{\leftleftarrows} & S_i^{(2)} \end{bmatrix}$$

and $\mathcal{C}_i$ be as in $\iota, \mathcal{D}, \mathcal{C}$ in Step 2 which are subject to the conditions (1) to (5). Note that we have the canonical equivalences of sites $X^{(j)}_{i,\mathrm{et}} \simeq Y^{(j)}_{i,\mathrm{et}}$. Noting this fact, one can construct the diagrams (for $i = 1, 2$)

$$\underline{\mathcal{D}}_i := \begin{bmatrix} X & \leftarrow & \underline{X}^{(0)} & \leftleftarrows & \underline{X}^{(1)} & \overset{\leftarrow}{\leftleftarrows} & \underline{X}^{(2)} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y_i & \leftarrow & \underline{Y}_i^{(0)} & \leftleftarrows & \underline{Y}_i^{(1)} & \overset{\leftarrow}{\leftleftarrows} & \underline{Y}_i^{(2)} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S & \leftarrow & \underline{S}^{(0)} & \leftleftarrows & \underline{S}^{(1)} & \overset{\leftarrow}{\leftleftarrows} & \underline{S}^{(2)} \end{bmatrix}$$

and the morphisms of diagrams

$$\gamma_i : \underline{\mathcal{D}}_i \longrightarrow \mathcal{D}_i \quad (i = 1, 2)$$

inducing the identity on $X \longrightarrow Y_i \longrightarrow S$ satisfying the following conditions:

(1) The diagram

$$\begin{matrix} X & \leftarrow & \underline{X}^{(0)} & \leftleftarrows & \underline{X}^{(1)} & \overset{\leftarrow}{\leftleftarrows} & \underline{X}^{(2)} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S & \leftarrow & \underline{S}^{(0)} & \leftleftarrows & \underline{S}^{(1)} & \overset{\leftarrow}{\leftleftarrows} & \underline{S}^{(2)} \end{matrix}$$

induced by $\underline{\mathcal{D}}_1$ and that induced by $\underline{\mathcal{D}}_2$ are the same. (We denote it by $\underline{\mathcal{D}}_0$.)

(2) Let

$$\underline{\mathcal{D}}'_i := \left[\begin{array}{ccccc} (\underline{X}^{(0)}, M) & \leftleftarrows & (\underline{X}^{(1)}, M) & \substack{\leftarrow \\ \leftarrow \\ \leftarrow} & (\underline{X}^{(2)}, M) \\ \downarrow & & \downarrow & & \downarrow \\ (\underline{Y}_i^{(0)}, J_i) & \leftleftarrows & (\underline{Y}_i^{(1)}, J_i) & \substack{\leftarrow \\ \leftarrow \\ \leftarrow} & (\underline{Y}_i^{(2)}, J_i) \\ \downarrow & & \downarrow & & \downarrow \\ (\underline{S}^{(0)}, N) & \leftleftarrows & (\underline{S}^{(1)}, N) & \substack{\leftarrow \\ \leftarrow \\ \leftarrow} & (\underline{S}^{(2)}, N) \end{array}\right]$$

be the diagram induced by $\underline{\mathcal{D}}_i$ and let $\gamma'_i : \underline{\mathcal{D}}'_i \longrightarrow \mathcal{D}'_i$ be the morphism of the diagrams induced by γ_i. Then we have a chart $\underline{\mathcal{C}}_i$ of the diagram $\underline{\mathcal{D}}'_i \xrightarrow{\gamma'_i} \mathcal{D}'_i$ extending the chart $\mathcal{C}_i$ of $\mathcal{D}'_i$.

(3) Let

$$\underline{\mathcal{D}}'_0 := \left[\begin{array}{ccccc} (\underline{X}^{(0)}, M) & \leftleftarrows & (\underline{X}^{(1)}, M) & \substack{\leftarrow \\ \leftarrow \\ \leftarrow} & (\underline{X}^{(2)}, M) \\ \downarrow & & \downarrow & & \downarrow \\ (\underline{S}^{(0)}, N) & \leftleftarrows & (\underline{S}^{(1)}, N) & \substack{\leftarrow \\ \leftarrow \\ \leftarrow} & (\underline{S}^{(2)}, N) \end{array}\right]$$

be the diagram induced by $\underline{\mathcal{D}}_0$. Then the restriction of the charts $\underline{\mathcal{C}}_1, \underline{\mathcal{C}}_2$ to $\underline{\mathcal{D}}'_0$ coincide.

(4) The formal schemes $\underline{Y}_i^{(j)}$ $(i = 1, 2, j = 0, 1, 2)$ are affine.

Let $\overline{\mathcal{D}}$ be the diagram

$$\begin{array}{ccccccc} X & \leftarrow & \underline{X}^{(0)} & \leftleftarrows & \underline{X}^{(1)} & \substack{\leftarrow \\ \leftarrow \\ \leftarrow} & \underline{X}^{(2)} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y_1 \times_S Y_2 & \leftarrow & \underline{Y}_1^{(0)} \times_{\underline{S}^{(0)}} \underline{Y}_2^{(0)} & \leftleftarrows & \underline{Y}_1^{(1)} \times_{\underline{S}^{(1)}} \underline{Y}_2^{(1)} & \substack{\leftarrow \\ \leftarrow \\ \leftarrow} & \underline{Y}_1^{(2)} \times_{\underline{S}^{(2)}} \underline{Y}_2^{(2)} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S & \leftarrow & \underline{S}^{(0)} & \leftleftarrows & \underline{S}^{(1)} & \substack{\leftarrow \\ \leftarrow \\ \leftarrow} & \underline{S}^{(2)} \end{array}$$

induced by $\underline{\mathcal{D}}_1$ and $\underline{\mathcal{D}}_2$ (by taking 'fiber product'), and let

$$\overline{\mathcal{D}}' := \left[\begin{array}{ccccc} (\underline{X}^{(0)}, M) & \leftleftarrows & (\underline{X}^{(1)}, M) & \substack{\leftarrow \\ \leftarrow \\ \leftarrow} & (\underline{X}^{(2)}, M) \\ \downarrow & & \downarrow & & \downarrow \\ (\underline{Y}_1^{(0)}, J_1) \times_{(\underline{S}^{(0)}, N)} (\underline{Y}_2^{(0)}, J_2) & \leftleftarrows & (\underline{Y}_1^{(1)}, J_1) \times_{(\underline{S}^{(1)}, N)} (\underline{Y}_2^{(1)}, J_2) & \substack{\leftarrow \\ \leftarrow \\ \leftarrow} & (\underline{Y}_1^{(2)}, J_1) \times_{(\underline{S}^{(2)}, N)} (\underline{Y}_2^{(2)}, J_2) \\ \downarrow & & \downarrow & & \downarrow \\ (\underline{S}^{(0)}, N) & \leftleftarrows & (\underline{S}^{(1)}, N) & \substack{\leftarrow \\ \leftarrow \\ \leftarrow} & (\underline{S}^{(2)}, N) \end{array}\right]$$

be the diagram induced by $\overline{\mathcal{D}}$. Then we have the natural morphisms of diagrams $\delta_i : \overline{\mathcal{D}} \longrightarrow \underline{\mathcal{D}}_i$, $\delta'_i : \overline{\mathcal{D}}' \longrightarrow \underline{\mathcal{D}}'_i\,(i = 1, 2)$. One can see that the diagram

$$\mathcal{D}'_1 \xleftarrow{\gamma_1} \underline{\mathcal{D}}'_1 \xleftarrow{\delta_1} \overline{\mathcal{D}}' \xrightarrow{\delta_2} \underline{\mathcal{D}}'_2 \xrightarrow{\gamma_2} \mathcal{D}'_2$$

admits a chart $\overline{\mathcal{C}}$ extending $\underline{\mathcal{C}}_1$ and $\underline{\mathcal{C}}_2$. Moreover, the diagram $\overline{\mathcal{D}}$ satisfies the conditions (1) to (3) in Step 2 (for $(Y, J) = (Y_1, J_1) \times_{(S,N)} (Y_2, J_2)$.) Let $\Delta_{\mathrm{HPDI},i}$ (resp. $\overline{\Delta}_{\mathrm{HPDI}}$) be the category Δ_{HPDI} in Step 2 for the diagram $\mathcal{D}_i$ (resp. $\overline{\mathcal{D}}$). Then the morphisms of diagrams

$$\mathcal{D}_1 \xleftarrow{\gamma_1\circ\delta_1} \overline{\mathcal{D}} \xrightarrow{\gamma_2\circ\delta_2} \mathcal{D}_2$$

and the chart $\overline{\mathcal{C}}$ induce the following commutative diagram of functors:

$$\begin{array}{ccccc}
I_{\mathrm{conv}}((X,M)/(S,N)) & \xrightarrow{\Psi_{(Y_1,J_1),\mathcal{D}_1,\mathcal{C}_1}} & \Delta_{\mathrm{HPDI},1} & \xleftarrow{\Lambda_{(Y_1,J_1),\mathcal{D}_1}} & I_{\mathrm{crys}}((X,M)/(S,N)) \\
\| & & \downarrow{\scriptstyle (\gamma_1\circ\delta_1)^*} & & \| \\
I_{\mathrm{conv}}((X,M)/(S,N)) & \xrightarrow{\Psi_{(Y_1,J_1)\times(Y_2,J_2),\overline{\mathcal{D}},\overline{\mathcal{C}}|_{\overline{\mathcal{D}}'}}} & \overline{\Delta}_{\mathrm{HPDI}} & \xleftarrow{\Lambda_{(Y_1,J_1)\times(Y_2,J_2),\overline{\mathcal{D}}}} & I_{\mathrm{crys}}((X,M)/(S,N)) \\
\| & & \uparrow{\scriptstyle (\gamma_2\circ\delta_2)^*} & & \| \\
I_{\mathrm{conv}}((X,M)/(S,N)) & \xrightarrow{\Psi_{(Y_2,J_2),\mathcal{D}_2,\mathcal{C}_2}} & \Delta_{\mathrm{HPDI},2} & \xleftarrow{\Lambda_{(Y_2,J_2),\mathcal{D}_2}} & I_{\mathrm{crys}}((X,M)/(S,N)),
\end{array}$$

where $\Lambda_{(Y_i,J_i),\mathcal{D}_i}$ $(i = 1, 2)$ are equivalence of categories and $\Lambda_{(Y_1,J_1)\times(Y_2,J_2),\overline{\mathcal{D}}}$ is fully-faithful. From the above diagram, we get the equivalence of functors

$$\Lambda^{-1}_{(Y_1,J_1),\mathcal{D}_1} \circ \Psi_{(Y_1,J_1),\mathcal{D}_1,\mathcal{C}_1} \simeq \Lambda^{-1}_{(Y_2,J_2),\mathcal{D}_2} \circ \Psi_{(Y_2,J_2),\mathcal{D}_2,\mathcal{C}_2}.$$

So the definition of $\Phi_{(X,M)}$ is independent of the choice of $\iota, \mathcal{D}$ and $\mathcal{C}$.

One can check the functoriality of the functor $\Phi_{(X,M)}$ with respect to (X, M) in a similar way.

Step 4. Let the notations be as in the statement of the theorem. We define the functor

$$\Phi : I_{\mathrm{conv}}((X,M)/(\mathrm{Spf}\, W, N)) \longrightarrow I_{\mathrm{crys}}((X,M)/(\mathrm{Spf}\, W, N)).$$

Let

$$X \leftarrow X^{(0)} \;\substack{\leftarrow\\ \leftarrow}\; X^{(1)} \;\substack{\leftarrow\\ \leftarrow\\ \leftarrow}\; X^{(2)} \tag{5.3.4}$$

be a Zariski open hypercovering such that each $X^{(i)}$ is a finite disjoint union of affine schemes. Let $p_i : X^{(1)} \longrightarrow X^{(0)}$ $(i = 1, 2)$ and $p_{ij} : X^{(2)} \longrightarrow X^{(1)}$ $(1 \leq i < j \leq 3)$ be projections and let Δ_{conv} (resp. Δ_{crys}) be the category of pairs $(\mathcal{E}, \varphi)$, where $\mathcal{E}$ is an object in $I_{\mathrm{conv}}((X^{(0)}, M)/(\mathrm{Spf}\, W, N))$ (resp. $I_{\mathrm{crys}}((X^{(0)}, M)/(\mathrm{Spf}\, W, N)))$ and φ is an isomorphism $p_2^*\mathcal{E} \xrightarrow{\sim} p_1^*\mathcal{E}$ in $I_{\mathrm{conv}}((X^{(1)}, M)/(\mathrm{Spf}\, W, N))$ (resp. $I_{\mathrm{crys}}((X^{(1)}, M)/(\mathrm{Spf}\, W, N)))$ satisfying $p_{12}^*(\varphi) \circ p_{23}^*(\varphi) = p_{13}^*(\varphi)$ in $I_{\mathrm{conv}}((X^{(2)}, M)/(\mathrm{Spf}\, W, N))$ (resp. $I_{\mathrm{crys}}((X^{(2)}, M)/(\mathrm{Spf}\, W, N)))$. Then, since both $I_{\mathrm{conv}}((X, M)/(\mathrm{Spf}\, W, N))$, $I_{\mathrm{crys}}((X, M)/(\mathrm{Spf}\, W, N))$ satisfy the descent property for Zariski covering by finite open sets, we have the equivalences of categories

$$e_{\mathrm{conv}} : I_{\mathrm{conv}}((X, M)/(\mathrm{Spf}\, W, N)) \xrightarrow{\sim} \Delta_{\mathrm{conv}},$$

$$e_{\mathrm{crys}} : I_{\mathrm{crys}}((X, M)/(\mathrm{Spf}\, W, N)) \xrightarrow{\sim} \Delta_{\mathrm{crys}}.$$

Let

$$\Phi_{(X^{(i)}, M)} : I_{\mathrm{conv}}((X^{(i)}, M)/(\mathrm{Spf}\, W, N)) \longrightarrow I_{\mathrm{crys}}((X^{(i)}, M)/(\mathrm{Spf}\, W, N)) \qquad (i = 0, 1, 2)$$

be the functor constructed in Step 3. Since the functors $\Phi_{(X^{(i)}, M)}$ $(i = 0, 1, 2)$ are funtorial with respect to the morphisms in the diagram (5.3.4), they induce the functor

$$\Phi_\Delta : \Delta_{\mathrm{conv}} \longrightarrow \Delta_{\mathrm{crys}}.$$

Then, the desired functor Φ is defined by $\Phi := e_{\mathrm{crys}}^{-1} \circ \Phi_\Delta \circ e_{\mathrm{conv}}$. It can be chacked easily that the functor Φ is independent of the choice of the hypercovering (5.3.4).

Step 5. Let Φ be the functor defined in Step 4. One can check easily that Φ is exact. Hence Φ induces the functor

$$\overline{\Phi} : \mathcal{N}I_{\mathrm{conv}}((X/W)^{\log}) \longrightarrow \mathcal{N}I_{\mathrm{crys}}((X/W)^{\log}).$$

Let us prove the categorical equivalence of $\overline{\Phi}$. Since the categories $I_{\mathrm{conv}}((X/W)^{\log})$, $I_{\mathrm{crys}}((X/W)^{\log})$ satisfy the descent property for finite open coverings, one can check that it suffices to prove the categorical equivalence Zariski locally. (One can prove this by the same argument as that in Lemma 5.2.2.) Hence we may assume that X is affine. Let $(X, M) \hookrightarrow (Y, J)$ be an exact closed immersion into a fine log formal W-scheme (Y, J) which is formally log smooth integral over $(\mathrm{Spf}\, W, N)$ satisfying $(Y, J) \times_{(\mathrm{Spf}\, W, N)} (\mathrm{Spec}\, k, N) = (X, M)$. Recall that we have the canonical equivalence of categories

$$b_1 : \mathcal{N}I_{\mathrm{conv}}((X/W)^{\log}) \xrightarrow{\sim} \mathcal{N}I_{\mathrm{inf}}((X/W)^{\log}) \simeq \mathcal{N}\hat{C}((Y, J)/(\mathrm{Spf}\, W, N)),$$

where the first functor is the functor $\bar{\iota}$ in Section 5.3 and the second equivalence follow from Theorem 3.2.15. On the other hand, we have an exact fully-faithful functor

$$b_2' : I_{\mathrm{crys}}((X/W)^{\log}) \longrightarrow \hat{C}((Y, J)/(\mathrm{Spf}\, W, N)).$$

which is defined in Proposition 4.3.7. Since b_2' is exact, it induces the functor

$$b_2 : \mathcal{N}I_{\mathrm{crys}}((X/W)^{\log}) \longrightarrow \mathcal{N}\hat{C}((Y, J)/(\mathrm{Spf}\, W, N)),$$

which is also fully-faithful. By constructions of these functors, one can check the commutativity of the following diagram of functors:

$$\begin{array}{ccc} \mathcal{N}I_{\mathrm{conv}}((X/W)^{\log}) & \xrightarrow{b_1} & \mathcal{N}\hat{C}((Y, J)/(\mathrm{Spf}\, W, N)) \\ {\scriptstyle \overline{\Phi}}\downarrow & & \| \\ \mathcal{N}I_{\mathrm{crys}}((X/W)^{\log}) & \xrightarrow{b_2} & \mathcal{N}\hat{C}((Y, J)/(\mathrm{Spf}\, W, N)). \end{array}$$

Since b_1 is equivalent and b_2 is fully-faithful, the functor $\overline{\Phi}$ gives an equivalence of categories. So the theorem is proved. □

Using the above theorem and the results in Sections 5.1 and 5.2, we can prove the Berthelot-Ogus theorem for fundamental groups.

THEOREM 5.3.2 (Berthelot-Ogus theorem for π_1). *Assume we are given the following commutative diagram of fine log schemes*

$$\text{(BO)} \qquad \begin{array}{ccccc} (X_k, M) & \hookrightarrow & (X, M) & \hookleftarrow & (X_K, M) \\ \downarrow & & f\downarrow & & \downarrow \\ (\mathrm{Spec}\, k, N) & \hookrightarrow & (\mathrm{Spec}\, V, N) & \hookleftarrow & (\mathrm{Spec}\, K, N) \\ & \searrow & \downarrow & & \\ & & (\mathrm{Spec}\, W, N), & & \end{array}$$

where the two squares are Cartesian, f is proper log smooth integral and X_k is reduced. Assume moreover that $H^0_{\mathrm{dR}}((X,M)/(\mathrm{Spf}\, V,N)) = V$ holds, and that we are given a V-valued point x of $X_{f\text{-triv}}$. Denote the special fiber (resp. generic fiber) of x by x_k (resp. x_K). Then there exists a canonical isomorphism of pro-algebraic groups

$$\pi_1^{\mathrm{crys}}((X_k,M)/(\mathrm{Spf}\, W,N),x_k) \times_{K_0} K \overset{\simeq}{=} \pi_1^{\mathrm{dR}}((X_K,M)/(\mathrm{Spec}\, K,N),x_K).$$

Proof. First, by Corollary 5.2.16, we have $H^0(((X_k,M)/(\mathrm{Spf}\, V,N))_{\mathrm{conv}}, \mathcal{K}_{X/V}) = K$ and there exists a canonical isomorphism

$$(5.3.5) \quad \pi_1^{\mathrm{conv}}((X_k,M)/(\mathrm{Spf}\, V,N),x_k) \overset{\simeq}{=} \pi_1^{\mathrm{dR}}((X_K,M)/(\mathrm{Spec}\, K,N),x_K).$$

Since $H^0(((X_k,M)/(\mathrm{Spf}\, V,N))_{\mathrm{conv}}, \mathcal{K}_{X/V}) = K$ holds, X_k is connected. Hence, by Propositions 4.1.7 and 5.1.11, we can define the crystalline fundamental group of (X_k,M) over $(\mathrm{Spf}\, W,N)$ and the convergent fundamental group of (X_k,M) over $(\mathrm{Spf}\, W,N)$ or $(\mathrm{Spf}\, V,N)$. Then, by Corollary 5.1.14, we have the isomorphism

(5.3.6)
$$\pi_1^{\mathrm{conv}}((X_k,M)/(\mathrm{Spf}\, W,N),x_k) \times_{K_0} K \overset{\simeq}{=} \pi_1^{\mathrm{conv}}((X_k,M)/(\mathrm{Spf}\, V,N),x_k).$$

Finally, by Theorem 5.3.1, we have the isomorphism

$$(5.3.7) \quad \pi_1^{\mathrm{crys}}((X_k,M)/(\mathrm{Spf}\, W,N),x_k) \overset{\simeq}{=} \pi_1^{\mathrm{conv}}((X_k,M)/(\mathrm{Spf}\, W,N),x_k).$$

Combining the isomorphisms (5.3.5), (5.3.6) and (5.3.7), we obtain the assertion. □

Finally we give two typical examples of the above theorem.

Example 5.3.3. Let X be a connected scheme which is proper smooth over $\mathrm{Spec}\, V$ and let D be a relative normal crossing divisor in X over $\mathrm{Spec}\, V$. Let M be the log structure on X associated to D, and let N be the trivial log structure on $\mathrm{Spec}\, W$. Denote the special fiber and the generic fiber of X by X_k, X_K, respectively. Let x be a V-valued point of $X - D$ and let x_k, x_K the special fiber and the generic fiber of x, respectively. Then we have the diagram (BO) which satisfies the conditions in Theorem 5.3.2. Hence we have the isomorphism

$$\pi_1^{\mathrm{crys}}((X_k, M)/\mathrm{Spf}\, W, x_k) \times_{K_0} K \cong \pi_1^{\mathrm{dR}}((X_K, M)/\mathrm{Spec}\, K, x_K),$$

and the right hand side is isomorphic to $\pi_1^{\mathrm{dR}}(X_K - D_K/\mathrm{Spec}\, K, x_K)$ by Proposition 3.1.8.

Example 5.3.4. Let X be a regular scheme and let $f : X \longrightarrow \mathrm{Spec}\, V$ be a proper flat morphism. Assume moreover that the special fiber X_k of X is a reduced normal crossing divisor. Let M be the log structure on X associated to X_k and let N be the log structure on $\mathrm{Spec}\, W$ associated to the closed point. Then, by Example 2.4.6, the morphism $f : (X, M) \longrightarrow (\mathrm{Spec}\, V, N)$ is log smooth and integral. Denote the special fiber and the generic fiber of X by X_k, X_K, respectively. Let x be a V-valued point of $X_{f\text{-triv}}$ and let x_k, x_K the special fiber and the generic fiber of x, respectively. Then we have the diagram (BO) which satisfies the conditions in Theorem 5.3.2. Hence we have the isomorphism

$$\pi_1^{\mathrm{crys}}((X_k, M)/(\mathrm{Spf}\, W, N), x_k) \times_{K_0} K \cong \pi_1^{\mathrm{dR}}(X_K/\mathrm{Spec}\, K, x_K).$$

(Note that M, N are trivial on $X_K, \mathrm{Spec}\, K$, respectively.)

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