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Sato-Kashiwara Determinant and Levi Conditions for Systems

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Abstract. We prove that a second-microlocal version of the Sato-Kashiwara determinant computes the Newton polygon of determined systems of linear partial differential operators with constant multiplicities. Applications are given to the Cauchy problem for hyperbolic systems with regular singularities.

Introduction

Let X be a complex manifold, and denote by T^*X its cotangent bundle. A basic microlocal invariant attached to a coherent $\mathcal{D}_X$-module $\mathcal{M}$ is its characteristic variety $\mathrm{char}(\mathcal{M})$, an involutive subset of T^*X. Let $\mathcal{A}$ be the determined $\mathcal{D}_X$-module represented by a square matrix A of partial differential operators. The construction of the Dieudonné determinant associates to A a meromorphic function $\det(A)$ on T^*X. Sato-Kashiwara [29] proved that $\det(A)$ is in fact holomorphic, and that $\mathrm{char}(\mathcal{A})$ is computed by its zero locus. As in the scalar case, where such determinant coincides with the principal symbol, one can read-off from $\det(A)$ other invariants, such as the multiplicity of $\mathcal{A}$. We recall these results in section 1.

Now, let $V \subset T^*X$ be an involutive submanifold. Second-microlocal invariants attached to $\mathcal{M}$ are the characteristic varieties $\mathrm{char}_V^{(r,s)}(\mathcal{M})$ (see [22]), which are subsets of the normal bundle $T_V(T^*X)$. If $\mathcal{M}$ is represented by a single differential operator, these data are summarized in its Newton polygon. We recall these constructions in section 2, where we also state that a second-microlocal version of the Sato-Kashiwara determinant defines and computes the Newton polygon of determined systems. Mimicking the original Sato-Kashiwara's argument, we give a proof of this fact in Appendix A. Let us mention that an algebraic approach to regularity of determinants in filtered rings is proposed in [1], where a weaker form of this result is also obtained.

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In the theory of (micro)differential operators, an important notion is that of regular singularities introduced in [16]. The regularity condition for systems with constant multiplicities coincides with the so-called Levi conditions. For determined systems, such conditions can be expressed in terms of the second-microlocal Sato-Kashiwara determinant. We show these facts in section 3.

Finally, consider a real analytic manifold M, of which X is a complexification. Let $\mathcal{A}$ be a determined system which is hyperbolic with respect to a hypersurface $N \subset M$, and has real constant multiplicities. Using a result of [9], we show in section 4 that Levi conditions are sufficient for the well-poseness of the C^∞ Cauchy problem for $\mathcal{A}$ with data on N. For matrices of Kovalevskayan type this fact was obtained in [33] and [25], using a totally different approach. (Commenting on a preliminary version of this paper, W. Matsumoto informed us that in an unpublished paper he also tried to express Levi conditions using a variation of the Sato-Kashiwara determinant.)

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1. Sato-Kashiwara Determinant

Following Sato-Kashiwara [29] (see [2] for an exposition), we recall in this section the notion of determinant for square matrices of differential operators.

Let X be a complex manifold, and $\pi\colon T^*X \to X$ its cotangent bundle. Denote by $\mathcal{O}_X$ the sheaf of holomorphic functions on X, and by $\mathcal{D}_X$ the sheaf of linear partial differential operators. Recall that coherent $\mathcal{D}_X$-modules represent systems of partial differential equations. More precisely, a coherent $\mathcal{D}_X$-module $\mathcal{M}$ is locally of the form $\mathcal{D}_X^{m_0}/\mathcal{D}_X^{m_1}M$, where M is an $m_1 \times m_0$ matrix with elements in $\mathcal{D}_X$.

The ring $\mathcal{D}_X$ is endowed with a $\mathbb{Z}$-filtration $\mathsf{F}\mathcal{D}_X$ by the order of the operators, and the associated graded ring $\mathsf{G}\mathcal{D}_X$ is identified to the subsheaf of $\pi_*\mathcal{O}_{T^*X}$ of functions which are polynomials in the fibers of π. One denotes

by $\sigma(P) \in \pi_*\mathcal{O}_{T^*X}$ the principal symbol of an operator $P \in \mathcal{D}_X$. Using the notion of good filtrations, one defines the characteristic variety $\mathrm{char}(\mathcal{M})$ of a coherent $\mathcal{D}_X$-module $\mathcal{M}$, a closed conic involutive subset of T^*X (see [15]). In particular, $\mathrm{char}(\mathcal{D}_X/\mathcal{D}_X P)$ is the zero locus of $\sigma(P)$.

Let $A = (A_{ij})$ be an $m \times m$ matrix with elements in $\mathcal{D}_X$, and denote by $\mathcal{A} = \mathcal{D}_X^m/\mathcal{D}_X^m A$ its associated $\mathcal{D}_X$-module. Following [24], one says that A is normal if there exists a family of integers r_i, s_j, called Leray-Volevich weights, such that

$$\text{(1.1)} \qquad \begin{cases} A_{ij} \in \mathsf{F}_{r_i+s_j}\mathcal{D}_X, \\ \det(\sigma_{r_i+s_j}(A_{ij})) \text{ is not identically zero,} \end{cases}$$

where $\sigma_r(\cdot)$ denotes the symbol of order r. In this case, one checks that $\mathrm{char}(\mathcal{A})$ is the zero locus of $\det(\sigma_{r_i+s_j}(A_{ij}))$. Even if A is not normal, Sato-Kashiwara [29] computed the characteristic variety of $\mathcal{A}$ using the notion of determinant in non commutative fields, as we now recall.

Let K be a field, not necessarily commutative, and set $\overline{\mathsf{K}} = (\mathsf{K}^\times/[\mathsf{K}^\times,\mathsf{K}^\times]) \cup \{0\}$, where $[\mathsf{K}^\times,\mathsf{K}^\times]$ denotes the commutator subgroup of the multiplicative group $\mathsf{K}^\times = \mathsf{K} \setminus \{0\}$. Denote by $\mathrm{Mat}_m(\mathsf{K})$ the ring of $m \times m$ matrices with elements in K. Dieudonné [11] (see [4] for an exposition) proved that there exists a unique multiplicative morphism

$$\mathrm{Det}\colon \mathrm{Mat}_m(\mathsf{K}) \to \overline{\mathsf{K}},$$

satisfying the axioms: (i) $\mathrm{Det}(B) = \overline{c}\,\mathrm{Det}(A)$ if B is obtained from A by multiplying one row of A on the left by $c \in \mathsf{K}$ (where $\overline{c}$ denotes the image of c by the map $\mathsf{K} \to \overline{\mathsf{K}}$); (ii) $\mathrm{Det}(B) = \mathrm{Det}(A)$ if B is obtained from A by adding one row to another; (iii) the unit matrix has determinant $\overline{1}$. Such a determinant satisfies natural properties as $\mathrm{Det}(AB) = \mathrm{Det}(A)\,\mathrm{Det}(B) = \mathrm{Det}(A \oplus B)$, and an $m \times m$ matrix A is invertible as a left (resp. right) K-linear endomorphism of K^m if and only if $\mathrm{Det}(A) \neq 0$.

Concerning the computation of $\mathrm{Det}(A)$, denote by $\mathrm{GL}_m(\mathsf{K})$ the group of non-singular matrices, and by $\mathrm{SL}_m(\mathsf{K})$ its subgroup generated by the matrices $U_{ij}(c)$, for $i \neq j$, obtained from the unit matrix I_m by replacing the zero in the i-th row and j-th column by c. The product $U_{ij}(c)A$ amounts to adding to the i-th row of A its j-th row multiplied on the left by c. The usual Gauss method shows that for any $A \in \mathrm{GL}_m(\mathsf{K})$ there exist $c \neq 0$ and $U \in \mathrm{SL}_m(\mathsf{K})$ such that $A = U\,D_m(c)$, where $D_m(c)$ is the matrix obtained

from I_m by replacing the 1 in the m-th row and m-th column by c. One then has $\operatorname{Det} A = \overline{c}$. Note that c is in fact unique only up to commutators, since one checks that $D_m(ded^{-1}e^{-1}) \in \mathrm{SL}_m(\mathsf{K})$ if $m \geq 2$.

Let K be endowed with a filtration FK. Assuming that the associated graded ring GK is commutative, the universal property of $\overline{\mathsf{K}}$ associates a multiplicative morphism $\overline{\sigma}\colon \overline{\mathsf{K}} \to \mathsf{GK}$ to the symbol map $\sigma\colon \mathsf{K} \to \mathsf{GK}$. One sets

$$\det(A) = \overline{\sigma}(\operatorname{Det}(A)) \in \mathsf{GK}. \tag{1.2}$$

Let now X be a complex manifold. The ring $\mathcal{D}_X$ admits a (e.g. left) field of fractions $\mathsf{K}(\mathcal{D}_X)$, and the above construction yields a notion of determinant for matrices of differential operators (see [13]) that extends the classical one for normal matrices

$$\det\colon \mathrm{Mat}_m(\mathcal{D}_X) \hookrightarrow \mathrm{Mat}_m(\mathsf{K}(\mathcal{D}_X)) \to \mathsf{GK}(\mathcal{D}_X) \hookrightarrow \pi_*\mathcal{M}_{T^*X},$$

where $\mathcal{M}_{T^*X}$ denotes the sheaf of meromorphic functions on T^*X. Sato-Kashiwara proved that such a determinant is in fact holomorphic, and that it computes the characteristic variety of the associated $\mathcal{D}_X$-module.

Theorem 1.1 (see [29]). *Let $A \in \mathrm{Mat}_m(\mathcal{D}_X)$, and denote by $\mathcal{A} = \mathcal{D}_X^m/\mathcal{D}_X^m A$ its associated $\mathcal{D}_X$-module. Then*

(o) $\det(A) \in \pi_*\mathcal{O}_{T^*X}$ *is a homogeneous polynomial in the fibers of π,*

(i) *if* $\det(A) \not\equiv 0$, *then* $\operatorname{char}(\mathcal{A})$ *is the zero locus of* $\det(A)$.

In fact, Sato-Kashiwara obtained the above result as a corollary of Theorem 1.2 below.

Let $\mathcal{E}_X$ be the sheaf of microdifferential operators on T^*X, introduced by Sato-Kawai-Kashiwara [30] (see [31] for an exposition). This is an analytic localization of $\mathsf{F}\mathcal{D}_X$, so that $P \in \mathcal{E}_X$ is invertible if and only if $\sigma(P) \in \mathcal{O}_{T^*X}$ does not vanish. The $\mathbb{Z}$-filtered ring $\mathcal{E}_X$ admits a field of fractions $\mathsf{K}(\mathcal{E}_X)$, and (1.2) defines the notion of determinant for square matrices with elements in $\mathcal{E}_X$. Denote by $\dot{T}^*X = T^*X \setminus X$ the complement of the zero-section in T^*X.

Theorem 1.2 (see [29]). *Let $A = (A_{ij})$ be a square matrix with elements in $\mathcal{E}_X(U)$, for an open subset $U \subset \dot{T}^*X$. Then*

(o) $\det(A)$ *is a homogeneous section of* $\mathcal{O}_{T^*X}(U)$,

(i) A *is invertible in* $\mathcal{E}_X(U)$ *if and only if* $\det(A)$ *vanishes nowhere in* U,

(ii) *if* A *is normal as in* (1.1), *then* $\det(A) = \det(\sigma_{r_i+s_j}(A_{ij}))$,

(iii) *if* $P \in \mathcal{E}_X$ *satisfies* $[P, A] = 0$, *then* $\{\sigma(P), \det(A)\} = 0$, *where* $\{\cdot,\cdot\}$ *denotes the Poisson bracket.*

Actually, Sato-Kashiwara's proof of the above result implies that the Dieudonné determinant in $\mathcal{E}_X$ is "almost regular", in the sense that there is a commutative diagram

(1.3)
$$\begin{array}{ccccc} & & \overset{\det}{\frown} & & \\ \mathrm{Mat}_m(\mathsf{K}(\mathcal{E}_X)) & \xrightarrow[\mathrm{Det}]{} & \overline{\mathsf{K}(\mathcal{E}_X)} & \xrightarrow{\overline{\sigma}} & \mathcal{M}_{T^*X} \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{Mat}_m(\mathcal{E}_X) & \overset{\text{up to codimension 2}}{\cdots\cdots\cdots\cdots} & \overline{\mathcal{E}_X} & \xrightarrow{\overline{\sigma}} & \mathcal{O}_{T^*X}, \end{array}$$

where $\overline{\mathcal{E}_X}$ denotes the subsheaf of $\overline{\mathsf{K}(\mathcal{E}_X)}$ whose germ at $p \in T^*X$ is given by

$$(\overline{\mathcal{E}_X})_p = \{\overline{Q^{-1}P} \colon P, Q \in (\mathcal{E}_X)_p,\ \sigma(Q)(p) \neq 0\}.$$

More precisely

Lemma 1.3. *Let* $U \subset \dot{T}^*X$ *be an open subset,* $V \subset U$ *a smooth hypersurface, and* $p \in V$. *Let* $A \in \mathrm{Mat}_n(\mathcal{E}_X(U))$. *Then, there exist an open neighborhood* $\Omega \ni p$, *a closed subset* $Z \subset V \cap \Omega$ *of codimension one in* V, *and an operator* $P \in \mathcal{E}_X(\Omega)$, *such that*

$$\mathrm{Det}(A)|_{\Omega\setminus Z} = \overline{P}|_{\Omega\setminus Z}.$$

Remark 1.4. Adjamagbo [1] proposed an algebraic approach to the regularity of determinants in filtered rings, based on the notion of normality recalled in (1.1). In this framework, he obtained the analogue of Theorems 1.1, 1.2, and A.1. However, the analogue of Theorem 2.2 is stated in loc. cit. in a weaker form. Our proof of Theorem 2.2 is based on Lemma 1.3

above. That is why in this paper we prefer to rely on the original Sato-Kashiwara's argument, that we recall in Appendix A.

Example 1.5. To conclude this section, let us give two examples of computation of the determinant (for matrices which are not normal).

(i) This is the original Sato-Kashiwara's example. Let $X = \mathbb{C}$ with holomorphic coordinate z, and consider

$$A = \begin{pmatrix} zD + \alpha(z) & D^2 + \beta(z)D + \gamma(z) \\ z^2 & zD + \delta(z) \end{pmatrix}.$$

Then

$$A = \begin{pmatrix} 1 & 0 \\ 0 & z^2 \end{pmatrix} \begin{pmatrix} 1 & zD + \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & Q \\ 1 & zD + \delta + 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z^{-2} \end{pmatrix},$$

where $Q = (\delta + \alpha - 1 - z\beta)zD + \alpha\delta - 2\delta + z\delta' - z^2\gamma$. Denoting by ζ the covariable in $T^*\mathbb{C}$, one then has

$$\det A = \begin{cases} (\delta + \alpha - 1 - z\beta)z\zeta & \text{if it is not zero,} \\ \alpha\delta - 2\delta + z\delta' - z^2\gamma & \text{otherwise.} \end{cases}$$

(ii) Let $X = \mathbb{C}^2$ with holomorphic coordinate (z_0, z_1), and consider the matrix

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} D_0^2 + D_0 & z_0z_1D_0 + z_0z_1 + z_1 \\ D_0D_1 & z_0z_1D_1 + z_0 + 1 \end{pmatrix}.$$

Noticing that

$$\begin{pmatrix} 1 & 0 \\ -A_{21} & A_{11} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{11} \end{pmatrix},$$

we have

$$\operatorname{Det} A = \overline{D_0^2 + D_0}.$$

2. Newton Polygon

In this section we recall the notion of Newton polygon for microdifferential operators as discussed in Laurent [22] (see also [21]). We then use it to state a second-microlocal version of Sato-Kashiwara's theorem.

Let V be an involutive submanifold of T^*X, and $\tau: T_V(T^*X) \to V$ its normal bundle. Besides the order filtration, the ring $\mathcal{E}_X|_V$ is naturally endowed with the so-called V-filtration, $\mathsf{V}\mathcal{E}_X = \bigcup_{j\in\mathbb{Z}} \mathsf{V}_j\mathcal{E}_X$.

Example 2.1. (a) Outside of the zero-section, one has

$$\mathsf{V}_j\mathcal{E}_X = (\mathsf{F}_j\mathcal{E}_X)\,\mathcal{E}_V,$$

where $\mathcal{E}_V$ denotes the sub-$\mathsf{F}_0\mathcal{E}_X$-algebra of $\mathcal{E}_X|_V$ generated by those operators in $\mathsf{F}_1\mathcal{E}_X$ whose symbol of degree 1 belongs to the defining ideal of V in $\mathcal{O}_{T^*X}$ (see [16]).

(b) Let $Y \subset X$ be a submanifold, and denote by $\mathcal{I}_Y$ its defining ideal in $\mathcal{O}_X$. If $V = T^*_Y X$, then $\mathsf{V}_j\mathcal{E}_X = \sum_{k+\ell\le j}(\mathsf{F}_k\mathcal{E}_X)\,\pi^{-1}(\mathsf{V}_\ell\mathcal{D}_X)$, where

$$\mathsf{V}_\ell\mathcal{D}_X = \{P \in \mathcal{D}_X|_Y : \forall k \in \mathbb{Z},\ P\,\mathcal{I}_Y^k \subset \mathcal{I}_Y^{k-\ell}\},$$

with $\mathcal{I}_Y^\ell = \mathcal{O}_X$ for $\ell \le 0$ (see [14]).

Let Σ be the set of pairs (r,s) of one of the three types:

(i) $r = \cdot$ and $s \in \mathbb{Q}$, $s \ge 1$,

(ii) $s = \cdot$ and $r \in \mathbb{Q} \cup \{\infty\}$, $r > 1$,

(iii) $r, s \in \mathbb{Q} \cup \{\infty\}$, and $1 \le s < r \le \infty$,

and let us consider the ordering of $\mathbb{Z}^2$ for which $(i',j') \underset{(r,s)}{\le} (i,j)$ reads, according to the type of $(r,s) \in \Sigma$,

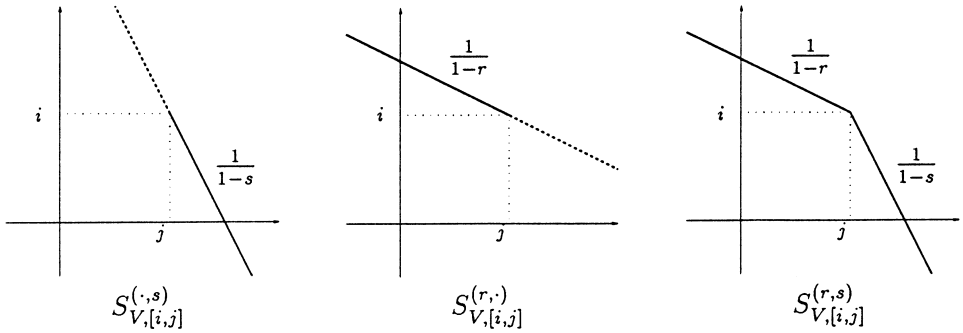
(i) $i' - i \le (1-s)(j'-j)$, the inequality being strict if $j' > j$,

(ii) $j' \le j + (i'-i)/(1-r)$, the inequality being strict if $i' > i$,

(iii) $j' \le j + (i'-i)/(1-r)$ and $i' - i \le (1-s)(j'-j)$.

Note that the ordering of $\mathbb{Z}^2$ given by (i) and (ii) is isomorphic to the lexicographical ordering, while the one given in (iii) is isomorphic to the product ordering.

Graphically, the set $S^{(r,s)}_{V,[i,j]} = \{(x,y) \in \mathbb{R}^2 : (x,y) \underset{(r,s)}{\leq} (i,j)\}$, lies to the left of the lines drawn in the following picture.



$S^{(\cdot,s)}_{V,[i,j]}$ $S^{(r,\cdot)}_{V,[i,j]}$ $S^{(r,s)}_{V,[i,j]}$

To the above ordering is associated the filtration of $\mathcal{E}_X|_V$ given by

$$\mathsf{F}^{(r,s)}_V \mathcal{E}_X = \bigcup_{i,j\in\mathbb{Z}} \mathsf{F}^{(r,s)}_{V,[i,j]} \mathcal{E}_X, \quad \mathsf{F}^{(r,s)}_{V,[i,j]} \mathcal{E}_X = \sum_{(i',j') \underset{(r,s)}{\leq} (i,j)} \mathsf{F}_{i'} \mathcal{E}_X \cap \mathsf{V}_{j'} \mathcal{E}_X.$$

For any $(r,s) \in \Sigma$, the graded ring $\mathsf{G}^{(r,s)}\mathcal{E}_X = \bigoplus_{i,j} \mathsf{F}^{(r,s)}_{V,[i,j]}\mathcal{E}_X / \sum_{[i',j'] \underset{(r,s)}{<} [i,j]} \mathsf{F}^{(r,s)}_{V,[i',j']}$ is identified to a subsheaf of $\tau_*\mathcal{O}_{T_V(T^*X)}$, the homogeneous elements being the functions which are homogeneous with respect to the $\mathbb{C}^\times$-actions on $T_V(T^*X)$ induced by π and τ. One denotes by $\sigma^{(r,s)}_V(P) \in \tau_*\mathcal{O}_{T_V(T^*X)}$ the principal symbol of type (r,s) of an operator $P \in \mathcal{E}_X|_V$. Let us also consider the filtration

$$\mathsf{F}^{(s)}_V \mathcal{E}_X = \begin{cases} \mathsf{F}^{(\cdot,s)}_V \mathcal{E}_X & \text{for } s \in \mathbb{Q},\ s \geq 1, \\ \mathsf{F}^{(\infty,\cdot)}_V \mathcal{E}_X & \text{for } s = \infty, \end{cases}$$

and denote by $\sigma^{(s)}_V(\cdot)$ the associated symbol.

The Newton polygon associated to $P \in \mathcal{E}_X|_V$ is the convex subset $N_V(P) \subset \mathbb{R}^2$, obtained by intersecting the sets $S^{(r,s)}_{V,[i,j]}$ such that $P \in \mathsf{F}^{(r,s)}_{V,[i,j]}\mathcal{E}_X$. It is easy to check that the boundary of $N_V(P)$ has a finite

number, say $e+2$, of edges with slopes $-\infty = m_0 < m_1 < \cdots < m_e < m_{e+1} = 0$, and has vertices with integral coordinates. Set $s_k = 1 - 1/m_k$ for $k = 0, \ldots, e$, so that $1 = s_0 < s_1 < \cdots < s_e < \infty$.

The Newton polygon $N_V(P)$ is a convenient way to keep track of the whole family of symbols associated to P. In fact, setting $s_{e+1} = \infty$, according to the type of $(r, s) \in \Sigma$ one has

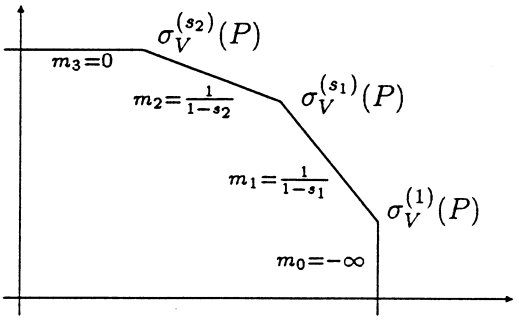
$$\text{(2.1)} \qquad \begin{cases} \sigma_V^{(\cdot,s)}(P) = \sigma_V^{(s_k)}(P) & \text{for } s_k \le s < s_{k+1}, \\ \sigma_V^{(r,\cdot)}(P) = \sigma_V^{(s_k)}(P) & \text{for } s_k < r \le s_{k+1}, \end{cases}$$

and

$$\text{(2.2)} \qquad \sigma_V^{(r,s)}(P) = \begin{cases} \sigma_V^{(s_k)}(P) & \text{for } s_k \le s < r \le s_{k+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the $\mathbb{Z}^2$-degree of P in $\mathsf{F}_V^{(r,s)}\mathcal{E}_X$ is given by the coordinates of the corresponding vertex of $N_V(P)$.

$N_V(P)$:



The characteristic variety $\mathrm{char}_V^{(r,s)}(\mathcal{M}) \subset T_V(T^*X)$ of a coherent $\mathcal{E}_X|_V$-module $\mathcal{M}$, is defined by using the notion of good $\mathsf{F}_V^{(r,s)}$-filtrations. Although the notion of Newton polygon for $\mathcal{M}$ loses its meaning, Laurent [23] associates to $\mathcal{M}$ a finite number of slopes $-\infty = m_0 < m_1 < \cdots < m_e < m_{e+1} = 0$, so that, setting $s_k = 1 - 1/m_k$, one has

$$\begin{cases} \mathrm{char}_V^{(\cdot,s)}(\mathcal{M}) = \mathrm{char}_V^{(s_k)}(\mathcal{M}) & \text{for } s_k \le s < s_{k+1}, \\ \mathrm{char}_V^{(r,\cdot)}(\mathcal{M}) = \mathrm{char}_V^{(s_k)}(\mathcal{M}) & \text{for } s_k < r \le s_{k+1}. \end{cases}$$

However, the analogue of property (2.2) does not seem to hold, in general.

The filtered ring $\mathsf{F}_V^{(s)}\mathcal{E}_X$ admits a field of fractions, and by formula (1.2) one may associate to $A \in \mathrm{Mat}_m(\mathcal{E}_X|_V)$ a determinant, that we denote by

$\det_V^{(s)}(A)$. Sato-Kashiwara's proof of regularity dealt with a $\mathbb{Z}$-filtration, and it is straightforward to adapt it for the $\mathsf{F}_V^{(s)}$ filtration, which is indexed by $\mathbb{Z}^2$ endowed with the lexicographical order. Then, we have the following analogue of Theorem 1.1.

Theorem 2.2. *Let $V \subset T^*X$ be a locally closed involutive submanifold, and denote by $\mathcal{A} = \mathcal{E}_X^m/\mathcal{E}_X^m A$ the $\mathcal{E}_X$-module associated to $A \in \mathrm{Mat}_m(\mathcal{E}_X|_V)$. Then,*

(o) $\det_V^{(s)}(A)$ *is a homogeneous element of $\tau_*\mathcal{O}_{T_V(T^*X)}$,*

(i) *if $\det_V^{(s)}(A) \not\equiv 0$, then $\mathrm{char}_V^{(s)}(\mathcal{A})$ is the zero locus of $\det_V^{(s)}(A)$.*

(ii) *Assume that V is a hypersurface. Then A admits a Newton polygon $N_V(A)$. More precisely, there exists a convex subset $N_V(A) \subset \mathbb{R}^2$ whose boundary has the same slopes $-\infty = m_0 < m_1 < \cdots < m_e < m_{e+1} = 0$ as $\mathcal{A}$, so that, setting $s_k = 1 - 1/m_k$, one has*

$$\text{(2.3)}\quad \begin{cases} \mathrm{char}_V^{(\cdot,s)}(\mathcal{A}) = \det_V^{(s_k)}(A)^{-1}(0) & \text{for } s_k \le s < s_{k+1}, \\ \mathrm{char}_V^{(r,\cdot)}(\mathcal{A}) = \det_V^{(s_k)}(A)^{-1}(0) & \text{for } s_k < r \le s_{k+1}. \end{cases}$$

Moreover, the $\mathbb{Z}^2$-degree of $\det_V^{(s_k)}(A)$ is given by the coordinates of the corresponding vertex of $N_V(A)$.

As pointed out in Remark 1.4, an algebraic approach to this result is proposed in [1]. However, the characteristic property (2.3) of the Newton polygon is stated in loc.cit. only for $\dim X = 1$.

Proof. A proof of statements (o) and (i) is given in Appendix A, and we only prove here assertion (ii).

Let us begin by noticing that if $P, Q \in \mathcal{E}_X|_V$ satisfy $\overline{P} = \overline{Q}$ in $\overline{\mathcal{E}_X}|_V$, then $N_V(P) = N_V(Q)$. In fact, in this case one has $\sigma_V^{(s)}(P) = \sigma_V^{(s)}(Q)$ for any s.

By Theorem 1.2 (iii), we may use the trick of the dummy variable, and assume that $V \subset \dot{T}^*X$. By Lemma 1.3, there exists $P \in \mathcal{E}_X$ such that $\mathrm{Det}(A)|_{V\setminus Z} = \overline{P}$, where Z is an analytic subset of codimension at least 1 in V. By the argument in the previous paragraph, the definition $N_V(A) = N_V(P)$ is well-posed. On $\tau^{-1}(V \setminus Z)$ we have $\det_V^{(s)}(A) = \overline{\sigma}_V^{(s)}(\overline{P}) = \sigma_V^{(s)}(P)$ for any s, and such equality still holds in $\tau^{-1}(Z)$ by analytic continuation. Then, (2.3) follows from (i) and (2.1). □

3. Systems with Regular Singularities

In this section we recall the notion of $\mathcal{E}_X$-module with regular singularities. In the case of determined systems, we show that regularity can be tested using the second-microlocal Sato-Kashiwara determinant.

Let $V \subset \dot{T}^*X$ be an involutive submanifold. As in Example 2.1, let $\mathcal{E}_V$ be the subalgebra of $\mathcal{E}_X$ generated over $\mathsf{F}_0\mathcal{E}_X$ by the sections P of $\mathsf{F}_1\mathcal{E}_X$ such that $\sigma_1(P)$ belongs to the annihilating ideal of V in T^*X. Following Kashiwara-Oshima [16], one says that a coherent $\mathcal{E}_X$-module $\mathcal{M}$ has regular singularities along V if locally there exists a coherent sub-$\mathsf{F}_0\mathcal{E}_X$-module $\mathcal{M}_0$ of $\mathcal{M}$ which generates it over $\mathcal{E}_X$, and such that $\mathcal{E}_V\mathcal{M}_0 \subset \mathcal{M}_0$. In particular, recall that a system with regular singularities along V is supported by V.

As proved by Laurent [22, Theorem 3.1.7] and Monteiro-Fernandes [28] (see [31] for an exposition), a coherent $\mathcal{E}_X$-module $\mathcal{M}$ has regular singularities along V if and only if

$$\operatorname{char}_V^{(\infty,1)}(\mathcal{M}) \subset V, \tag{3.1}$$

the zero-section of $T_V(T^*X)$. Moreover, one says that $\mathcal{M}$ satisfies the Levi conditions along V if

$$\operatorname{char}_V^{(\infty,1)}(\mathcal{M}) = \operatorname{char}_V^{(\cdot,1)}(\mathcal{M}). \tag{3.2}$$

LEMMA 3.1. *Let $V \subset \dot{T}^*X$ be a locally closed regular involutive submanifold, and $\mathcal{M}$ a coherent $\mathcal{E}_X$-module defined in a neighborhood of V, with $\operatorname{char}(\mathcal{M}) = V$. Then, the following conditions are equivalent*

(i) *$\mathcal{M}$ has regular singularities along V,*

(ii) *$\mathcal{M}$ satisfies the Levi conditions along V.*

PROOF. By [22, Proposition 3.1.2] $\operatorname{char}_V^{(\cdot,1)}(\mathcal{M})$ is the Whitney normal cone of $\operatorname{char}(\mathcal{M})$ along V, and hence $\operatorname{char}_V^{(\cdot,1)}(\mathcal{M}) = \operatorname{char}(\mathcal{M})$. □

Combining Theorem 2.2 and Lemma 3.1, we get

THEOREM 3.2. *Let $\mathcal{A}$ be the $\mathcal{E}_X$-module associated to $A \in \operatorname{Mat}_m(\mathcal{E}_X)$, and set $V = \dot{T}^*X \cap \det(A)^{-1}(0)$. Assume that V is a smooth regular hypersurface. Then, the following conditions are equivalent*

(i) *$\mathcal{A}$ has regular singularities along V,*

(ii) *$\mathcal{A}$ satisfies the Levi conditions along V,*

(iii) *$N_V(A)$ is reduced to a quadrant.*

For instance, the system associated to the matrix A of Example 1.5 (ii) has regular singularities along his characteristic variety $\{\zeta_0 = 0\}$.

4. Determined Cauchy Problem

As we now recall, Levi conditions are sufficient for the well-poseness of the C^∞ Cauchy problem for hyperbolic systems with real constant multiplicities. Here, we discuss in particular the case of determined systems.

Let Y be a submanifold of X. To a coherent $\mathcal{D}_X$-module $\mathcal{M}$ one associates its pull-back $\mathcal{M}_Y = \mathcal{D}_{Y\to X} \otimes_{\mathcal{D}_X|_Y} \mathcal{M}|_Y$, where $\mathcal{D}_{Y\to X} = \mathcal{O}_Y \otimes_{\mathcal{O}_X|_Y} \mathcal{D}_X|_Y$ denotes the transfer bi-module. One says that Y is non-characteristic for $\mathcal{M}$ if

$$\operatorname{char}(\mathcal{M}) \cap T^*_Y X \subset T^*_X X.$$

By [15], if Y is non-characteristic for $\mathcal{M}$, then $\mathcal{M}_Y$ is coherent, and the Cauchy-Kovalevskaya-Kashiwara theorem states the isomorphism

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)|_Y \xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y).$$

Let $N \subset M$ be real analytic manifolds of which $Y \subset X$ is a complexification. One says that N is hyperbolic for $\mathcal{M}$ if

$$T^*_N M \cap C_{T^*_M X}(\operatorname{char}(\mathcal{M})) \subset T^*_M M,$$

where $C_{T^*_M X}(\cdot)$ denotes the Whitney normal cone, and we used the embedding $T^*M \to T_{T^*_M X} T^*X$ (see [18, §6.2]). By [5] and [17], hyperbolicity is a sufficient condition for the well-poseness of the Cauchy problem in the framework of Sato hyperfunctions. Concerning C^∞ functions, one has

Theorem 4.1 (see [9]). *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module, and set $V = \operatorname{char}(\mathcal{M}) \cap \dot{T}^*X$. Assume*

(o) *$V \cap \dot{T}^*_M X$ is a smooth regular involutive submanifold of $\dot{T}^*_M X$, of which V is a complexification,*

(i) *N is hyperbolic for $\mathcal{M}$,*

(ii) *$\mathcal{M}$ satisfies the Levi conditions along V.*

Then, the C^∞ Cauchy problem for $\mathcal{M}$ is well-posed, i.e.

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}^\infty_M)|_N \xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{C}^\infty_N).$$

Remark 4.2.

(i) In [9] this result was stated for distributions. However, as pointed out in §7.1 of loc. cit., using the functor of formal microlocalization introduced in [7, 8] one can easily adapt the proof to yield the well-poseness for the C^∞ Cauchy problem.

(ii) Using the results of [20] instead of those of [9], one gets propagation for C^∞ solutions instead of the well-poseness for the Cauchy problem.

Example 4.3. Let $(x) = (x_1, x')$ be a local system of coordinates in M, and let N be the hypersurface $x_1 = 0$. Let $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P$ be associated to a single differential operator $P \in \mathcal{D}_X$ of order r. Then, hypotheses (o) and (i) of Theorem 4.1 are equivalent to say that $\sigma(P)$ is a hyperbolic polynomial with respect to N, with real constant multiplicities, which means that, for x and η' real,

$$\sigma(P)(x; \tau, \eta') \text{ has } r \text{ real roots } \tau, \text{ with constant multiplicities.}$$

Moreover, condition (ii) coincides with the usual Levi conditions. In this case the above result is classical, and Levi conditions are also known to be necessary for the well-poseness of the Cauchy problem (see [12], [10] and [6]).

Combining Theorems 4.1 and 3.2, we get

Theorem 4.4. *Let $N \subset M$ be a hypersurface, and $\mathcal{A} = \mathcal{D}_X^m/\mathcal{D}_X^m A$ the $\mathcal{D}_X$-module associated to $A \in \mathrm{Mat}_m(\mathcal{D}_X)$. Denote by V the zero locus of* $\det(A)$ *in $\dot{T}^*X$, and assume*

(i) $\det(A)$ *is a hyperbolic polynomial with respect to* N*, with real constant multiplicities,*

(ii) *the equivalent conditions in Theorem* 3.2 *are satisfied.*

Then

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{A}, \mathcal{C}^\infty_M)|_N \xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{A}_Y, \mathcal{C}^\infty_N). \tag{4.1}$$

If A is of Kovalevskayan type (see below), the Cauchy problem can be stated in a classical formalism—i.e. without the use of $\mathcal{D}$-module theory—and has been studied by many authors. In particular, Vaillant [33] and Matsumoto [25] gave necessary and sufficient conditions for well-poseness of the hyperbolic Cauchy problem for systems with real constant multiplicities. (See also [34] and [26].) Let us discuss the relation with our approach.

Let r be the degree of $\det(A)$. By (i), Y is non-characteristic for $\mathcal{A}$. Under this assumption, Andronikof [3] proved that $\mathcal{A}_Y$ is projective of rank r, and that $\mathcal{A}_Y$ is locally free if and only if A is normal in the sense of (1.1). As we now recall, this means that (4.1) is equivalent to a Cauchy problem with free traces if and only if A is normal.

By definition, $\mathcal{A}$ admits the presentation

$$0 \to \mathcal{D}_X^m \underset{\cdot A}{\to} \mathcal{D}_X^m \to \mathcal{A} \to 0.$$

Since $\mathcal{A}_Y$ is projective, the isomorphism (4.1) is equivalent to

$$\begin{cases} \operatorname{coker}(\mathcal{C}^\infty_M|_N \underset{A\cdot}{\to} \mathcal{C}^\infty_M|_N) = 0, \\ \ker(\mathcal{C}^\infty_M|_N \underset{A\cdot}{\to} \mathcal{C}^\infty_M|_N) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{A}_Y, \mathcal{C}^\infty_N). \end{cases} \tag{4.2}$$

Assume that A is normal, and let $\gamma : \mathcal{A}_Y \xleftarrow{\sim} \mathcal{D}_Y^r$ be a local isomorphism. Then (4.2) is equivalent to the well-poseness of the Cauchy problem

$$\begin{cases} A\,u = v, & v \in (\mathcal{C}^\infty_M)^m, \\ \gamma(u) = w, & w \in (\mathcal{C}^\infty_N)^r. \end{cases} \tag{4.3}$$

Let us state a lemma which is useful in order to compute explicitly the trace morphism γ. (Refer to [35] for a discussion of possible choices of the trace morphism, in a classical formalism.) Denote by $\mathcal{O}_X\hat{|}_Y$ the formal restriction of $\mathcal{O}_X$ to Y.

LEMMA 4.5. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module, $\mathcal{N}$ a coherent $\mathcal{D}_Y$-module, and $\gamma{:}\mathcal{N} \to \mathcal{M}_Y$ a $\mathcal{D}_Y$-linear morphism. Assume that the morphism*

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X\hat{|}_Y) \to R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N}, \mathcal{O}_Y) \tag{4.4}$$

associated to γ is an isomorphism. Then γ is an isomorphism.

PROOF. By construction, there is a commutative diagram

$$\begin{array}{ccc} & R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X\hat{|}_Y) & \\ \swarrow^{\sim} & & \searrow^{\sim} \\ R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N}, \mathcal{O}_Y) & \longleftarrow & R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y), \end{array}$$

where the isomorphism on the right hand side is the formal version of the Cauchy-Kovalevskaya-Kashiwara theorem (see [19, Theorem 7.2]), and the horizontal line is induced by γ. It follows that $R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N}, \mathcal{O}_Y) \xleftarrow{\sim} R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y)$. Considering a distinguished triangle $\mathcal{N} \underset{\gamma}{\to} \mathcal{M}_Y \to \mathcal{P} \underset{+1}{\to}$, the last isomorphism is equivalent to $R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{P}, \mathcal{O}_Y) = 0$. Since $\mathcal{P}$ is coherent, this implies $\mathcal{P} = 0$ (see [27]). Finally, this gives $\mathcal{N} \underset{\gamma}{\xrightarrow{\sim}} \mathcal{M}_Y$. □

Let $(x) = (x_1, x')$ be a local system of coordinates in M, and let N be the hypersurface $x_1 = 0$. One says that A is of Kovalevskayan type if

$$A(x, D) = I_m D_1 + B(x, D').$$

Note that $\mathcal{A}_Y$ is a quotient of $\mathcal{D}^m_{Y\to X}$, and let $\gamma : \mathcal{A}_Y \xleftarrow{\sim} \mathcal{D}^m_Y$ be induced by the natural morphism $\mathcal{D}_Y \to \mathcal{D}_{Y\to X}$ sending 1 to $1_{Y\to X}$. The isomorphism (4.4) is then equivalent to the well-poseness of the formal Cauchy problem

$$\begin{cases} A\,u = v, & v \in (\mathcal{O}_X\hat{|}_Y)^m, \\ u|_Y = w, & w \in (\mathcal{O}_Y)^m, \end{cases} \tag{4.5}$$

which is easily proved by computing recurrently the coefficients in the formal power series. By Lemma 4.5, we get $\mathcal{M}_Y \simeq (\mathcal{D}_Y)^m$. Hence, the Cauchy problem (4.3) is written as

$$\begin{cases} A\,u = v, & v \in (\mathcal{C}^\infty_M)^m, \\ u|_Y = w, & w \in (\mathcal{C}^\infty_N)^m. \end{cases} \tag{4.6}$$

Necessary and sufficient conditions for the well-poseness of (4.6) are described in Vaillant [33] and Matsumoto [25] (by [32], these conditions are equivalent for matrices with small Jordan blocks). Besides applying only to systems of Kovalevskayan type, each of these descriptions has some drawbacks with respect to Theorem 4.4. Namely, conditions in [33] have been tested only for matrices with small Jordan blocks, while conditions in [25] are expressed in terms of a normal form for the matrix A. Let us briefly recall how Levi conditions are presented in [25], and note that they are actually equivalent to those in Theorem 4.4.

In the framework of what he calls "meromorphic formal symbol class", Matsumoto [25] shows that A can be reduced to a direct sum of first order systems. Each such system has only one characteristic root, its "principal symbol" is in Jordan form, and the lower order terms are in Sylvester form. Assuming that the characteristic root is $\zeta_1 = 0$, the model of such a Jordan block is the matrix

$$J(x,D) = I_k D_1 + \begin{pmatrix} 0 & |D'| & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & |D'| \\ b_1 & \cdots & b_{k-1} & b_k \end{pmatrix}, \quad \text{with } b_j = b_j(x,D') \in \mathsf{F}_0\mathcal{E}_X.$$

Then, Levi conditions in [25] are stated by asking that the elements b_j of each Jordan block (say, of size $k \times k$) have a degree not bigger than $j - k$.

In the language of $\mathcal{E}_X$-modules, reductions in the meromorphic formal symbol class correspond to quantized contact transformations with polar singularities along a hypersurface transversal to the characteristic variety $V = \mathrm{char}(\mathcal{A})$. By analytic continuation, condition (ii) in Theorem 4.4 is invariant under such transformations. One then checks that the above conditions are precisely those under which $\mathrm{det}_V^{(\infty,\cdot)}(J) = \zeta_1^k$, so that $\mathrm{det}_V^{(\infty,\cdot)}(A)$ is a local equation for V.

Remark 4.6. Even without the assumption of A being of Kovalevskayan type, one might prove that Levi conditions (ii) in Theorem 4.4 are also necessary for the well-poseness of the C^∞ Cauchy problem for $\mathcal{A}$. Since such conditions are known to be necessary for single differential operators, this could be done by reducing A to triangular form. In fact, Sato-Kashiwara proof of Lemma 1.3, that we recall below, shows that for

any $p \in V = \mathrm{char}(\mathcal{A})$, there exist an open conic neighborhood $\Omega \ni p$, and an analytic subset $W \subset \Omega$, such that $W \cap V$ has codimension at least 2 in Ω, and $A|_{\Omega\setminus W} = E\,T$, for $E, T \in \mathrm{Mat}_m(\mathcal{E}_X(\Omega \setminus W))$, with E invertible, and T lower triangular (here W is the union of the hypersurfaces $\sigma(A_{11})/\zeta_1^v = 0$ obtained at each induction step of the proof). Technically, one should use the functor of formal microlocalization of [8] along the lines of [9], but we will not do it here.

Appendix A. Sato-Kashiwara's Argument

The aim of this Appendix is to recall the original Sato-Kashiwara's argument, and to obtain along their lines a proof of Theorem 2.2.

Proof of Lemma 1.3 (see [29]). By definition of Dieudonné determinant, $\mathrm{Det}(A) = \overline{Q^{-1}P} \in \overline{\mathsf{K}(\mathcal{E}_X)}$ for some $P, Q \in \mathcal{E}_X$. In particular, $\mathrm{Det}(A) \in \overline{\mathcal{E}}_X$ outside of the characteristic hypersurface $S = \sigma(Q)^{-1}(0)$. Let us then consider $\mathrm{Det}(A)|_S$. Denote by α the canonical one-form on T^*X, and let $S_{\mathrm{reg}} \subset S$ be the open subset where S is smooth and $\alpha \neq 0$. Since $S \setminus S_{\mathrm{reg}}$ has codimension at least 2 in T^*X, it is not restrictive to assume $S = S_{\mathrm{reg}}$. Recall that $\mathcal{E}_X$ is invariant by homogeneous symplectic transformations of T^*X (i.e. transformations which preserve the canonical one-form). One may then assume that S is defined by the equation $\zeta_1 = 0$, in a system of local symplectic coordinates $(z;\zeta) \in T^*X$.

The proof will be by induction on the size m of A. Let v_{ij} be the multiplicity of $\sigma(A_{ij})$ at $\{\zeta_1 = 0\}$, and set $v = \min\{v_{ij} \colon A_{ij} \neq 0\}$. If $A = 0$ there is nothing to prove, and hence one may assume $v \geq 0$. One proceeds by induction on the multiplicity v. Without loss of generality one may assume $v = v_{11}$. Up to a subset of codimension at least 2, one may also assume that $\sigma(A_{11})/\zeta_1^v$ never vanishes on $\{\zeta_1 = 0\}$. By Weierstrass division theorem, one may write $A_{1j} = A_{11}Q_j + R_j$, for $R_j = \sum_{k<v} R_{j,k}D_1^k$ with $R_{j,k}$ independent of D_1. Hence $A = \widetilde{A} \cdot E$, with

$$\widetilde{A} \;=\; \begin{pmatrix} A_{11} & R_2 & \cdots & R_m \\ A_{21} & A_{22} - A_{21}Q_2 & \cdots & A_{2m} - A_{21}Q_m \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{22} - A_{m1}Q_m & \cdots & A_{mm} - A_{m1}Q_m \end{pmatrix},$$

$$E \;=\; \begin{pmatrix} 1 & Q_2 & \cdots & Q_m \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

One has $\mathrm{Det}(A) = \mathrm{Det}(\widetilde{A})$. If one of the R_j's is not zero, then the multiplicity of some $\sigma(R_j)$ at $\{\zeta_1 = 0\}$ is strictly less than v, and the induction proceeds. If $R_j = 0$ for any j, then $\mathrm{Det}(\widetilde{A})$ is the product of $\overline{A_{11}}$ with the determinant of a matrix of size $m-1$, and again the induction proceeds. □

Proof of Theorem 1.2 (see [29]). Since subsets of codimension at least 2 are removable singularities for $\mathcal{O}_{T^*X}$, it follows from Lemma 1.3 that $\det(A) \in \mathcal{O}_{T^*X}$. One can get the rest of the statement along the lines of the proof of Lemma 1.3 (see [2] for an exposition). □

Proof of Theorem 1.1 (see [29]). For a coherent $\mathcal{D}_X$-module $\mathcal{M}$ one has

$$\mathrm{char}(\mathcal{M}) = \mathrm{supp}(\mathcal{E}_X\mathcal{M}),$$

where $\mathcal{E}_X\mathcal{M} = \mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{M}$. Using Theorem 1.2 (iii) one may apply the trick of the dummy variable to work outside of the zero-section, and get the statement as a corollary of Theorem 1.2 (o) and (i). □

Let us now deal with the second microlocal version of Sato-Kashiwara determinant. We will set $\Xi = T^*X$ for short.

For $1 \le s \le \infty$, Laurent [22] introduced the sheaf $\mathcal{E}_V^{2(s)}$ of second microdifferential operators on $T_V\Xi$ (denoted $\mathcal{E}_V^{2(s,s)}$ in loc. cit.), which is a Gevrey localization of $\mathsf{F}_V^{(s)}\mathcal{E}_X$. The filtered ring $\mathsf{F}_V^{(s)}\mathcal{E}_V^{2(s)}$ admits a field of fractions, so that (1.2) defines a determinant for matrices with elements in $\mathcal{E}_V^{2(s)}$. Before stating the second microlocal analogue of Theorem 1.2, let us begin by recalling some notions of second microlocal geometry.

Recall that the canonical one-form $\alpha\colon \Xi \to T^*\Xi$ is the restriction to the diagonal $\Xi \subset \Xi \times_X \Xi$ of the map ${}^t\pi'\colon \Xi \times_X \Xi \to T^*\Xi$ associated to the projection π. The pull-back $\alpha_V = \tau^*(\alpha|_V)$ is a degenerate one-form on $T_V\Xi$, and one sets $T^*_{\mathrm{rel}}(T_V\Xi) = T^*(T_V\Xi)/\ker \alpha_V$. Denote by $\perp$ the orthogonal with respect to the symplectic form $d\alpha$, and by $H_f = \langle d\alpha, df \wedge \cdot \rangle$ the Hamiltonian vector field of $f \in \mathcal{O}_\Xi$. The map $df \mapsto H_f$ gives an identification

$T^*_V\Xi \xrightarrow{\sim} (TV)^\perp$. The inclusion $(TV)^\perp \subset TV$ induces by duality a projection $T^*V \to T_V\Xi$. The diagram

$$\begin{array}{ccc} T_V\Xi \times_V T^*V & \xrightarrow{{}^t\tau'} & T^*(T_V\Xi) \\ \downarrow & & \downarrow \\ T_V\Xi \times_V T_V\Xi & \xrightarrow{\tilde\tau} & T^*_{\mathrm{rel}}(T_V\Xi) \end{array}$$

defines a map $\tilde\tau$. The relative one form $\alpha^{\mathrm{rel}}_V : T_V\Xi \to T^*_{\mathrm{rel}}(T_V\Xi)$ is the restriction of $\tilde\tau$ to the diagonal, and the pair $(\alpha_V, \alpha^{\mathrm{rel}}_V)$ give $T_V\Xi$ a structure of homogeneous bisymplectic manifold. For example, let V be described by the equations $\xi = z = 0$ in a local system of symplectic coordinates $(x, y, z; \xi, \eta, \zeta) \in \Xi$. Then $T^*_{\mathrm{rel}}(T_V\Xi) \ni (x, y, \eta, \zeta, \tilde x, \tilde\zeta; x^*, \zeta^*)$, and $\alpha^{\mathrm{rel}}_V = \sum \tilde x_i\, dx_i + \sum \tilde\zeta_j\, d\zeta_j$, $\alpha_V = \sum \eta_k\, dy_k$.

THEOREM A.1. *Let $V \subset \Xi$ be a locally closed involutive submanifold, and $A = (A_{ij})$ a square matrix with elements in $\mathcal{E}^{2(s)}_V(\Omega)$, where $1 \le s \le \infty$, and Ω is an open subset of $\dot T_V\Xi$. Then*

(o) $\det^{(s)}_V(A)$ *is a homogeneous section of $\mathcal{O}_{T_V\Xi}(\Omega)$,*

(i) *A is invertible in $\mathcal{E}^{2(s)}_V(\Omega)$ if and only if $\det^{(s)}_V(A)$ vanishes nowhere in Ω,*

(ii) *if A is normal in the obvious sense, its determinant $\det^{(s)}_V(A)$ can be computed using Leray-Volevich weights,*

(iii) *if $P \in \mathcal{E}^{2(s)}_V$ satisfies $[P, A] = 0$, then $\{\sigma^{(s)}_V(P), \det^{(s)}_V(A)\}^{\mathrm{rel}} = 0$, where $\{\cdot,\cdot\}^{\mathrm{rel}}$ denotes the Poisson bracket associated to α^{rel}_V.*

(A similar result was obtained in [1].)

PROOF. As in the proof of Theorem 1.2, it is enough to show that $\det^{(s)}_V$ is "almost regular". The line of the proof is the same as that of Lemma 1.3, and we only sketch it here.

By definition, $\mathrm{Det}(A) = \overline{Q^{-1}P}$ for some $P, Q \in \mathcal{E}^{2(s)}_V$. It is then enough to prove that $\mathrm{Det}(A) \in \overline{\mathcal{E}}^{2(s)}_V$ outside of a hypersurface in $S = \sigma^{(s)}_V(Q)^{-1}(0)$. Let $S' \subset S$ be the set of smooth points. There are two possibilities

(a) $\alpha_V^{\mathrm{rel}}|_{S'}$ does not vanish identically. In this case, we denote by $S_{\mathrm{reg}} \subset S'$ the open subset where $\alpha_V^{\mathrm{rel}}|_{S'} \neq 0$. By [22, Theorem 2.9.11], the ring $\mathcal{E}_V^{2(s)}$ is invariant by homogeneous bisymplectic transformations (i.e. transformations preserving the forms α_V and α_V^{rel}), and one may assume that S_{reg} is defined by the equation $\tilde{\zeta}_1 = 0$, in a system of local bisymplectic coordinates $(x, y, \eta, \zeta, \tilde{x}, \tilde{\zeta}) \in T_V\Xi$.

(b) $\alpha_V^{\mathrm{rel}}|_{S'} \equiv 0$. Then, $\alpha_V|_{S'}$ does not vanish identically. In this case, we denote by $S_{\mathrm{reg}} \subset S'$ the open subset where $\alpha_V \neq 0$. After a homogeneous bisymplectic transformation, one may assume that S_{reg} is defined by the equation $\eta_1 = 0$.

Denote by v_{ij} the multiplicity of $\sigma_V^{(s)}(A_{ij})$ at S_{reg}. By [22, Theorem 2.7.1], Weierstrass division theorem holds in $\mathcal{E}_V^{2(s)}$. The proof then proceeds as the one of Lemma 1.3. (Though Sato-Kashiwara's proof dealt with a $\mathbb{Z}$-filtration, it is straightforward to adapt it for a filtration which is indexed by $\mathbb{Z}^2$, endowed with the lexicographical order.) □

Proof of Theorem 2.2 (o) and (i). By Theorem A.1 (iii), we may use the trick of the dummy variable on $T_V\Xi$ to work outside of the zero-section. Recall that, for a coherent $\mathcal{E}_X$-module $\mathcal{M}$,

$$\mathrm{char}_V^{(s)}(\mathcal{M}) = \mathrm{supp}(\mathcal{E}_V^{2(s)} \otimes_{\tau^{-1}\mathcal{E}_X|_V} \mathcal{M}|_V).$$

The claim is then a corollary of Theorem A.1 (o) and (i). □

References

[1] Adjamagbo, K., *Panorama de la théorie des déterminants sur un anneau non commutatif*, Bull. Sci. Math. (2) **117** (1993), no. 3, 401–420, Announcement of the author's Ph.D. thesis, Univ. Paris VI, 1991; see also C. R. Acad. Sci. Paris Sér. I Math. **293** (1981), no. 9, 447–449; Ibid. **294** (1982), no. 21, 681–684.

[2] Andronikof, E., *Déterminant, selon M. Sato et M. Kashiwara*, Seminar on Hyperbolic and Holomorphic Partial Differential Equations, 1979–1980, Univ. Paris VI, Paris, 1980, pp. Exp. No. 1, 18.

[3] Andronikof, E., Sur les filtrations de Cohen-Macaulay des modules microdifférentiels, J. Math. Pures Appl. (9) **66** (1987), no. 4, 369–384.

[4] Artin, E., *Geometric algebra*, John Wiley & Sons Inc., New York, 1988, Reprint of the 1957 original, A Wiley-Interscience Publication.

[5] Bony, J.-M. and P. Schapira, *Solutions hyperfonctions du problème de Cauchy*, Hyperfunctions and pseudo-differential equations, Springer, 1973, pp. 82–98. Lecture Notes in Math., Vol. 287.

[6] Chazarain, J., Opérateurs hyperboliques a caractéristiques de multiplicité constante, Ann. Inst. Fourier (Grenoble) **24** (1974), no. 1, 173–202.

[7] Colin, V., Spécialisation du foncteur de Whitney, C. R. Acad. Sci. Paris Sér. I Math. **323** (1996), no. 4, 383–388.

[8] Colin, V., Formal microlocalization, C. R. Acad. Sci. Paris Sér. I Math. **327** (1998), no. 3, 289–293.

[9] D'Agnolo, A. and F. Tonin, Cauchy problem for hyperbolic $\mathcal{D}$-modules with regular singularities, Pacific J. Math. **184** (1998), no. 1, 1–22.

[10] de Paris, J.-C., Problème de Cauchy oscillatoire pour un opérateur différentiel à caractéristiques multiples; lien avec l'hyperbolicité, J. Math. Pures Appl. (9) **51** (1972), 231–256.

[11] Dieudonné, J., Les déterminants sur un corps non commutatif, Bull. Soc. Math. France **71** (1943), 27–45.

[12] Flaschka, H. and G. Strang, The correctness of the Cauchy problem, Advances in Math. **6** (1971), 347–379 (1971).

[13] Hufford, G., On the characteristic matrix of a matrix of differential operators, J. Differential Equations **1** (1965), 27–38.

[14] Kashiwara, M., *Vanishing cycle sheaves and holonomic systems of differential equations*, Algebraic geometry (Tokyo/Kyoto, 1982), Springer, Berlin, 1983, pp. 134–142.

[15] Kashiwara, M., *Algebraic study of systems of partial differential equations*, Mém. Soc. Math. France (N.S.) (1995), no. 63, xiv+72, Master's Thesis submitted to Tokyo University in December 1970, translated from the Japanese by A. D'Agnolo and J.-P. Schneiders.

[16] Kashiwara, M. and T. Oshima, Systems of differential equations with regular singularities and their boundary value problems, Ann. Math. (2) **106** (1977), no. 1, 145–200.

[17] Kashiwara, M. and P. Schapira, Micro-hyperbolic systems, Acta Math. **142** (1979), no. 1–2, 1–55.

[18] Kashiwara, M. and P. Schapira, *Sheaves on manifolds*, Springer-Verlag, Berlin, 1990.

[19] Kashiwara, M. and P. Schapira, *Moderate and formal cohomology associated with constructible sheaves*, Mém. Soc. Math. France (N.S.) (1996), no. 64, iv+76.

[20] Koshimizu, H. and K. Takeuchi, *Extension theorems for the distribution solutions to $\mathcal{D}$-modules with regular singularities*, Proc. Amer. Math. Soc. **128** (2000), no. 6, 1685–1690.

[21] Laumon, G., *Transformations canoniques et spécialisation pour les $\mathcal{D}$-modules filtrés*, Astérisque (1985), no. 130, 56–129, Differential systems and singularities (Luminy, 1983).

[22] Laurent, Y., *Théorie de la deuxième microlocalisation dans le domaine complexe*, Birkhäuser Boston Inc., Boston, Mass., 1985.

[23] Laurent, Y., Polygône de Newton et b-fonctions pour les modules microdifférentiels, Ann. Sci. École Norm. Sup. (4) **20** (1987), no. 3, 391–441.

[24] Leray, J., *Hyperbolic differential equations*, The Institute for Advanced Study, Princeton, N. J., 1955.

[25] Matsumoto, W., Normal form of systems of partial differential and pseudo-differential operators in formal symbol classes, J. Math. Kyoto Univ. **34** (1994), no. 1, 15–40.

[26] Matsumoto, W., *The Cauchy problem for systems—through the normal form of systems and theory of weighted determinant*, Séminaire sur les Équations aux Dérivées Partielles, 1998–1999 Exp. No. XVIII, 27, École Polytech., Palaiseau.

[27] Mebkhout, Z., Théorème de dualité pour les $\mathcal{D}_{\mathcal{X}}$-modules cohérents, C. R. Acad. Sci. Paris Sér. A-B **285** (1977), no. 12, A785–A787.

[28] Monteiro Fernandes, T., *Problème de Cauchy pour les systèmes microdifférentiels*, vol. 140–141, pp. 135–220, Astérisque, 1986.

[29] Sato, M. and M. Kashiwara, The determinant of matrices of pseudo-differential operators, Proc. Japan Acad. **51** (1975), 17–19.

[30] Sato, M., Kawai, T. and M. Kashiwara, *Microfunctions and pseudo-differential equations*, Hyperfunctions and pseudo-differential equations, Springer, Berlin, 1973, pp. 265–529. Lecture Notes in Math., Vol. 287.

[31] Schapira, P., *Microdifferential systems in the complex domain*, Grundlehren der mathematischen Wissenschaften, no. 269, Springer, Berlin Heidelberg New York Tokyo, 1985.

[32] Taglialatela, G. and J. Vaillant, Conditions invariantes d'hyperbolicité des systèmes et réduction des systèmes, Bull. Sci. Math. **120** (1996), no. 1, 19–97.

[33] Vaillant, J., Conditions d'hyperbolicité des systèmes, C. R. Acad. Sci. Paris Sér. I Math. **313** (1991), no. 5, 227–230.

[34] Vaillant, J., Invariants des systèmes d'opérateurs différentiels et sommes formelles asymptotiques, Japan. J. Math. (N.S.) **25** (1999), no. 1, 1–153.

[35] Wagschal, C., Diverses formulations du problème de Cauchy pour un système d'équations aux dérivées partielles, J. Math. Pures Appl. (9) **53** (1974), 51–69.

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