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Freeness of Adjoint Linear Systems on Threefolds with Terminal Gorenstein Singularities or Some Quotient Singularities

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Abstract. We generalize the result of Kawamata concerning the strong version of Fujita's freeness conjecture for smooth 3-folds to some singular cases, namely, Gorenstein terminal singularities, Gorenstein $\mathbb{Q}$-factorial terminal singularities and quotient singularities of type $1/r(1,1,1)$ and of type $1/r(1,1,-1)$. We generalize furthermore the result of that to projective threefolds with only canonical singularities for canonical and not terminal singularities. It turns out that the estimates in the first three cases are better than the one for the smooth case, while it is not in the fourth case. We also give explicit examples which show the estimate in the fourth case is necessarily worse than the one for the smooth case.

0. Introduction

We recall related results which are previously known. T.Fujita conjectured that, for a smooth projective variety X and an ample divisor L on X, the linear system $|K_X + mL|$ is free if $m \geq \dim X + 1$. Reider [Rdr] proved that this is the case if $\dim X = 2$. Ein and Lazarsfeld [EL] proved that this is the case if $\dim X = 3$. Kawamata [K3] proved that this is the case if $\dim X = 4$. For a projective variety X of dimension 2 with some singularities, Ein and Lazarsfeld [EL] and Matsushita [M] extended the result of Reider [Rdr] to singular cases. Kawachi [KM] obtained the effective estimates for a normal surface X. For a projective variety X of dimension 3 with some singularities, Oguiso and Peternell [OP] proved that $|K_X + 5L|$ is free if X is a projective threefold with only $\mathbb{Q}$-factorial terminal Gorenstein singularities. Ein, Lazarsfeld and Maşek [ELM] and Matsushita [M] extended some of the results of Ein and Lazarsfeld [EL] to projective threefolds with terminal singularities.

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A strong version of Fujita's freeness conjecture:
Let X be a normal projective variety of dimension n, $x_0 \in X$ a smooth point, and L an ample Cartier divisor. Assume that there exist positive numbers σ_p for $p = 1, 2, \cdots, n$ which satisfy the following conditions: (1) $\sqrt[p]{(L)^p \cdot W} \geq \sigma_p$ for any subvariety W of dimension p which contains x_0, (2)$\sigma_p \geq n$ for all p, and $\sigma_n > n$. Then $|K_X + L|$ is free at x_0.

Fujita [F] proved that, if $\sigma_1 \geq 3$, $\sigma_2 \geq \sqrt{7}$, and $\sigma_3 > \sqrt[3]{51}$, then $|K_X + L|$ is free at x_0. Our results are generalizations of the following result of Kawamata [K3]: Let X be a normal projective variety of dimension 3, L an ample Cartier divisor, and $x_0 \in X$ a smooth point. Assume that there are positive numbers σ_p for $p = 1, 2, 3$ which satisfy the following conditions: (1) $\sqrt[p]{(L)^p \cdot W} \geq \sigma_p$ for any subvariety W of dimension p which contains x_0, (2) $\sigma_1 \geq 3$, $\sigma_2 \geq 3$, and $\sigma_3 > 3$. Then $|K_X + L|$ is free at x_0.

We shall prove the following results in this paper: Let X be a normal projective variety of dimension 3, $x_0 \in X$ a point, and L an ample $\mathbb{Q}$-Cartier divisor such that $K_X + L$ is a Cartier divisor at x_0. Assume that $\sqrt[p]{(L)^p \cdot W} \geq \sigma_p$ for any subvariety W of dimension p which contains x_0.
(1)(Theorem 3.1) Let $x_0 \in X$ be a Gorenstein terminal singular point. Assume that $\sigma_1 > (\sqrt[3]{2} + \sqrt{3})/\sqrt[3]{2}$, $\sigma_2 > \sqrt[3]{2} + \sqrt{3}$, and $\sigma_3 > \sqrt[3]{2} + \sqrt{3}$. Then $|K_X + L|$ is free at x_0. (Note that $3 > \sqrt[3]{2} + \sqrt{3} > 2.99$)
(2)(Theorem 3.3) Let $x_0 \in X$ be a Gorenstein terminal singular $\mathbb{Q}$-factorial point. Assume that $\sigma_1 \geq 2$, $\sigma_2 \geq 2\sqrt{2}$, and $\sigma_3 > 2\sqrt[3]{2}$. Then $|K_X + L|$ is free at x_0.
(3)(Theorem 3.4) Let $x_0 \in X$ be a quotient singular point of type $(1/r, 1/r, 1/r)$ for an integer r. Assume that $\sigma_1 \geq 3/r$, $\sigma_2 \geq 3/\sqrt{r}$, and $\sigma_3 > 3/\sqrt[3]{r}$. Then $|K_X + L|$ is free at x_0.
(4)(Theorem 3.6) Let $x_0 \in X$ be a quotient singular point of type $(1/r, 1/r, -1/r)$ for an integer $r \geq 3$. Assume that $\sigma_1 \geq 1 + (1/r)$, $\sigma_2 \geq (1 + (1/r))\sqrt{r+3}$, and $\sigma_3 > (1 + (1/r))\sqrt[3]{r+2}$. Then $|K_X + L|$ is free at x_0.
(5)(Theorem 3.8) Let X be a projective threefold with only canonical singularities and $x_0 \in X$ be a canonical and not terminal singular point. Assume that $\sigma_1 \geq 3$, $\sigma_2 \geq 3$, and $\sigma_3 > 3$. Then $|K_X + L|$ is free at x_0.

The result (1), the result (5), and the result for the smooth case [K3] imply the following: Corollary (Corollary 3.9). Let X be a projective variety of dimension 3 and H an ample Cartier divisor on X. Assume that X has at most canonical Gorenstein singularities. Then $|K_X + mH|$ is free if $m \geq 4$. Moreover, if $(H^3) \geq 2$, then $|K_X + 3H|$ is also free. Note that Lee[L1, L2] also obtained a result similar to this corollary independently.

We also give a series of examples (Example 3.5) which shows that our estimate in (3) is optimal for each r. Note that the estimates for σ_p in cases $(1), (2)$ and (3) are better than the one in the smooth case. On the other hand, our estimate in (4) is worse than in the smooth case, especially when the indices r being large, indeed, $\sigma_3 \to +\infty$ if $r \to +\infty$. In Example 3.7, we construct an infinite sequence $(X_r, L_r)(r = 1, 2, 3, \cdots)$ consisting of a 3-fold X_r having a singular point of type $1/r(1, 1, -1)$ and an ample $\mathbb{Q}$-Cartier divisor L_r on X_r which satisfies that $(L_r^3) \to +\infty$ if $r \to +\infty$, $K_{X_r} + L_r$ is Cartier at the point, but $|K_{X_r} + L_r|$ is not free at the point for each r. This sequence shows that there exists no uniform estimate for σ_3 independent of the indices r and also explains the reason why $\sigma_3 \to +\infty$ under $r \to +\infty$ in our estimate (4).

Our proof is very similar to the one for the smooth case given in [K3]. However this involves more careful and detailed analysis of multiplicities and discrepancies. For example, important differences for proof in between the case (1) and the smooth case are: the discrepancy coefficient for K_X under the blow up $x_0 \in X$ is 1 in (1), while it is 2 in the smooth case; $\text{mult}_{x_0}X = 2$ in (1), while it is 1 in the smooth case and the multiplicity $\text{mult}_{x_0}S$ of the minimal center S (Definition 1.2) at x_0 is 1, 2 or 3 in (1) while $\text{mult}_{x_0}S = 1$ or 2 in the smooth case if S is a surface. We also notice that in cases (3) and (4) the discrepancy coefficients for K_X under the blow up $x_0 \in X$ are no more integers; it is $3/r - 1$ in (3) and $1/r$ in (4). By this reason, we need to treat $\mathbb{Q}$-Cartier divisors whose orders $\text{ord}_{x_0}D$ at x_0 is of the form d/r and especially for Theorem 3.4, the case (3), we need Lemma 2.3, a generalization of [K3, Theorem 2.2] to $\mathbb{Q}$-Cartier divisors of fractional orders at x_0.

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1. Preliminaries

Most of the results of this paper are the applications of the following vanishing theorem:

Theorem 1.1 ([K1, V]). *Let X be a smooth projective variety and D a $\mathbb{Q}$-divisor. Assume that D is nef and big, and that the support of the difference $\ulcorner D \urcorner - D$ is a normal crossing divisor. Then $H^p(X, K_X + \ulcorner D \urcorner) = 0$ for $p > 0$.*

We recall notation of [K3](cf [KMM]).

Definition 1.2. Let X be a normal variety and $D = \Sigma_i d_i D_i$ an effective $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. If $\mu : Y \to X$ is an embedded resolution of the pair (X, D), then we can write

$$K_Y + F = \mu^*(K_X + D)$$

with $F = \mu_*^{-1} D + \Sigma_j e_j E_j$ for the exceptional divisors E_j.

The pair (X, D) is said to have only *log canonical singularities* (*LC*) (resp. *kawamata log terminal singularities* (*KLT*)) if $d_i \leq 1$(resp. < 1) for all i and $e_j \leq 1$(resp. < 1) for all j.

A subvariety W of X is said to be a *center of log canonical singularities* for the pair (X, D), if there is a birational morphism from a normal variety $\mu : Y \to X$ and a prime divisor E on Y with the coefficient $e \geq 1$ such that $\mu(E) = W$. The set of all the centers of log canonical singularities is denoted by $CLC(X, D)$. The union of all the subvarieties in $CLC(X, D)$ is denoted by $LLC(X, D)$ and called the *locus of log canonical singularities* for (X, D). For a point $x_0 \in X$, we define $CLC(X, x_0, D) = \{W \in CLC(X, D); x_0 \in W\}$.

We shall use the following propositions proved by Kawamata [K3]. Then we shall control the singularities of the minimal center of log canonical singularities and replace the minimal center of log canonical singularities by a smaller subvariety.

Proposition 1.3 ([K3, 1.5,1.6]). *Let X be a normal variety and D an effective $\mathbb{Q}$-Cartier divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. Assume* *that X is KLT and (X, D) is LC. If $W_1, W_2 \in CLC(X, D)$ and W is an irreducible component of $W_1 \cap W_2$, then $W \in CLC(X, D)$. If (X, D) is not KLT at a point $x_0 \in X$, then there exists the unique minimal element W_0 of $CLC(X, x_0, D)$. Moreover, W_0 is normal at x_0.*

Proposition 1.4 ([K3, 1.9]). *Let $x_0 \in X$, D and W_0 be as in Proposition 1.3. Assume that $\dim W_0 = 2$. Then W_0 has at most a rational singularity at x_0. Moreover, if W_0 is singular at x_0, and if D' is an effective $\mathbb{Q}$-Cartier divisor on X such that $\mathrm{ord}_{x_0} D'|_{W_0} \geq 1$, then $\{x_0\} \in CLC(X, x_0, D + D')$.*

Remark 1.5. The result in Proposition 1.4 is extended to higher dimensions ([K4]).

Proposition 1.6 ([K3, 1.10]). *Let $x_0 \in X$, D and W_0 be as in Proposition 1.3. Let D_1 and D_2 be effective $\mathbb{Q}$-Cartier divisors on X whose supports do not contain W_0 and which induce the same $\mathbb{Q}$-Cartier divisor on W_0. Assume that $(X, D + D_1)$ is LC at x_0 and there exists an element of $CLC(X, x_0, D + D_1)$ which is properly contained in W_0. Then the similar statement holds for the pair $(X, D + D_2)$.*

2. General Method

We can construct divisors which have high multiplicity at a given point from the following lemma.

Lemma 2.1. *Let X be a normal and complete variety of dimension n, L a nef and big $\mathbb{Q}$-Cartier divisor, $x_0 \in X$ a point, and t,t_0 rational numbers such that $t > t_0 > 0$. Then there exists an effective $\mathbb{Q}$-Cartier divisor D such that $D \sim_{\mathbb{Q}} tL$ and*

$$\mathrm{ord}_{x_0} D \geq (t_0 + \epsilon) \sqrt[n]{\frac{(L^n)}{\mathrm{mult}_{x_0} X}}$$

which is a rational number for $0 \leq \epsilon \ll \sqrt[n]{\mathrm{mult}_{x_0} X / (L^n)}$.

Proof. We take $r \in \mathbb{Q}$ such that rL is Cartier and $t' \in \mathbb{Q}$ such that $t' = t/t_0 > 1$. By [K3, 2.1], there exists an effective $\mathbb{Q}$-Cartier divisor rD'

such that

$$\mathrm{ord}_{x_0} rD' \geq r \sqrt[n]{\frac{(L^n)}{\mathrm{mult}_{x_0} X}}.$$

Therefore we may take $D = t_0 D'$. □

We generalize [K3, 2.3] in which a given point is assumed to be a Gorenstein KLT point. For our purpose, we need to consider the case where a given point is a KLT point. The following proposition is the key of the proofs of our main results.

Proposition 2.2. *Let X be a normal projective variety of dimension n, $x_0 \in X$ a KLT point, and L an ample $\mathbb{Q}$-Cartier divisor such that $K_X + L$ is Cartier at x_0. Assume that there exists an effective $\mathbb{Q}$-Cartier divisor D which satisfies the following conditions:*
(1) *$D \sim_{\mathbb{Q}} tL$ for a rational number $t < 1$,*
(2) *(X, D) is LC at x_0,*
(3) *$\{x_0\} \in CLC(X, D)$.*
Then $|K_X + L|$ is free at x_0.

Proof (cf [K3 2.3]). Let $r = \min\{s \in \mathbb{N}; sL \text{ is Cartier}\}$ and D' be a general member of $|mrL|$ for $m \gg 0$ which passes through x_0. Replacing D by $(1 - \epsilon_1)(D + \epsilon_2 D')$ for some $0 < \epsilon_i \ll 1/(rm)$, we may assume that x_0 is an isolated point of $LLC(X, D)$. Let $\mu : Y \to X$ be an embedded resolution of the pair (X, D). Then

$$K_Y + E + F_1 + F_2 = \mu^*(K_X + D)$$

where E is a reduced divisor such that $\mu(E) = \{x_0\}$, F_1 is a divisor of the form $\sum_j f_{1j} F_{1j}$ such that $f_{1j} < 1$ and $x_0 \in \mu(F_{1j})$, and F_2 is a divisor of the form $\sum_i f_{2i} F_{2i}$ such that $x_0 \notin \mu(F_{2i})$. Then

$$K_Y + (1 - t)\mu^* L \sim_{\mathbb{Q}} \mu^*(K_X + L) - E - F_1 - F_2.$$

Thus

$$H^1(Y, \mu^*(K_X + L) - E + \lceil -F_1 \rceil - F_2') = 0$$

where $F_2' = \mu^*(K_X + L) - \lceil \mu^*(K_X + L) - F_2 \rceil$ and we obtain a surjection

$$H^0(Y, \mu^*(K_X + L) + \lceil -F_1 \rceil - F_2') \to H^0(E, \mu^*(K_X + L)) \cong \mathbb{C}.$$

Since $\lceil -F_1 \rceil - F_2'$ is effective and exceptional over a neighborhood of x_0,

$$H^0(X, K_X + L) \to H^0(E, \mu^*(K_X + L))$$

is also surjective. □

We generalize [K3, 2.2] in which $\mathbb{Q}$-divisors have integral order at x_0. For our purpose, we need to treat $\mathbb{Q}$-Cartier divisors of fractional orders at x_0.

LEMMA 2.3. *Let X be a normal projective variety of dimension 3, $x_0 \in X$ a quotient singular point of type $(1/r, 1/r, 1/r)$ for an integer r, L an ample $\mathbb{Q}$-Cartier divisor such that $K_X + L$ is Cartier at x_0, W a prime divisor with $\mathrm{ord}_{x_0} W = d/r \geq 1/r$ for an integer d, and e,k positive rational numbers such that $de \leq 1$ and $(k/r)^3 < (L)^3/r^2$. Assume that there exists an effective $\mathbb{Q}$-divisor D such that $D \sim_{\mathbb{Q}} L$ and $\mathrm{ord}_{x_0} D \geq k/r$, and moreover that $D \geq ekW$ for any such D. Then there exists a real number λ with $0 \leq \lambda < 1$ and $\lambda \leq max\{1 - de, (3de)^{-1/2}\}$ which satisfies the following condition: if k' is a positive rational number such that $k' > k$ and*

$$(\lambda\frac{k}{r})^3+(\frac{1-de-\lambda}{1-\lambda})^2\{(\frac{k'}{r}+\frac{\lambda de}{1-de-\lambda}\frac{k}{r})^3-(\frac{\lambda k}{r}+\frac{\lambda de}{1-de-\lambda}\frac{k}{r})^3\} < \frac{(L)^3}{r^2},$$

then there exists an effective $\mathbb{Q}$-divisor D such that $D \sim_{\mathbb{Q}} L$ and $\mathrm{ord}_{x_0} D \geq k'/r$. (If $\lambda = 1 - de$, then the left hand side of the above inequality should be taken as a limit.)

PROOF. (cf [K3, 2.2]). We have $\mathrm{mult}_{x_0} X = r^2$. Let $\bar{k} = \sup\ \{q$; there exists an effective $\mathbb{Q}$-divisor D such that $D \sim_{\mathbb{Q}} L$ and $\mathrm{ord}_{x_0} D = q/r\ \}$. Let us define a function $\phi(q)$ for $q \in \mathbb{Q}$ with $0 \leq q < \bar{k}$ to be the largest real number such that $D \geq \phi(q)W$ whenever $D \geq 0$, $D \sim_{\mathbb{Q}} L$ and $\mathrm{ord}_{x_0} D = q/r$. Then ϕ is a convex function. In fact, if $\mathrm{ord}_{x_0} D_i = q_i/r$ and $D_i = (\phi(q_i) + \epsilon_i)W+$ other components for $0 \leq \epsilon_i \ll 1$ and $i = 1$, 2, then $\mathrm{ord}_{x_0}(tD_1 + (1-t)D_2) = t(q_1/r) + (1-t)(q_2/r)$ and $tD_1 + (1-t)D_2 = (t(\phi(q_1) + \epsilon_1) + (1-t)(\phi(q_2) + \epsilon_2))W+$ other components, hence $\phi(tq_1 + (1-t)q_2) \leq t\phi(q_1) + (1-t)\phi(q_2)$. Since $\phi(k) \geq ek$, there exists a real number λ such that $0 \leq \lambda < 1$ and $\phi(q) \geq e(q - \lambda k)/(1-\lambda)$ for any q.

Let m be a large and sufficiently divisible integer and $\nu : H^0(X, mL) \to O_{X,x_0}(mL) \cong O_{X,x_0}$ the evaluation homomorphism. We consider subspaces

$V_i = \nu^{-1}({m_{x_0}}^i)$ of $H^0(X, mL)$ for integers i such that $\lambda km/r \leq i \leq k'm/r$. First, we have

$$\dim V_{\lceil \lambda km/r \rceil} \geq \dim H^0(X, mL) - r^2 \frac{(\lambda km/r)^3}{3!} + \text{lower terms in } m.$$

Let $D \in |mL|$ be a member corresponding to $h \in V_i$ for some i. Since we have $D \geq \phi(ir/m)mW$, the number of conditions in order for $h \in V_{i+1}$ is at most (the number of homogeneous polynomials of order $i - \phi(ir/m)m(d/r)$ in 3 variables) $\times$ $\mathrm{mult}_{x_0} X$, i.e.,

$$r^2 \frac{(i - \phi(ir/m)md/r)^2}{2!} + \text{lower terms in } m.$$

Therefore, we have $k' < \bar{k}$ because

$$\begin{aligned} & r^2 \frac{(\lambda km/r)^3}{3!} + \sum_{i=\lceil \lambda km/r \rceil}^{k'm/r-1} r^2 \frac{(i - \frac{e(ir/m - \lambda k)}{(1-\lambda)} md/r)^2}{2!} + \text{lower terms in } m \\ & = r^2 \frac{(\lambda km/r)^3}{3!} \\ & \quad + r^2 \frac{(\frac{1-de-\lambda}{1-\lambda})^2 \{(k'/r + \frac{\lambda de}{1-de-\lambda} k/r)^3 - (\lambda k/r + \frac{\lambda de}{1-de-\lambda} k/r)^3\} m^3}{3!} \\ & \quad + \text{lower terms in } m \\ & \quad < \frac{m^3(L^3)}{3!} + \text{lower terms in } m. \end{aligned}$$

By [K3 2.2 last part], we have that $\lambda \leq \max\{1 - de, (3de)^{-1/2}\}$. □

The following theorem plays the important role in the proofs of our main results.

Theorem 2.4 ([A, Corollary 6]). *Let S be a normal surface, x_0 be a point of S. Suppose S has a rational singularity at x_0. Let Z be the fundamental cycle.*

Then

$$\mathrm{mult}_{x_0} S = -Z^2$$

for all integers k,

$$\dim m_{S,x_0}^k / m_{S,x_0}^{k+1} = k\,\mathrm{mult}_{x_0} S + 1$$

where m_{S,x_0} is the maximal ideal of x_0 in S and

$$\mathrm{embdim}_{x_0} S = \mathrm{mult}_{x_0} S + 1.$$

The following lemma plays the supporting role in the proof of Theorem 3.6.

LEMMA 2.5. *Let (X, x_0) be a quotient singularity $\mathbb{C}^3/\mathbb{Z}_r(a/r, -a/r, 1/r)$ with $0 < a < r$. Let $S_i = \mathbb{C}^2/\mathbb{Z}_r(i/r, 1/r)$ for $i = a, -a$. Then*

$$\mathrm{embdim}_{x_0} X = \mathrm{mult}_{x_0} X + 2.$$

$$\mathrm{mult}_{x_0} X = \mathrm{mult}_{x_0} S_a + \mathrm{mult}_{x_0} S_{-a}.$$

$$\mathrm{embdim}_{x_0} X = \mathrm{embdim}_{x_0} S_a + \mathrm{embdim}_{x_0} S_{-a}.$$

PROOF. Let $(xy)^w x^s z^u, (xy)^w y^t z^u \in m^n_{X,x_0}/m^{n+1}_{X,x_0}$ where m_{X,x_0} is the maximal ideal of x_0 in X. Then $x^s z^u, y^t z^u \in m^{n-w}_{X,x_0}/m^{n-w+1}_{X,x_0}$. We consider that $x^s z^u \in m^{n-w}_{S_a,x_0}/m^{n-w+1}_{S_a,x_0}$ where m_{S_a,x_0} is the maximal ideal of x_0 in S_a and that $y^t z^u \in m^{n-w}_{S_{-a},x_0}/m^{n-w+1}_{S_{-a},x_0}$ where m_{S_{-a},x_0} is the maximal ideal of x_0 in S_{-a}. By Theorem 2.4., for $i = a, r-a$

$$\dim m^{n-w}_{S_i,x_0}/m^{n-w+1}_{S_i,x_0} = (n-w)\mathrm{mult}_{x_0} S_i + 1.$$

$m^{n-w}_{S_a,x_0}/m^{n-w+1}_{S_a,x_0} \cap m^{n-w}_{S_{-a},x_0}/m^{n-w+1}_{S_{-a},x_0} = (z^r)^{n-w}$ for $0 \le w < n$. Then

$$\begin{aligned}
&\dim m^n_{X,x_0}/m^{n+1}_{X,x_0} \\
&= \sum_{w=0}^{n-1} \{\dim m^{n-w}_{S_a,x_0}/m^{n-w+1}_{S_a,x_0} + \dim m^{n-w}_{S_{-a},x_0}/m^{n-w+1}_{S_{-a},x_0} - 1\} + 1 \\
&= \sum_{w=0}^{n-1} \{(n-w)(\mathrm{mult}_{x_0} S_a + \mathrm{mult}_{x_0} S_{-a}) + 1\} + 1 \\
&= (\mathrm{mult}_{x_0} S_a + \mathrm{mult}_{x_0} S_{-a})\frac{n^2}{2} + (\mathrm{mult}_{x_0} S_a + \mathrm{mult}_{x_0} S_{-a} + 2)\frac{n}{2} + 1
\end{aligned}$$

Hence

$$\mathrm{mult}_{x_0} X = \mathrm{mult}_{x_0} S_a + \mathrm{mult}_{x_0} S_{-a}.$$

$$\mathrm{embdim}_{x_0} X = \mathrm{mult}_{x_0} S_a + \mathrm{mult}_{x_0} S_{-a} + 2. \square$$

3. Main Theorem

First we consider Gorenstein terminal singular points.

Theorem 3.1. *Let X be a normal projective variety of dimension 3, $x_0 \in X$ a Gorenstein terminal singular point, and L an ample $\mathbb{Q}$-Cartier divisor such that L is Cartier at x_0. Assume that there are positive numbers σ_p for $p = 1, 2, 3$ which satisfy the following conditions:*
(1) $\sqrt[p]{(L)^p \cdot W} \geq \sigma_p$ for any subvariety W of dimension p which contains x_0,
(2) $\sigma_1 > (\sqrt[3]{2} + \sqrt{3})/\sqrt[3]{2}$, $\sigma_2 > \sqrt[3]{2} + \sqrt{3}$, and $\sigma_3 > \sqrt[3]{2} + \sqrt{3}$.
Then $|K_X + L|$ is free at x_0.

Proof. Since x_0 is a Gorenstein terminal singularity of dimension 3, x_0 is an isolated hypersurface singularity of multiplicity 2 ([R, (1.1) Theorem]). Let U be a neighborhood at x_0. Let $U \subset V$ be an embedding of U as a hypersurface in a smooth fourfold V, and let $g : \widetilde{V} \longrightarrow V$ be the blowing-up of V at x_0. Let $\widetilde{U}$ be the proper transform of U in $\widetilde{V}$, $f : \widetilde{U} \longrightarrow U$ the restriction of g to $\widetilde{U}$, and $F \subset \widetilde{V}$ the exceptional divisor of g. Then

$$K_{\widetilde{U}} = (K_{\widetilde{V}} + \widetilde{U})|_{\widetilde{U}} = (g^*K_V + 3F + g^*U - 2F)|_{\widetilde{U}} = f^*K_U + (F|_{\widetilde{U}}).$$

(cf.[ELM, Lemma 2.2 's proof, l.8–l.13])

Step 0. Let t be a rational number such that $t > 2\sqrt[3]{2}/\sqrt[3]{(L^3)}$. Since $\sigma_3 > 2\sqrt[3]{2}$, we can take $t < 1$. Let t_0 be a rational number such that $t_0 = 2\sqrt[3]{2}/\sqrt[3]{(L^3)} - \epsilon$ for $0 \leq \epsilon \ll \sqrt[3]{2/(L^3)}$. By Lemma 2.1, there exists an effective $\mathbb{Q}$-Cartier divisor D such that $D \sim_{\mathbb{Q}} tL$ and $\mathrm{ord}_{x_0} D \geq (t_0 + \epsilon)\sqrt[3]{(L^3)/2}$. Hence $\mathrm{ord}_{x_0} D = 2$.

Let c be the *log canonical threshold* of (X, D) at x_0:

$$c = \sup\{t \in \mathbb{Q};\ K_X + tD \text{ is } LC \text{ at } x_0\ \}.$$

Then $c \leq 1$. Let W be the minimal element of $CLC(X, x_0, cD)$. If $W = \{x_0\}$, then $|K_X + L|$ is free at x_0 by Proposition 2.2, since $ct < 1$.

Step 1. We consider the case in which $W = C$ is a curve. By Proposition 1.3, C is normal at x_0, i.e., smooth at x_0. Since $t < 1$, we have $ct + (1 - c) < 1$. Since $\sigma_1 \geq 2$, there exists a rational number t' with

$ct + (1 - c) < t' < 1$ and an effective $\mathbb{Q}$-Cartier divisor D'_C on C such that $D'_C \sim_{\mathbb{Q}} (t' - ct)L|_C$ and $\mathrm{ord}_{x_0} D'_C = 2(1 - c)$. As in [K3, 3.1 Step1], there exists an effective $\mathbb{Q}$-Cartier divisor D' on X such that $D' \sim_{\mathbb{Q}} (t' - ct)L$ and $D'|_C = D'_C$. Let D'_1 be a general effective $\mathbb{Q}$-Cartier divisor on an affine neighborhood U of x_0 in X such that $D'_1|_{C\cap U} = D'_C|_{C\cap U}$ and $\mathrm{ord}_{x_0} D'_1 = 2(1-c)$. Then we have $\mathrm{ord}_{x_0}(cD + D'_1) = 2$, hence $\{x_0\} \in CLC(U, cD + D'_1)$. Let

$$c' = \sup\{t \in \mathbb{Q};\ K_X + (cD + tD'_1) \text{ is } LC \text{ at } x_0\ \}.$$

Since D'_1 is chosen to be general, we have $c' > 0$. We have an element W' such that $W' \in CLC(X, x_0, cD + c'D'_1)$ and $W' \not\supseteq C$. By Proposition 1.3, $CLC(X, x_0, cD + c'D'_1)$ has an element which is properly contained in C. By Proposition 1.6, we conclude that $(X, cD + c'D')$ is LC at x_0, and $CLC(X, x_0, cD + c'D')$ has an element which is properly contained in C, i.e., $\{x_0\}$.

Step 2. We consider the case in which $W = S$ is a surface. By Proposition 1.4, S has at most a rational singularity at x_0.

Step 2-1. We assume first that S is smooth at x_0. As in Step 1, we take a rational number t', an effective $\mathbb{Q}$-Cartier divisor D' on X and a positive number c' such that $ct + (1 - c) < t' < 1$, $D' \sim_{\mathbb{Q}} (t' - ct)L$, $\mathrm{ord}_{x_0} D'|_S, = 2(1 - c)$, $(X, cD + c'D')$ is LC at x_0, and that the minimal element W' of $CLC(X, x_0, cD + c'D')$ is properly contained in S. Thus we have the theorem when $W' = \{x_0\}$.

We consider the case in which $W' = C$ is a curve. Since $t, t' < 1$, we have $ct + c'(t' - ct) + (1 - c)(1 - c') < 1$. As in Step 1, we take a rational number t'', an effective $\mathbb{Q}$-Cartier divisor D'' on X and a positive number c'' such that $ct + c'(t' - ct) + (1 - c)(1 - c') < t'' < 1$, $D'' \sim_{\mathbb{Q}} (t'' - ct - c'(t' - ct))L$, $\mathrm{ord}_{x_0} D''|_C = 2(1-c)(1-c')$, $(X, cD + c'D' + c''D'')$ is LC at x_0 and that the minimal element W'' of $CLC(X, x_0, cD + c'D' + c''D'')$ is properly contained in C, i.e., $\{x_0\} \in CLC(X, x_0, cD + c'D' + c''D'')$.

Step 2-2. We assume that S has a rational singularity at x_0. Since the embedding dimension of X at x_0 is 4 and x_0 is also a singular point of S, the embedding dimension of S at x_0 is 3 or 4. Therefore $d := \mathrm{mult}_{x_0} S = 2$ or 3, because $x_0 \in S$ is a rational singular point([A, Corollary 6]). Since

$t > 2\sqrt[3]{2}/\sqrt[3]{(L^3)}$ and $\sqrt[3]{(L^3)} > \sqrt[3]{2}+\sqrt{3}$, we can take $t < 2\sqrt[3]{2}/(\sqrt[3]{2}+\sqrt{3})$, so $t/2+\sqrt{3}/\sigma_2 < 1$. Therefore, if $c \geq 1/2$, then

$$ct + \frac{2\sqrt{d}(1-c)}{\sigma_2} < 1.$$

In this case, we can take a rational number t' and an effective $\mathbb{Q}$-Cartier divisor D' on X such that $ct + 2\sqrt{d}(1-c)/\sigma_2 < t' < 1$, $D' \sim_{\mathbb{Q}} (t'-ct)L$ and $\mathrm{ord}_{x_0} D'|_S = 2(1-c)$, and proceed as in Step 2-1.

On the other hand, if $c \leq 1/2$, then

$$ct + \frac{\sqrt{d}}{\sigma_2} < 1.$$

We take t' and D' with $D' \sim_{\mathbb{Q}} (t'-ct)L$ and $\mathrm{ord}_{x_0} D'|_S = 1$, by Proposition 1.4, $\{x_0\} \in CLC(X, x_0, cD + D')$. As in Step 1, there exists c' such that $1 \geq c' > 0$, $(X, x_0, cD + c'D')$ is LC at x_0, and that the minimal element of W' of $CLC(X, x_0, cD + c'D')$ is properly contained in S. If $W' = \{x_0\}$, we have the theorem.

We consider the case in which $W' = C$ is a curve. We have

$$\mathrm{ord}_{x_0}(cD + c'D')|_S \geq 2c + c' = 2 - 2(1 - c - \frac{c'}{2}).$$

Since $\sqrt{d}/\sigma_2 < t' - ct$, we can take $t' - ct < \sqrt{d}/(\sqrt[3]{2}+\sqrt{3})$. Then,

$$\begin{aligned} & ct + c'(t'-ct) + \frac{2}{\sigma_1}(1 - c - \frac{c'}{2}) \\ & \quad < c(\frac{2\sqrt[3]{2}}{\sqrt[3]{2}+\sqrt{3}}) + \frac{2\sqrt[3]{2}}{\sqrt[3]{2}+\sqrt{3}}(1-c) + c'(\frac{\sqrt{d}}{\sqrt[3]{2}+\sqrt{3}} - \frac{\sqrt[3]{2}}{\sqrt[3]{2}+\sqrt{3}}) \leq 1 \end{aligned}$$

The rest is the same as before. □

The following example shows that the condition $\sigma_3 > 2\sqrt[3]{2}$ in Theorem 3.1 is optimal.

Example 3.2. Let $X = \{xy + z^2 + t^2 = 0\} \subset \mathbb{P}^4$ and $x_0 = (0:0:0:0:1)$. Then x_0 is a Gorenstein terminal singular point and $K_X = \mathcal{O}(-3)$. If $L = \mathcal{O}(3)$, then $|K_X + L|$ is free at x_0. If $L = \mathcal{O}(2)$, then $|K_X + L|$ is not free at x_0 and $L^3 X = 16$.

If we assume furthermore that X is $\mathbb{Q}$-factorial at x_0, we obtain better estimates for σ_p.

THEOREM 3.3. *Let X be a normal projective variety of dimension 3, L an ample $\mathbb{Q}$-Cartier divisor, $x_0 \in X$ a Gorenstein terminal $\mathbb{Q}$-factorial singular point. Assume that there are positive numbers σ_p for $p = 1, 2, 3$ which satisfy the following conditions:*
(1) $\sqrt[p]{(L)^p \cdot W} \geq \sigma_p$ for any subvariety W of dimension p which contains x_0,
(2) $\sigma_1 \geq 2$, $\sigma_2 \geq 2\sqrt{2}$, and $\sigma_3 > 2\sqrt[3]{2}$.
Then $|K_X + L|$ is free at x_0.

PROOF. Since $\sigma_1 \geq 2$ and $\sigma_3 > 2\sqrt[3]{2}$, Steps 0,1 are the same as Steps 0,1 of the proof of Theorem 3.1.

Step 2. We consider the case in which $W = S$ is a surface. Since X is $\mathbb{Q}$-factorial terminal Gorenstein at x_0, X is factorial at x_0([K2, Lemma 5.1]). In particular, S is a Cartier divisor at x_0. Then we have $2 > \mathrm{ord}_{x_0} cD \geq \mathrm{ord}_{x_0} S$. Hence we have $\mathrm{ord}_{x_0} S = 1$ and $\mathrm{mult}_{x_0} S = 2$. There exists a rational number t' with $ct + (1 - c) < t' < 1$. By Lemma 2.1 and $\sigma_2 \geq 2\sqrt{2}$, there exists an effective $\mathbb{Q}$-Cartier divisor D_S on S such that $D_S \sim_{\mathbb{Q}} (t' - ct)L|_S$ and $\mathrm{ord}_{x_0} D_S = 2(1 - c)$.

The rest is the same as Step 2-1 of the proof of Theorem 3.1. □

We consider non Gorenstein singular points in the following.

The following theorem is generalization to dimension of 3 of the following result of Kawachi [KM]: Let S be a normal projective surface, $x_0 \in S$ a quotient singular point of type $(1/r, 1/r)$ for an integer r and L a nef and big $\mathbb{Q}$-Cartier divisor such that $K_X + L$ is Cartier at x_0. If $LC \geq 2/r$ for any curve C through x_0 and $\sqrt{L^2} > 2/\sqrt{r}$, then $|K_X + L|$ is free at x_0.

THEOREM 3.4. *Let X be a normal projective variety of dimension 3, $x_0 \in X$ a quotient singular point of type $(1/r, 1/r, 1/r)$ for an integer r, and L an ample $\mathbb{Q}$-Cartier divisor such that $K_X + L$ is Cartier at x_0. Assume that there are positive numbers σ_p for $p = 1, 2, 3$ which satisfy the following conditions:*
(1) $\sqrt[p]{(L)^p \cdot W} \geq \sigma_p$ for any subvariety W of dimension p which contains

x_0,
(2) $\sigma_1 \geq 3/r$, $\sigma_2 \geq 3/\sqrt{r}$, *and* $\sigma_3 > 3/\sqrt[3]{r}$.
Then $|K_X + L|$ *is free at* x_0.

Proof. Since $X = \mathbb{C}^3/\mathbb{Z}_r(1/r, 1/r, 1/r)$ around x_0, setting $\mathcal{O}_{\mathbb{C}^3,0} = \mathbb{C}\{x, y, z\}$, we have $\mathcal{O}_{X,x_0} = \mathcal{O}_{\mathbb{C}^3,0}{}^{\mathbb{Z}_r(1/r,1/r,1/r)} = \mathbb{C}\{x^a y^b z^c | a+b+c = r, a \geq 0, b \geq 0, c \geq 0\}$ and $m_{X,x_0}^n = (x^a y^b z^c | a+b+c = nr)$ where m_{X,x_0} is the maximal ideal of x_0 in X. Therefore

$$\dim_{\mathbb{C}} \mathcal{O}_{X,x_0}/m_{X,x_0}^n = \sum_{k=0}^{n-1} (kr+2)(kr+1)/2 = (r^2/3!)n^3 + \text{lower terms in } n.$$

Hence $\mathrm{mult}_{x_0} X = r^2$.

Step 0. Let t be a rational number such that $t > (3/\sqrt[3]{r})/\sqrt[3]{(L^3)}$. Since $\sigma_3 > 3/\sqrt[3]{r}$, we can take $t < 1$. Let t_0 be a rational number such that $t_0 = (3/\sqrt[3]{r})/\sqrt[3]{(L^3)} - \epsilon$ for $0 \leq \epsilon \ll \sqrt[3]{r^2/(L^3)}$. By Lemma 2.1, there exists an effective $\mathbb{Q}$-Cartier divisor D such that $D \sim_{\mathbb{Q}} tL$ and $\mathrm{ord}_{x_0} D \geq (t_0 + \epsilon)\sqrt[3]{(L^3)/r^2}$. Hence $\mathrm{ord}_{x_0} D = 3/r$.

Let $f : \bar{X} \to X$ be the blowing-up of X at x_0, $E \subset \bar{X}$ the exceptional divisor of f, and $\bar{D}$ the proper transform of D in $\bar{X}$. Then

$$K_{\bar{X}} = f^* K_X + (3/r - 1)E.$$

$$f^* D = \bar{D} + (3/r)E.$$

Let c be the log canonical threshold of (X, D) at x_0:

$$c = \sup\{t \in \mathbb{Q};\ K_X + tD \text{ is } LC \text{ at } x_0\ \}.$$

Then $c \leq 1$. Let W be the minimal element of $CLC(X, x_0, cD)$. If $W = \{x_0\}$, then $|K_X + L|$ is free at x_0 by Proposition 2.2, since $ct < 1$.

Step 1. We consider the case in which $W = C$ is a curve. By Proposition 1.3, C is normal at x_0, i.e., smooth at x_0. Since $\sigma_1 \geq 3/r$, as in Step 1 of the proof of Theorem 3.1, we take a rational number t', an effective $\mathbb{Q}$-Cartier divisor D' on X and a positive number c' such that $ct + (1-c) < t' < 1$, $D' \sim_{\mathbb{Q}} (t' - ct)L$, $\mathrm{ord}_{x_0} D'|_C = (3/r)(1-c)$, $(X, cD + c'D')$ is LC at x_0, and that the minimal element W' of $CLC(X, x_0, cD+c'D')$ is properly contained in $C, i.e., \{x_0\}$.

Step 2. We consider the case in which $W = S$ is a surface. Let U be a neighborhood at x_0, and $h : \widetilde{U} \longrightarrow U$ its closure of local universal cover, i.e., $U = \mathbb{C}^3/\mathbb{Z}_r(1/r, 1/r, 1/r)$ and $\widetilde{U} = \mathbb{C}^3$. Let $\widetilde{S}$ be $h^*(S|_U)$, and x_1 be $h^{-1}(x_0)$. Since $x_0 \in S$ is a KLT point [K3, 1.7] and h is ramified only over x_0, $\widetilde{S} \subset \mathbb{C}^3$ is also irreducible and $x_1 \in \widetilde{S}$ is at most KLT. Moreover $\widetilde{S} \subset \mathbb{C}^3$ is a hypersurface, $\widetilde{S}$ is also Gorenstein. Therefore, $x_1 \in \widetilde{S}$ is a rational Gorenstein point, that is, $x_1 \in \widetilde{S}$ is a smooth point or a rational double point. If x_1 is a smooth point of $\widetilde{S}$, then $S|_U \cong \mathbb{C}^2/\mathbb{Z}_r(1/r, 1/r)$. Hence we have $\mathrm{mult}_{x_0} S = r$. If x_1 is a rational double point of $\widetilde{S}$, that $\widetilde{S}$ is A_{rn+1} type ($n \in \mathbb{Z}_{\geq 0}$), i.e., $\widetilde{S} \cong \mathrm{Spec}\mathbb{C}[x^{rn+2}, y^{rn+2}, xy]$, since $\widetilde{S}$ is invariant by the action of $\mathbb{Z}_r(1/r, 1/r, 1/r)$. Since the action of $\mathbb{Z}_r$ on $\mathbb{C}^3$ is a scalar multiplication by ξ, we may assume that the equation of $x_1 \in \widetilde{S}$ is of the standard form under the same coordinates of $\mathbb{C}^3$. Then by looking the action of $\mathbb{Z}_r$ on the equation and using the fact that the equation must be semi-invariant, we conclude the result. Since

$$\begin{aligned} S|_U = \mathrm{Spec}\mathbb{C}[&(x^{rn+2})^r, (x^{rn+2})^{r-1}(xy), \ldots, (x^{rn+2})(xy)^{r-1}, \\ &(xy)^r, (y^{rn+2})^r, \ldots, (y^{rn+2})(xy)^{r-1}], \end{aligned}$$

the embedding dimension of $S|_U$ is $2r+1$. Hence we have $\mathrm{mult}_{x_0} S = 2r$ ([A, Corollary 6]). Hence $\mathrm{mult}_{x_0} S = 2r$ and $\mathrm{ord}_{x_0} S = 2/r$ or $\mathrm{mult}_{x_0} S = r$ and $\mathrm{ord}_{x_0} S = 1/r$. Let $d := \mathrm{mult}_{x_0} S/r = 1$ or 2.

Step 2-1. We assume first that $d = 1$. As in Step 1, there exists a rational number t' with $ct + (1-c) < t' < 1$. By Lemma 2.1 and $\sigma_2 \geq 3/\sqrt{r}$, there exists an effective $\mathbb{Q}$-Cartier divisor D_S on S such that $D_S \sim_{\mathbb{Q}} (t' - ct)L|_S$ and $\mathrm{ord}_{x_0} D_S = (3/r)(1-c)$.

The rest of Step 2-1 is the same as Step 2-1 of the proof of Theorem 3.1.

Step 2-2. We assume that $d = 2$. As in Step 2-1, we take a rational number t' with $ct + \sqrt{2}(1-c) < t'$ and an effective $\mathbb{Q}$-Cartier divisor D' on X with $D' \sim_{\mathbb{Q}} (t' - ct)L$ and $\mathrm{ord}_{x_0} D'|_S = (3/r)(1-c)$. Here we need the factor $\sqrt{2}$ because S has multiplicity $2r$ at x_0. Then we take $0 < c' \leq 1$

such that $(X, cD + c'D')$ is LC and $CLC(X, x_0, cD + c'D')$ has an element which is properly contained in S.

We shall prove that we may assume $ct + \sqrt{2}(1-c) < 1$. Then we can take $t' < 1$ as in Step 2-1, and the rest of the proof is the same. For this purpose, we apply Lemma 2.3. In argument of Steps 0 through 2-1, the number t was chosen under the only condition that $t < 1$. So we can take $t = 1 - \epsilon_1$, where the ϵ_n for $n = 1, 2, ...$ will stand for very small positive rational numbers. Then $k/r = 3/(r(1-\epsilon_1)) = (3/r) + \epsilon_2$ and $e = 1/(3c)$. This means the following: for any effective $D \sim_{\mathbb{Q}} tL$, if $\mathrm{ord}_{x_0} D \geq 3/r$, then $cD \geq S$. We look for $k' > 6/(3-\sqrt{2})$ so that there exists an effective $\mathbb{Q}$-Cartier divisor $D \sim_{\mathbb{Q}} tL$ with $t < (3-\sqrt{2})/2$ and $\mathrm{ord}_{x_0} D \geq 3/r$. The equation for k' becomes

$$\lambda^3 + (\frac{1-2e-\lambda}{1-\lambda})^2 \{(\frac{2}{3-\sqrt{2}} + \frac{2\lambda e}{1-2e-\lambda})^3 - (\lambda + \frac{2\lambda e}{1-2e-\lambda})^3\} < 1.$$

We have $\lambda \leq 1/\sqrt{6e}$, $1/3 \leq e \leq 1/2$, and in particular, $0 \leq \lambda \leq 1/\sqrt{2}$. By [K3, 3.1 Step2-2], we obtain a desired D, and can choose a new t such that $t < (3-\sqrt{2})/2$. Then we repeat the preceding argument from Step 0. If we arrive at Step 2-2 again, then we have $2/3 \leq c \leq 1$ and $ct + \sqrt{2}(1-c) < 1$. □

The following example shows that the conditions in Theorem 3.4 is best possible.

Example 3.5. Let $X = \mathbb{P}(1,1,1,r)$ and $x_0 = (0:0:0:1)$. Then x_0 is a quotient singular point of type $(1/r, 1/r, 1/r)$ and $K_X = \mathcal{O}(-3-r)$. If $K_X + L$ is Cartier and L is effective, we have $L = \mathcal{O}(rk+3)$ $(k \in \mathbb{Z}, rk+3 \geq 0)$. If $L = \mathcal{O}(3)$, then $|K_X + L|$ is not free at x_0. Hence the condition $\sigma_3 > 3/\sqrt[3]{r}$ is necessary in Theorem 3.4.

We consider non Gorenstein terminal singular points in the following.

Theorem 3.6. *Let X be a normal projective variety of dimension 3, $x_0 \in X$ a quotient singular point of type $(1/r, 1/r, -1/r)$ for an integer $r \geq 3$, and L an ample $\mathbb{Q}$-Cartier divisor such that $K_X + L$ is Cartier at x_0. Assume that there are positive numbers σ_p for $p = 1, 2, 3$ which satisfy the following conditions:*

(1) $\sqrt[p]{(L)^p \cdot W} \geq \sigma_p$ *for any subvariety W of dimension p which contains*

x_0,
(2) $\sigma_1 \geq 1 + (1/r)$, $\sigma_2 \geq (1 + (1/r))\sqrt{r+3}$, *and* $\sigma_3 > (1 + (1/r))\sqrt[3]{r+2}$.
Then $|K_X + L|$ *is free at* x_0.

PROOF. Since

$$\mathbb{C}^3/\mathbb{Z}_r(1/r, 1/r, -1/r) = \mathrm{Spec}\mathbb{C}[x^r, x^{r-1}y, \ldots, xy^{r-1}, y^r, xz, yz, z^r],$$

by Lemma 2.5, $\mathrm{mult}_{x_0}X = \mathrm{embdim}_{x_0}X - 2 = r+2$. Hence $\mathrm{mult}_{x_0}X = r+2$.
Step 0. Let t be a rational number such that $t > (1+(1/r))\sqrt[3]{r+2}/\sqrt[3]{(L^3)}$. Since $\sigma_3 > (1+(1/r))\sqrt[3]{r+2}$, we can take $t < 1$. Let t_0 be a rational number such that $t_0 = (1+(1/r))\sqrt[3]{r+2}/\sqrt[3]{(L^3)} - \epsilon$ for $0 \leq \epsilon \ll \sqrt[3]{(r+2)/(L^3)}$. By Lemma 2.1, there exists an effective $\mathbb{Q}$-Cartier divisor D such that $D \sim_{\mathbb{Q}} tL$ and $\mathrm{ord}_{x_0}D \geq (t_0 + \epsilon)\sqrt[3]{(L^3)/(r+2)}$. Hence $\mathrm{ord}_{x_0}D = 1 + (1/r)$.
Let $f : \bar{X} \to X$ be the weighted blowing-up of X at x_0, $E \subset \bar{X}$ the exceptional divisor of f, and $\bar{D}$ be the proper transform of D in $\bar{X}$. Then

$$K_{\bar{X}} = f^*K_X + (1/r)E.$$

$$f^*D = \bar{D} + eE, e \geq 1 + (1/r).$$

Let c be the log canonical threshold of (X, D) at x_0:

$$c = \sup\{t \in \mathbb{Q};\ K_X + tD \text{ is } LC \text{ at } x_0\ \}.$$

Then $c \leq 1$. Let W be the minimal element of $CLC(X, x_0, cD)$. If $W = \{x_0\}$, then $|K_X + L|$ is free at x_0 by Proposition 2.2, since $ct < 1$.

Step 1. We consider the case in which $W = C$ is a curve. By Proposition 1.3, C is normal at x_0, i.e., smooth at x_0. Since $\sigma_1 \geq 1 + (1/r)$, as in Step 1 of the proof of Theorem 3.1, we take a rational number t', an effective $\mathbb{Q}$-Cartier divisor D' on X and a positive number c' such that $ct + (1-c) < t' < 1$, $D' \sim_{\mathbb{Q}} (t' - ct)L$, $\mathrm{ord}_{x_0}D'|_C = (1 + (1/r))(1-c)$, $(X, cD + c'D')$ is LC at x_0, and that the minimal element W' of $CLC(X, x_0, cD + c'D')$ is properly contained in $C, i.e., \{x_0\}$.

Step 2. We consider the case in which $W = S$ is a surface. Since the embedding dimension of X at $x_0 = r + 4$, the embedding dimension of S at $x_0 \leq r + 4$. Hence we have $\mathrm{mult}_{x_0}S \leq r + 3$ ([A, Corollary 6]). There exists a rational number t' with $ct + (1-c) < t' < 1$. By Lemma 2.1 and $\sigma_2 \geq (1+(1/r))\sqrt{r+3}$, there exists an effective $\mathbb{Q}$-Cartier D_S on S such that $D_S \sim_{\mathbb{Q}} (t'-ct)L|_S$ and $\mathrm{ord}_{x_0} D_S = (1+(1/r))(1-c)$.

The rest of Step 2 is the same as Step 2-1 of the proof of Theorem 3.1. □

The following example shows that the estimates in some terminal singular cases is necessarily worse than the one for the smooth case.

Example 3.7. Let $X = \mathbb{P}(1,1,r-1,r)$ and $x_0 = (0:0:0:1)$. Then x_0 is a quotient singular point of type $(1/r, 1/r, -1/r)$ and $K_X = \mathcal{O}(-2r-1)$. If $K_X + L$ is Cartier at x_0 and L is effective, we have $L = \mathcal{O}(rk+1)$ $(k \in \mathbb{Z}, rk+1 \geq 0)$. If $L = \mathcal{O}(r+1)$, then $|K_X + L|$ is not free at x_0 and $L^3 = (r+1)^3/r(r-1)$. Hence if $r \geq 23$, then the condition $\sigma_3 > 3$.

We obtain the following theorem based on [K3 Theorem 3.1] and [R, Main Theorem].

Theorem 3.8. *Let X be a projective variety of dimension 3 with only canonical singularities, $x_0 \in X$ be a canonical and not terminal singular point, L an ample $\mathbb{Q}$-Cartier divisor on X such that $K_X + L$ is Cartier at x_0. Assume that there are positive numbers σ_p for $p = 1,2,3$ which satisfy the following conditions:*
(1) $\sqrt[p]{(L)^p \cdot W} \geq \sigma_p$ for any subvariety W of dimension p which contains x_0,
(2) $\sigma_1 \geq 3$, $\sigma_2 \geq 3$, and $\sigma_3 > 3$.
Then $|K_X + L|$ is free at x_0.

Proof. By a theorem of Reid [R, Main Theorem], a partial resolution $f: Y \to X$ such that $K_Y = f^*K_X$, and Y has only terminal singularity points. There exists a smooth point $y_0 \in f^{-1}(x_0)$.

Step 0. Let t be a rational number such that $t > 3/\sqrt[3]{(f^*L^3)}$. Since $\sigma_3 > 3$, we can take $t < 1$. Let t_0 be a rational number such that $t_0 = 3/\sqrt[3]{(f^*L^3)} - \epsilon$ for $0 \leq \epsilon \ll \sqrt[3]{1/(L^3)}$. By Lemma 2.1, there exists an effective $\mathbb{Q}$-Cartier divisor D such that $D \sim_{\mathbb{Q}} tL$ and $\mathrm{ord}_{y_0} f^*D \geq (t_0 + \epsilon)\sqrt[3]{(L^3)}$. Hence $\mathrm{ord}_{y_0} f^*D = 3$.

Let $g : \bar{Y} \to Y$ be the blowing-up of Y at y_0, $E \subset \bar{Y}$ the exceptional divisor of g, and $\bar{f^*D}$ the proper transform of f^*D in $\bar{Y}$. Then

$$K_{\bar{Y}} = g^*K_Y + 2E.$$

$$g^*f^*D = \bar{f^*D} + 3E.$$

$$K_{\bar{Y}} + \bar{f^*D} + E = g^*f^*(K_X + D).$$

Let c be the log canonical threshold of (X, D) at x_0:

$$c = \sup\{t \in \mathbb{Q};\ K_X + tD \text{ is } LC \text{ at } x_0\ \}.$$

Then $c \leq 1$. Let W be the minimal element of $CLC(X, x_0, cD)$. If $W = \{x_0\}$, then $|K_X + L|$ is free at x_0 by Proposition 2.2, since $ct < 1$.

Step 1. We consider the case in which $W = C$ is a curve. By Proposition 1.3, C is normal at x_0, i.e., smooth at x_0. Since $\sigma_1 \geq 3$, as in Step 1 of the proof of Theorem 3.1, we take a rational number t', an effective $\mathbb{Q}$-Cartier divisor D' on X and a positive number c' such that $ct + (1-c) < t' < 1$, $D' \sim_{\mathbb{Q}} (t' - ct)L$, $\mathrm{ord}_{x_0} D'|_C = 3(1-c)$, $(X, cD + c'D')$ is LC at x_0, and that the minimal element W' of $CLC(X, x_0, cD + c'D')$ is properly contained in C, *i.e.*, $\{x_0\}$.

Step 2. We consider the case in which $W = S$ is a surface. We have $S' = f^{-1}S \in CLC(Y, y_0, f^*cD)$. By Proposition 1.4, S' has at most a rational singularity at y_0.

Step 2-1. We assume first that S' is smooth at y_0. As in Step 1, there exists a rational number t' with $ct + (1-c) < t' < 1$. By Proposition 2.1 and $\sigma_2 \geq 3$, there exists an effective $\mathbb{Q}$-Cartier divisor $f^*(D_S)$ on S' such that $D_S \sim_{\mathbb{Q}} (t' - ct)L|_S$ and $\mathrm{ord}_{y_0} f^*(D_S) = 3(1-c)$.

The rest of Step 2-1 is the same as Step 2-1 of the proof of Theorem 3.1.

Step 2-2. We assume that $d = 2$. As in Step 2-1, we take a rational number t' with $ct + \sqrt{2}(1-c) < t'$ and an effective $\mathbb{Q}$-Cartier divisor D' on X with $D' \sim_{\mathbb{Q}} (t' - ct)L$ and $\mathrm{ord}_{y_0} f^*(D'|_S) = 3(1-c)$. Here we need the factor $\sqrt{2}$ because S' has multiplicity 2 at y_0. Then we take $0 < c' \leq 1$ such that $(X, cD + c'D')$ is LC and $CLC(X, x_0, cD + c'D')$ has an element which is properly contained in S.

We shall prove that we may assume $ct + \sqrt{2}(1-c) < 1$. Then we can take $t' < 1$ as in Step 2-1, and the rest of the proof is the same. For this purpose, we apply Lemma 2.3. In argument of Steps 0 through 2-1, the number t was chosen under the only condition that $t < 1$. So we can take $t = 1 - \epsilon_1$, where the ϵ_n for $n = 1, 2, ...$ will stand for very small positive rational numbers. Then $k = 3/(1-\epsilon_1) = 3 + \epsilon_2$ and $e = 1/(3c)$. This means the following: for any effective $D \sim_{\mathbb{Q}} tL$, if $\mathrm{ord}_{y_0} f^*D \geq 3$, then $cf^*D \geq S'$. We look for $k' > 6/(3-\sqrt{2})$ so that there exists an effective $\mathbb{Q}$-Cartier divisor $D \sim_{\mathbb{Q}} tL$ with $t < (3-\sqrt{2})/2$ and $\mathrm{ord}_{y_0} f^*D \geq 3$. The equation for k' becomes

$$\lambda^3 + (\frac{1-2e-\lambda}{1-\lambda})^2\{(\frac{2}{3-\sqrt{2}} + \frac{2\lambda e}{1-2e-\lambda})^3 - (\lambda + \frac{2\lambda e}{1-2e-\lambda})^3\} < 1.$$

We have $\lambda \leq 1/\sqrt{6e}$, $1/3 \leq e \leq 1/2$, and in particular, $0 \leq \lambda \leq 1/\sqrt{2}$. By [K2, 3.1 Step2-2], we obtain a desired f^*D, and can choose a new t such that $t < (3-\sqrt{2})/2$. Then we repeat the preceding argument from Step 0. If we arrive at Step 2-2 again, then we have $2/3 \leq c \leq 1$ and $ct + \sqrt{2}(1-c) < 1$. □

Theorem 3.1, Theorem 3.8, and the result for the smooth case [K3 Theorem 3.1] imply the following corollary in which the estimate is better than the one in [OP, Theorem 2]

COROLLARY 3.9. *Let X be a projective variety of dimension 3 with only Gorenstein canonical singularities, and H an ample Cartier divisor. Then $|K_X + mH|$ is free if $m \geq 4$. Moreover, if $(H^3) \geq 2$, then $|K_X + 3H|$ is also free.*

We obtain the following corollary from Corollary 3.9 and [OP Theorem 1,3] (cf. [OP Theorem I,II]).

COROLLARY 3.10. *Let (X, L) be a polarized canonical Calabi-Yau threefolds.* (1) *$|mL|$ gives a birational map when $m \geq 5$.* (2) *$|mL|$ is free if $m \geq 4$.* (3) *$|mL|$ is very ample when $m \geq 10$.*

References

[A] Artin, M., On isolated rational singularities of surfaces, Amer. J. Math. **88** (1966), 129–136.

[EL] Ein, L. and R. Lazarsfeld, Global generation of pluricanonical and adjoint linear series on smooth projective threefolds, J. Amer. Math. Soc. **6** (1993), 875–903.

[ELM] Ein, L., Lazarsfeld, R. and V. Maşek, Global generation of linear series on terminal threefolds, Internat. J. Math. **6** (1995), 1–18.

[F] Fujita, T., Remarks on Ein-Lazarsfeld criterion of spannedness of adjoint bundles of polarized threefold, Preprint, e-prints/alg-geom/9311013.

[KM] Kawachi, T. and V. Maşek, Reider-type theorems on normal surface, J. Alg. Geom. **7** (1998), 239–249.

[K1] Kawamata, Y., A generalization of Kodaira-Ramanujam's vanishing theorem, Math. Ann. **261** (1982), 43–46.

[K2] Kawamata, Y., Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces, Ann. of Math. **127** (1988), 93–163.

[K3] Kawamata, Y., On Fujita's freeness conjecture for 3-folds and 4-folds, Math. Ann. **308** (1997), 491–505.

[K4] Kawamata, Y., Subadjunction of log canonical divisors II, Amer. J. Math. **120** (1998), 893–899.

[KMM] Kawamata, Y., Matsuda, K. and K. Matsuki, Introduction to the minimal model problem, Adv. St. Pure Math. **10** (1987), 283–360.

[M] Matsushita, D., Effective base point freeness, Kodai. Math. J. **19** (1996), 87–116.

[OP] Oguiso, K. and T. Peternell, On polarized canonical Calabi-Yau threefolds, Math. Ann. **301** (1995), 237–248.

[L1] Lee, S., Remarks on the pluricanonical and the adjoint linear series on projective threefolds, Comm. Alg. **27** (1999), 4459–4476.

[L2] Lee, S., Quartic-canonical systems on canonical threefolds of index 1, preprint.

[R] Reid, M., Minimal models of canonical threefolds (Pagoda), In S. Iitaka, editor, Algebraic Varieties and Analytic Varieties, volume 1 of Adv. Stud. Pure Math., 131–180, Kinokuniya and North-Holland, 1983.

[Rdr] Reider, I., Vector bundles of rank 2 and linear systems on algebraic surface, Ann. Math. **127** (1988), 309–316.

[V] Viehweg, E., Vanishing theorems, J. reine angew. Math. **335** (1982), 1–8.

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