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Spherical Functions in a Certain Distinguished Model

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Abstract. We study unramified principal series representations of general linear groups over p-adic fields, distinguished with respect to the fixator of the Galois involution. We give a certain condition for the unramified principal series to be distinguished, and give a formula for spherical vectors in the distinguished models of GL_n, following the method of Kato and Hironaka. We give explicit results for GL_2 and GL_3.

0. Introduction

Let E/F be a quadratic extension of $\mathfrak{p}$-adic fields, $G = GL_n(E)$ and $H = GL_n(F)$. Regard H as the fixator of the Galois involution of E/F acting on G. It is known ([F1]) that the symmetric variety $H\backslash G$ is multiplicity free; for any irreducible smooth representation π of G, the dimension of the space of all G-morphisms of π into the space $\mathcal{C}^\infty(H\backslash G)$ of locally constant functions on $H\backslash G$ is at most one. If the above space of morphisms is non-zero, then π is said to be H-distinguished, and we call the realization of π in $\mathcal{C}^\infty(H\backslash G)$ the H-distinguished model of π.

Distinguished representations are of particular importance for the connection with functoriality principle ([F1, 2]), and for the appearance in a certain Rankin-Selberg method ([R]; the quadratic extension version of the doubling method). Motivated by the latter, we study the unramified H-distinguished models and the spherical vectors therein. In the GL_2-case, an explicit formula for such spherical functions was already given by W. Banks ([B]). Also, several examples of spherical functions on $\mathfrak{p}$-adic symmetric varieties were studied by Y. Hironaka, S. Kato and F. Sato ([H1-4], [HS], [K]).

Now we summarize the contents of the present paper. Throughout this paper we assume that E/F is unramified. Let P be the Borel subgroup consisting of upper triangular matrices, T be the maximal torus consisting

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of diagonal matrices, W be the Weyl group of (G,T), identified with the subgroup consisting of permutation matrices, and $w_0 \in W$ be the longest element, that is, the anti-diagonal monomial matrix. We use the Galois involution θ on G twisted by w_0 (see the Notation section). Let H be the fixator of θ in G. It is isomorphic to $GL_n(F)$. Let $\mathcal{O}_E$ be the valuation ring of E, ϖ be a prime element of $\mathcal{O}_E$, $K = GL_n(\mathcal{O}_E)$ and B be the Iwahori subgroup consisting of elements of K whose entries below the diagonals belong to $\varpi\mathcal{O}_E$. In §1, we give a parametrization of $H\backslash G/K$; let m be the integer part of $n/2$, and set

$$\Lambda_m = \{ \lambda = (\lambda_1, \cdots, \lambda_m) \in \mathbb{Z}^m \; ; \; \lambda_1 \leqslant \cdots \leqslant \lambda_m \leqslant 0 \}.$$

For $\lambda \in \Lambda_m$, put

$$\varpi_0^\lambda = \mathrm{diag}(\varpi^{\lambda_1}, \cdots, \varpi^{\lambda_m}, \underbrace{1, \cdots, 1}_{n-m}).$$

We show that $\{\varpi_0^\lambda \; ; \; \lambda \in \Lambda_m\}$ gives a complete set of representatives of $H\backslash G/K$ (see **(1.4)**).

In §2, we study H-distinguished models for unramified principal series. Let $X_{\mathrm{ur}}(T)_\theta$ be the set of all unramified characters on T which is trivial on $T \cap H$. We show that if χ is regular and the unramified principal series $I(\chi)$ has an H-distinguished model, then ${}^w\chi \in X_{\mathrm{ur}}(T)_\theta$ for some $w \in W$. At the same time we prove the uniqueness of the model, and determine the support of the H-invariant functionals (see **(2.2)**).

At the end of §2, we construct a non-zero H-invariant linear functional L_χ on $I(\chi)$ for $\chi \in X_{\mathrm{ur}}(T)_\theta$, using complex powers of relative P-invariants, as was proposed in [K]. We define the spherical function Q_χ on $H\backslash G/K$ by $Q_\chi(g) = \langle L_\chi, \pi_\chi(g)\phi_{K,\chi} \rangle$. Here $\phi_{K,\chi}$ is the unique K-fixed vector in $I(\chi)$ such that $\phi_{K,\chi|K} \equiv 1$. In §3, following the method of [H4] we prove a formula for Q_χ, which is the main result of this paper (**(3.4)**);

THEOREM. *Let $\chi \in X_{\mathrm{ur}}(T)_\theta$ be regular and assume that $c_w(\chi^{-1})c_{w^{-1}}({}^w\chi) \neq 0$ for all $w \in W$. Then, for $\lambda \in \Lambda_m$,*

$$Q_\chi(\varpi_0^\lambda) = \mathrm{vol}(Bw_0B) \sum_{w \in W_\theta} \frac{c_{w_0}({}^w\chi)b_w(\chi)}{c_w(\chi^{-1})} {}^w\chi\delta_P^{1/2}(\varpi_0^\lambda).$$

Here, $c_w(\chi)$ is the usual c-function given by (14) *of* §*3, $b_w(\chi)$ is the factor determined by the functional equation of invariant functionals* **(3.3)***, δ_P is the modulus of P and $W_\theta = W \cap H$.*

Note that the sum is taken over the *little* Weyl group W_θ, not over the full Weyl group W as in [H4]. The vanishing of terms associated to $w \notin W_\theta$ follows from the determination of the support of invariant functionals in §2.

If $Q_\chi(1) \neq 0$, put $\widetilde{Q}_\chi = Q_\chi(1)^{-1} \cdot Q_\chi$. Then the above formula can be rewritten as

$$\widetilde{Q}_\chi(\varpi_0^\lambda) = \mathrm{vol}(Bw_0B) \sum_{w \in W_\theta} \frac{c_{w_0}({}^w\chi)}{Q_{{}^w\chi}(1)} \cdot {}^w\chi\delta_P^{1/2}(\varpi_0^\lambda),$$

provided that $Q_{{}^w\chi}(1) \neq 0$ for all $w \in W_\theta$(see **(3.5)**). In §4, we compute the value $Q_\chi(1)$ directly for $n = 2$ and $n = 3$ and give the explicit formulae of $\widetilde{Q}_\chi$ in these cases (**(4.1)** and **(4.3)**).

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Notation

Let E/F be a quadratic unramified extension of $\mathfrak{p}$-adic fields, with the absolute values $|\cdot|_E$, $|\cdot|_F$ respectively. Let $\mathcal{O}_E$ be the valuation ring, k_E the residue class field, q_E the residue order, of E. Similarly $\mathcal{O}_F$, k_F and q_F are defined for F. We may fix a prime element ϖ of E which is also a prime element of F. We shall drop the subscript E and write $\mathcal{O} = \mathcal{O}_E$, $q = q_E$, etc, when there is no fear of confusion. For $x \in E$, the conjugate of x over F is denoted by $\bar{x}$.

Let G be the group $GL_n(E)$ and P, T, K, B be the subgroups of G given in the Introduction. Let P^- be the Borel subgroup opposite to P and N, N^- be the unipotent radical of P, P^- respectively. Put $N_0 = N \cap B$, $T_0 = T \cap B$, $N_1^- = N^- \cap B$ and $N_1 = w_0 N_1^- w_0^{-1}$ where w_0 is the anti-diagonal monomial matrix in G. The Iwahori factorization asserts that $B = N_0 T_0 N_1^-$, uniquely decomposed in this order.

The Weyl group W of (G,T), isomorphic to the symmetric group of n-letters, is identified with the subgroup of G consisting of permutation matrices. W acts on quasi-characters χ of T by ${}^w\chi(t) = \chi(w^{-1}tw)$ for $w \in W$, $t \in T$. We say that χ is regular if ${}^w\chi = \chi$ implies $w = 1$.

For $g = (g_{ij}) \in G$, we write $\bar{g}$ for the matrix $(\bar{g}_{ij})$. Define the involution θ on G by

$$\theta(g) = w_0 \bar{g} w_0^{-1} \quad \text{for } g \in G.$$

Then θ leaves K and T stable, and $\theta(N) = N^-$. Let H be the fixator of θ in G;

$$H = \{\, h \in G\,;\, \theta(h) = h \,\}.$$

Note that H is isomorphic to $GL_n(F)$. We write W_θ for $W \cap H = \{w \in W; \theta(w) = w\}$, which is the centralizer of w_0 in W.

Any closed subgroups of G and any homogeneous spaces of them are all regarded as totally disconnected Hausdorff spaces. For such a space Y, topological notions are used with respect to this Hausdorff topology unless otherwise stated. For a subset Z of Y, the closure of Z in Y is denoted by $Z^{c\ell}$. Let us write $\mathcal{C}^\infty(Y)$ for the space of all locally constant $\mathbb{C}$-valued functions on Y, $\mathcal{C}_c^\infty(Y)$ for the subspace of those with compact support. Linear functionals on $\mathcal{C}_c^\infty(Y)$ are called distributions on Y.

Fix a Haar measure dg of G, normalized so that $\int_K dg = 1$. Also fix a left Haar measure $d_\ell p$ of P so that $\int_{P\cap K} d_\ell p = 1$. Let δ_P be the modulus character of P. It is given explicitly by

$$\delta_P\left(\operatorname{diag}(t_1, \cdots, t_n)\right) = \prod_{1\leqslant i<j\leqslant n} |t_i t_j^{-1}| = \prod_{1\leqslant i\leqslant n} |t_i|^{n-2i+1} \tag{1}$$

on T. On the space of all locally constant functions $f : G \to \mathbb{C}$ such that $f(pg) = \delta_P(p) f(g)$ ($p \in P$, $g \in G$) and that $P\backslash \operatorname{supp}(f)$ is compact, there is a unique (up to constant) non-zero right G-invariant linear functional, which is denoted by $f \mapsto \oint_{P\backslash G} f(\dot{g}) d\dot{g}$. We may normalize this so that $\oint_{P\backslash G} f(\dot{g}) d\dot{g} = \int_K f(k) dk$.

For a representation (π, V) of G, we denote by (π^*, V^*) the dual representation and by $(\tilde{\pi}, \tilde{V})$ the smooth contragredient, that is, the smooth part of (π^*, V^*). For a subgroup U of G, the subspace of all $\pi(U)$-fixed vectors in V is denoted by V^U.

§1. Double Coset Decompositions

In this section first we recall the description of the double cosets $P\backslash G/H$ and prepare some properties of them for our later use. Then we give a parametrization of the double cosets $H\backslash G/K$ on which our spherical functions are defined.

Put

$$X = \{ x \in G ; \theta(x)x = 1 \}.$$

G acts on X (from the right) by the θ-twisted conjugation

$$(x, g) \mapsto x * g := \theta(g)^{-1} x g \quad \text{for } x \in X,\, g \in G.$$

Put $\tau(g) = 1 * g = \theta(g)^{-1} g$. By the Hilbert Theorem 90, τ induces a G-equivariant homeomorphism $H\backslash G \xrightarrow{\sim} X$ (see e.g. [F1]). Similarly, the mapping $g \mapsto \tau(g^{-1})$ induces $G/H \xrightarrow{\sim} X$. For each $v \in W \cap X$, we may fix an element $\eta_v \in G$ such that

$$\theta(\eta_v)\eta_v^{-1} = \tau(\eta_v^{-1}) = v$$

by the surjectivity of τ. In particular we take $\eta_1 = 1$.

For $x \in G$ and $1 \leqslant i \leqslant n$, let $d_i(x)$ be the determinant of the upper left i by i block of x. Then for $p \in P$, $p' \in P^-$ and $x \in G$ we have $d_i(p'xp) = d_i(p')d_i(p)d_i(x)$. So $d_{i|X}$ gives a relative P-invariant polynomial function on X;

$$d_i(x * p) = d_i\left(\theta(p)^{-1}p\right) d_i(x) \quad \text{for all } p \in P,\, x \in X. \tag{2}$$

Note that $p \mapsto d_i\left(\theta(p)^{-1}p\right)$ is an E-rational character of P. Put $m = [n/2]$, the integer part of $n/2$, and set

$$X^0 = \{ x \in X ; d_i(x) \neq 0 \quad \text{for all } 1 \leqslant i \leqslant m \}.$$

Clearly X^0 is an open dense, P-stable subset of X under the $*$-action.

(1.1) Lemma. *G decomposes into the disjoint union of the double cosets $P\eta_v H$, where v runs over $W \cap X$, and $P \cdot H$ is the unique open dense (P, H)-double coset in G.*

PROOF. The Bruhat decomposition for G implies that

$$X=\bigcup_{w\in W}(P^-wP\cap X),\quad \text{and}\quad P^-wP\cap X\neq\emptyset\quad \text{if and only if}\quad w\in W\cap X.$$

As in [F2, p.421], one has $P^-vP\cap X=v*P$ for every $v\in W\cap X$. Pulling back through $\tau\big((\,\cdot\,)^{-1}\big)$ we have the first assertion. To prove the second assertion, it is enough to see that $1*P=X^0$. If $v\in W$ satisfies $d_i(v)\neq 0$ for all $1\leqslant i\leqslant m$, then the upper left m by m block of v must be the identity matrix. If moreover $v\in X$, i.e., if $\theta(v)v=w_0vw_0v=1$, then the lower right m by m block of v also must be the identity, hence $v=1$. This shows that $W\cap X^0=\{1\}$, which implies that $1*P=X^0$. □

For $v\in W\cap X$ we define an involution θ_v on G by

$$\theta_v(g)=v^{-1}\theta(g)v\quad \text{for } g\in G.$$

Put $P_v=P\cap v^{-1}P^-v$. Then θ_v leaves P_v stable. Let R_v be the fixator of θ_v in P_v;

$$R_v=\{\,r\in P\,;\,v^{-1}\theta(r)v=r\,\}. \tag{3}$$

R_v is identified with the the stabilizer in $P\times H$ of the representative η_v as follows;

$$\begin{aligned}\big\{\,(p,h)\in P\times H\,;\,p\eta_vh^{-1}=\eta_v\,\big\}&=\big\{\,(p,h)\,;\,p\in P,\ \eta_v^{-1}p\eta_v=h\in H\,\big\}\\ =\big\{\,(p,\eta_v^{-1}p\eta_v)\,;\,\theta_v(p)=p\in P\,\big\}&=\big\{\,(r,\eta_v^{-1}r\eta_v)\,;\,r\in R_v\,\big\}.\end{aligned}$$

Regarding R_v as a subgroup of $P\times H$ as above, the double coset $P\eta_vH$ is homeomorphic to $(P\times H)/R_v$. We have the following semi-direct product decomposition;

$$R_v=(T\cap R_v)\ltimes(N\cap R_v). \tag{4}$$

Now, for $\mu=(\mu_1,\dots,\mu_n)\in\mathbb{Z}^n$, put

$$\varpi^\mu=\mathrm{diag}(\varpi^{\mu_1},\dots,\varpi^{\mu_n})\quad \text{and}\quad \varpi^{-\mu}=(\varpi^\mu)^{-1}.$$

Also, for $\lambda = (\lambda_1, \cdots, \lambda_m) \in \mathbb{Z}^m$, put

$$\varpi_0^\lambda = \mathrm{diag}(\varpi^{\lambda_1}, \cdots, \varpi^{\lambda_m}, \underbrace{1, \cdots, 1}_{n-m}), \quad \text{and} \quad \varpi_0^{-\lambda} = (\varpi_0^\lambda)^{-1}.$$

The following lemma is easily shown by a direct matrix calculation and the ultrametric inequality.

(1.2) Lemma. *For $n \in N_1$, $n' \in N_1^-$ and $\mu \in \mathbb{Z}^n$ with $\mu_1 \leqslant \ldots \leqslant \mu_n$, one has*

$$|d_i(n\varpi^\mu n')| = |d_i(\varpi^\mu)|.$$

For $\mu \in \mathbb{Z}^n$, ϖ^μ belongs to X if and only if $\mu_{n-i+1} = -\mu_i$ for all i. If moreover $\mu_1 \leqslant \cdots \leqslant \mu_n$ is assumed, we must have $\mu_1 \leqslant \cdots \leqslant \mu_m \leqslant 0$. Now set

$$\Lambda_m = \{ \lambda = (\lambda_1, \cdots, \lambda_m) \in \mathbb{Z}^m \ ; \ \lambda_1 \leqslant \cdots \leqslant \lambda_m \leqslant 0 \}.$$

Then we have

$$\{ \varpi^\mu \in X \ ; \ \mu \in \mathbb{Z}^n, \ \ \mu_1 \leqslant \ldots \leqslant \mu_n \} = \{ \tau(\varpi_0^\lambda) \ ; \ \lambda \in \Lambda_m \}.$$

The following corollary, which is similar to [H4, (2.2)], is important for our later use;

(1.3) Corollary. *For $b \in B$ and $\lambda \in \Lambda_m$, one has*

$$|d_i\left(\tau(\varpi_0^\lambda) * b\right)| = |d_i\left(\tau(\varpi_0^\lambda)\right)| \neq 0.$$

Consequently, $B\varpi_0^{-\lambda} \subset P \cdot H$ holds for all $\lambda \in \Lambda_m$.

Proof. This follows directly from **(1.2)**, using the Iwahori factorization for B. Note that we pull back through $\tau\left((\,\cdot\,)^{-1}\right)$ to get $B\varpi_0^{-\lambda} \subset P \cdot H$. $\square$

In the rest of this section we give a parametrization of $H\backslash G/K$.

(1.4) Proposition. *G decomposes into the disjoint union of the double cosets $H\varpi_0^\lambda K$, where λ runs over Λ_m.*

PROOF. The assertion is equivalent to the decomposition

$$X = \bigcup_{\lambda \in \Lambda_m} \tau(\varpi_0^\lambda) * K \quad \text{(disjoint union)}$$

of X into K-orbits. First, by the Cartan decomposition for G, one has

$$X = \bigcup_{\substack{\mu=(\mu_1,\cdots,\mu_n)\in\mathbb{Z}^n \\ \mu_1 \leqslant \cdots \leqslant \mu_n}} (K\varpi^\mu K \cap X) \quad \text{(disjoint union)}$$

and it is easy to see that $K\varpi^\mu K \cap X \neq \emptyset$ for $\mu_1 \leqslant \cdots \leqslant \mu_n$ if and only if $\varpi^\mu \in X$. In this case we may replace ϖ^μ by $\tau(\varpi_0^\lambda)$, where $\lambda \in \Lambda_m$ as above. We show that

$$K\tau(\varpi_0^\lambda)K \cap X = \tau(\varpi_0^\lambda) * K \qquad \text{for all } \lambda \in \Lambda_m.$$

Since $k_1\tau(\varpi_0^\lambda)k_2 = \big(\tau(\varpi_0^\lambda)k_2\theta(k_1)\big) * \theta(k_1^{-1})$, it is enough to show that, for $\lambda \in \Lambda_m$,

(*) For any $k \in K$ with $\tau(\varpi_0^\lambda)k \in X$,
there is a $k' \in K$ such that $\tau(\varpi_0^\lambda)k = \tau(\varpi_0^\lambda) * k$.

Put

$$C_\lambda = \varpi_0^\lambda K \varpi_0^{-\lambda} \cap \theta(\varpi_0^\lambda K \varpi_0^{-\lambda}).$$

Then, one can show that C_λ is a θ-stable subgroup contained in K. In fact, if we understand that $\lambda_{m+1} = \cdots = \lambda_n = 0$, then C_λ is given by

$$\begin{aligned} C_\lambda &= \Big\{ (c_{ij}) \in G \;;\; \det(c_{ij}) \in \mathcal{O}^\times,\ |c_{ij}| \leqslant \min(q^{-\lambda_i+\lambda_j}, q^{-\lambda_{n-i+1}+\lambda_{n-j+1}}) \Big\} \\ &= \Big\{ (c_{ij}) \in K \;;\; |c_{ij}| \leqslant \min(q^{-\lambda_i+\lambda_j}, q^{-\lambda_{n-i+1}+\lambda_{n-j+1}}) \Big\}. \end{aligned}$$

Now (*) follows from the assertion

(**) $$H^1(\{1,\theta\}, C_\lambda) = \{1\}.$$

Indeed, if $\tau(\varpi_0^\lambda)k \in X$, $k \in K$, then $\varpi_0^\lambda k \varpi_0^{-\lambda} = \theta(\varpi_0^\lambda k \varpi_0^{-\lambda})^{-1} \in C_\lambda$. So (**) implies that there is a $c \in C_\lambda$ such that $\varpi_0^\lambda k \varpi_0^{-\lambda} = \theta(c)^{-1}c$, which leads to the equation $\tau(\varpi_0^\lambda)k = \tau(\varpi_0^\lambda) * k'$, with $k' = \varpi_0^{-\lambda} c \varpi_0^\lambda \in K$.

To prove (**), first let $\rho_0 : K \to GL_n(k_E)$ be the mod-ϖ map and $\tilde{\theta}$ be the involution on $GL_n(k_E)$ defined by $\tilde{\theta}(g) = w_0 \bar{g} w_0^{-1}$, where bar denotes the Galois involution of k_E/k_F. Regard $\tilde{\theta}$ as a k_F-involution on $GL_n(k_E)$. It is clear that $\rho_0 \circ \theta = \tilde{\theta} \circ \rho_0$.

Put $M_\lambda = \rho_0(C_\lambda)$. We observe that M_λ is the group of k_F-rational points of a Zariski-connected group over k_F. Let l be the largest number such that $\lambda_l \neq 0$ and assume that λ is of the form

$$\underbrace{\lambda_1 = \cdots = \lambda_{i_1}}_{i_1} < \underbrace{\lambda_{i_1+1} = \cdots = \lambda_{i_1+i_2}}_{i_2} < \cdots < \underbrace{\lambda_{i_1+\cdots+i_{k-1}+1} = \cdots = \lambda_l}_{i_k} < 0.$$

Then, M_λ consists of all matrices of the form $\begin{pmatrix} p_1 & x & 0 \\ & g & \\ 0 & y & p_2 \end{pmatrix}$ where $p_1 \in GL_l(k_E)$ is upper quasi-triangular of type $(i_1, \cdots, i_k)$, $p_2 \in GL_l(k_E)$ is lower quasi-triangular of type $(i_k, \cdots, i_1)$, $g \in GL_{n-2l}(k_E)$ and $x, y \in \mathrm{Mat}_{l,n-2l}(k_E)$. So M_λ is a semi-direct product of rational points of Zariski-connected groups $\left\{\begin{pmatrix} p_1 & & 0 \\ & g & \\ 0 & & p_2 \end{pmatrix}\right\}$ and $\left\{\begin{pmatrix} 1 & x & 0 \\ & 1 & \\ 0 & y & 1 \end{pmatrix}\right\}$.

Put $C_\lambda^{(1)} = \ker(\rho_0) \cap C_\lambda$. Then, one has an exact sequence of 1-cohomology sets;

$$H^1(\{1, \theta\}, C_\lambda^{(1)}) \longrightarrow H^1(\{1, \theta\}, C_\lambda) \longrightarrow H^1(\{1, \tilde{\theta}\}, M_\lambda).$$

The last set is trivial by Lang's theorem. So (**) follows from

$$(***) \qquad H^1(\{1, \theta\}, C_\lambda^{(1)}) = \{1\}.$$

To prove (***), put $K(N) = 1 + \varpi^N \mathrm{Mat}_n(\mathcal{O})$ for each integer $N \geqslant 1$ and define $\rho_N : K(N) \to \mathrm{Mat}_n(k_E)$ by

$$\rho_N(1 + \varpi^N a) = a \mod \varpi \quad \text{for } a \in \mathrm{Mat}_n(\mathcal{O}).$$

Then ρ_N is a homomorphism onto the additive group of $\mathrm{Mat}_n(k_E)$ and $K(N+1) = \ker(\rho_N)$, $K(1) = \ker(\rho_0)$. Note that $K(N)$ is normal in K and is θ-stable. Define the involution $\tilde{\theta}$ on the additive group of $\mathrm{Mat}_n(k_E)$ by the same as before. Then the relation $\rho_N \circ \theta = \tilde{\theta} \circ \rho_N$ holds. Put

$C_\lambda^{(N)} = C_\lambda \cap K(N)$ and $A_\lambda^{(N)} = \rho_N(C_\lambda^{(N)})$. Then $A_\lambda^{(N)}$ is an additive subgroup of $\mathrm{Mat}_n(k_E)$ over k_F. As before one has an exact sequence

$$H^1(\{1,\theta\}, C_\lambda^{(N+1)}) \longrightarrow H^1(\{1,\theta\}, C_\lambda^{(N)}) \longrightarrow H^1(\{1,\tilde{\theta}\}, A_\lambda^{(N)}).$$

By the additive version of the Hilbert Theorem 90, the last term vanishes. So the vanishing of $H^1(\{1,\theta\}, C_\lambda^{(N)})$ follows from that of $H^1(\{1,\theta\}, C_\lambda^{(N+1)})$. Inductively, (***) follows from the vanishing of $H^1(\{1,\theta\}, C_\lambda^{(N)})$ for some large N.

Now, if $N \geqslant -\lambda_1$, one has $C_\lambda \supset K(N)$, thus $C_\lambda^{(N)} = K(N)$. By the argument exactly as in [PR, pp. 292–294], one can show that

$$H^1(\{1,\theta\}, K(N)) = \{1\}$$

for any integer N, hence the proof is completed. □

§2. Invariant Functionals on Unramified Principal Series

Let $X_{\mathrm{ur}}(T)$ be the set of all unramified quasi-characters of T. We regard $\chi \in X_{\mathrm{ur}}(T)$ also as a quasi-character of P by letting $\chi_{|N} \equiv 1$. For $\chi \in X_{\mathrm{ur}}(T)$ let $(\pi_\chi, I(\chi))$ be the unramified principal series attached to χ. Thus $I(\chi)$ is the space of all locally constant $\mathbb{C}$-valued functions φ on G satisfying

$$\varphi(pg) = \chi(p)\delta_P(p)^{1/2}\varphi(g) \quad \text{for } p \in P,\ g \in G,$$

and π_χ is the right translation of G on $I(\chi)$.

Let λ, ρ be respectively the left, right translations of G on $\mathcal{C}_c^\infty(G)$. For $\chi \in X_{\mathrm{ur}}(T)$, let $\mathcal{D}_\chi(G)$ be the space of all distributions D on G satisfying

$$(5) \qquad \langle D, \lambda(p)f \rangle = \chi(p)^{-1}\delta_P(p)^{1/2}\langle D, f \rangle \quad \text{for all } p \in P,\ f \in \mathcal{C}_c^\infty(G),$$

and $\mathcal{D}_\chi(G)^H$ be the space of all $D \in \mathcal{D}_\chi(G)$ satisfying

$$(6) \qquad \langle D, \rho(h)f \rangle = \langle D, f \rangle \quad \text{for all } h \in H,\ f \in \mathcal{C}_c^\infty(G).$$

Define $p_\chi : \mathcal{C}_c^\infty(G) \to I(\chi)$ as usual by

$$(p_\chi(f))(g) = \int_P \chi^{-1}(p)\delta_P(p)^{1/2} f(pg)\, d_\ell p$$

for $f \in \mathcal{C}_c^\infty(G)$, $g \in G$. As is shown in [H4, (1.2)], the dual map p_χ^* of p_χ gives rise to a right G-isomorphism $I(\chi)^* \xrightarrow{\sim} \mathcal{D}_\chi(G)$. Therefore we have

$$\mathrm{Hom}_H(I(\chi), \mathbb{C}) = (I(\chi)^*)^H \xrightarrow[\sim]{p_\chi^*} \mathcal{D}_\chi(G)^H. \tag{7}$$

By this we may regard H-invariant linear functionals as $P \times H$-relatively invariant distributions on G. From now on we study the space $\mathcal{D}_\chi(G)^H$ by the standard method, so called *the Bruhat theory* for $P\backslash G/H$.

For any locally closed subset Ω of G satisfying $P\Omega H \subset \Omega$, define $\mathcal{D}_\chi(\Omega)^H$ to be the space of all distributions on Ω having the same $P \times H$-equivariance as those in $\mathcal{D}_\chi(G)^H$ (i.e., the relations (5) and (6)). Put $\Omega_v = P\eta_v H$ for $v \in W \cap X$, where η_v is as in §1. Recall that $G = \bigcup_{v \in W\cap X} \Omega_v$ (disjoint union) and $\Omega_v \simeq (P \times H)/R_v$, where R_v is defined by (3).

(2.1) Lemma. *Let $\chi \in X_{\mathrm{ur}}(T)$ and $v \in W \cap X$.*

(i) *The modulus character δ_v of R_v is trivial on $N \cap R_v$, and on $T \cap R_v$ it is given by*

$$\delta_v(t) = \delta_P(t)^{1/2} \quad \textit{for all } t \in T \cap R_v.$$

(ii) *One has*

$$\dim \mathcal{D}_\chi(\Omega_v)^H = \begin{cases} 1 & \textit{if } \chi_{|T\cap R_v} \equiv 1, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. (i) By the semi-direct product decomposition (4) of R_v, δ_v is directly computed as the Jacobian of the adjoint action of $T \cap R_v$ on $N \cap R_v$. Let $x = (x_{ij}) \in N \cap R_v$. Since $N \cap R_v \subset N \cap v^{-1}N^-v$, we may only look at the entries x_{ij} with $i < j$, $v(i) > v(j)$. (Here and henceforth we regard elements of W also as permutations of indices.) Moreover since $\theta_v(x) = x$, we must have $x_{n-v(i)+1, n-v(j)+1} = \bar{x}_{ij}$. In particular if $i = n - v(i) + 1$ and $j = n - v(j) + 1$, then $x_{ij} \in F$. Similarly, for $t = \mathrm{diag}(t_1, \cdots, t_n) \in T \cap R_v$, we have $t_{n-v(i)+1} = \bar{t}_i$ and in particular, $t_i \in F^\times$ if $n - v(i) + 1 = i$. Now, $\delta_v(t)$ is computed as;

$$\begin{aligned}\delta_v(t) = &\prod |t_i t_j^{-1}|_F \text{ (product over } i = n - v(i) + 1 < j = n - v(j) + 1) \\ &\times \left(\prod |t_i t_j^{-1}|\right)^{1/2} \ (i < j,\ v(i) > v(j) \text{ and,} \\ &\qquad\qquad\qquad i \neq n - v(i) + 1 \text{ or } j \neq n - v(j) + 1) \\ = &\prod_{i<j, v(i)>v(j)} |t_i t_j^{-1}|^{1/2}\end{aligned}$$

since $|\cdot|_F = |\cdot|^{1/2}$. Comparing with (1), it is enough to see that

$$\prod_{i<j, v(i)<v(j)} |t_i t_j^{-1}| = 1 \quad \text{for } t = \mathrm{diag}(t_1, \cdots, t_n) \in T \cap R_v.$$

As is well-known (e.g. [M, p.289]),

$$\prod_{i<j, v(i)<v(j)} |t_i t_j^{-1}| = \delta_P(t)^{1/2} \delta_P(vtv^{-1})^{1/2}.$$

If $t \in T \cap R_v$ then $vtv^{-1} = w_0 \bar{t} w_0^{-1}$, thus

$$\delta_P(t)^{1/2} \delta_P(vtv^{-1})^{1/2} = \delta_P(t)^{1/2} \delta_P(w_0 \bar{t} w_0^{-1})^{1/2} = \delta_P(t)^{1/2} \delta_P(\bar{t})^{-1/2} = 1.$$

(ii) Recall the following criterion for the existence of relative invariant distributions on homogeneous spaces ([BZ, (1.21)]); let G_1 be a totally disconnected locally compact group, H_1 be a closed subgroup of G_1, ω be a quasi-character of G_1. Then, there is a non-zero distribution D on G_1/H_1 satisfying $\langle D, \lambda(g) f \rangle = \omega(g) \langle D, f \rangle$ for all $g \in G_1$, $f \in \mathcal{C}_c^\infty(G_1/H_1)$ if and only if

$$\omega_{|H_1} = \delta_{G_1|H_1} \cdot \delta_{H_1}^{-1}.$$

Here, δ_{G_1}, δ_{H_1} are the modulus characters of G_1, H_1 respectively. If such a non-zero distribution D exists, it is unique up to constant multiples. Applying this to $G_1 = P \times H$, $H_1 = R_v$ and $\omega = \chi^{-1} \delta_P^{1/2} \times 1$, (ii) is a direct consequence of (i). $\square$

For each $v \in W \cap X$ put

$$X_{\mathrm{ur}}(T)_{\theta,v} = \left\{ \chi \in X_{\mathrm{ur}}(T)\,;\, \chi_{|T\cap R_v} \equiv 1 \right\} \quad \text{and} \quad X_{\mathrm{ur}}(T)_\theta = X_{\mathrm{ur}}(T)_{\theta,1}.$$

(2.2) Proposition.

(i) *If $\mathcal{D}_\chi(G)^H \neq (0)$, then $\chi \in X_{\mathrm{ur}}(T)_{\theta,v}$ for some $v \in W \cap X$.*

(ii) *If χ is regular, then $\mathcal{D}_\chi(G)^H$ is at most one dimensional. If χ is regular and $\mathcal{D}_\chi(G)^H \neq (0)$, then $v \in W \cap X$ in* (i) *is uniquely determined, and for any non-zero $D \in \mathcal{D}_\chi(G)^H$, the support* supp$(D)$ *of D is given by the closure $\Omega_v^{c\ell}$ of Ω_v.*

(iii) *If χ is regular and $\mathcal{D}_\chi(G)^H \neq (0)$, then there is a $w \in W$ such that ${}^w\chi \in X_{\mathrm{ur}}(T)_\theta$.*

Proof. (i) follows immediately from **(2.1)** (ii).

(ii) Assume that χ is regular and $\mathcal{D}_\chi(G)^H \neq (0)$. The uniqueness of $v \in W \cap X$ such that $\chi \in X_{\mathrm{ur}}(T)_{\theta,v}$ follows immediately from the regularity of χ. Thus, if $\chi \in X_{\mathrm{ur}}(T)_{\theta,v}$, Ω_v is the unique $P \times H$-orbit such that $\mathcal{D}_\chi(\Omega_v)^H \neq (0)$, hence one has $\mathcal{D}_\chi(\Omega)^H = (0)$ for any $P \times H$-stable subset Ω of G such that $\Omega \cap \Omega_v = \emptyset$.

Now since Ω_v is open in its closure $\Omega_v^{c\ell}$, we have the following two exact sequences;

$$(0) \longrightarrow \mathcal{D}_\chi(\Omega_v^{c\ell})^H \xrightarrow{\text{ext}} \mathcal{D}_\chi(G)^H \xrightarrow{\text{res}} \mathcal{D}_\chi(G - \Omega_v^{c\ell})^H,$$
$$(0) \longrightarrow \mathcal{D}_\chi(\Omega_v^{c\ell} - \Omega_v)^H \xrightarrow{\text{ext}} \mathcal{D}_\chi(\Omega_v^{c\ell})^H \xrightarrow{\text{res}} \mathcal{D}_\chi(\Omega_v)^H.$$

As was noticed above, one has $\mathcal{D}_\chi(G - \Omega_v^{c\ell})^H = \mathcal{D}_\chi(\Omega_v^{c\ell} - \Omega_v)^H = (0)$, thus

$$\mathcal{D}_\chi(G)^H \underset{\text{ext}}{\overset{\sim}{\longleftarrow}} \mathcal{D}_\chi(\Omega_v^{c\ell})^H \underset{\text{res}}{\hookrightarrow} \mathcal{D}_\chi(\Omega_v)^H. \tag{8}$$

Since $\mathcal{D}_\chi(G)^H$ is non-zero and $\mathcal{D}_\chi(\Omega_v)^H$ is one dimensional, all three spaces in (8) are isomorphic and one dimensional. This completes the proof of (ii).

(iii) Let χ be regular, $\mathcal{D}_\chi(G)^H \neq (0)$ and $\chi \in X_{\mathrm{ur}}(T)_{\theta,v}$. Put $v_0 = w_0 v$. Then $v_0^2 = 1$, so v_0 is a product of disjoint transpositions. Put

$$\chi_i(x) = \chi(\mathrm{diag}(1, \cdots, 1, \overset{i}{\check{x}}, 1, \cdots, 1)).$$

If i is an index such that $v_0(i) = i$, then $\chi \in X_{\mathrm{ur}}(T)_{\theta,v}$ implies that $\chi_{i|F^\times} \equiv 1$. Since χ_i and E/F are unramified, one must have $\chi_i \equiv 1$. By the regularity of χ, one can conclude that there is no pair $\{i, j\}$ of indices such that $i = v_0(i) \neq j = v_0(j)$, so v_0 is a product of $m = [n/2]$-disjoint transpositions, say,

$$v_0 = (i_1, j_1) \cdots (i_m, j_m), \quad i_1 < \cdots < i_m \quad \text{and} \quad i_k < j_k \text{ for all } k.$$

Take $w \in W$ so that

$$w(i_k) = k, \quad w(j_k) = n - k + 1 \quad \text{for all } k.$$

Then one has $wv_0w^{-1} = w_0$, that is, $v = \theta(w)^{-1}w$. This shows that $w^{-1}(T \cap H)w = T \cap R_v$, hence ${}^w\chi_{|T\cap H} \equiv 1$. □

REMARK. Actually we do not need the assumption that χ is unramified in **(2.2)** (i) and (ii). Essentially the same statement as **(2.2)** (i) is proved in [JLR].

According to the above proposition, to study unramified principal series $I(\chi)$ with $(I(\chi)^*)^H \neq (0)$, we may restrict ourselves (at least generically) to the case $\chi \in X_{\mathrm{ur}}(T)_\theta$. In the following we construct a non-zero H-invariant linear functional L_χ on $I(\chi)$ explicitly for $\chi \in X_{\mathrm{ur}}(T)_\theta$.

Write $\chi \in X_{\mathrm{ur}}(T)$ as

(9) $$\chi\left(\mathrm{diag}(t_1, \cdots, t_n)\right) = \prod_{i=1}^{n} |t_i|^{s_i}, \quad s_i \in \mathbb{C}.$$

If $\chi \in X_{\mathrm{ur}}(T)_\theta$, we may assume that $s_{n-i+1} = -s_i$ for all i since χ is trivial on $T \cap H$. So $\chi \in X_{\mathrm{ur}}(T)_\theta$ is of the form

(10) $$\chi\left(\mathrm{diag}(t_1, \cdots, t_n)\right) = \prod_{i=1}^{m} |t_i t_{n-i+1}^{-1}|^{s_i}$$

for $s_1, \cdots, s_m \in \mathbb{C}$. Define a $\mathbb{C}$-valued function Δ_χ on $P \cdot H$ by

(11) $$\Delta_\chi(g) = \prod_{i=1}^{m} |d_i\left(\theta(g)g^{-1}\right)|^{s_i'} \quad \text{for } g \in P \cdot H,$$

where $s_1', \cdots, s_m' \in \mathbb{C}$ are related with $s_1, \cdots, s_m \in \mathbb{C}$ by

$$(12) \qquad \begin{cases} s_i' = s_i - s_{i+1} - 1 & \text{for } i < m, \\ s_m' = s_m - \frac{n-2m+1}{2}. \end{cases}$$

If $\mathrm{Re}(s_i') > 0$ for all i, Δ_χ can be extended to a continuous function on G (but not locally constant on G in general). For $p \in P$ (with diagonal entries $t_1, \cdots, t_n$), $g \in G$ and $h \in H$ we have

$$\begin{aligned} \Delta_\chi(pgh) &= \left(\prod_{i=1}^{m} |d_i(\theta(p)p^{-1})|^{s_i'} \right) \Delta_\chi(g) \quad \text{by (2)} \\ &= \left(\prod_{i=1}^{m} |t_i|^{-\sum_{j=i}^{m} s_j'} |\bar{t}_{n-i+1}|^{\sum_{j=i}^{m} s_j'} \right) \Delta_\chi(g) \\ &= \chi^{-1}\delta_P^{1/2}(p)\Delta_\chi(g) \quad \text{by (10) and (12)} . \end{aligned}$$

For $\chi \in X_{\mathrm{ur}}(T)_\theta$ such that $\mathrm{Re}(s_i') > 0$, define a linear functional L_χ on $I(\chi)$ by

$$(13) \qquad \langle L_\chi, \varphi \rangle = \oint_{P\backslash G} \Delta_\chi(\dot{g})\varphi(\dot{g})d\dot{g} \quad \left(\ = \int_K \Delta_\chi(k)\varphi(k)dk \ \right)$$

for $\varphi \in I(\chi)$. As in [H4, Remark(1.1)], the functional L_χ, which is initially defined for $\mathrm{Re}(s_i') > 0$, is analytically continued to whole of $(s_1, \cdots, s_m) \in \mathbb{C}^m$. By the right H-invariance of Δ_χ, L_χ belongs to $(I(\chi)^*)^H$.

(2.3) PROPOSITION. *If $\chi \in X_{\mathrm{ur}}(T)_\theta$ is regular, then $(I(\chi)^*)^H$ is one dimensional. In fact, L_χ defined above gives a non-zero H-invariant linear functional on $I(\chi)$, which is unique up to constant multiples.*

PROOF. By **(2.2)**(ii), it is enough to see that $L_\chi \neq 0$. Taking $\lambda = 0$ in **(1.3)**, we have $\Delta_\chi(b) = 1$ for all $b \in B$. Therefore, $\langle L_\chi, p_\chi(\mathrm{ch}_B) \rangle = \mathrm{vol}(B)$, which is non-zero. $\square$

§3. A Formula for Our Spherical Functions

Let $\chi \in X_{\mathrm{ur}}(T)_\theta$ and assume that χ is regular. We have observed in §2 that $\dim(I(\chi)^*)^H = 1$. By the Frobenius reciprocity,

$$(I(\chi)^*)^H \simeq \mathrm{Hom}_G\left(I(\chi), \mathcal{C}^\infty(H\backslash G)\right),$$

hence there is a unique realization of $I(\chi)$ in $\mathcal{C}^\infty(H\backslash G)$. Let L_χ be as in (13) of §2, and define

$$Q_\chi(g) = \langle\, L_\chi, \pi_\chi(g)\phi_{K,\chi}\,\rangle$$

where $\phi_{K,\chi}$ is the unique element of $I(\chi)$ such that $\phi_{K,\chi}(k) = 1$ for all $k \in K$. The function Q_χ on G is then the unique (up to constant) right K-invariant function in the realization of $I(\chi)$ in $\mathcal{C}^\infty(H\backslash G)$. By the description **(1.4)** of $H\backslash G/K$, Q_χ is completely determined by their values at $\varpi_0^\lambda \in G$, for $\lambda \in \Lambda_m$. In this section, following [H4] we give an expression of Q_χ as a linear combination of quasi-characters ${}^w\chi\delta_P^{1/2}$, where w varies in the *little* Weyl group $W_\theta = W \cap H$.

For $\chi \in X_{\mathrm{ur}}(T)$, identify $I(\chi)^\sim$ with $I(\chi^{-1})$ by the natural pairing

$$\langle\langle\, \phi, \psi\,\rangle\rangle = \oint_{P\backslash G} \phi(\dot g)\psi(\dot g)d\dot g \quad \left(\;= \int_K \phi(k)\psi(k)dk \;\right)$$

for $\phi \in I(\chi)$, $\psi \in I(\chi^{-1})$. Let $p_B : I(\chi) \to I(\chi)^B$ be defined by

$$p_B(\phi)(g) = \mathrm{vol}(B)^{-1}\int_B \phi(gb)db$$

and $p_B^*(\ell) := \ell \circ p_B$ for $\ell \in I(\chi)^*$. Then p_B is the identity on $I(\chi)^B$ and $p_B^*(\ell)$ is fixed by B for all $\ell \in I(\chi)^*$. Since B is open and compact, $p_B^*(\ell)$ is a smooth linear form on $I(\chi)$. By the above identification $I(\chi)^\sim \simeq I(\chi^{-1})$, $p_B^*(\ell)$ is regarded as an element of $I(\chi^{-1})^B$ so that $\langle\, p_B^*(\ell), \phi\,\rangle = \langle\, \ell, p_B(\phi)\,\rangle = \langle\langle\, \phi, p_B^*(\ell)\,\rangle\rangle$ for all $\phi \in I(\chi)$.

(3.1) LEMMA. *Let $\ell \in I(\chi)^*$. As an element of $I(\chi^{-1})^B$, $p_B^*(\ell)$ is given by*

$$p_B^*(\ell) = \sum_{w\in W} \mathrm{vol}(BwB)^{-1}\langle\, \ell, \phi_{w,\chi}\,\rangle\phi_{w,\chi^{-1}}.$$

Here, $\phi_{w,\chi} = p_\chi(\mathrm{ch}_{BwB})$ *for* $w \in W$.

Proof. $\{\phi_{w,\chi^{-1}}\}_{w\in W}$ forms a basis of $I(\chi^{-1})^B$ (see [C, (2.1)]). Write

$$p_B^*(\ell) = \sum_{w\in W} a_w \phi_{w,\chi^{-1}} \quad (\, a_w \in \mathbb{C}\,).$$

Then, for $w \in W$, taking $\langle\langle\, \phi_{w,\chi},\ \cdot\ \rangle\rangle$ on both sides,

$$\langle\langle\, \phi_{w,\chi}, p_B^*(\ell)\,\rangle\rangle = a_w \mathrm{vol}(BwB).$$

On the other hand,

$$\langle\, p_B^*(\ell), \phi_{w,\chi}\,\rangle = \langle\, \ell, p_B(\phi_{w,\chi})\,\rangle = \langle\, \ell, \phi_{w,\chi}\,\rangle$$

by definition. Thus $a_w = \mathrm{vol}(BwB)^{-1}\langle\, \ell, \phi_{w,\chi}\,\rangle$. □

Let $\chi \in X_{\mathrm{ur}}(T)$ be regular and for $w \in W$, let $T_w^\chi : I(\chi) \to I({}^w\chi)$ be the standard intertwining operator ([C, §3]) and $c_w(\chi)$ be defined by the relation $T_w^\chi(\phi_{K,\chi}) = c_w(\chi)\phi_{K,{}^w\chi}$. If χ is of the form (9), then $c_w(\chi)$ is given by

$$c_w(\chi) = \prod_{i<j, w(i)>w(j)} \frac{1-q^{-s_i+s_j-1}}{1-q^{-s_i+s_j}} \tag{14}$$

(see [C, (3.1), (3.3)]).

(3.2) Lemma [H4, Prop.(1.6),(1.7)]. *For* $w \in W$ *and a regular* $\chi \in X_{\mathrm{ur}}(T)$, *assume that* $c_w(\chi^{-1})c_{w^{-1}}({}^w\chi) \neq 0$. *Define an intertwining map* $\widetilde{T}_w^\chi : I(\chi)^* \to I({}^w\chi)^*$ *by*

$$\widetilde{T}_w^\chi = \frac{c_w(\chi^{-1})}{c_{w^{-1}}({}^w\chi)} \cdot \left(T_{w^{-1}}^{w\chi}\right)^*.$$

Then,

(i) $\widetilde{T}_w^\chi$ *is an extension of* $T_w^{\chi^{-1}} : I(\chi^{-1}) \to I({}^w\chi^{-1})$, *regarding* $I(\chi^{-1}) \subset I(\chi)^*$, $I({}^w\chi^{-1}) \subset I({}^w\chi)^*$ *and,*

(ii) $p_B^* \circ \widetilde{T}_w^\chi = T_w^{\chi^{-1}} \circ p_B^*$.

Now let $\chi \in X_{\mathrm{ur}}(T)_\theta$ be regular and $L_\chi \in (I(\chi)^*)^H$ be defined by (13). Observe that for $w \in W$, ${}^w\chi \in X_{\mathrm{ur}}(T)_{\theta,v}$ where $v = \theta(w)w^{-1}$. In particular ${}^w\chi \in X_{\mathrm{ur}}(T)_\theta$ if and only if $w \in W_\theta$, so $L_{{}^w\chi} \in (I({}^w\chi)^*)^H$ is defined for $w \in W_\theta$ as before. For each $w \in W$, we choose $L_\chi^{(w)} \in (I({}^w\chi)^*)^H$ as follows;

$$\begin{cases} \text{If } w \in W_\theta,\ L_\chi^{(w)} = L_{{}^w\chi}. \\ \text{If } w \notin W_\theta \text{ and } (I({}^w\chi)^*)^H \neq (0), \\ \qquad \text{then } \textit{fix} \text{ a non-zero } L_\chi^{(w)} \in (I({}^w\chi)^*)^H \ \textit{arbitrarily}. \\ \text{If } w \notin W_\theta \text{ and } (I({}^w\chi)^*)^H = (0), \text{ then } L_\chi^{(w)} = 0. \end{cases}$$

Then, by **(2.2)**(ii),

(3.3) LEMMA. *For each $w \in W$, there is a constant $b_w(\chi) \in \mathbb{C}$ such that*

$$\widetilde{T}_w^\chi(L_\chi) = b_w(\chi) L_\chi^{(w)}.$$

Now we give an expression of Q_χ which is analogous to the formula for zonal spherical functions.

(3.4) THEOREM. *Let $\chi \in X_{\mathrm{ur}}(T)_\theta$ be regular and assume that $c_w(\chi^{-1})c_{w^{-1}}({}^w\chi) \neq 0$ for all $w \in W$. Then, for $\lambda \in \Lambda_m$,*

$$Q_\chi(\varpi_0^\lambda) = \mathrm{vol}(Bw_0B) \sum_{w \in W_\theta} \frac{c_{w_0}({}^w\chi) b_w(\chi)}{c_w(\chi^{-1})} {}^w\chi \delta_P^{1/2}(\varpi_0^\lambda).$$

Here, $c_w(\chi)$ is given by (14)*, $b_w(\chi)$ is determined by the functional equation of invariant functionals in the above lemma, and $W_\theta = W \cap H$.*

PROOF. By definition,

$$\begin{aligned} Q_\chi(\varpi_0^\lambda) &= \langle\, \pi_\chi^*(\varpi_0^{-\lambda}) L_\chi, \phi_{K,\chi} \,\rangle = \langle\, \pi_\chi^*(\varpi_0^{-\lambda}) L_\chi, p_B(\phi_{K,\chi}) \,\rangle \\ &= \langle\, p_B^* \left(\pi_\chi^*(\varpi_0^{-\lambda}) L_\chi \right), \phi_{K,\chi} \,\rangle. \end{aligned} \tag{15}$$

Let $\{f_{w,\chi^{-1}}\}_{w\in W}$ be the Casselman basis of $I(\chi^{-1})^B$ (see [C, p.402]) and write

$$p_B^*\left(\pi_\chi^*(\varpi_0^{-\lambda})L_\chi\right) = \sum_{w\in W} \alpha_w \cdot f_{w,\chi^{-1}},$$

regarding $p_B^*\left(\pi_\chi^*(\varpi_0^{-\lambda})L_\chi\right)$ as an element of $I(\chi^{-1})^B$. Applying $T_w^{\chi^{-1}}(\,\cdot\,)(1)$ on both sides we have

$$\begin{aligned}\alpha_w &= T_w^{\chi^{-1}}\left(p_B^*\left(\pi_\chi^*(\varpi_0^{-\lambda})L_\chi\right)\right)(1)\\ &= p_B^*\left(\widetilde{T}_w^{\chi}\left(\pi_\chi^*(\varpi_0^{-\lambda})L_\chi\right)\right)(1) \quad \text{by } \mathbf{(3.2)}\text{(ii)}\\ &= p_B^*\left(\pi_{{}^w\chi}^*(\varpi_0^{-\lambda})\widetilde{T}_w^{\chi}(L_\chi)\right)(1)\\ &= b_w(\chi)p_B^*\left(\pi_{{}^w\chi}^*(\varpi_0^{-\lambda})L_\chi^{(w)}\right)(1) \quad \text{by } \mathbf{(3.3)}.\end{aligned}$$

Using **(3.1)** here for $\ell = \pi_{{}^w\chi}^*(\varpi_0^{-\lambda})L_\chi^{(w)}$, this is equal to

$$\begin{aligned}&b_w(\chi)\sum_{w'\in W}\mathrm{vol}(Bw'B)^{-1}\langle\, \pi_{{}^w\chi}^*(\varpi_0^{-\lambda})L_\chi^{(w)}, \phi_{w',{}^w\chi}\,\rangle\phi_{w',{}^w\chi^{-1}}(1)\\ &\quad = b_w(\chi)\mathrm{vol}(B)^{-1}\langle\, L_\chi^{(w)}, \pi_{{}^w\chi}(\varpi_0^{\lambda})\phi_{1,{}^w\chi}\,\rangle.\end{aligned}$$

Returning to (15),

$$Q_\chi(\varpi_0^\lambda) = \mathrm{vol}(B)^{-1}\sum_{w\in W} b_w(\chi)\langle\, L_\chi^{(w)}, \pi_{{}^w\chi}(\varpi_0^{\lambda})\phi_{1,{}^w\chi}\,\rangle \cdot \langle\langle\, \phi_{K,\chi}, f_{w,\chi^{-1}}\,\rangle\rangle. \tag{16}$$

Here, we have

$$\begin{aligned}\langle\, L_\chi^{(w)}, \pi_{{}^w\chi}(\varpi_0^{\lambda})\phi_{1,{}^w\chi}\,\rangle &= \langle\, L_\chi^{(w)}, p_{{}^w\chi}(\mathrm{ch}_{B\varpi_0^{-\lambda}})\,\rangle\\ &= \begin{cases}\mathrm{vol}(B)\cdot {}^w\chi\delta^{1/2}(\varpi_0^\lambda) & \text{if } w\in W_\theta,\\ 0 \quad \text{otherwise.}\end{cases}\end{aligned}$$

Indeed, if $w \in W_\theta$, replacing ${}^w\chi$ by χ it is enough to see this for $w = 1$. By definition,

$$\begin{aligned}\langle\, L_\chi, p_\chi(\mathrm{ch}_{B\varpi_0^{-\lambda}})\,\rangle &= \int_B \Delta_\chi(b\varpi_0^{-\lambda})db = \int_B \prod_{i=1}^m |d_i\left(\tau(\varpi_0^\lambda) * b^{-1}\right)|^{s_i'} db \\ &= \mathrm{vol}(B)\cdot \prod_{i=1}^m |d_i\left(\tau(\varpi_0^\lambda)\right)|^{s_i'} \quad \text{by } \mathbf{(1.3)} \\ &= \mathrm{vol}(B)\cdot \chi\delta^{1/2}(\varpi_0^\lambda) \quad \text{by (10) and (12).}\end{aligned}$$

On the other hand, if $w \notin W_\theta$ and $L_\chi^{(w)} \neq 0$, then the support of the distribution $p^*_{{}^w\chi}(L_\chi^{(w)})$ is $\Omega_v^{c\ell} = (P\eta_v H)^{c\ell}$ with $v = \theta(w)^{-1}w \neq 1$, by **(2.2)**(ii). Since $\Omega_v^{c\ell} \cap \Omega_1 = \emptyset$ for $v \neq 1$ by **(1.1)** and $B\varpi_0^{-\lambda} \subset \Omega_1$ by **(1.3)**, we have $\mathrm{supp}\left(p^*_{{}^w\chi}(L_\chi^{(w)})\right) \cap B\varpi_0^{-\lambda} = \emptyset$, hence $\langle\, L_\chi^{(w)}, p_{{}^w\chi}(\mathrm{ch}_{B\varpi_0^{-\lambda}})\,\rangle = 0$.

Now (16) reduces to

$$Q_\chi(\varpi_0^\lambda) = \sum_{w\in W_\theta} b_w(\chi)\langle\langle\, \phi_{K,\chi}, f_{w,\chi^{-1}}\,\rangle\rangle\, {}^w\chi\delta^{1/2}(\varpi_0^\lambda).$$

By the formula for zonal spherical functions in [C, (4.2)], it is known that

$$\langle\langle\, \phi_{K,\chi}, f_{w,\chi^{-1}}\,\rangle\rangle = \mathrm{vol}(Bw_0B)\frac{c_{w_0}({}^{w_0w}\chi^{-1})}{c_w(\chi^{-1})}.$$

Finally, if $w \in W_\theta$ and $\chi \in X_{\mathrm{ur}}(T)_\theta$, then one has ${}^{w_0w}\chi^{-1} = {}^w({}^{w_0}\chi^{-1}) = {}^w\chi$. This completes the proof. □

For a regular $\chi \in X_{\mathrm{ur}}(T)_\theta$, assume that

$$Q_{{}^w\chi}(1) \neq 0 \quad \text{for all } w \in W_\theta.$$

Then, for each $w \in W_\theta$, applying both sides of **(3.3)** to $\phi_{K,{}^w\chi}$, one has

$$\begin{aligned}\langle\, \widetilde{T}_w^\chi(L_\chi), \phi_{K,{}^w\chi}\,\rangle &= \frac{c_w(\chi^{-1})}{c_{w^{-1}}(\chi)}\cdot\langle\, L_\chi, T_{w^{-1}}^{{}^w\chi}(\phi_{K,{}^w\chi})\,\rangle = c_w(\chi^{-1})\langle\, L_\chi, \phi_{K,{}^w\chi}\,\rangle \\ &= b_w(\chi)\langle\, L_{{}^w\chi}, \phi_{K,{}^w\chi}\,\rangle.\end{aligned}$$

Thus

$$b_w(\chi) = \frac{Q_\chi(1)}{Q_{^w\chi}(1)} \cdot c_w(\chi^{-1}).$$

Put $\widetilde{Q}_\chi = Q_\chi(1)^{-1} \cdot Q_\chi$. Then we have

(3.5) Corollary. *For a regular* $\chi \in X_{\mathrm{ur}}(T)_\theta$ *such that* $c_w(\chi^{-1})c_{w^{-1}}({}^w\chi) \neq 0$ *for all* $w \in W$ *and that* $Q_{^w\chi}(1) \neq 0$ *for all* $w \in W_\theta$*,*

$$\widetilde{Q}_\chi(\varpi_0^\lambda) = \mathrm{vol}(Bw_0B) \sum_{w \in W_\theta} \frac{c_{w_0}({}^w\chi)}{Q_{^w\chi}(1)} \cdot {}^w\chi\delta_P^{1/2}(\varpi_0^\lambda).$$

§4. Explicit Computations for $n = 2$ and 3

In this section we give an explicit formula of $\widetilde{Q}_\chi$ for $n = 2$ and $n = 3$. By **(3.5)** it is enough to compute $Q_\chi(1)$ for $\chi \in X_{\mathrm{ur}}(T)_\theta$.

(I) n=2

In this case, $W_\theta = W = \{1, w_0\}$. Write $\chi \in X_{\mathrm{ur}}(T)_\theta$ as

$$\chi \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} = |t_1|^s |t_2|^{-s}, \quad s \in \mathbb{C}.$$

Then χ is regular if and only if $q^{-s} \neq \pm 1$, which we assume below. The function Δ_χ defined by (11) is then of the form

$$\Delta_\chi(g) = |d_1(\theta(g)g^{-1})|^{s-\frac{1}{2}}.$$

Note that $d_1(x)$ is just the $(1,1)$-entry of the matrix x. By the decomposition $K = B \cup Bw_0B$ (disjoint),

$$Q_\chi(1) = \int_K \Delta_\chi(k)dk = \int_B \Delta_\chi(b)db + \int_{Bw_0B} \Delta_\chi(y)dy. \tag{17}$$

The integral over B is $\mathrm{vol}(B)$ by **(1.3)**. Also, by the Iwahori factorization,

$$\int_{Bw_0B} \Delta_\chi(y)dy = \mathrm{vol}(Bw_0B) \int_{\mathcal{O}} \Delta_\chi\left(w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) dx$$

where dx is the additive Haar measure of $\mathcal{O}$ normalized so that $\mathrm{vol}(\mathcal{O}) = 1$. Now the $(1,1)$-entry of $\theta\left(w_0\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)\right)\left(w_0\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)\right)^{-1}$ is $1 - x\bar{x}$, by a direct calculation. We have to compute the integral

$$\int_{\mathcal{O}} |1 - x\bar{x}|^{s-\frac{1}{2}} dx = \int_{|x|<1} dx + \int_{|x|=1} |1 - x\bar{x}|^{s-\frac{1}{2}} dx \tag{18}$$

but the latter integral, over $|x| = 1$, is already computed in [FH, pp. 705–706] as follows (see also **(4.2)** of this section); the volume of the set

$$\{\, x \in \mathcal{O}\,;\, |x| = 1, \quad |1 - x\bar{x}| = 1 \,\}$$

is $1 - q_F^{-1} - 2q^{-1}$, and for $i \geqslant 1$ the volume of the set

$$\{\, x \in \mathcal{O}\,;\, |x| = 1, \quad |1 - x\bar{x}| = q^{-i} \,\}$$

is $q_F^{-i}(1 - q^{-1})$, both are with respect to our additive Haar measure. Thus (18) is computed as

$$q^{-1} + (1 - q_F^{-1} - 2q^{-1}) + \sum_{i \geqslant 1} q_F^{-i}(1 - q^{-1}) q^{-i(s-\frac{1}{2})}$$

$$= 1 - q_F^{-1} - q^{-1} + (1 - q^{-1}) \cdot \frac{q^{-s}}{1 - q^{-s}} \quad \text{for } \mathrm{Re}(s) > 0. \tag{19}$$

Since $\mathrm{vol}(B) = (q+1)^{-1}$, $\mathrm{vol}(Bw_0B) = q(q+1)^{-1}$, returning to (17) we have

$$\begin{aligned} Q_\chi(1) &= \frac{1}{q+1} \cdot (1 + q \times (19)) = \frac{1}{q+1} \cdot \left(q - q_F + (q-1) \cdot \frac{q^{-s}}{1 - q^{-s}} \right) \\ &= \frac{q - q_F}{q+1} \cdot \frac{1 + q^{-s-\frac{1}{2}}}{1 - q^{-s}}. \end{aligned}$$

So $Q_{^w\chi}(1) \neq 0$ for all $w \in W_\theta$ if and only if $q^{-s} \neq -q^{\pm 1/2}$.

(4.1) Theorem. *For $n = 2$, assume that $\chi \in X_{\mathrm{ur}}(T)_\theta$ is of the form $\chi(\mathrm{diag}(t_1, t_2)) = |t_1|^s |t_2|^{-s}$, $s \in \mathbb{C}$, $q^{-s} \neq \pm 1$, $-q^{1/2}$, $\pm q^{-1/2}$. Then $\widetilde{Q}_\chi$ is given by*

$$\widetilde{Q}_\chi \begin{pmatrix} \varpi^\lambda & 0 \\ 0 & 1 \end{pmatrix} = \frac{q_F}{q_F - 1} \cdot \left(\frac{1 - q^{-s-\frac{1}{2}}}{1 + q^{-s}} \cdot q^{-\lambda(s+\frac{1}{2})} + \frac{1 - q^{s-\frac{1}{2}}}{1 + q^{s}} \cdot q^{\lambda(s-\frac{1}{2})} \right)$$

for $\lambda \in \mathbb{Z}$, $\lambda \leqslant 0$.

PROOF. For our choice of χ,

$$c_{w_0}(\chi) = \frac{1 - q^{-2s-1}}{1 - q^{-2s}}$$

(see (14)). Therefore, $c_{w_0}(\chi^{-1}) c_{w_0^{-1}}({}^{w_0}\chi) \neq 0$ holds if and only if $q^{-s} \neq \pm q^{-1/2}$. Also, by the above computation of $Q_\chi(1)$,

$$\frac{c_{w_0}(\chi)}{Q_\chi(1)} = \frac{q+1}{q - q_F} \cdot \frac{(1 - q^{-2s-1})(1 - q^{-s})}{(1 - q^{-2s})(1 + q^{-s-\frac{1}{2}})} = \frac{q+1}{q - q_F} \cdot \frac{1 - q^{-s-\frac{1}{2}}}{1 + q^{-s}}$$

which proves the above formula, by **(3.5)**. $\square$

REMARK. If we replace λ ($\leqslant 0$) by $-\lambda$ ($\geqslant 0$), then **(4.1)** coincides with the formula given by Banks [B].

(II) n=3

Put $w_1 = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix}$, $w_2 = \begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}$. Then $W = \{ 1, w_1, w_2, w_1 w_2, w_2 w_1, w_0 \}$, $W_\theta = \{ 1, w_0 \}$. Write $\chi \in X_{\mathrm{ur}}(T)_\theta$ as

$$\chi \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} = |t_1|^s |t_3|^{-s}, \quad s \in \mathbb{C}.$$

Then χ is regular if and only if $q^{-s} \neq \pm 1$, which again we assume below. The function Δ_χ is of the form

$$\Delta_\chi(g) = |d_1(\theta(g) g^{-1})|^{s-1}.$$

As in the case $n = 2$, we use $K = \cup_{w \in W} BwB$ (disjoint union) and the Iwahori factorization $B = N_1^- T_0 N_0$ to compute $Q_\chi(1)$;

$$Q_\chi(1) = \int_K \Delta_\chi(k) dk = \sum_{w \in W} \int_{BwB} \Delta_\chi(g) dg$$

$$= \sum_{w \in W} \mathrm{vol}(BwB) \int_{N_1^-} \int_{N_0} \Delta_\chi(wn'n) dn' dn. \tag{20}$$

Here the Haar measures dn', dn of N_1^-, N_0 are normalized so that $\mathrm{vol}(N_1^-) = \mathrm{vol}(N_0) = 1$. We shall compute the six integrals in (20).

(II-1) $w = 1$. By **(1.3)**,

$$\int_{N_1^-}\int_{N_0} \Delta_\chi(n'n)dn'dn = 1. \tag{21}$$

(II-w_1) $w = w_1 = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix}$. In this case, using

$$w_1 \begin{pmatrix} 1 & 0 & * \\ & 1 & * \\ & & 1 \end{pmatrix} w_1^{-1} \subset N, \quad w_1 \begin{pmatrix} 1 & & \\ * & 1 & \\ 0 & 0 & 1 \end{pmatrix} w_1^{-1} \subset N$$

we obtain

$$\begin{aligned}
&\int_{N_1^-}\int_{N_0} \Delta_\chi(w_1 n'n)dn'dn \\
&\quad = \int_{\mathcal{O}}\int_{\mathcal{O}}\int_{\mathcal{O}} \Delta_\chi\left(w_1 \begin{pmatrix} 1 & & \\ 0 & 1 & \\ \varpi x & \varpi y & 1 \end{pmatrix} \begin{pmatrix} 1 & z & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}\right) dxdydz.
\end{aligned}$$

Here and henceforth, the additive Haar measures are normalized so that $\mathrm{vol}(\mathcal{O}) = 1$. By a direct calculation of the matrix inside Δ_χ, this is equal to

$$\begin{aligned}
&\int_{\mathcal{O}}\int_{\mathcal{O}}\int_{\mathcal{O}} |-z + \varpi\bar{z}\bar{x} + \varpi\bar{y} - \varpi^2\bar{x}y|^{s-1} dxdydz \\
&\quad = \int_{\mathcal{O}}\int_{\mathcal{O}}\int_{\mathcal{O}} |(\varpi\bar{y} - z) + \varpi\bar{x}(\bar{z} - \varpi y)|^{s-1} dxdydz \\
&\quad = \int_{\mathcal{O}}\int_{\mathcal{O}} |z + \varpi\bar{x}\bar{z}|^{s-1} dxdz \quad \text{(by replacing } z \text{ by } z + \varpi\bar{y}) \\
&\quad = \int_{\mathcal{O}} |z|^{s-1} dz \quad \text{(since } |z| > |\varpi\bar{x}\bar{z}| \text{ for all } x,\, z \in \mathcal{O}) \\
&\quad = (1 - q^{-1}) \cdot \frac{1}{1 - q^{-s}}. \qquad (22)
\end{aligned}$$

(II-w_2) $w = w_2 = \begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}$. As in **(II-$w_1$)**, we may write

$$\int_{N_1^-}\int_{N_0} \Delta_\chi(w_2 n' n)dn'dn$$
$$= \int_{\mathcal{O}^3} \Delta_\chi\left(w_2 \begin{pmatrix} 1 & & \\ \varpi x & 1 & \\ \varpi y & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ & 1 & z \\ & & 1 \end{pmatrix}\right) dxdydz,$$

and by a matrix calculation inside Δ_χ, this is equal to

$$\int_{\mathcal{O}^3} |\bar{z} - \varpi x + \varpi yz - \varpi^2 \bar{x} y|^{s-1} dxdydz$$
$$= \int_{\mathcal{O}^2} |\bar{z} + \varpi yz|^{s-1} dydz \quad (\text{by } z \rightsquigarrow z + \varpi\bar{x})$$
$$= (22).$$

(II-w_1w_2) $w = w_1w_2 = \begin{pmatrix} & & 1 \\ 1 & 0 & \\ 0 & 1 & \end{pmatrix}$. We may change the order of the Iwahori factorization to make computations easier. By

$$w_1w_2 \begin{pmatrix} 1 & & \\ 0 & 1 & \\ * & * & 1 \end{pmatrix} w_2^{-1}w_1^{-1} \subset N, \quad w_1w_2 \begin{pmatrix} 1 & * & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} w_2^{-1}w_1^{-1} \subset N$$

we have

$$\int_{N_1^-}\int_{N_0} \Delta_\chi(w_1w_2 n' n)dn'dn$$
$$= \int_{\mathcal{O}^3} \Delta_\chi\left(w_1w_2 \begin{pmatrix} 1 & 0 & x \\ & 1 & y \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ \varpi z & 1 & \\ 0 & 0 & 1 \end{pmatrix}\right) dxdydz.$$

Computing the matrix inside Δ_χ, this is equal to

$$\begin{aligned}
&\int_{\mathcal{O}^3} |-x\bar{y}+\varpi xz-y+\varpi\bar{z}|^{s-1}dxdydz \\
&\qquad = \int_{\mathcal{O}^2} |y+x\bar{y}|^{s-1}dxdy \quad (\text{by } y \rightsquigarrow -y+\varpi\bar{z}) \\
&\qquad = \int_{\mathcal{O}^2} |y+xy|^{s-1}dxdy \quad (\text{by } x \rightsquigarrow (\bar{y}^{-1}y)x) \\
&\qquad = \int_{\mathcal{O}} |1+x|^{s-1}dx \cdot \int_{\mathcal{O}} |y|^{s-1}dy \\
&\qquad = (1-q^{-1})^2 \cdot \frac{1}{(1-q^{-s})^2}. \qquad (23)
\end{aligned}$$

(II-w_2w_1) $w = w_2w_1 = \begin{pmatrix} & 1 & 0 \\ & 0 & 1 \\ 1 & & \end{pmatrix}$. As in **(II-$w_1w_2$)**, we have

$$\begin{aligned}
&\int_{N_1^-}\int_{N_0} \Delta_\chi(w_2w_1n'n)dn'dn \\
&\quad = \int_{\mathcal{O}^3} \Delta_\chi\left(w_2w_1 \begin{pmatrix} 1 & x & y \\ & 1 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & \varpi z & 1 \end{pmatrix} \right) dxdydz \\
&\quad = \int_{\mathcal{O}^3} |-x\bar{y}+\bar{x}+\varpi\bar{y}\bar{z}-\varpi z|^{s-1}dxdydz \\
&\quad = \int_{\mathcal{O}^2} |\bar{x}-x\bar{y}|^{s-1}dxdy \quad (\text{by } x \rightsquigarrow x+\varpi\bar{z}) \\
&\quad = (23).
\end{aligned}$$

(II-w_0) $w = w_0$. Since $w_0N_1^-w_0^{-1} \subset N$,

$$\begin{aligned}
&\int_{N_1^-}\int_{N_0} \Delta_\chi(w_0n'n)dn'dn \\
&\quad = \int_{\mathcal{O}^3} \Delta_\chi\left(w_0 \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} \right) dxdydz \\
&\quad = \int_{\mathcal{O}^3} |\bar{y}(xz-y)-\bar{x}z+1|^{s-1}dxdydz \\
&\quad = \int_{\mathcal{O}^3} |\bar{x}z(\bar{y}-1)+(1-y\bar{y})|^{s-1}dxdydz \quad (\text{by } y \rightsquigarrow (\bar{x}^{-1}x)y). \qquad (24)
\end{aligned}$$

We divide $y \in \mathcal{O}$ into $\{|y| < 1\}$ and $\{|y| = 1\}$. The integral over $\{|y| < 1\}$ is easy to compute; since $|\bar{y} - 1| = 1$ for $|y| < 1$,

$$\begin{aligned}
&\int_{|y|<1} \int_{x,z\in\mathcal{O}} |\bar{x}z(\bar{y}-1) + (1-y\bar{y})|^{s-1} dxdydz \\
&\quad = \int_{|y|<1} \int_{x,z\in\mathcal{O}} |\bar{x}z + (1-y\bar{y})|^{s-1} dxdydz \\
&\quad = \int_{|y|<1} \int_{|x|<1} \int_{z\in\mathcal{O}} |\bar{x}z + (1-y\bar{y})|^{s-1} dxdydz \\
&\qquad + \int_{|y|<1} \int_{|x|=1} \int_{z\in\mathcal{O}} |\bar{x}z + (1-y\bar{y})|^{s-1} dxdydz.
\end{aligned}$$

The integrand in the first term is 1. Replacing z by $\bar{x}^{-1}z - \bar{x}^{-1}(1-y\bar{y})$ in the second, the above is equal to

$$(25)\quad q^{-2} + q^{-1}(1-q^{-1}) \cdot \int_{z\in\mathcal{O}} |z|^{s-1} dz = q^{-2} + q^{-1}(1-q^{-1})^2 \cdot \frac{1}{1-q^{-s}}.$$

Next, the integral over $\{|y| = 1\}$ is;

$$\begin{aligned}
&\int_{|y|=1} \int_{x,z\in\mathcal{O}} |\bar{x}z(\bar{y}-1) + (1-y\bar{y})|^{s-1} dxdydz \\
&= \sum_{i=0}^{\infty} \int_{|x|=q^{-i}} \int_{|y|=1} q^i \cdot \int_{|z|\leqslant q^{-i}} |z(\bar{y}-1) + (1-y\bar{y})|^{s-1} dxdydz \\
&= \sum_{i=1}^{\infty} q^{-i}(1-q^{-1}) \int_{|y|=1} q^i \cdot \int_{|z|\leqslant q^{-i}} |\bar{x}z(\bar{y}-1) + (1-y\bar{y})|^{s-1} dydz \\
&= (1-q^{-1}) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \int_{|y|=1, |\bar{y}-1|=q^{-j}} q^j \int_{|z|\leqslant q^{-i-j}} |z + (1-y\bar{y})|^{s-1} dydz \\
&= (1-q^{-1}) \sum_{i,j\geqslant 0} q^j \cdot \Bigg\{ \int_{\substack{|y|=1, |\bar{y}-1|=q^{-j} \\ |1-y\bar{y}|\leqslant q^{-i-j}}} \int_{|z|\leqslant q^{-i-j}} |z + (1-y\bar{y})|^{s-1} dydz \\
&\qquad\qquad + \int_{\substack{|y|=1, |\bar{y}-1|=q^{-j} \\ |1-y\bar{y}|> q^{-i-j}}} \int_{|z|\leqslant q^{-i-j}} |z + (1-y\bar{y})|^{s-1} dydz \Bigg\}.
\end{aligned}$$

In the former integral in $\{\cdots\}$, we may replace z by $z-(1-y\bar{y})$. In the latter, the integrand is $|1-y\bar{y}|^{s-1}$. So, this is written as

$$(26)\quad (1-q^{-1})\sum_{i,j\geqslant 0} q^j \left\{ v_{i,j}(1-q^{-1})\cdot\frac{q^{-i-j}}{1-q^{-s}} + q^{-i-j}\sum_{k=0}^{i-1} v'_{k,j}q^{-(k+j)(s-1)} \right\}$$

where, for $i,\ j,\ k\geqslant 0$,

$$v_{i,j} = \mathrm{vol}\left(\{y\in\mathcal{O}\,;\ |y|=1, |\bar{y}-1|=q^{-j}, |1-y\bar{y}|\leqslant q^{-i-j}\}\right),$$
$$v'_{k,j} = \mathrm{vol}\left(\{y\in\mathcal{O}\,;\ |y|=1, |\bar{y}-1|=q^{-j}, |1-y\bar{y}|= q^{-k-j}\}\right),$$

both measured by the additive Haar measure as before. Note that if $|y|=1$, $|\bar{y}-1|=q^{-j}$ then $|1-y\bar{y}|=|1-\bar{y}+\bar{y}(1-y)|\leqslant q^{-j}$.

(4.2) LEMMA.
(i)

$$v_{0,0}=1-2q^{-1},\qquad v_{0,j}=q^{-j}(1-q^{-1})\quad (j\geqslant 1),$$
$$v_{i,0}=q^{-\frac{i}{2}}\quad (i\geqslant 1),\qquad v_{i,j}=q^{-\frac{i}{2}-j}(1-q^{-\frac{1}{2}})\quad (i,j\geqslant 1).$$

(ii)

$$v'_{0,0}=1-q^{-\frac{1}{2}}-2q^{-1},\qquad v'_{0,j}=q^{-j}(1-q^{-\frac{1}{2}})\quad (j\geqslant 1),$$
$$v'_{k,0}=q^{-\frac{k}{2}}(1-q^{-\frac{1}{2}})\quad (k\geqslant 1),\qquad v'_{k,j}=q^{-\frac{k}{2}-j}(1-q^{-\frac{1}{2}})^2\quad (i,j\geqslant 1).$$

PROOF. Put $U^{(m)}=1+\varpi^m\mathcal{O}$ for $m\geqslant 1$ and $U^{(0)}=\mathcal{O}^\times$. At first, it is easy to see that $v_{0,j}$ is computed as $\mathrm{vol}(U^{(j)})-\mathrm{vol}(U^{(j+1)})$, so the first two of (i) are easy. To compute $v_{i,j}$ for $i\geqslant 1$, set

$$U_{i,j}=\{\,y\in\mathcal{O}\,;\ |y|=1, |\bar{y}-1|\leqslant q^{-j}, |1-y\bar{y}|\leqslant q^{-i-j}\,\}$$

for $i,\ j\geqslant 0$. Then, $U_{i,j}$ is a multiplicative subgroup of $\mathcal{O}^\times$, lying between $U^{(j)}$ and $U^{(i+j)}$. Moreover it is easy to observe that

$$U_{i,j}=\bigcup_{\substack{\varepsilon\in U^{(j)}/U^{(i+j)}\\ \varepsilon\bar{\varepsilon}\equiv 1 \mod U^{(i+j)}}} \varepsilon\cdot U^{(i+j)}.$$

Since E/F is unramified, the norm map $N_{E/F} : E^\times \to F^\times$ gives a surjection $U^{(m)} \twoheadrightarrow U_F^{(m)}$ for each m, hence induces $U^{(m)}/U^{(n)} \twoheadrightarrow U_F^{(m)}/U_F^{(n)}$ for $m \leqslant n$. Here $U_F^{(m)} = U^{(m)} \cap F$. Now the volume of $U_{i,j}$ is computed as;

$$\begin{aligned}\mathrm{vol}(U_{i,j}) &= \#\ker[U^{(j)}/U^{(i+j)} \twoheadrightarrow U_F^{(j)}/U_F^{(i+j)}] \times \mathrm{vol}(U^{(i+j)})\\ &= q^{-i-j}\frac{\#(U^{(j)}/U^{(i+j)})}{\#(U_F^{(j)}/U_F^{(i+j)})} = \begin{cases} q^{-\frac{i}{2}}(1+q^{-\frac{1}{2}}) & (j=0),\\ q^{-\frac{i}{2}-j} & (j \geqslant 1).\end{cases}\end{aligned}$$

Using this, $v_{i,j}$ is computed as $\mathrm{vol}(U_{i,j}) - \mathrm{vol}(U_{i-1,j+1})$ for $i \geqslant 1$ so (i) follows immediately. Also, (ii) follows from (i) since $v'_{k,j} = v_{k.j} - v_{k+1,j}$. □

Applying **(4.2)** and by an earnest computation, (26) is equal to

$$-\frac{q^{-1}(1-q^{-1})^2}{1-q^{-s}} + \frac{(1-q^{-1})^2(1-q^{-1}+q^{-s-\frac{1}{2}})}{(1-q^{-s})^2} - q^{-\frac{3}{2}}(1+q^{-\frac{1}{2}}) + \frac{q^{-1}(1-q^{-1})}{1-q^{-s}}. \tag{27}$$

Returning to (24),

$$\begin{aligned}(24) &= (25) + (27)\\ &= -q^{-\frac{3}{2}} + \frac{q^{-1}(1-q^{-1})}{1-q^{-s}} + \frac{(1-q^{-1})^2(1-q^{-1}+q^{-s-\frac{1}{2}})}{(1-q^{-s})^2}.\end{aligned} \tag{28}$$

Finally, since $\mathrm{vol}(BwB) = \mathrm{vol}(B) \times q^{\ell(w)}$, returning to (20),

$$\begin{aligned}Q_\chi(1) &= \mathrm{vol}(B) \times \{(21) + 2 \times q \times (22) + 2 \times q^2 \times (23) + q^3 \times (28)\}\\ &= \mathrm{vol}(B)\Big\{(1-q^{3/2}) + \big(2q(1-q^{-1}) + q^2(1-q^{-1})\big)\cdot\frac{1}{1-q^{-s}}\\ &\quad + \Big(2q^2(1-q^{-1})^2 + q^3(1-q^{-1})^2(1-q^{-1}++q^{-s-\frac{1}{2}})\Big)\\ &\quad \cdot\frac{1}{(1-q^{-s})^2}\Big\}\end{aligned}$$

which is simplified as

$$Q_\chi(1) = \mathrm{vol}(B)(q^3 - q_F^3)\frac{(1-q^{-s-1})(1+q^{-s-\frac{1}{2}})}{(1-q^{-s})^2}. \tag{29}$$

Note that $Q_{^w\chi}(1) \neq 0$ for all $w \in W_\theta$ if and only if $q^{-s} \neq q^{\pm 1}$, $-q^{\pm 1/2}$.

(4.3) THEOREM. *For $n = 3$, assume that $\chi \in X_{\rm ur}(T)_\theta$ is of the form $\chi(\mathrm{diag}(t_1, t_2, t_3)) = |t_1|^s |t_3|^{-s}$, $s \in \mathbb{C}$, $q^{-s} \neq \pm 1$, $q^{\pm 1}$, $-q^{1/2}$, $\pm q^{-1/2}$. Then, $\widetilde{Q}_\chi$ is given by*

$$\widetilde{Q}_\chi(\varpi_0^\lambda) = \frac{qq_F}{qq_F - 1}\left(\frac{(1-q^{-s-1})(1-q^{-s-\frac{1}{2}})}{1-q^{-2s}} q^{-\lambda(s+1)} + \frac{(1-q^{s-1})(1-q^{s-\frac{1}{2}})}{1-q^{2s}} q^{\lambda(s-1)}\right)$$

for $\lambda \in \mathbb{Z}$, $\lambda \leqslant 0$.

PROOF. First, by (14) it is known that $c_w(\chi^{-1})c_{w^{-1}}(^w\chi) \neq 0$ for all $w \in W$ if and only if $q^{-s} \neq q^{-1}$, $\pm q^{-1/2}$. Also, for our choice of χ,

$$c_{w_0}(\chi) = \left(\frac{1-q^{-s-1}}{1-q^{-s}}\right)^2 \cdot \frac{1-q^{-2s-1}}{1-q^{-2s}}.$$

Therefore, by (29),

$$\begin{aligned}\frac{c_{w_0}(\chi)}{Q_\chi(1)} &= \frac{1}{\mathrm{vol}(B)q^3(1-q_F^{-3})} \cdot \frac{(1-q^{-s})^2(1-q^{-s-1})^2(1-q^{-2s-1})}{(1-q^{-s-1})(1+q^{-s-\frac{1}{2}})(1-q^{-s})^2(1-q^{-2s})} \\ &= \frac{1}{\mathrm{vol}(Bw_0B)} \frac{qq_F}{qq_F-1} \cdot \frac{(1-q^{-s-1})(1-q^{-s-\frac{1}{2}})}{(1-q^{-2s})}\end{aligned}$$

which proves the above formula, by **(3.5)**. □

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