J. Math. Sci. Univ. Tokyo
7 (2000), 325–345.

Remarks on Traces of H^1-functions Defined in a Domain with Corners

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Abstract. The set of traces of $H^1(\Omega)$-functions on a part γ of the boundary $\partial\Omega$ is considered, where Ω is a bounded domain in $\mathbb{R}^2$ with a certain singularity, particularly, with corners at the end points of γ. The aim of the present paper is to show that the set of all traces of functions in $H^1(\Omega)$ is equal algebraically and topologically to the domain of a certain fractional power of minus Laplacian on γ with the zero boundary condition. The result is expected to be of use for the mathematical analysis of the DDM (domain decomposition method) applied to such Ω.

1. Introduction and Main Results

The present paper is concerned with the relationship between the set of boundary values of functions of a certain Sobolev class defined in a two-dimensional domain Ω and fractional powers of an elliptic partial differential operator on a part of the boundary of Ω. Although a considerable part of our result may be said to be a new version of the known result (cf. Grisvard [8]), our method of analysis, which is a combination of the spectral approach and elementary transformations of Ω, could give a better prospect for some generalizations, for instance, to the case of solenoidal vector fields mentioned in Section 4. We also note that our result plays some foundational role in analysis of the domain decomposition method ([4], [5], [12], [13]). We shall later give a remark about this issue (Remark 1.4), which clarifies our motivation.

1991 *Mathematics Subject Classification.* 46E35, 46C15.
Key words: trace theorem, non-smooth domain, fractional power of operator, domain decomposition method.

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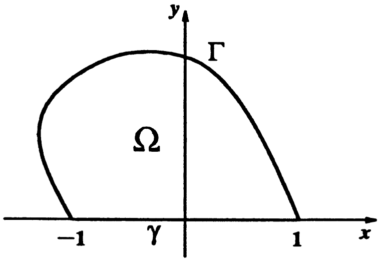

Figure 1(a). Example of Ω.

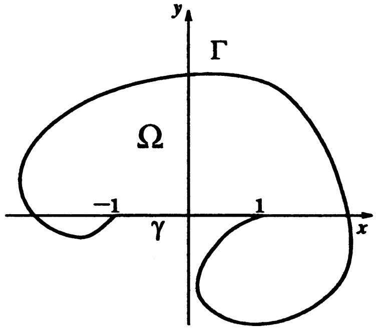

Figure 1(b). Example of Ω.

In order to present the result and our approach as clearly as possible, we restrict our attention to a simple situation. We consider a bounded domain Ω in $\mathbb{R}^2$, the xy-plane, and assume that a part γ of the boundary $\partial\Omega$ is a line segment. Without loss of generality, we suppose that $\gamma = \{(x, y);\ |x| < 1,\ y = 0\}$. In addition, we put $\Gamma = \partial\Omega \backslash \gamma$ and assume that Γ is a piecewise smooth curve. Our interest lies in the case where Γ intersects $\overline{\gamma}$ transversally. Admissible geometries of Ω are exemplified in Figure 1.

As for function spaces and their norms, we follow the notation of Lions-Magenes [11]. The trace operator from $H^1(\Omega)$ to $H^{1/2}(\gamma)$ is denoted by γ_0, and the boundary value (trace) $\gamma_0 v$ of $v \in H^1(\Omega)$ will be written conveniently as $v|_\gamma$. The meaning of $v|_\Gamma$ is similar. Then we introduce a closed subspace of $H^1(\Omega)$ by setting

$$K^1(\Omega) = \{v \in H^1(\Omega); v|_\Gamma = 0\}.$$

We recall that the usual $H^1(\Omega)$-norm is equivalent to the Dirichlet norm

$$\|\nabla v\|^2_{L^2(\Omega)} = \left\{ \iint_\Omega |\nabla v|^2 dxdy \right\}^{1/2} = \left\{ \iint_\Omega (v_x^2 + v_y^2) dxdy \right\}^{1/2}$$

in $K^1(\Omega)$ by virtue of the Poincaré inequality. Here subscripts mean partial derivatives. The focus of the present paper is on the set of boundary values $v|_\gamma$ of $v \in K^1(\Omega)$.

The basic Hilbert space in our consideration is $L^2(\gamma)$. The usual $L^2(\gamma)$ inner product and norm are written as $(\cdot, \cdot)$ and $\|\cdot\|$, respectively. The symbol L denotes the minus Laplacian on γ with the zero Dirichlet boundary condition. More precisely, L is the Friedrichs extension in $L^2(\gamma)$ of the symmetric operator $-d^2/dx^2$ defined in $C_0^\infty(\gamma)$. The domain $\mathfrak{D}(L)$ of L is $H^2(\gamma) \cap H_0^1(\gamma)$. Since, as is well-known, L is a positive-definite, self-adjoint operator, its fractional power L^α, for $0 < \alpha < 1$, can be defined in a standard way by the spectral resolution. The domain $\mathfrak{D}(L^\alpha)$ of L^α forms a Hilbert space equipped with the graph norm

$$\|f\|_{\mathfrak{D}(L^\alpha)} = \Big\{\|f\|^2 + \|L^\alpha f\|^2\Big\}^{1/2}.$$

The purpose of this paper is to show that the set $\{v|_\gamma;\ v \in K^1(\Omega)\}$ is equal to $\mathfrak{D}(L^{1/4})$ algebraically and topologically. That is, we are going to prove

Theorem 1.1. *The following two claims hold true:*
A. (Trace). Let $v \in K^1(\Omega)$ and put $f = v|_\gamma$. Then we have $f \in \mathfrak{D}(L^{1/4})$ and

$$\|f\|_{\mathfrak{D}(L^{1/4})} \le C\|v\|_{H^1(\Omega)}, \tag{1.1}$$

where C is a positive constant depending only on Ω.
B. (Extension). Let $f \in \mathfrak{D}(L^{1/4})$. Then there exists a function $v \in K^1(\Omega)$ such that $v|_\gamma = f$ and

$$\|v\|_{H^1(\Omega)} \le C'\|f\|_{\mathfrak{D}(L^{1/4})}, \tag{1.2}$$

where C' is a positive constant depending only on Ω.

At this stage, we recall the following result due to Fujiwara [6] which gives a concrete characterization of $\mathfrak{D}(L^{1/4})$:

$$\mathfrak{D}(L^{1/4}) = \Big\{f \in H^{1/2}(\gamma);\ \int_\gamma \rho^{-1} f^2 dx < \infty\Big\}, \tag{1.3}$$

where $\rho = \rho(x)$ stands for the distance from the end points of γ (i.e., $\rho(x) = 1 - |x|$). Let V denote the function space of the right-hand side of (1.3). Then, in view of the closed graph theorem (e.g. Kato [10] or Yosida [14]), the V-norm defined by

$$\|f\|_V = \Big\{\|f\|^2_{H^{1/2}(\gamma)} + \int_\gamma \rho^{-1} f^2 dx\Big\}^{1/2}$$

is equivalent to $\|f\|_{\mathfrak{D}(L^{1/4})}$ in V. Combining Theorem 1.1 with Fujirawa's result, we have

COROLLARY 1.1. *There exist positive constants $\tilde{C}$ and $\tilde{C}'$ depending only on Ω such that the following two assertions hold true:*

(i) *For every $v \in K^1(\Omega)$, we have $f = v|_\gamma \in V$ and $\|f\|_V \leq \tilde{C}\|v\|_{H^1(\Omega)}$,*

(ii) *For every $f \in V$, there exists $v \in K^1(\Omega)$ such that $v|_\gamma = f$ and $\|v\|_{H^1(\Omega)} \leq \tilde{C}'\|f\|_V$.*

REMARK 1.1. Theorem 1.1 (therefore Corollary 1.1) remains true if L is replaced by a more general elliptic partial differential operator of the second order. In fact, our theorem is still valid with an arbitrary regularly accretive operator L whose domain is $\mathfrak{D}(L) = H^2(\gamma) \cap H_0^1(\gamma)$ (Fujiwara [6]). Concerning the definition of such operator, we refer to Kato [9].

REMARK 1.2. Other characterizations of the space V are possible, which are useful in some contexts. For instance,

- V is equal to the real interpolation space $[H_0^1(\gamma), L^2(\gamma)]_{1/2}$, which is denoted by $H_{00}^{1/2}(\gamma)$ in Lions-Magenes [11],
- $V = \{f \in H^{1/2}(\gamma);\ \tilde{f} \in H^{1/2}(\partial\Omega)\}$, where $\tilde{f}$ is the zero extension of f onto $\partial\Omega$ (Grisvard [8]).

REMARK 1.3. It should be noted that Corollary 1.1 is a particular case of Theorem 1.5.2.3 in Grisvard [8]. Conversely, combining Grisvard's result with Fujiwara's one, we can derive Theorem 1.1. However, as we mentioned previously, our objective of this paper is to clarify a direct relationship between the space $\{v|_\gamma; v \in K^1(\Omega)\}$ and $\mathfrak{D}(L^{1/4})$.

REMARK 1.4. As was described above, results of this paper are of importance in analysis of the domain decomposition method. Here we explain this matter with the aid of an example. We consider the Poisson equation

$$-\triangle u = g \tag{1.4}$$

in a two-dimensional bounded domain D with the boundary condition

$$u = 0 \text{ on } \partial D. \tag{1.5}$$

We divide the *whole domain* D into two disjoint *subdomains* D_1 and D_2 by a line segment γ which we call the *artificial boundary.* Then, under a certain regularity assumption, the problem (1.4)(1.5) is equivalent to the transmission problem:

$$\begin{cases} -\triangle u_1 = g \text{ in } D_1, \quad u_1 = 0 \text{ on } \partial D_1 \backslash \gamma, \quad u_1 = \mu \text{ on } \gamma, \\ -\triangle u_2 = g \text{ in } D_2, \quad u_2 = 0 \text{ on } \partial D_2 \backslash \gamma, \quad u_2 = \mu \text{ on } \gamma, \end{cases}$$

where μ is chosen such that $\partial u_1/\partial n_1 = -\partial u_2/\partial n_2$ (n_j is the unit normal to γ outgoing from D_j, $j = 1, 2$). Several iterative algorithms to obtain such μ are proposed by several authors. See, for more detail, the monograph [12] by A. Quarteroni and A. Valli. Consequently, the problem

$$\triangle w = 0 \text{ in } D_j,\ w = 0 \text{ on } \partial D_j \backslash \gamma,\ w = \xi \text{ on } \gamma, \quad (j = 1, 2) \tag{1.6}$$

appears. As is well-known, if we are going to deal with this problem in the framework of the $H^1(\Omega_j)$-space, we must regard ξ as an element of V. However, in view of Theorem 1.1, we can also regard ξ as an element of $\mathfrak{D}(L^{1/4})$. This allows us to apply some properties or materials related to L, for example the eigenvalues and the eigenfunctions of L, to the domain decomposition analysis. Furthermore, we believe that an analysis carried out in this paper gives a better view of the domain decomposition analysis.

The proof of Theorem 1.1 is established in Sections 2 and 3. In Section 4, we present a corresponding result concerning the solenoidal vector fields. The final section 5 is devoted to a remark on higher dimensional cases.

Acknowledgment. The authors wish to express their heartfelt thanks to Professor Hisashi Okamoto for his valuable advise.

2. Proof of Theorem 1.1

2.1. Plan of the proof

First of all, we show Theorem 1.1 in the case where Ω is a rectangle $Q_T = \gamma \times (0, T)$ with T a parameter. Theorem 1.1 in this case is easily proved by the Fourier expansion.

Next, the case of a trapezoid $E_{T,b}$, T and b being parameters, is considered in Subsection 2.3. The results obtained there are used in Subsection

2.4. The key point of analysis is to introduce a certain transformation from $E_{T,b}$ to Q_T. This allows us to reduce the problem to the case of Q_T.

Subsection 2.4 is devoted to the situation that Ω lies in the upper half plane. Namely, we shall prove Theorem 1.1 under this additional assumption. In fact, the claim A, which appears shortly, concerning traces of $K^1(\Omega)$-functions can be proved by taking a trapezoid $E_{T,b}$ including Ω. On the other hand, the proof of the claim B below concerning $K^1(\Omega)$-extensions of functions of $\mathfrak{D}(L^{1/4})$ can be accomplished by taking a trapezoid $E_{T,b}$ which is included in Ω.

The general case which is exemplified in Figure 1(b) will be considered in Section 3.

2.2. The case of a rectangle

Let T be an arbitrary positive constant and Q_T be a rectangular domain defined by $Q_T = \{(x,y); -1 < x < 1,\ 0 < y < T\}$. In this subsection, we are going to show

LEMMA 2.1. *Theorem 1.1 is true if $\Omega = Q_T$. In particular, we can take C as unity.*

Before proving this lemma, we review a characterization of L^α, $0<\alpha<1$, in terms of the eigenvalues and the eigenfunctions of L. Let $\{\lambda_n\}_{n=1}^\infty$ be the set of eigenvalues of L and $\phi_n = \phi_n(x)$ be the eigenfunction corresponding to λ_n which is normalized as $\|\phi_n\| = 1$. In the present case, we actually have $\lambda_n = n^2\pi^2/4$, $\phi_n(x) = \sin n\pi(x-1)/2$. However, we use the generic symbols λ_n and ϕ_n since it is convenient when we consider higher dimensional cases. Then L^α can be expressed as

$$\begin{cases} \mathfrak{D}(L^\alpha) = \Big\{ f = \displaystyle\sum_{n=1}^\infty c_n\phi_n \in L^2(\gamma);\ \ \sum_{n=1}^\infty c_n^2\lambda_n^{2\alpha} < \infty \Big\}, \\ L^\alpha f = \displaystyle\sum_{n=1}^\infty c_n\lambda_n^\alpha\phi_n, \quad \text{for } f = \sum_{n=1}^\infty c_n\phi_n \in \mathfrak{D}(L^\alpha) \text{ with } c_n \in \mathbb{R}, \end{cases}$$

and the following equality holds good

$$\|L^\alpha f\| = \Big(\sum_{n=1}^\infty c_n^2\lambda_n^{2\alpha}\Big)^{1/2}, \qquad \text{for } f = \sum_{n=1}^\infty c_n\phi_n \in \mathfrak{D}(L^\alpha) \text{ with } c_n \in \mathbb{R}.$$

Proof of Lemma 2.1 *A. (Trace).* Let $v \in K^1(Q_T)$. By the density argument, we may assume that v is continuous in $\overline{Q_T}$. According to this, we can write v by using the Fourier series as follows

$$v(x,y) = \sum_{n=1}^{\infty} a_n(y)\phi_n(x), \qquad a_n(y) = \int_{-1}^{1} v(x,y)\phi_n(x)dx.$$

Thanks to this expression, we get

$$\int_0^T \int_{-1}^1 \frac{\partial v}{\partial x}(x,y)^2 dxdy = \sum_{n=1}^{\infty} \lambda_n \int_0^T a_n(y)^2 dy,$$
$$\int_0^T \int_{-1}^1 \frac{\partial v}{\partial y}(x,y)^2 dxdy = \sum_{n=1}^{\infty} \int_0^T a_n'(y)^2 dy,$$

where $a_n'(y)$ denotes $da_n(y)/dy$. These yield

$$\|\nabla v\|^2_{L^2(Q_T)} = \sum_{n=1}^{\infty} \lambda_n \int_0^T a_n(y)^2 dy + \sum_{n=1}^{\infty} \int_0^T a_n'(y)^2 dy. \tag{2.1}$$

On the other hand, we have

$$\begin{aligned} \lambda_n^{1/2} a_n(0)^2 &= -\lambda_n^{1/2} \int_0^T \frac{d}{dy}\Big\{a_n(y)^2\Big\}dy \\ &= -\int_0^T 2\lambda_n^{1/2} a_n(y)a_n'(y)dy \\ &\le \int_0^T \Big\{\lambda_n a_n(y)^2 + a_n'(y)^2\Big\}dy. \end{aligned} \tag{2.2}$$

Then noting that $v|_\gamma = f = \sum_{n=1}^{\infty} a_n(0)\phi_n$, by virtue of (2.1) and (2.2), we deduce

$$\|L^{1/4} f\|^2 = \sum_{n=1}^{\infty} \lambda_n^{1/2} a_n(0)^2 \le \|\nabla v\|^2_{L^2(Q_T)}.$$

Thus we arrive at

$$\|f\|_{\mathfrak{D}(L^{1/4})} \le \|v\|_{H^1(Q_T)}.$$

B. (Extension). Let $f = \sum_{n=1}^{\infty} c_n \phi_n \in \mathfrak{D}(L^{1/4})$ with $c_n \in \mathbb{R}$. First we put

$$w(x,y) = \sum_{n=1}^{\infty} c_n e^{-\sqrt{\lambda_n} y} \phi_n(x). \tag{2.3}$$

Then, by elementary calculations, we get

$$\int_0^T \int_{-1}^1 \frac{\partial w}{\partial x}(x,y)^2 dxdy \leq \frac{1}{2} \sum_{n=1}^{\infty} c_n^2 \sqrt{\lambda_n},$$
$$\int_0^T \int_{-1}^1 \frac{\partial w}{\partial y}(x,y)^2 dxdy \leq \frac{1}{2} \sum_{n=1}^{\infty} c_n^2 \sqrt{\lambda_n}.$$

Hence we have

$$\|\nabla w\|^2_{L^2(Q_T)} \leq \sum_{n=1}^{\infty} c_n^2 \sqrt{\lambda_n} = \|L^{1/4} f\|^2. \tag{2.4}$$

On the other hand, it is easy to verify that $w(\pm 1, y) = 0$, $y > 0$, holds.

Therefore, the function w satisfies all requested properties except for the boundary condition on $y = T$.

Take a smooth function $\zeta = \zeta(y)$, $0 \leq y \leq T$, such that $0 \leq \zeta \leq 1$ and

$$\zeta(y) = \begin{cases} 1 & (0 \leq y \leq T/4) \\ 0 & (T/2 \leq y \leq T). \end{cases}$$

Then the function $v = \zeta w \in K^1(Q_T)$ satisfies $v|_\gamma = \zeta(0) w|_\gamma = w|_\gamma = f$ and

$$\begin{aligned} \|\nabla v\|^2_{L^2(Q_T)} &= \iint_{Q_T} (\zeta' w)^2 dxdy + \iint_{Q_T} \zeta^2 |\nabla w|^2 dxdy \\ &\leq \max\{\max_{0 \leq y \leq T} \zeta'(y)^2, 1\} \|w\|^2_{H^1(Q_T)}. \end{aligned}$$

This proves the lemma. □

We conclude this subsection with another proof of the latter half of Lemma 2.1 which adopts the semi-group generated by $-L^{1/2}$.

Another Proof of Lemma 2.1 *B. (Extension).* By using the semi-group $\{e^{-y\sqrt{L}}\}_{y\geq 0}$, the function $w = w(x, y)$ defined in (2.3) can be written as

$$w(y) = e^{-y\sqrt{L}}f, \qquad (y > 0).$$

Here $y \mapsto w(y)$ is regarded as an element in $L^2((0,\infty); L^2(\gamma))$. Now we are going to derive the inequality (2.4) with the aid of this expression. Noting that $w(y) \in \mathfrak{D}(L^{1/2})$ for $y > 0$ and $f \in \mathfrak{D}(L^{1/4})$, we can calculate as

$$\begin{aligned} \|L^{1/2}w(y)\|^2 &= (L^{1/4}e^{-y\sqrt{L}}L^{1/4}f, L^{1/4}e^{-y\sqrt{L}}L^{1/4}f) \\ &= (L^{1/2}e^{-y\sqrt{L}}L^{1/4}f, e^{-y\sqrt{L}}L^{1/4}f) \\ &= -\frac{1}{2}\frac{d}{dy}\|e^{-y\sqrt{L}}L^{1/4}f\|^2. \end{aligned} \tag{2.5}$$

Here we have made use of the fundamental property of the semi-group

$$\frac{d}{dy}e^{-y\sqrt{L}}f = -L^{1/2}e^{-y\sqrt{L}}f, \qquad (f \in L^2(\gamma)). \tag{2.6}$$

Integrating (2.5) over $(0, T)$, we obtain

$$\int_0^T \|L^{1/2}w(y)\|^2 \, dy = \frac{1}{2}\Big(\|L^{1/4}f\|^2 - \|e^{-T\sqrt{L}}L^{1/4}f\|^2\Big) \leq \frac{1}{2}\|L^{1/4}f\|^2.$$

This, together with (2.6), gives

$$\iint_{Q_T} \left(\frac{\partial w}{\partial y}\right)^2 dxdy = \int_0^T \|L^{1/2}w(y)\|^2 \, dy \leq \frac{1}{2}\|L^{1/4}f\|^2. \tag{2.7}$$

On the other hand, it is immediate to see from the definition of $L^{1/2}$ that

$$\|L^{1/2}w(y)\|^2 = \int_{-1}^1 \left(\frac{\partial w}{\partial x}\right)^2 dx.$$

Therefore we have

$$\iint_{Q_T} \left(\frac{\partial w}{\partial x}\right)^2 dxdy \leq \frac{1}{2}\|L^{1/4}f\|^2. \tag{2.8}$$

By virtue of (2.7) and (2.8), we deduce

$$\|\nabla w\|^2_{L^2(Q_T)} \leq \|L^{1/4}f\|^2. \ \square$$

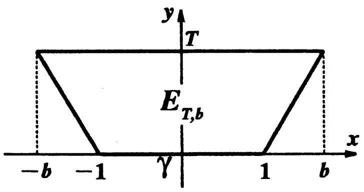

Figure 2(a). Example of $E_{T,b}$ $(b > 1)$.

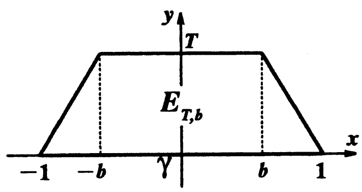

Figure 2(b). Example of $E_{T,b}$ $(b < 1)$.

2.3. The case of a trapezoid

Our next geometry of Ω is a trapezoid defined by

$$E_{T,b} = \bigcup_{0<\eta<T} \{(x,y);\ |x| < \theta(\eta),\ y = \eta\}, \tag{2.9}$$

where T, b are arbitrary positive constants and

$$\theta = \theta_{T,b}(\eta) = 1 + \frac{(b-1)}{T}\eta.$$

See Figure 2. We note that $E_{T,1} = Q_T$ holds.

The goal of this subsection is to prove

LEMMA 2.2. *Theorem 1.1 is true if* $\Omega = E_{T,b}$.

REMARK 2.1. In what follows, $C_{p_1,\cdots,p_m}$ denote positive constants depending only on parameters $p_1,\cdots,p_m$. The value of $C_{p_1,\cdots,p_m}$ may change in the same context.

PROOF OF LEMMA 2.2 *A. (Trace).* Let $v \in K^1(E_{T,b})$, and put $f = v|_\gamma$. We consider the transformation Φ from Q_T onto $E_{T,b}$ defined by

$$\Phi:\ (x,y) \mapsto (\theta(y)x, y).$$

Obviously, Φ is bijective, and by putting $\xi = \theta(y)x$ and $\eta = y$, we have

$$dxdy = \theta(y)^{-1} d\xi d\eta \le \max(1, b^{-1}) d\xi d\eta.$$

We introduce the function $\tilde{v} \in K^1(Q_T)$ as the pullback of v by Φ, that is,

$$\tilde{v}(x,y) = (v \circ \Phi)(x,y) = v(\theta(y)x, y), \qquad (x,y) \in Q_T.$$

Then, we can calculate as

(2.10)
$$
\begin{aligned}
&\|\nabla\tilde{v}\|^2_{L^2(Q_T)}\\
&= \iint_{Q_T}\Big\{\Big(\frac{\partial\tilde{v}}{\partial x}\Big)^2+\Big(\frac{\partial\tilde{v}}{\partial y}\Big)^2\Big\}dxdy\\
&= \iint_{Q_T}\Big\{\theta(y)^2 v_\xi(\theta(y)x,y)^2\\
&\qquad\qquad + [x\theta'(y)v_\xi(\theta(y)x,y)+v_\eta(\theta(y)x,y)]^2\Big\}dxdy\\
&\le C_{T,b}\iint_{E_{T,b}}[v_\xi(\xi,\eta)^2+v_\eta(\xi,\eta)^2]d\xi d\eta = C_{T,b}\|\nabla v\|^2_{L^2(E_{T,b})}.
\end{aligned}
$$

On the other hand, since $\theta(0)=1$, we have $f=\tilde{v}|_\gamma$. Therefore, on account of Lemma 2.1 A, $\|f\|_{\mathfrak{D}(L^{1/4})}\le\|\tilde{v}\|_{H^1(Q_T)}$ holds. This, together with (2.10), yields

$$\|f\|_{\mathfrak{D}(L^{1/4})}\le C_{T,b}\|v\|_{H^1(E_{T,b})}.$$

B. (Extension). Let $f\in\mathfrak{D}(L^{1/4})$. Suppose that $\tilde{v}\in K^1(Q_T)$ is an extension of f into Q_T as Lemma 2.1 B. Then, we introduce the function $v\in K^1(E_{T,b})$ as the pullback of $\tilde{v}$ by Ψ;

$$v(x,y)=(\tilde{v}\circ\Psi)(x,y)=\tilde{v}(\theta(y)^{-1}x,y),\qquad (x,y)\in E_{T,b},$$

where Ψ is the inverse of Φ and is defined by

$$\Psi:\ (x,y)\mapsto(\theta(y)^{-1}x,y).$$

Since Ψ is bijective and

$$dxdy=\theta(y)d\xi d\eta\le\max(1,b)d\xi d\eta\qquad\text{with } \xi=\theta(y)x,\ \eta=y,$$

in the same manner as in the derivation of the inequality (2.10), we can get

$$\|\nabla v\|_{L^2(E_{T,b})}\le C_{T,b}\|\nabla\tilde{v}\|_{L^2(Q_T)}.$$

This leads to

$$\|v\|_{H^1(E_{T,b})}\le C_{T,b}\|f\|_{\mathfrak{D}(L^{1/4})}.$$

Moreover it is clear from the definition that $v|_\gamma=f$. Hence we completes the proof. $\square$

2.4. The case where Ω lies in the upper half plane

In this subsection, we consider the case in which

$$\left\{\begin{array}{l}\Omega \subset \mathbb{R}^2_+ = \{(x,y);\ y>0\} \text{ and} \\ \Gamma \text{ intersects the } x\text{-axis transversally.}\end{array}\right. \tag{2.11}$$

By virtue of results proved in the preceding two subsections, we can prove the following lemma.

LEMMA 2.3. *Under the additional assumption (2.11), Theorem 1.1 is true.*

PROOF *A. (Trace).* Let $v \in K^1(\Omega)$ and take a trapezoid which includes Ω, specifically, choose positive constants T and b subject to $\Omega \subseteq E_{T,b}$, where $E_{T,b}$ is the trapezoid defined by (2.9). Then the zero extension $\tilde{v}$ of v into $E_{T,b}$ belongs to $K^1(E_{T,b})$, and $\|\tilde{v}\|_{H^1(E_{T,b})} = \|v\|_{H^1(\Omega)}$ holds. Therefore, by virtue of Lemma 2.2 A, we deduce $v|_\gamma = \tilde{v}|_\gamma \in \mathfrak{D}(L^{1/4})$ and (1.1).

B. (Extension). Let $f \in \mathfrak{D}(L^{1/4})$. We then take a trapezoid $E_{T,b}$ which is included in Ω. Firstly, we consider the extension $\tilde{v} \in K^1(E_{T,b})$ of f into $E_{T,b}$ subject to $\tilde{v}|_\gamma = f$ and $\|\tilde{v}\|_{H^1(E_{T,b})} \le C_{T,b}\|f\|_{\mathfrak{D}(L^{1/4})}$. Such a function $\tilde{v}$ can be chosen by Lemma 2.2 B. Next, let v denote the zero extension of $\tilde{v}$ into Ω. Then v is the desired function. $\square$

3. Proof of Theorem 1.1 (continued)

3.1. General cases

In this section, the proof of Theorem 1.1 in the general case is presented. Firstly, we consider the claim B concerning extensions. We note that the proof of Lemma 2.3 B in the preceding subsection remains to be valid without the assumption (2.11). This means that we have already compeleted the proof of the claim B.

We proceed to prove the claim A concerning traces. In doing so, we introduce an auxiliary domain $S_{T,b,\beta_1,\beta_2} \supset \Omega$, where T, b are parameters and β_1, β_2 are some suitable functions. Specifically, we introduce smooth functions $\beta_1 = \beta_1(y)$ and $\beta_2 = \beta_2(y)$, $-T \le y \le 0$, satisfying

$$\left\{\begin{array}{l}\beta_1(0)=1, \quad \beta_2(0)=-1, \quad -\infty < \beta_1'(0) < 0, \quad 0 < \beta_2'(0) < \infty, \\ -1 < \beta_2(y) < \beta_1(y) < 1, \quad (-T \le y \le 0).\end{array}\right.$$

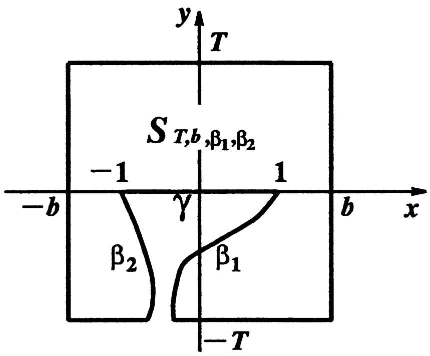

Figure 3. Example of S_{T,b,β_1,β_2}.

Then put

$$S_{T,b,\beta_1,\beta_2} = \{(x,y);\ |x| < b,\ |y| < T\} \backslash \overline{S^C_{T,b,\beta_1,\beta_2}}, \tag{3.1}$$

where

$$\overline{S^C_{T,b,\beta_1,\beta_2}} = \bigcup_{-T \le \eta \le 0} \{(x,y);\ \beta_2(\eta) \le x \le \beta_1(\eta),\ y = \eta\}.$$

See Figure 3. Then we have

LEMMA 3.1. *The claim A of Theorem 1.1 is true if* $\Omega = S_{T,b,\beta_1,\beta_2}$.

The claim A of Theorem 1.1 immediately follows from the above lemma. In fact, since T, b, β_1 and β_2 can be so taken that $\Omega \subset S_{T,b,\beta_1,\beta_2}$ holds, the same proof of Lemma 2.3 A works. Therefore it remains to prove Lemma 3.1.

3.2. Proof of Lemma 3.1

The proof is done in the following four steps, I)-IV).

I) Let $v \in K^1(S_{T,b,\beta_1,\beta_2})$. Assume that there exists a function $h \in K^1(Q_T)$ satisfying $h|_\gamma = v|_\gamma$ and

$$\|h\|_{H^1(Q_T)} \le C_{T,b,\beta_1,\beta_2} \|v\|_{H^1(S_{T,b,\beta_1,\beta_2})}. \tag{3.2}$$

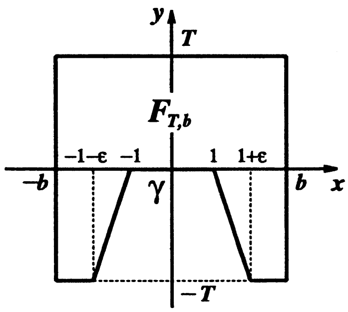

Figure 4. Example of $F_{T,b}$.

Then, in view of Lemma 2.1 A, we establish the proof. Hence it is enough to show the existence of the function h above. Actually, in the similar manner as in the proof of Lemma 2.2 A, we would like to construct h as the pullback v by some suitable transformation from Q_T to S_{T,b,β_1,β_2}. However, a direct construction of h seems troublesome, and we prefer to take a detour. Namely, we shall intermediately introduce functions h_1 and h_2 which are defined in special classes $F_{T,b}$ and $G_{T,b}$, respectively, and then define h in terms of h_2. Specifically, the remainder of the proof is the following:

II) definition of h_1 as the pullback of v;
III) definition of h_2 as the pullback of h_1;
IV) definition of h as the pullback of h_2.

II) Set $\sigma(y) = 1 - (\varepsilon/T)y$ with $\varepsilon = (b-1)/2$. Then we introduce

$$F_{T,b} = \{(x,y);\ |x| < b,\ |y| < T\} \backslash \overline{F^C_{T,b}}, \tag{3.3}$$

where

$$\overline{F^C_{T,b}} = \{(x,y);\ |x| \le \sigma(y),\ -T \le y \le 0\}.$$

See Figure 4. We consider the transformation Φ_1 of $F_{T,b} \to S_{T,b,\beta_1,\beta_2}$ defined by

$$\Phi_1 :\ (x,y) \mapsto (\phi(x,y), y), \qquad (x,y) \in F_{T,b}.$$

Here we have put

$$\phi(x,y) = \begin{cases} \dfrac{b-\beta_1(y)}{b-1}(x-\sigma(y))+b & (x>0, y<0), \\ \dfrac{b-\beta_2(y)}{b-1}(x-\sigma(y))+b & (x<0, y<0), \\ x & (y \geq 0). \end{cases}$$

Note that, for a fixed y_0 in $-T < y_0 < 0$, the line segment $I(y_0) = \{(x, y_0);\ \sigma(y_0) < x < b\}$ is mapped onto the line segment $\{(x, y_0);\ \beta_1(y_0) < x < b\}$ by the restriction of Φ_1 on $I(y_0)$, and this mapping is bijective. The corresponding fact is also true for the left hand side of $F_{T,b}$. Therefore, Φ_1 is bijective. On the other hand, noting that

$$1 \leq \frac{\partial \phi}{\partial x}(x,y) \leq C_{T,b}, \qquad (x,y) \in F_{T,b},$$

it is easy to verify that

$$dxdy = \left(\frac{\partial \phi}{\partial x}\right)^{-1} d\xi d\eta \leq d\xi d\eta \quad \text{with } \xi = \phi(x,y), \eta = y. \tag{3.4}$$

Now the function $h_1 \in K^1(F_{T,b})$ is defined as the pullback of v by Φ_1 as follows:

$$h_1(x,y) = (v \circ \Phi_1)(x,y) = v(\phi(x,y), y), \qquad (x,y) \in F_{T,b}.$$

Then, taking (3.4) in mind, in the same way as in the derivation of (2.10), we can obtain

$$\|\nabla h_1\|_{L^2(F_{T,b})} \leq C_{T,b,\beta_1,\beta_2} \|\nabla v\|_{L^2(S_{T,b,\beta_1,\beta_2})}. \tag{3.5}$$

Moreover $h_1|_\gamma = v|_\gamma$ is obvious by the definition.

III) Next we deal with the following domain

$$G_{T,b} = \bigcup_{0<\eta<T} \{(x,y);\ |x| < \kappa(\eta),\ y = \eta\}, \tag{3.6}$$

with

$$\kappa(\eta) = \begin{cases} \dfrac{2(b-1)}{T}\eta + 1 & (0 < \eta < T/2), \\ b & (T/2 \leq \eta < T). \end{cases}$$

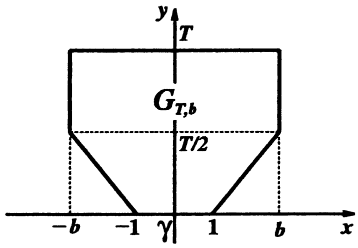

Figure 5. Example of $G_{T,b}$.

See Figure 5. We introduce the transformation Φ_2 of $G_{T,b} \to F_{T,b}$ by

$$\Phi_2 : \ (x, y) \mapsto (x, \psi(x, y)), \qquad (x, y) \in G_{T,b}.$$

Here we have put

$$\psi(x,y) = \begin{cases} \dfrac{2T}{T-\tau(x)}(y-T)+T & (1+\varepsilon < x < b), \\ \dfrac{T-\omega(x)}{T-\tau(x)}(y-T)+T & (1 < x \le 1+\varepsilon), \\ y & (-1 \le x \le 1) \\ \dfrac{T-\omega(-x)}{T-\tau(-x)}(y-T)+T & (-1-\varepsilon \le x < -1), \\ \dfrac{2T}{T-\tau(-x)}(y-T)+T & (-b < x < -1-\varepsilon), \end{cases}$$

where

$$\omega(x) = \frac{T}{\varepsilon}(1-x), \qquad \tau(x) = \frac{T}{2(b-1)}(x-1).$$

We note that Φ_2 is continuous and bijective. In fact, for a fixed x_0 in $1+\varepsilon < x_0 < b$, the line segment $J(x_0) = \{(x_0, y);\ \tau(x_0) < y < T\}$ is mapped onto the line segment $\{(x_0, y);\ \omega(x_0) < y < T\}$ and this mapping is bijective. The same is true if $1 < x_0 < 1+\varepsilon$. We define the function $h_2 \in K^1(G_{T,b})$ as the pullback of h_1 by Φ_2;

$$h_2(x, y) = (h_1 \circ \Phi_2)(x, y) = h_1(x, \psi(x, y)), \qquad (x, y) \in G_{T,b}.$$

Then, in the same way as in the derivation of (2.10), we can deduce

$$\|\nabla h_2\|_{L^2(G_{T,b})} \le C_{T,b} \|\nabla h_1\|_{L^2(F_{T,b})}. \tag{3.7}$$

In order to derive this inequality, we have made use of

$$\left|\frac{\partial\psi}{\partial x}(x,y)\right| \le C_{T,b}, \quad 1 \le \frac{\partial\psi}{\partial y}(x,y) < C_{T,b}, \quad (x,y) \in G_{T,b}$$

and

$$dxdy = \Big(\frac{\partial\psi}{\partial y}\Big)^{-1} d\xi d\eta \le d\xi d\eta \quad \text{with } \xi = x, \eta = \psi(x,y).$$

Moreover, by the definition, we have $h_2|_\gamma = h_1|_\gamma$.

IV) We now define $h \in K^1(Q_T)$ by

$$h(x,y) = (h_2 \circ \Phi_0)(x,y) = h_2(\Theta(y)x, y), \qquad (x,y) \in Q_T,$$

where Φ_0 is defined in an obvious way and

$$\Theta(y) = \begin{cases} 2\dfrac{b-1}{T}y + 1 & (0 < y < T/2), \\ b & (T/2 \le y < T). \end{cases}$$

Then we have

$$\|\nabla h\|_{L^2(Q_T)} \le C_{T,b}\|\nabla h_2\|_{L^2(G_{T,b})} \tag{3.8}$$

and $h_2|_\gamma = h|_\gamma$. Then, evidently, h is the function that we want to get. In fact, it follows from the definitions of h_1, h_2 and h that $v|_\gamma = h|_\gamma$ and, by virtue of (3.5), (3.7) and (3.8), we have (3.2). This completes the proof. □

4. The Case of Solenoidal Vector Fields

In this section, we present corresponding results for the space of solenoidal vector functions. The facts described here are of use in analysis of the domain decomposition method for the Stokes equations (for instance, see Saito [13]). We are concerned with the following function spaces:

$$\begin{aligned}
\mathbf{V} &= \mathfrak{D}(L^{1/4}) \times \mathfrak{D}(L^{1/4}); \\
\mathbf{V}_\sigma &= \Big\{\mathbf{f} \in \mathbf{V};\ \int_\gamma \mathbf{f}\cdot\mathbf{n}\, dx\ = 0\Big\}; \\
\mathbf{H}^1(\Omega) &= H^1(\Omega) \times H^1(\Omega); \\
\mathbf{H}^1_0(\Omega) &= \Big\{\mathbf{v} \in \mathbf{H}^1(\Omega);\ \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega\Big\}; \\
\mathbf{K}^1(\Omega) &= \{\mathbf{v} \in \mathbf{H}^1(\Omega);\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma\}; \\
\mathbf{K}^1_\sigma(\Omega) &= \Big\{\mathbf{v} \in \mathbf{K}^1(\Omega);\ \operatorname{div}\mathbf{v} = \mathbf{0} \text{ in } \Omega\Big\},
\end{aligned}$$

where $\mathbf{n}$ stands for the unit outward normal to γ, i.e., $\mathbf{n} = \{0, -1\}$, and $\mathbf{f} \cdot \mathbf{n}$ means the usual inner product of $\mathbf{f}$ and $\mathbf{n}$ in $\mathbb{R}^2$. The following notation is employed

$$\begin{aligned} \|\mathbf{f}\|_{\mathbf{V}} &= \{\|f_1\|^2_{\mathfrak{D}(L^{1/4})} + \|f_2\|^2_{\mathfrak{D}(L^{1/4})}\}^{1/2}, \qquad \text{for } \mathbf{f} = \{f_1, f_2\} \in \mathbf{V}; \\ \|\mathbf{v}\|_{\mathbf{H}^1} &= \{\|v_1\|^2_{H^1(\Omega)} + \|v_2\|^2_{H^1(\Omega)}\}^{1/2}, \qquad \text{for } \mathbf{v} = \{v_1, v_2\} \in \mathbf{H}^1(\Omega). \end{aligned}$$

THEOREM 4.1. *The following two claims hold true:*
A. (Trace). Let $\mathbf{v} \in \mathbf{K}^1_\sigma(\Omega)$ *and put* $\mathbf{f} = \mathbf{v}|_\gamma$. *Then we have* $\mathbf{f} \in \mathbf{V}_\sigma$ *and*

$$\|\mathbf{f}\|_{\mathbf{V}} \leq C_\Omega \|\mathbf{v}\|_{\mathbf{H}^1}. \tag{4.1}$$

B. (Extension). Let $\mathbf{f} \in \mathbf{V}_\sigma$. *Then there exists a* $\mathbf{v} \in \mathbf{K}^1_\sigma(\Omega)$ *such that* $\mathbf{v}|_\gamma = \mathbf{f}$ *and*

$$\|\mathbf{v}\|_{\mathbf{H}^1} \leq C_\Omega \|\mathbf{f}\|_{\mathbf{V}}. \tag{4.2}$$

The claim A is a direst consequence of Theorem 1.1 A and the Gauss divergence theorem

$$0 = \iint_\Omega \operatorname{div} \mathbf{v} \, dxdy = \int_\gamma \mathbf{v} \cdot \mathbf{n} \, dx.$$

On the other hand, we can prove the claim B by the standard device[1] which apply the following lemma due to I. Babuška and A.K. Aziz ([2]):

LEMMA 4.1. *For any* $F \in L^2(\Omega)$ *satisfying*

$$\iint_\Omega F \, dxdy = 0,$$

there is a vector function $\mathbf{u} \in \mathbf{H}^1_0(\Omega)$ *such that* $\operatorname{div} \mathbf{u} = F$ *in* Ω *and*

$$\|\mathbf{u}\|_{\mathbf{H}^1} \leq C_\Omega \|F\|_{L^2(\Omega)}. \tag{4.3}$$

[1] For example, we refer to Arnold et al. [1].

Remark 4.1. Babuška and Aziz [2] proved this lemma when the boudary $\partial\Omega$ is smooth. However, it is easy to extend it to the case of a piecewise smooth boundary. For exmaple, we refer to Girault and Raviart [7].

The claim B is proved as follows: Let $\mathbf{f} \in \mathbf{V}_\sigma$. Take a vector function $\mathbf{w} \in \mathbf{K}^1(\Omega)$ such that $\mathbf{w}|_\gamma = \mathbf{f}$ and

$$\|\mathbf{w}\|_{\mathbf{H}^1} \le C_\Omega \|\mathbf{f}\|_{\mathbf{V}}.$$

Putting $F = -\mathrm{div}\,\mathbf{w}$, we have

$$\iint_\Omega F\ dxdy = -\iint_\Omega \mathrm{div}\,\mathbf{w}\ dxdy = -\int_\gamma \mathbf{f}\cdot\mathbf{n}\ dx = 0.$$

Therefore, by virtue of Lemma 4.1, there is a $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ such that $\mathrm{div}\,\mathbf{u} = F$ in Ω and (4.3). We then put $\mathbf{v} = \mathbf{w} + \mathbf{u}$. This is the desired function.

5. A Remark on the Case of $\Omega \subset \mathbb{R}^N$

When Ω is a domain in $\mathbb{R}^N$, $N \ge 3$, similar results hold true. Here we shall state only a result about a simple geometry. Let γ be a bounded domain in $\mathbb{R}^{N-1}$ whose boundary $\partial\gamma$ is sufficiently smooth, and let L be an operator defined by

$$L = -\triangle = -\Big(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{N-1}^2}\Big) \quad \text{with } \mathfrak{D}(L) = H^2(\gamma) \cap H_0^1(\gamma). \tag{5.1}$$

We consider a finite cylinder

$$\Omega_T = \{(x_1, \cdots, x_N);\ (x_1, \cdots, x_{N-1}) \in \gamma,\ 0 < x_N < T\}$$

with $T > 0$. Then the same proof as of Lemma 2.1 still remains valid for this case. Namely, we have

Theorem 5.1. *The following two claims hold good:*
A. (Trace). Let $v \in K^1(\Omega_T)$ and put $f = v|_\gamma$. Then we have $f \in \mathfrak{D}(L^{1/4})$ and

$$\|f\|_{\mathfrak{D}(L^{1/4})} \le C_{T,\gamma} \|v\|_{H^1(\Omega_T)}.$$

B. (Extension). Let $f \in \mathfrak{D}(L^{1/4})$. Then there exists a function $u \in K^1(\Omega_T)$ such that $u|_\gamma = f$ and

$$\|v\|_{H^1(\Omega)} \leq C_{T,\gamma}\|f\|_{\mathfrak{D}(L^{1/4})}.$$

REMARK 5.1. If γ is not smooth, as long as the operator L in (5.1) can be defined, Theorem 5.1 remains true. This is, for example, the case where γ is a two-dimensional convex polygon (Grisvard [8]). Moreover, in this case, we know the concrete characterization of $\mathfrak{D}(L^{1/4})$; if γ is a two-dimensional convex polygon, we have

$$\mathfrak{D}(L^{1/4}) = \left\{\xi \in H^{1/2}(\gamma);\ \int_\gamma \rho_j^{-1}\xi^2 d\gamma < \infty\ (j = 1, \cdots, m)\right\}, \tag{5.2}$$

where the boundary $\partial\gamma$ is written as $\partial\gamma = \cup_{j=1}^m G_j$ with the line segment G_j and ρ_j stands for the distance from G_j. In fact, Zolesio [15] proved that the real interpolation space $[K^2(\gamma), L^2(\gamma)]_{3/4}$ coincides with the space of the right-hand side of (5.2). Here we have put $K^2(\gamma) = \{f \in H^2(\gamma);\ f|_{\partial\gamma} = 0\}$. This, together with the well-known relation $\mathfrak{D}(L^{1/4}) = [K^2(\gamma), L^2(\gamma)]_{3/4}$, implies (5.2). However, the concrete characterization of $\mathfrak{D}(L^{1/4})$ in a higher dimensional domain γ with the non-smooth boundary seems to have room for further study.

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(Received May 10, 1999)

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