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A Limit Formula for a Class of Gibbs Measures with Long Range Pair Interactions

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Abstract. Let $X_i,\ i = 1, 2, \dots$ be real-valued *i.i.d.* variables with a compactly supported density. Under certain assumptions on V, we give an asymptotic evaluation of $E[\exp(-\frac{1}{2}\sum_{i,j=1}^{n} V(X_i, X_j))]$ up to the factor $(1 + o(1))$. As an application of this result, we prove a limit formula for a class of Gibbs measures with long range pair interactions.

1. Introduction

Let P_n be the probability measure on $\mathbf{R}^n$ given by

$$P_n(d\underline{t}) = \frac{1}{Z_n} \cdot \exp\left(-\frac{1}{2} \left\{ \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \log |t_i - t_j|^{-1} + n \sum_{i=1}^{n} t_i^2 \right\} \right) d\underline{t} \tag{1.1}$$

Here $d\underline{t} = dt_1 dt_2 \cdots dt_n$ and Z_n is the normalizing constant. In [J], K. Johansson showed the following asymptotic formula:

$$\lim_{n\to\infty} \frac{1}{n} \log \int \exp\left(\sum_{i=1}^{n} g(t_i) \right) P_n(d\underline{t}) = \int g(t)\mu_0(dt) \tag{1.2}$$

for every good test function g. Here μ_0 is the semi-circle distribution, which minimizes the functional

$$J[\mu] = \int \int V(s,t)\mu(ds)\mu(dt)$$

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of all the probability measures on $\mathbf{R}$ where

$$V(s,t) = \log |s-t|^{-1} + \frac{1}{2}(s^2+t^2).$$

Based on this result, in [J], not only the logarithmic asymptotics but the limiting value of

$$\int_{\mathbf{R}^n} \exp\left(\sum_{i=1}^{n} g(t_i)\right) P_n(d\underline{t}), \tag{1.3}$$

up to the factor $(1+o(1))$ is also derived. Altohough the method employed in the article is based on the theory of the orthogonal functions and is quite analytic, the results seems to have the strong connection with the large deviation principle for P_n established by G. Ben Arous and A. Guionnet [BG].

In this context, the objective of the present article is to evaluate the integral (1.3) up to the factor $(1+o(1))$ for

$$P_n(d\underline{t}) = \frac{1}{Z_n} \cdot \exp\left(-\frac{1}{2}\sum_{i,j=1}^{n} V(t_i,t_j)\right) dt_1 dt_2 \cdots dt_n,$$

in the case V has a good regularity, applying the ideas and techniques in the probability theory. Since first initiated by [KT] and [B], many results have been obtained on the precise estimate of Laplace-type integrals based on the principle of large deviation and have been applied to study some limiting behavior of the Gibbs measures with the mean field interactions. However, since the interaction in P_n is not of ordinary large deviation order, the method using the large deviation principle is not applicable here.

Now let us state the precise setting. Let $I = [0,1] \subset \mathbf{R}$ and $\mathcal{M}_1(I)$ be the space of the probability measures on I. Let V be a real-valued functional on $I \times I$, for which we assume the following conditions:

(V.1) $$V(s,t) = V(t,s) \quad \text{for all} \quad (s,t) \in I \times I.$$

(V.2) There is an $\alpha \in (0,1]$ and a $C > 0$ such that

$$|V(s,t) - V(s',t)| \leq C|s-s'|^{\alpha} \quad \text{for all} \quad (s,s',t) \in I \times I \times I.$$

(V.3) There is only one $\mu_0 \in \mathcal{M}_1(I)$ which minimizes the real-valued functional

$$J[\mu] = \int_I \int_I V(s,t)\mu(ds)\mu(dt)$$

on $\mathcal{M}_1(I)$, i.e., there is only one $\mu_0 \in \mathcal{M}_1(I)$ such that

$$J[\mu_0] = \inf\{J[\mu], \quad \mu \in \mathcal{M}_1(I)\}.$$

(V.4) $\mu_0(dt)$ is mutually absolutely continuous with respect to dt and the Radon-Nikodym derivative satisfies that

$$\frac{\mu_0(dt)}{dt} \geq c_{\mu_0} \text{ on } I \text{ for some } c_{\mu_0} > 0 \text{ and } \quad \int_I \log \frac{\mu_0(dt)}{dt}\mu_0(dt) < \infty.$$

For this μ_0 let $L_0^2(\mu_0) = \left\{ f \in L^2(\mu_0); \int_I f(t)\mu_0(dt) = 0 \right\}$ with the norm $\|\cdot\|_{L_0^2(\mu_0)}$, and let $V_0(s,t) = V(s,t) - J[\mu_0]$ on $I \times I$. Then, in view of (V.1), (V.2) and (V.3), we are able to define a non-negative, symmetric and compact operator V_0 on $L_0^2(\mu_0)$ given by

$$\langle u, V_0 v\rangle_{L^2(\mu_0)} = \int_I \int_I V_0(s,t)u(s)v(t)\mu_0(ds)\mu_0(dt) \tag{1.4}$$

for all $u, v \in L_0^2(\mu_0)$. Let $\{\lambda_k\}_{k=1,2\ldots}$ and $\{\phi_k\}_{k=1,2\ldots}$ be its eigenvalues and eigenfunctions.

We further assume the following:

(V.5) V_0 is strictly positive, i.e., $\lambda_k > 0$ for all $k \geq 1$.

(V.6)

$$\sum_{k=1}^{\infty} \lambda_k^p < \infty \quad \text{for some} \quad p < \frac{1}{3} \cdot \frac{\alpha}{\alpha+1}.$$

with the exponent α appear in (V.2) and

(V.7)

$$\sup_{k \geq 1} \lambda_k^{\frac{1-p}{2}} \sup_{t \in I} |\phi_k(t)| < \infty.$$

(V.8)

$$\xi_0(\cdot) = \log \frac{\mu_0(dt)}{dt}(\cdot) - \int_I \log \frac{\mu_0(dt)}{dt}\mu_0(dt) \in H_1.$$

Here, H_1 is defined to be the domain of the operator $V_0^{-\frac{1}{2}}$ on $L_0^2(\mu_0)$, equipped with the norm

$$\|g\|_{H_1} = \|V_0^{-\frac{1}{2}} g\|_{L_0^2(\mu_0)}.$$

For each $n \in N$, define

$$Z_n = \int_I \cdots \int_I \exp\left(-\frac{1}{2}\sum_{i,j=1}^n V(t_i, t_j)\right) dt_1 dt_2 \cdots dt_n \tag{1.5}$$

and define a probability measure on $I^{\otimes n}$ by

$$P_n(d\underline{t}) = \frac{1}{Z_n} \cdot \exp\left(-\frac{1}{2}\sum_{i,j=1}^n V(t_i, t_j)\right) dt_1 dt_2 \cdots dt_n. \tag{1.6}$$

Here, $d\underline{t}$ stands for $dt_1 dt_2 \cdots dt_n$.

Our main theorems are the following:

THEOREM 1. *Under the assmption (V.1)-(V.7), we have*

$$Z_n = \exp\left(-\frac{1}{2}n^2 J[\mu_0] - n\int_I \log\frac{\mu_0(dt)}{dt}\mu_0(dt) + \frac{1}{2}\|V_0^{-\frac{1}{2}}\xi_0\|^2_{L_0^2(\mu_0)}\right) \cdot \{\det(I + nV_0)\}^{-\frac{1}{2}}(1 + o(1))$$

as $n \to \infty$. *Here,* $\det(I + nV_0)$ *is the determinant of the operator* $I + nV_0$ *on* $L_0^2(\mu_0)$.

THEOREM 2. *Under the assmption (V.1)-(V.7), for any* $f \in H_1$,

$$\int_{I^{\otimes n}} \exp\left(\sum_{j=1}^n f(t_j)\right) P_n(d\underline{t}) = \exp\left(n\int_I f(t)\mu_0(dt) + \langle V_0^{-\frac{1}{2}}\xi_0, V_0^{-\frac{1}{2}} f\rangle_{L_0^2(\mu_0)} - \frac{1}{2}\|V_0^{-\frac{1}{2}} f\|^2_{L_0^2(\mu_0)}\right) \cdot (1 + o(1))$$

as $n \to \infty$.

Remark. (i) By (V.6), V_0 is of trace class, and so $\det(I + nV_0)$ exists. Moreover, under the assumption (V.6), $\{\det(I + nV_0)\}^{-\frac{1}{2}}$ converges to 0 slower than $\exp(-c \cdot n^p)$ for any positive constant c.
(ii) The method we adopt to prove the theorems is not applicable in the case I consisits of isolated points, say $I = \{-1, 1\}$. The sample path regularity of the Gaussian process Π, which has mean 0 and V_0 as its covariance, plays the crucial role in the proof.

The organization of this article is as follows. In the next section we intruduce the Gaussian process and give some useful formulas for later use. In section 3, we study the asymptotic behavior of $\log\det(I + nV_0)$. With these preliminaries, in section 4 and 5, we show Theorem(5.6). After this theorem is established, we show Theorem 1 and Theorem 2 rather straightforwardly in section 6 and 7. We sketch an example of V in section 8.

2. Preliminary Results I

Let H_{-1} be the completion of $L^2_0(\mu_0)$ with respect to the norm

$$\|f\|_{H_{-1}} = \|V_0^{\frac{1}{2}} f\|_{L^2_0(\mu_0)},$$

then, by (V.5), H_1 is a dense subspace of $L^2_0(\mu_0)$ and so is $L^2_0(\mu_0)$ in H_{-1}. Note that $V_0^{-1} : H_1 \to H_{-1}$ is an isometric isomorphism. The mapping $i : H_{-1} \to H^*_{1,0} = \{\nu \in H^*_1,\ \langle \nu, 1 \rangle = 0\}$ defined by

$$\langle i(\nu), f \rangle = \langle V_0^{\frac{1}{2}} \nu, V_0^{-\frac{1}{2}} f \rangle_{L^2_0(\mu_0)}$$

is also an isometric isomorphism, and thus we identify $\nu \in H_{-1}$ with $i(\nu) \in H^*_{1,0}$.

For any $w : t \in I \to w_t \in \mathbf{R}$, let $\|\cdot\|_\beta$ be the Hölder norm of w, i.e.,

$$\|w\|_\beta = \sup_{s,t \in I} \frac{|w_s - w_t|}{|t - s|^\beta} \tag{2.1}$$

and let

$$W^\beta = \{\ w \in C(I);\ \|w\|_\beta < \infty\ \}.$$

Then, in view of (V.1), (V.2) and (V.5), there is a unique Gaussian measure Π on W^β, for any $\beta < \frac{\alpha}{2}$, for which

$$E^\Pi[w_t] = 0, \qquad t \in I$$

and

$$E^\Pi[w_s \cdot w_t] = V_0(s,t) \qquad s,\ t \in I$$

hold. This is showed by applying the result of Ciesielski([C]) of the path regularity the Gaussian processes. Moreover, since

$$\begin{aligned} E^\Pi\left[\left(\int_I w_s \mu_0(ds)\right)^2\right] &= \int_I \int_I V_0(s,t)\mu_0(ds)\mu_0(dt) \\ &= 0, \end{aligned}$$

we see that

$$\Pi\left(\ w \in W^{\beta,0}\ \right) = 1. \tag{2.2}$$

where

$$W^{\beta,0} = \left\{ w \in W^\beta;\ \int_I w_s \mu_0(ds) = 0 \right\}.$$

Accordingly, the following Landau-Shepp-Fernique-type estimate holds. We refer to [Kuo] for its proof.

PROPOSITION (2.3). *For any $\beta < \frac{\alpha}{2}$, there is a constant $c = c_\beta > 0$ such that*

$$E^\Pi\left[\ e^{c\|w\|_\beta^2}\ \right] < \infty.$$

As an application of this proposition, we have the following estimate, which will be useful in the coming sections.

COROLLARY (2.4). *For any $\beta < \frac{\alpha}{2}$, there is a constant $c = c_\beta > 0$ and $C = C_\beta > 0$ such that for all $r > 0$,*

$$\Pi(w \in W^{\beta,0};\ \|w\|_\beta \geq r\ ) \leq C \cdot e^{-c \cdot r^2}.$$

Now observe that, for all $f \in L_0^2(\mu_0)$,

$$E^{\Pi}\left[\left|\langle w, f\rangle_{L_0^2(\mu_0)}\right|^2\right] = \langle f, V_0 f\rangle_{L_0^2(\mu_0)}.$$

Thus, we can define $\langle w, \nu\rangle \in L^2(W^{\beta,0})$ for all $\nu \in H_{-1}$ and we have

$$E^{\Pi}\left[\exp\left(-\sqrt{-1}\,\langle w, \nu\rangle\right)\right] = \exp\left(-\frac{1}{2}\langle \nu, V_0\nu\rangle\right). \tag{2.5}$$

For each $\underline{t} = (t_1, t_2, \dots) \in I^{\otimes\infty}$ and $n \in \mathbf{N}$, we denote by $\rho_n(\underline{t})$ the probability measure on I given by

$$\rho_n(\underline{t})(dy) = \frac{1}{n}\sum_{j=1}^{n} \delta_{t_j}(dy)$$

and by $\eta_n(\underline{t})$ the signed measure on I given by

$$\eta_n(\underline{t})(dy) = n(\rho_n(\underline{t}) - \mu_0).$$

Notice that, since

$$\sup_{t\in I}\|\delta_t\|_{H_{-1}}^2 = \sup_{t\in I}\sum_{k=1}^{\infty}\lambda_k\,\phi_k(t)^2 < \infty,$$

by virtue of (V.6) and (V.7), we see that $\rho_n \in H_{-1}$ and $\eta_n \in H_{-1}$. As a result of (2.5), we have the following.

LEMMA (2.6). *For all* $\nu \in H_{-1}$,

$$\begin{aligned}\int_{I^{\otimes n}} \exp\left(-\frac{1}{2}\langle \eta_n(\underline{t}) - \nu, V_0(\eta_n(\underline{t}) - \nu)\rangle\right)\mu_0^{\otimes n}(dt) \\ = \int_{W^{\beta,0}} \Pi(dw)\exp\left(\sqrt{-1}\,\langle w, \nu\rangle\right)F(w)^n\end{aligned} \tag{2.7}$$

where

$$F(w) = \int_I \exp\left(-\sqrt{-1}w_t\right) \mu_0(dt). \tag{2.8}$$

PROOF. By (2.5) and Fubini's theorem,

$$\begin{aligned}
&\int_{I^{\otimes n}} \exp\left(-\frac{1}{2}\langle \eta_n(\underline{t}) - \nu, V_0(\eta_n(\underline{t}) - \nu)\rangle\right) \mu_0^{\otimes n}(dt) \\
&\quad = \int_{I^{\otimes n}} \mu_0^{\otimes n}(dt) \int_{W^{\beta,0}} \Pi(dw) \exp\left(-\sqrt{-1}\,\langle w, \eta_n(\underline{t}) - \nu\rangle\right) \\
&\quad = \int_{W^{\beta,0}} \Pi(dw) \exp\left(\sqrt{-1}\,\langle w, \nu\rangle\right) \int_{I^{\otimes n}} \mu_0^{\otimes n}(dt) \exp\left(-\sqrt{-1}\,\langle w, \eta_n(\underline{t})\rangle\right) \\
&\quad = \int_{W^{\beta,0}} \Pi(dw) \exp\left(\sqrt{-1}\,\langle w, \nu\rangle\right) F(w)^n.
\end{aligned}$$

where

$$F(w) = \int_I \exp\left(-\sqrt{-1}\,\langle w, \delta_t - \mu_0\rangle\right) \mu_0(dt) = \int_I \exp\left(-\sqrt{-1}w_t\right) \mu_0(dt).$$

The last equality comes from (2.2). □

3. Preliminary Results II

In this section we investigate the behavior of $\det(I + nV_0)$ under our assumptions. In the remainder of this article, we write $a_n \prec b_n$ if the two positive sequences $\{a_n\}$ and $\{b_n\}$ satisfy that $\lim_{n\to\infty} a_n/b_n = 0$.

For any $p \in (0,1)$, we define

$$\mathcal{L}^p = \left\{ V : L_0^2(\mu_0) \to L_0^2(\mu_0) : \begin{array}{c} \text{positive, symmetric and compact} \\ \text{operator with the eigenvalues} \\ \lambda_1 \geq \lambda_2 \geq \cdots > 0 \quad \text{satisfying} \\ \sum_{k=1}^{\infty} \lambda_k^p < \infty. \end{array} \right\}.$$

For any $V \in \mathcal{L}^p$ with the eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots > 0$, let $N(\lambda) = {}^\sharp\{k;\ \lambda_k \geq \lambda\}$, where ${}^\sharp\{A\}$ stands for the number of the elements of the set A.

Lemma (3.1). *If $V \in \mathcal{L}^p$, then $N(\lambda) \prec \lambda^{-p}$ as $\lambda \to 0$.*

Proof. Let
$$f(t) = \frac{1}{t^{1/p}}$$
on $(0, \infty)$ and $g : (0, \infty) \to (0, \infty)$ be given by

$$g(t) = \ell\,\lambda_{k-1} + (1-\ell)\,\lambda_k$$

if $t = \ell(k-1) + \ell k$ for some $k \in \mathbf{N}$ and $0 \leq \ell \leq 1$. Then, by the assumption $\lambda_k \prec \frac{1}{k^{1/p}}$ as $k \to \infty$, and so $\varepsilon(t) = \frac{g(t)}{f(t)} \to 0$ as $t \to \infty$.

Let $f^{-1}(\lambda) = \lambda^{-p}$ and $g^{-1}(\lambda) = \max\{s : g(s) = \lambda\}$ for $\lambda > 0$. Then

$$g^{-1}(\lambda) = \varepsilon(g^{-1}(\lambda))^p \cdot f^{-1}(\lambda)$$

and so
$$\lim_{\lambda \to 0} \frac{g^{-1}(\lambda)}{f^{-1}(\lambda)} = 0,$$
which completes the proof noting $N(\lambda)$ and $g^{-1}(\lambda)$ have the same order as $\lambda \to 0$. $\square$

Lemma (3.2). *For every $V \in \mathcal{L}^p$ and $n \in \mathbf{N}$,*

$$\log\det(I + nV) = \int_0^{\lambda_1} \frac{n}{1 + n\,\lambda} N(\lambda) d\,\lambda\,.$$

Proof. For each $\varepsilon > 0$, let k_ε denote the smallest integer k such that $\lambda_k \leq \varepsilon$. Then

$$\log\det(I + nV) = \sum_{k=1}^{\infty} \log(1 + n\,\lambda_k) = \lim_{\varepsilon \to 0} \sum_{k=1}^{k_\varepsilon} \log(1 + n\,\lambda_k)$$

and

$$\sum_{k=1}^{k_\varepsilon} \log(1+n\,\lambda_k) = -\int_\varepsilon^{\lambda_1} \log(1+n\,\lambda) N(d\,\lambda)$$
$$= \log(1+n\,\varepsilon)N(\varepsilon) + \int_\varepsilon^{\lambda_1} \frac{n}{1+n\,\lambda} N(\lambda) d\,\lambda\,.$$

By Lemma(3.1),

$$\log(1+n\,\varepsilon)N(\varepsilon) = n\,\varepsilon \cdot N(\varepsilon)(1+o(1)) \prec n\,\varepsilon^{1-p}$$

as $\varepsilon \to 0$ and thus

$$\lim_{\varepsilon\to 0} \log(1+n\,\varepsilon)N(\varepsilon) = 0,$$

from which (3.2) follows. □

PROPOSITION (3.3). *For any* $0 < p < 1$, *if* $V \in \mathcal{L}^p$, *then*

$$\log\det(I+nV) \prec n^p$$

as $n \to \infty$.

PROOF. For any $\delta > 0$, $\displaystyle\int_\delta^{\lambda_1} \frac{n}{1+n\,\lambda} d\,\lambda$ is bounded in n, whereas $\displaystyle\lim_{n\to\infty} \int_0^\delta \frac{n}{1+n\,\lambda} d\,\lambda = \infty$. Thus, noting that $N(\lambda)$ is decreasing in λ,

$$\lim_{n\to\infty} \frac{\displaystyle\int_\delta^{\lambda_1} \frac{n}{1+n\,\lambda} N(\lambda) d\,\lambda}{\displaystyle\int_0^\delta \frac{n}{1+n\,\lambda} N(\lambda) d\,\lambda} \le \lim_{n\to\infty} \frac{\displaystyle\int_\delta^{\lambda_1} \frac{n}{1+n\,\lambda} d\,\lambda}{\displaystyle\int_0^\delta \frac{n}{1+n\,\lambda} d\,\lambda} = 0. \tag{3.4}$$

Moreover, by Lemma(3.1), for every small $\zeta > 0$, there is a $\delta > 0$ such that $N(\lambda) \le \zeta \cdot \lambda^{-p}$ for all $\lambda \in [0,\delta]$, and thus

$$\int_0^\delta \frac{n}{1+n\,\lambda} N(\lambda) d\,\lambda \le \zeta \int_0^\delta \frac{n\,\lambda^{-p}}{1+n\,\lambda} d\,\lambda \tag{3.5}$$
$$\le \zeta \cdot n^p \int_0^\infty \frac{\lambda^{-p}}{1+\lambda} d\,\lambda\,.$$

Hence, by (3.4) and (3.5),

$$\int_0^{\lambda_1} \frac{n}{1+n\,\lambda} N(\lambda)d\,\lambda \prec n^p$$

as $n \to \infty$, and thus the Proposition follows from Lemma(3.2). □

4. Evaluation of the Integral (2.7), I

For each $n \in \mathbf{N}$, define c_n and d_n by

(4.1) $c_n = n^{-\frac{2-p}{4}} \cdot \{\log\det(I+nV_0)\}^{\frac{1}{4}}$ and $d_n = n^{\frac{p}{4}} \cdot \{\log\det(I+nV_0)\}^{\frac{1}{4}}$.

By assumption (V.6), we are able to pick a $\beta < \frac{\alpha}{2}$ such that $V_0 \in \mathcal{L}^p$ for some $p < \frac{1}{3} \cdot \frac{2\beta}{2\beta+1}$. By (V.6) and Proposition (3.3), we see that $\log\det(I+nV_0) \prec n^p$, and thus we see easily that $\lim_{n\to\infty} c_n = 0$, $\lim_{n\to\infty} d_n = \infty$,

(4.2) $\log\det(I+nV_0) \prec nc_n^2 \prec n^p$ and $\log\det(I+nV_0) \prec d_n^2 \prec n^p$.

Note also that, since $p < \frac{1}{3} \cdot \frac{2\beta}{2\beta+1}$,

$$\lim_{n\to\infty} n \cdot c_n^{2+\frac{2\beta}{2\beta+1}} \cdot d_n^{\frac{1}{2\beta+1}} = 0. \tag{4.3}$$

For every $n \in \mathbf{N}$, let

$$\Gamma_n = \left\{ w \dot{\in} W^{\beta,0}; \quad \|w\|_{L_0^2(\mu_0)} \le c_n \quad \text{and} \quad \|w\|_\beta \le d_n \right\}.$$

LEMMA (4.4). *For the choice of $\{c_n\}$ and $\{d_n\}$ given by (4.1),*

$$\lim_{n\to\infty} n \cdot \sup_{w\in\Gamma_n} \int |w_t|^3 \mu_0(dt) = 0 \tag{4.5}$$

and

$$\lim_{n\to\infty} n \cdot \sup_{w\in\Gamma_n} \left(\int |w_t|^2 \mu_0(dt) \right)^2 = 0. \tag{4.6}$$

Proof. Recall that the Hölder norm $\|\cdot\|_\beta$ is defined in (2.1). The following interpolation inequality holds: there is a $C>0$ such that for all $g \in L_0^2(\mu_0) \cap W^{\beta,0}$,

$$\|g\|_\infty \le C \left\{ \|g\|_{L_0^2(\mu_0)}^{2\beta} \cdot \|g\|_\beta \right\}^{\frac{1}{2\beta+1}}.$$

We refer to [T] for the proof. Thus, for all $w \in \Gamma_n$,

$$\begin{aligned} n \cdot \int |w_t|^3 \mu_0(dt) &\le n\|w\|_\infty \int_I |w_t|^2 \mu_0(dt) \\ &\le Cn \cdot c_n^{2+\frac{2\beta}{2\beta+1}} \cdot d_n^{\frac{1}{2\beta+1}} \end{aligned}$$

and so, by (4.3), (4.5) follows. Also, since $nc_n^2 \prec n^p$, for every $w \in \Gamma_n$,

$$n \cdot \left(\int |w_t|^2 \mu_0(dt) \right)^2 \le nc_n^4 \prec n^{2p-1},$$

and thus (4.6) follows noting that $p < \frac{1}{6}$. □

Now recall that $F : W^{\beta,0} \to \mathbf{C}$ is defined by (2.8).

Lemma (4.7).

$$\lim_{n\to\infty} \sup_{w\in\Gamma_n} \left| F(w)^n \cdot \exp\left(\frac{n}{2} \|w\|_{L_0^2(\mu_0)}^2 \right) - 1 \right| = 0.$$

Proof. Since

$$\left| e^{-\sqrt{-1}x} - \left(1 - \sqrt{-1}x - \frac{1}{2}x^2 \right) \right| \le |x|^3,$$

for all $x \in \mathbf{R}$, it follows that

$$\left| F(w) - \left(1 - \frac{1}{2}\|w\|_{L_0^2(\mu_0)}^2 \right) \right| \le \int_I |w_t|^3 \mu_0(dt) \tag{4.8}$$

and thus

$$|F(w)-1| \le \frac{1}{2}\|w\|^2_{L^2_0(\mu_0)} + \int_I |w_t|^3 \mu_0(dt) \tag{4.9}$$

for all $w \in W^{\beta,0}$. By (4.9) and Lemma(4.4), $|F(w)-1| < 1/2$ for all $w \in \Gamma_n$ for sufficiently large n. Note also that for any $z \in \mathbf{C}$ safisfying $|z-1| < 1/2$,

$$|\operatorname{Log} z - (z-1)| \le |z-1|^2. \tag{4.10}$$

Here, $\operatorname{Log} z$ is the principal value of $\log z$. Thus, by (4.8) and (4.10),

$$\begin{aligned}\left|\operatorname{Log} F(w) + \frac{1}{2}\|w\|^2_{L^2_0(\mu_0)}\right| &\le |\operatorname{Log} F(w) - (F(w)-1)| \\ &\quad + \left|(F(w)-1) + \frac{1}{2}\|w\|^2_{L^2_0(\mu_0)}\right| \\ &\le |F(w)-1|^2 + \int_I |w_t|^3 \mu_0(dt).\end{aligned}$$

Thus, by (4.5), (4.6) and (4.9),

$$\lim_{n\to\infty} n \cdot \sup_{w\in\Gamma_n} \left|\operatorname{Log} F(w) + \frac{1}{2}\|w\|^2_{L^2_0(\mu_0)}\right| = 0,$$

from which the assertion follows immediately. □

Lemma (4.11). *For all* $\nu \in H_{-1}$,

$$\begin{aligned} &E^{\Pi}\left[\exp\left(\sqrt{-1}\,\langle w, \nu\rangle\right)\exp\left(-\frac{n}{2}\|w\|^2_{L^2_0(\mu_0)}\right);\ w \in \Gamma_n\right] \\ &\quad = \{\det(I+nV_0)\}^{-\frac{1}{2}}(1+o(1)). \end{aligned} \tag{4.12}$$

Proof. First, notice that if a positive sequence $\{a_n\}$ satisfies that

$$\log\det(I+nV_0) \prec a_n,$$

then

$$e^{-c\cdot a_n} \prec \{\det(I+nV_0)\}^{-\frac{1}{2}}$$

for any constant $c > 0$. Thus, by (4.2), we see that

$$E^{\Pi}\left[\exp\left(-\frac{n}{2}\|w\|^2_{L^2_0(\mu_0)}\right);\ \|w\|_{L^2_0(\mu_0)} \geq c_n\right] \leq \exp\left(-\frac{n \cdot c_n^2}{2}\right) \prec \{\det(I + nV_0)\}^{-\frac{1}{2}}$$

as $n \to \infty$. Also, by Corollary(2.4) and (4.2),

$$E^{\Pi}\left[\exp\left(-\frac{n}{2}\|w\|^2_{L^2_0(\mu_0)}\right);\ \|w\|_{\beta} \geq d_n\right] \leq \Pi(\|w\|_{\beta} \geq d_n) \prec \{\det(I + nV_0)\}^{-\frac{1}{2}}.$$

Thus,

$$E^{\Pi}\left[\exp\left(-\frac{n}{2}\|w\|^2_{L^2_0(\mu_0)}\right);\ w \in \Omega \setminus \Gamma_n\right] \prec \{\det(I + nV_0)\}^{-\frac{1}{2}} \tag{4.13}$$

as $n \to \infty$.

Now, recall $\{\phi_k\}_{k=1,2,\cdots}$ be the eigenfunctions of V_0 corresponding to the eigenvalues $\{\lambda_k\}_{k=1,2,\cdots}$. Notice that, for all $\nu \in H_{-1}$,

$$\sum_{k=1}^{\infty} \lambda_k \,|\,\langle\, \nu, \phi_k \,\rangle\,|^2 < \infty$$

and $(n\,\lambda_k + 1)^{-1} < 1$ for all n and k and $\lim_{n\to\infty}(n\,\lambda_k + 1)^{-1} = 0$ for all $k \geq 1$ by (V.5). Hence, by virtue of the bounded convergence theorem, we obtain

$$\lim_{n\to\infty} \langle\, (I + nV_0)^{-1}\nu, V_0\nu \,\rangle = \lim_{n\to\infty} \sum_{k=1}^{\infty} (n\,\lambda_k + 1)^{-1}\,\lambda_k\,\langle\, \nu, \phi_k \,\rangle^2 = 0$$

and so

$$\begin{aligned} E^{\Pi}&\left[\exp\left(\sqrt{-1}\,\langle\, w, \nu \,\rangle\right)\exp\left(-\frac{n}{2}\|w\|^2_{L^2_0(\mu_0)}\right)\right] \\ &= \{\det(I + nV_0)\}^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\ \langle\, (I + nV_0)^{-1}\nu, V_0\nu \,\rangle\right) \\ &= \{\det(I + nV_0)\}^{-\frac{1}{2}}(1 + o(1)) \end{aligned} \tag{4.14}$$

as $n \to \infty$. Hence, the Lemma follows from (4.13) and (4.14). □

PROPOSITION (4.15). *For all* $\nu \in H_{-1}$,

$$E^{\Pi}[\ \exp\left(\sqrt{-1}\,\langle\, w, \nu\,\rangle\right) F(w)^n,\ \ w \in \Gamma_n\ ] = \{\det(I + nV_0)\}^{-\frac{1}{2}}(1 + o(1))$$

as $n \to \infty$.

PROOF. For each $n \in \mathbf{N}$, let

$$G_n(w) = F(w)^n \cdot \exp\left(\frac{n}{2}\|w\|^2_{L^2_0(\mu_0)}\right) - 1$$

for all $w \in \Gamma_n$. Then,

$$\begin{aligned}
&\left|\frac{E^{\Pi}[\ \exp\left(\sqrt{-1}\,\langle\, w, \nu\,\rangle\right) F(w)^n;\ \ w \in \Gamma_n\ ]}{E^{\Pi}\left[\exp\left(\sqrt{-1}\,\langle\, w, \nu\,\rangle\right) \exp\left(-\frac{n}{2}\|w\|^2_{L^2_0(\mu_0)}\right);\ w \in \Gamma_n\right]} - 1\right| \\
&= \left|\frac{E^{\Pi}\left[\exp\left(\sqrt{-1}\,\langle\, w, \nu\,\rangle\right) G_n(w) \exp\left(-\frac{n}{2}\|w\|^2_{L^2_0(\mu_0)}\right);\ w \in \Gamma_n\right]}{E^{\Pi}\left[\exp\left(\sqrt{-1}\,\langle\, w, \nu\,\rangle\right) \exp\left(-\frac{n}{2}\|w\|^2_{L^2_0(\mu_0)}\right);\ w \in \Gamma_n\right]}\right| \\
&\le \sup_{w \in \Gamma_n} |G_n(w)| \cdot \frac{E^{\Pi}\left[\exp\left(-\frac{n}{2}\|w\|^2_{L^2_0(\mu_0)}\right);\ w \in \Gamma_n\right]}{\left|E^{\Pi}\left[\exp\left(\sqrt{-1}\,\langle\, w, \nu\,\rangle\right) \exp\left(-\frac{n}{2}\|w\|^2_{L^2_0(\mu_0)}\right);\ w \in \Gamma_n\right]\right|}.
\end{aligned}$$

Thus, by Lemma(4.7) and Lemma(4.11), we see

$$\begin{aligned}
&E^{\Pi}[\ \exp\left(\sqrt{-1}\,\langle\, w, \nu\,\rangle\right) F(w)^n;\ \ w \in \Gamma_n\ ] \\
&= E^{\Pi}\left[\exp\left(\sqrt{-1}\,\langle\, w, \nu\,\rangle\right) \exp\left(-\frac{n}{2}\|w\|^2_{L^2_0(\mu_0)}\right);\ w \in \Gamma_n\right](1 + o(1))
\end{aligned}$$

and so the Proposition follows from Lemma(4.11). □

5. Evaluation of the Integral (2.7), II

LEMMA (5.1). *For any $\beta < \frac{\alpha}{2}$, there is a constant $c > 0$ and $C > 0$ such that*

$$\Pi\left(w \in W^{\beta,0}; \quad |F(w)| \geq 1 - \tau \quad \textit{and} \quad \|w\|_{L_0^2(\mu_0)} > \sqrt{\frac{\pi^2 \tau}{2}} \right) \leq C \exp\left(-\frac{c}{\tau^{2\beta}} \right)$$

for all $0 < \tau < 1$.

PROOF. For $j = 0, 1, 2, \dots, 15$, let $I_j = \cup_{m=-\infty}^{\infty} [(2m + \frac{j}{8})\pi, (2m + \frac{j+1}{8})\pi)$. Then I_j's are disjoint and $\cup_{j=0}^{15} I_j = \mathbf{R}$. For each $w \in W^{\beta,0}$, let $\theta_j = \mu_0(t; w_t \in I_j)$. Since $\sum_{j=0}^{15} \theta_j = 1$, there is at least one $j_0 \in \{0, 1, 2, \dots, 15\}$ such that $\theta_{j_0} \geq \frac{1}{16}$.

Let $\tau > 0$. If $w \in W^{\beta,0}$ satisfies that $\|w\|_\beta \leq \frac{\pi}{8} \frac{1}{\tau^\beta}$, then,

$$\mu_0(t;\ w_t \in [a, a + \pi/8)\ ) \geq c_{\mu_0} \cdot |t; w_t \in [a, a + \pi/8)| \geq c_{\mu_0} \cdot \tau$$

in the case the path w passes $[a, a + \pi/8)$, i.e., there are $t_0, t_1 \in I$ such that $w_{t_0} = a$ and $w_{t_1} = a + \pi/8$. Here, c_{μ_0} is the one given in the assumption (V.4).

Noting this, let us assume that $w \in W^{\beta,0}$ satisfies $\|w\|_\infty > \frac{\pi}{2}$ and $\|w\|_\beta \leq \frac{\pi}{8} \frac{1}{\tau^\beta}$. Then, inevitably $w \in W^{\beta,0}$ passes the interval $[0, \pi/2]$ or $[-\pi/2, 0]$. Thus there is at least one I_{j_1} such that $\mathrm{dis}(I_{j_1}, I_{j_0}) \geq \frac{\pi}{8}$ and $\theta_{j_1} \geq c_{\mu_0} \cdot \tau$. Therefore,

$$\begin{aligned} |F(w)|^2 &= \int_I \int_I e^{\sqrt{-1}(w_s - w_t)} \mu_0(ds) \mu_0(dt) \\ &= \int_I \int_I \cos(w_s - w_t) \mu_0(ds) \mu_0(dt) \\ &\leq (1 - 2\,\theta_{j_0}\,\theta_{j_1}) + 2\,\theta_{j_0}\,\theta_{j_1} \cos(\frac{\pi}{8}) \\ &\leq 1 - 2c_{\mu_0} \left(1 - \cos(\frac{\pi}{8}) \right) \cdot \frac{\tau}{16}. \end{aligned}$$

and thus

$$|F(w)| \leq 1 - C \cdot \tau$$

where $C = \frac{c_{\mu_0}}{8}\left(1 - \cos(\frac{\pi}{8})\right) < 1$.

Hence, if $\|w\|_\infty > \pi/2$ and $|F(w)| > 1 - \tau$, then $\|w\|_\beta \geq \frac{\pi}{8C^\beta} \cdot \frac{1}{\tau^\beta}$, and thus, applying Corollary(2.4), for $0 < \tau < 1$,

$$(5.2) \qquad \Pi\left(w \in W^{\beta,0};\ |F(w)| > 1 - \tau \quad \text{and} \quad \|w\|_\infty \geq \frac{\pi}{2}\right) \\ \leq \Pi\left(w \in W^{\beta,0};\ \|w\|_\beta \geq \frac{C'}{\tau^\beta}\right) \leq C \cdot \exp\left(-\frac{c}{\tau^{2\beta}}\right),$$

where $C' = \frac{\pi}{8} \cdot C^{-\beta}$ and $c = c_\beta \cdot C'^2$.

Next, let us assume that $w \in W^{\beta,0}$ satisfies $|F(w)| \geq 1 - \tau$ and $\|w\|_\infty \leq \frac{\pi}{2}$. Then, since

$$\cos x \leq 1 - \frac{2}{\pi^2}x^2$$

for all $x \in [-\pi, \pi]$,

$$|F(w)|^2 \leq 1 - \frac{2}{\pi^2}\int_I\int_I (w_t - w_s)^2 \mu_0(ds)\mu_0(dt) \\ = 1 - \frac{4}{\pi^2}\int_I |w_t|^2 \mu_0(dt).$$

Thus, if $|F(w)| \geq 1 - \tau$ and $\|w\|_\infty \leq \frac{\pi}{2}$, then

$$\int_I |w_t|^2 \mu_0(dt) \leq \frac{\pi^2\tau}{2}$$

and thus

$$(5.3) \qquad \Pi\left(w \in W^{\beta,0}; |F(w)| \geq 1 - \tau, \quad \|w\|_\infty \leq \frac{\pi}{2} \quad \text{and} \right. \\ \left. \|w\|_{L^2_0(\mu_0)} > \sqrt{\frac{\pi^2\tau}{2}}\right) = 0.$$

Hence, by (5.2) and (5.3), we have proved Lemma(5.1). □

PROPOSITION (5.4). *As* $n \to \infty$,

$$E^\Pi[\ |F(w)|^n;\ w \in \Omega \setminus \Gamma_n] \prec \{\det(I + nV_0)\}^{-\frac{1}{2}}.$$

Proof. Note that

$$
\begin{aligned}
E^{\Pi}[\ |F(w)|^n, \|w\|_{L_0^2(\mu_0)} > c_n] & \\
&= \int_0^1 \tau^n \Pi(w;\ |F(w)| \in d\tau, \|w\|_{L_0^2(\mu_0)} > c_n) \\
&= n \int_0^1 (1-\tau)^{n-1} \Pi(w;\ |F(w)| \geq 1-\tau, \|w\|_{L_0^2(\mu_0)} > c_n) d\tau.
\end{aligned}
$$

Let $t_n = \frac{2c_n^2}{\pi^2}$, where c_n is the one defined by (4.1), and we apply Lemma(5.1) to obtain,

$$
\begin{aligned}
&\int_0^{t_n} (1-\tau)^{n-1} \Pi(w;\ |F(w)| \geq 1-\tau, \|w\|_{L_0^2(\mu_0)} > c_n) d\tau \\
&\quad \leq C \int_0^{t_n} (1-\tau)^{n-1} \exp\left(-\frac{c}{\tau^{2\beta}}\right) d\tau \\
&\quad = \frac{C}{(n-1)^{1/(1+2\beta)}} \\
&\quad \cdot \int_0^{(n-1)^{1/(1+2\beta)} \cdot t_n} \left(1 - \frac{\tau}{(n-1)^{1/(1+2\beta)}}\right)^{n-1} \\
&\qquad\qquad \cdot \exp\left(-\frac{c \cdot (n-1)^{2\beta/(1+2\beta)}}{\tau^{2\beta}}\right) d\tau \\
&\quad \leq C \int_0^{(n-1)^{-\beta'+p}} \exp\left(-(n-1)^{\beta'} \cdot \tau - \frac{c(n-1)^{\beta'}}{\tau^{2\beta}}\right) d\tau \\
&\quad \leq C \int_0^1 \exp\left(-(n-1)^{\beta'}(\tau + \frac{c}{\tau^{2\beta}})\right) d\tau \\
&\quad \leq C \cdot e^{-A \cdot (n-1)^{\beta'}}
\end{aligned}
$$

for sufficiently large n. Here,

$$
A = \inf_{\tau \in [0,1]} \left(\tau + \frac{c}{\tau^{2\beta}}\right)
$$

is a positive constant and $\beta' = \frac{2\beta}{1+2\beta}$. The second inequality comes by applying $s = \tau/(n-1)^{1/(1+2\beta)}$ to the inequality $(1-s)^{n-1} \le e^{-(n-1)s}$ for all $s < 1$ and $n \in \mathbf{N}$. Since $p < \frac{1}{3}\beta' < \beta'$, $\log\det(I+nV_0) \prec (n-1)^p \prec (n-1)^{\beta'}$ and so

$$e^{-A\cdot(n-1)^{\beta'}} \prec \{\det(I+nV_0)\}^{-\frac{1}{2}}.$$

Also,

$$\begin{aligned}\int_{t_n}^1 (1-\tau)^{n-1}\Pi(w;\ |F(w)| \ge 1-\tau, \|w\|_{L_0^2(\mu_0)} > c_n)d\tau &\le \int_{t_n}^1 (1-\tau)^{n-1}d\tau \\ &\le (1-t_n)^{n-1} \\ &\le e^{-(n-1)t_n}.\end{aligned}$$

By the choice of $\{t_n\}$,

$$e^{-(n-1)t_n} \prec \{\det(I+nV_0)\}^{-\frac{1}{2}}.$$

and thus

$$E^{\Pi}[\ |F(w)|^n;\ \|w\|_{L_0^2(\mu_0)} > c_n] \prec \{\det(I+nV_0)\}^{-\frac{1}{2}}. \tag{5.5}$$

On the other hand, since $|F(w)| \le 1$, by Corollary(2.4),

$$\begin{aligned}E^{\Pi}[\ |F(w)|^n;\ \|w\|_\beta > d_n] &\le \Pi(\ \|w\|_\beta > d_n\ ) \\ &\prec \{\det(I+nV_0)\}^{-\frac{1}{2}}.\end{aligned} \tag{5.6}$$

as $n \to \infty$. Thus the proposition follows from (5.5) and (5.6). □

Before we close this section we summarize what we have established so far. By Proposition(4.15) and Proposition(5.4), along with Lemma(2.6), we obtain the following.

THEOREM (5.7). *For all $\nu \in H_{-1}$,*

$$\begin{aligned}&\int \exp\left(-\frac{1}{2}\langle\,\eta_n(\underline{t}) - \nu, V_0(\eta_n(\underline{t}) - \nu)\,\rangle\right)\mu_0^{\otimes n}(d\underline{t}) \\ &\quad = \{\det(I+nV_0)\}^{-\frac{1}{2}}(1+o(1)).\end{aligned}$$

6. Proof of Theorem 1

Let Λ be the set of bounded measurable functions on I. Then by (V.3), we see that for any $h \in L_0^2(\mu_0) \cap \Lambda$,

$$\begin{aligned} 0 &= \frac{d}{dt} J[\mu_0 + th\mu_0]|_{t=0} \\ &= \int_I \int_I V(s,t)\mu_0(ds)h(t)\mu_0(dt). \end{aligned}$$

Since $L_0^2(\mu_0) \cap \Lambda$ is dense in $L_0^2(\mu_0)$, we see that there exists a $\lambda \in \mathbf{R}$ for which

$$\int_I V(s,t)\mu_0(ds) = \lambda\,. \tag{6.1}$$

Integrating the both sides of the above identity by μ_0, we obtain

$$J[\mu_0] = \lambda\,. \tag{6.2}$$

Thus, by (6.1) and (6.2), we see that V_0 defined in the Introduction satisfies that

$$\int_I V_0(s,t)\mu_0(ds) = 0$$

and so the operator V_0 defined by (1.4) is an operator on $L_0^2(\mu_0)$. Moreover, by (6.1), we have

$$J[\mu] - J[\mu_0] = \int_I \int_I V_0(s,t)(\mu - \mu_0)(ds)(\mu - \mu_0)(dt)$$

for all $\mu \in \mathcal{M}_1(I)$, and thus

$$\begin{aligned} n^2(J[\rho_n(\underline{\mathrm{t}})] - J[\mu_0]) &= \int_I \int_I V_0(s,t)\eta_n(\underline{\mathrm{t}})(ds)\eta_n(\underline{\mathrm{t}})(dt) \\ &= \langle\, \eta_n(\underline{\mathrm{t}}), V_0\eta_n(\underline{\mathrm{t}})\,\rangle\,. \end{aligned} \tag{6.3}$$

Now, noting (V.4), we have

$$Z_n = \int_{I^{\otimes n}} \exp\left(-\frac{1}{2}n^2 J[\rho_n(\underline{t})]\right) dt_1 \cdots dt_n$$
$$= \int_{I^{\otimes n}} \exp\left(-\frac{1}{2}n^2 J[\rho_n(\underline{t})] - \sum_{j=1}^{n} \log \frac{\mu_0(dt)}{dt}(t_j)\right) \mu_0^{\otimes n}(d\underline{t}),$$

and thus, by (6.3),

$$\text{(6.4)}\quad \exp\left(\frac{1}{2}n^2 J[\mu_0] + n\int_I \log \frac{\mu_0(dt)}{dt}\mu_0(dt)\right) Z_n$$
$$= \int_{I^{\otimes n}} \exp\Bigg(-\frac{1}{2}n^2 (J[\rho_n(\underline{t})] - J[\mu_0]) - n\int_I \log\frac{\mu_0(dt)}{dt}(\rho_n(\underline{t}) - \mu_0)(dt)\Bigg)\mu_0^{\otimes n}(d\underline{t})$$
$$= \int_{I^{\otimes n}} \exp\left(-\frac{1}{2}\langle \eta_n(\underline{t}), V_0\eta_n(\underline{t})\rangle - \langle \xi_0, \eta_n(\underline{t})\rangle\right)\mu_0^{\otimes n}(d\underline{t})$$
$$= \exp\left(\frac{1}{2}\langle V_0^{-1}\xi_0, \xi_0\rangle\right) \cdot \int_{I^{\otimes n}} \exp\left(-\frac{1}{2}\langle \eta_n(\underline{t}) + V_0^{-1}\xi_0, V_0(\eta_n(\underline{t}) + V_0^{-1}\xi_0)\rangle\right)\mu_0^{\otimes n}(d\underline{t}).$$

Here, $V_0^{-1}\xi_0 \in H_{-1}$ and $\langle V_0^{-1}\xi_0, \xi_0\rangle = \|V_0^{-\frac{1}{2}}\xi_0\|^2_{L_0^2(\mu_0)}$ by (V.8). Therefore, Theorem 1 follows from Theorem(5.7). □

7. Proof of Theorem 2

In view of (6.4), we observe

$$P_n(d\underline{t}) = \frac{1}{Z_n'} \cdot \exp\left(-\frac{1}{2}\langle \eta_n(\underline{t}), V_0\eta_n(\underline{t})\rangle - \langle \xi_0, \eta_n(\underline{t})\rangle\right)\mu_0^{\otimes n}(d\underline{t})$$

where

$$Z_n' = \int_{I^{\otimes n}} \exp\left(-\frac{1}{2}\langle \eta_n(\underline{t}), V_0\eta_n(\underline{t})\rangle - \langle \xi_0, \eta_n(\underline{t})\rangle\right)\mu_0^{\otimes n}(d\underline{t}).$$

By Theorem(5.7),

$$\begin{aligned} Z_n' =& \exp\left(\frac{1}{2}\langle V_0^{-1}\xi_0, \xi_0 \rangle\right) \\ & \cdot \int_{I^{\otimes n}} \exp\left(-\frac{1}{2}\langle \eta_n(\underline{\mathrm{t}}) + V_0^{-1}\xi_0, V_0(\eta_n(\underline{\mathrm{t}}) + V_0^{-1}\xi_0) \rangle\right) \mu_0^{\otimes n}(d\underline{\mathrm{t}}) \\ =& \exp\left(\frac{1}{2}\langle V_0^{-1}\xi_0, \xi_0 \rangle\right) \{\det(I + nV_0)\}^{-\frac{1}{2}}(1 + o(1)) \end{aligned}$$

and, for any $f \in H_1$,

$$\begin{aligned} & \int_{I^{\otimes n}} \exp\left(-\frac{1}{2}\langle \eta_n(\underline{\mathrm{t}}), V_0\eta_n(\underline{\mathrm{t}}) \rangle - \langle \xi_0 - f, \eta_n(\underline{\mathrm{t}}) \rangle\right) \mu_0^{\otimes n}(d\underline{\mathrm{t}}) \\ & = \exp\left(\frac{1}{2}\langle V_0^{-1}(\xi_0 - f), \xi_0 - f \rangle\right) \\ & \quad \cdot \int_{I^{\otimes n}} \exp\Bigg(-\frac{1}{2}\langle \eta_n(\underline{\mathrm{t}}) + V_0^{-1}(\xi_0 - f), \\ & \qquad\qquad V_0(\eta_n(\underline{\mathrm{t}}) + V_0^{-1}(\xi_0 - f)) \rangle\Bigg) \mu_0^{\otimes n}(d\underline{\mathrm{t}}) \\ & = \exp\left(\frac{1}{2}\langle V_0^{-1}(\xi_0 - f), \xi_0 - f \rangle\right) \{\det(I + nV_0)\}^{-\frac{1}{2}}(1 + o(1)). \end{aligned}$$

Hence

$$\begin{aligned} & \exp\left(-n\int_I f d\mu_0\right) \int_{I^{\otimes n}} \exp\left(\sum_{j=1}^n f(t_j)\right) P_n(d\underline{\mathrm{t}}) \\ & = \int_{I^{\otimes n}} \exp(\langle f, \eta_n(\underline{\mathrm{t}}) \rangle) P_n(d\underline{\mathrm{t}}) \\ & = \frac{1}{Z_n'} \cdot \int_{I^{\otimes n}} \exp\left(-\frac{1}{2}\langle \eta_n(\underline{\mathrm{t}}), V_0\eta_n(\underline{\mathrm{t}}) \rangle - \langle \xi_0 - f, \eta_n(\underline{\mathrm{t}}) \rangle\right) \mu_0^{\otimes n}(d\underline{\mathrm{t}}) \\ & = \exp\left(\frac{1}{2}\langle V_0^{-1}(\xi_0 - f), \xi_0 - f \rangle - \frac{1}{2}\langle V_0^{-1}\xi_0, \xi_0 \rangle\right)(1 + o(1)). \end{aligned}$$

This completes the proof. □

8. An Example

Assume that $v : \mathbf{R} \to \mathbf{R}$ satisfies that

(i) $v(t) = v(t+1)$ and $v(t) = v(-t)$ for all $t \in \mathbf{R}$.
(ii) v is 6 times differentiable and $v^{(6)} \in C(\mathbf{R})$.
(iii) v is positive-definite.

Then, $V : I \times I \to \mathbf{R}$ given by

$$V(s,t) = v(t-s)$$

satisfies the assumptions (V.1) to (V.8) with $\mu_0(dt) = dt$ and $\alpha = 1$.

We briefly check this. Let $\{\lambda_k\}_{k \in \mathbf{Z}}$ be the Fourier coefficients of v, i.e.,

$$\lambda_k = \int_{-1/2}^{1/2} e^{2k\pi\sqrt{-1}t} v(t)dt.$$

Since v is real-valued and $v(t) = v(-t)$, $\lambda_k \in \mathbf{R}$ and $\lambda_{-k} = \lambda_k$ for all $k \in \mathbf{Z}$.

For all probability measure on I,

$$J[\mu] = 1 + 2\sum_{k=1}^{\infty} \lambda_k \left| \int_I e^{2k\pi\sqrt{-1}t} \mu(dt) \right|^2,$$

and thus $\mu_0(dt) = dt$ is the only probability measure which minimizes J. By (ii) and by the standard theory of Fourier analysis, we have

$$\sum_{k=1}^{\infty} k^{12}\, \lambda_k^2 < \infty,$$

and so by Hölder's inequality,

$$\sum_{k=1}^{\infty} \lambda_k^{\frac{1}{6+\delta}} \le \left(\sum_{k=1}^{\infty} k^{12}\, \lambda_k^2 \right)^{\frac{1}{12+2\delta}} \cdot \left(\sum_{k=1}^{\infty} k^{-\frac{12}{11+2\delta}} \right)^{\frac{11+2\delta}{12+2\delta}} < \infty$$

if $\delta < \frac{1}{2}$. Thus, this V satisfies the assumptions.

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References

[B] Bolthausen, E., Laplace Approximations for Sums of Independent Random Vectors, Probability Theory and Related Fields **72** (1986), 305–318.

[BG] Ben Arous, G. and A. Guionnet, Large deviations for Wigner's law and Voiculescu's non-commutative entropy, Probability Theory and Related Fields **108** (1997), 517–542.

[C] Ciesielski, Z., Hölder conditions for realizations of Gaussian processes, Trans. Amer. Math. Soc. **99** (1961), 403–413.

[J] Johansson, K., On Fluctuations of Eigenvalues of Random Hermitian Matrices, Duke Math. J. **91** (1998), no. 1, 151–204.

[Kuo] Kuo, H.-H., *Gaussian Measures in Banach Spaces*, Lect. Notes in Math. Vol. 463, Springer, 1975.

[KT] Kusuoka, S. and Y. Tamura, Gibbs measures for mean field potentials, J. Fac. Sci. Univ. Tokyo Sect. IA, Math. **31** (1984), 223–245.

[T] Triebel, H., *Theory of Function Spaces*, Birkhäuser, Berlin, 1992.

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