J. Math. Sci. Univ. Tokyo
7 (2000), 311–323.

On a Variety of Minimal Surfaces Invariant under a Screw Motion

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Abstract. In this paper, we will prove that a certain class of branched multi-valued minimal surfaces invariant under a translation or a screw motion becomes a real analytic variety via their Weierstrass data. We also prove that the class contains complex analytic variety and give a lower bound of its dimension.

1. Introduction

The purpose of this paper is to discuss the possibility to deform minimal surfaces invariant under a translation or a screw motion.

A moduli space of branched complete minimal surfaces of finite total curvature in $\mathbb{R}^n$ is studied by J. Pérez and A. Ros [10, 11], A. Ros [13], X. Mo [15], R. Kusner and N. Schmitt [5], and G. P. Pirola [12] in the case of $\mathbb{R}^3$ and by K. Moriya [7] in the case of $\mathbb{R}^4$. In [8, 9] there are explicit examples of moduli spaces of Weierstrass data for branched complete minimal annuli of finite total curvature in $\mathbb{R}^3$ or $\mathbb{R}^3/T$, where $T = T(v)$ is the discrete group of isometries generated by a translation by v. In [10], [11], and [9], a geometric structure of a moduli space is discussed, too.

In this paper, we will advance the study of the moduli space of minimal surfaces in $\mathbb{R}^3$ to that in a flat 3-space $\mathbb{R}^3/S$ by a modified Weierstrass representation, where S is a screw motion.

We will call a group $S = S(u, v)$ a screw motion if it is the discrete group of isometries generated by a transformation $s(u, v)\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$s(u,v)(x_1,x_2,x_3) = (x_1,x_2,x_3)^t R(u) + (0,0,v), \tag{1.1}$$

$$R(u) = \begin{pmatrix} \cos u & -\sin u & 0 \\ \sin u & \cos u & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1.2}$$

1991 *Mathematics Subject Classification.* Primary 53A10.

where $u \geq 0$ and $v \neq 0$. In the case where $u = 0$, $s(0, v)$ is the nontrivial translation $t(v)\colon \mathbb{R}^3 \to \mathbb{R}^3$ by $(0, 0, v) \in \mathbb{R}^3$. Hence we may see $S(0, v) = T(0, 0, v)$.

It is known that a flat, noncompact, nonsimply-connected three-manifold is finitely covered by $\mathbb{T} \times \mathbb{R}$ or $\mathbb{R}^3/S$, where $\mathbb{T}$ is a flat torus (cf. [14]). Surfaces invariant under a screw motion are considered as surfaces in a flat 3-space of the latter case.

If a branched conformal minimal surface $f\colon M \to \mathbb{R}^3/S$ is complete and of finite total curvature, then M is compactified conformally, that is, M becomes biholomorphic to a compact Riemann surface $\bar{M}$ with finitely many puncture points removed. We investigate only the case where $\bar{M}$ is $\mathbb{C}P^1$.

We will consider the following diagram:

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{f}} & \mathbb{R}^3 \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \Pi} \\ M & \xrightarrow[f]{} & \mathbb{R}^3/S, \end{array}$$

where $\pi\colon \tilde{M} \to M$ is a universal covering, Π is the natural projection, and $\tilde{f}\colon \tilde{M} \to \mathbb{R}^3$ is a branched conformal minimal surface such that $\Pi \circ \tilde{f}$ is well-defined. The above diagram becomes commutative if and only if f is well-defined. We will define a multi-valued minimal surface $\check{f}\colon M \to \mathbb{R}^3$ by $\check{f} := \tilde{f} \circ \pi^{-1}$. We can identify $\tilde{f}$ with $\check{f}$. We will discuss a class $\{\check{f}\}$ of branched multi-valued complete minimal surfaces. We will use a Weierstrass representation studied by H. Karcher [4], M. Callahan, D. Hoffman, and H. Karcher [1] and W. H. Meeks III and H. Rosenberg [6] and prove that a set of Weierstrass data corresponding to a class $\{\check{f}\}$ becomes a real analytic variety and contains a complex analytic variety.

In Section 2, we will give a definition of a certain class of multi-valued functions. In Section 3, we will describe a representation formula for minimal surfaces invariant under a screw motion and specify the classes of Weierstrass data corresponding to a certain class of multi-valued minimal surfaces $\{f\colon M \to \mathbb{R}^3/S\}$, denoted by $\mathcal{U}$ and a certain class of minimal surfaces $\{f\colon M \to \mathbb{R}^3/T\}$, denoted by $\mathcal{W}$. We will prove that if $\mathcal{U}$ and $\mathcal{W}$ are not empty, then they become real analytic varieties and, moreover, $\mathcal{U}$ with rational number u and $\mathcal{W}$ contain a complex analytic variety in Section 4.

Finally, we explain about $\mathcal{U}$ and $\mathcal{W}$ containing the Scherk's saddle tower and helicoidal saddle tower in Section 5.

The author would like to thank Y. Ohnita for his encouragement and the referee for informing him the reference [14].

2. A Multi-valued Function and a Meromorphic 1-form

In this section, we will give some definitions of a class of functions which play an important roll in this paper.

Fix a point z_0 on a compact Riemann surface $\bar{M}$. Let $A(z_0)$ be the set $\{(G, c)\}$ of pairs consisting of a nonzero complex number c and a meromorphic 1-form G on $\bar{M}$ such that any pole of G is of order -1 and different from z_0 and the residue of G is a real number at any point. Let $B(z_0)$ be the set of multi-valued functions $\{\check{g}\}$ obtained by

$$\check{g}(z) = c \exp \int_{z_0}^{z} G, \tag{2.1}$$

where $(G, c) \in \mathcal{A}(z_0)$. By the above relation, we can see that there exists a bijective correspondence between $A(z_0)$ and $B(z_0)$. We will extend the definition of the divisor on $\bar{M}$ as follows:

Definition 2.1. For an element $\check{g} \in B(z_0)$ corresponding to $(G, c) \in A(z_0)$, we will call the *multiplicity* of $\check{g}$ at $p \in \bar{M}$ the residue of G at p and denote it by $\operatorname{mult}_p \check{g}$.

Definition 2.2. For $I_i \in \mathbb{R}$ and $p_i \in \bar{M}$, we will call a formal finite sum $\sum I_i \cdot p_i$ a *divisor* on $\bar{M}$. For a divisor $E = \sum I_i \cdot p_i$ on $\bar{M}$, we will call $\{p_i\}$ the *support* of E and denote by $\operatorname{supp} E$. If any I_i is positive, then we call E *positive.*

Definition 2.3. For $\check{g} \in B(z_0)$, we will define the divisor $(\check{g})$ of $\check{g}$ by

$$(\check{g}) := \sum_{p \in \bar{M}} (\operatorname{mult}_p \check{g}) \cdot p. \tag{2.2}$$

When we write $(\check{g})$ by the difference of two positive divisors $(\check{g})_0$ and $(\check{g})_\infty$ so that $(\check{g}) = (\check{g})_0 - (\check{g})_\infty$, we will call $(\check{g})_0$ the *zero divisor* of $\check{g}$ and $(\check{g})_\infty$ the *polar divisor*.

3. A Representation Formula

In this section, we will describe the representation formula for minimal surfaces in $\mathbb{R}^3/S$ used in this paper, where $S = S(u, v)$ (cf. M. Callahan, D. Hoffman, and H. Karcher [1] and W. H. Meeks III and H. Rosenberg [6]).

Let $f\colon M \to \mathbb{R}^3/S$ be a branched minimal surface. Then we can obtain a multi-valued meromorphic function $\check{g}$ and a holomorphic 1-form $\check{\omega}$ corresponding to the multi-valued minimal surface $\check{f} = (\check{f}_1, \check{f}_2, \check{f}_3)$ by usual Weierstrass representation (cf. [2]):

$$\check{g} = \frac{\check{\Psi}_3}{\check{\Psi}_1 - \sqrt{-1}\check{\Psi}_2}, \ \check{\eta} = \check{\Psi}_3, \tag{3.1}$$

where $\check{\Psi}_i = (\partial \check{f}_i/\partial \check{z})d\check{z}$, $i = 1, 2, 3$ and $\check{z}$ is a local holomorphic coordinate on M. We will note that the function $\check{g}$ is the stereographic projection of the normal Gauss map of $\check{f}$ and $\check{\eta} = d\check{f}_3 + \sqrt{-1}d\check{f}_3^*$, where $\check{f}_3^*$ is a local conjugate harmonic function of $\check{f}_3$. Therefore if we define a meromorphic or holomorphic 1-form on M by $\mathfrak{g} := d \log \check{g}$ or $\eta := d\check{f}_3 + \sqrt{-1}d\check{f}_3^*$ respectively, both of them are well-defined.

For c and $c' \in \mathbb{C}^* = \mathbb{C} - \{0\}$, we will denote by $c \sim c'$ if $c = \exp[\sqrt{-1}nu]c'$ for some $n \in \mathbb{Z}$. Let $[c] \in \mathbb{C}^*/\sim$ be the equivalent class which c belongs to. Then $[\check{g}(z)]$ is well-defined. Hence, if we fix a suitable $z_0 \in M$, then we can obtain a pair $(\mathfrak{g}, \eta, [c])$ from f, where $[c] = [\check{g}(z_0)]$.

DEFINITION 3.1. We will call the pair $(\mathfrak{g}, \eta, [c])$ the *Weierstrass data* of f.

We can prove that a meromorphic 1-form $\mathfrak{g}$ and a holomorphic 1-form η on M become meromorphic or holomorphic 1-forms on a certain compact Riemann surface $\bar{M}$ in a similar way as in the case of unbranched minimal surfaces (cf. W. H. Meeks III and H. Rosenberg [6, Theorem 7]):

LEMMA 3.2. *For a branched complete conformal minimal surface $f\colon M \to \mathbb{R}^3/S$ of finite total curvature, there exists a holomorphic compactification $\bar{M}$ of M. Two 1-forms $\mathfrak{g}$ and η are considered as meromorphic or holomorphic 1-forms on $\bar{M}$.*

We may see that the set $\bar{M} \setminus M$ is a finite set and consists of puncture points defined in Definition 3.6. We can prove the following lemma:

LEMMA 3.3. *Any pole of* $\mathfrak{g}$ *is simple and the residue of* $\mathfrak{g}$ *at any point on* $\bar{M}$ *is a real number.*

PROOF. Since $\breve{g}$ is locally a meromorphic function on M, any pole of $\mathfrak{g}$ on M is simple and any residue of $\mathfrak{g}$ at any point on M is a real number. Hence, we will prove that the residue of $\mathfrak{g}$ at any puncture point is simple and a real number.

Let A be an end, or a neighborhood of a puncture point. We may assume that A is a punctured disk $D^* := \{z \in \mathbb{C} \,|\, 0 < |z| \leq 1\}$ centered at origin and 0 is the puncture point.

Let γ be a simple closed curve around 0 whose orientation is counter-clockwise. Since $\int_\gamma \mathfrak{g}$ is equal to $\sqrt{-1}$ times the rotational angle of $\breve{g}$ along γ, we can see that there exist two kinds of ends. One is the end such that

$$\int_\gamma \mathfrak{g} = 2n\pi\sqrt{-1}, \tag{3.2}$$

where $n \in \mathbb{Z}$. Another is the end such that

$$\int_\gamma \mathfrak{g} = (u_0 + 2n\pi)\sqrt{-1}, \tag{3.3}$$

where $n \in \mathbb{Z}$ and $|u_0| = u$, $u_0 \in \mathbb{R}$. Hence, if $\mathfrak{g}$ has a pole at a puncture point, then it has a simple pole at the puncture point whose residue is a real number. □

We will denote by $(f, \bar{M}, \mathbb{R}^3/S)$ a branched complete conformal minimal surface of finite total curvature $f\colon M \to \mathbb{R}^3/S$ such that $\bar{M}$ is the compactified Riemann surface from M.

DEFINITION 3.4. We will define the *divisor* $(\check{\Psi})$ *of* $\check{\Psi} = (\check{\Psi}_1, \check{\Psi}_2, \check{\Psi}_3)$ on M by

$$(\check{\Psi}) := -(\breve{g})_0 - (\breve{g})_\infty + (\eta). \tag{3.4}$$

Remark 3.5. In [8] and [9], a branched complete conformal minimal surface $f\colon M \to \mathbb{R}^3$ or $\mathbb{R}^3/T$ of finite total curvature was considered. In this case, we can see that $\check{g}$, $\check{\eta}$ and $\check{\Psi}_i$, $i = 1, 2, 3$, are well-defined on $\bar{M}$. The divisor $(\check{\Psi})$ is defined by

$$(\check{\Psi}) = \sum_{p \in \bar{M}} \left(\min_{i=1,2,3} \operatorname{mult}_p \check{\Psi}_i \right) \cdot p. \tag{3.5}$$

Then we may see that the relation (3.4) holds.

Assume that the following relation holds:

$$(\check{\Psi}) = \sum_{j=1}^{s} B_j \cdot b_j - \sum_{i=1}^{r} P_i \cdot p_i, \tag{3.6}$$

where B_j, $j = 1, \dots, s$ and P_i, $i = 1, \dots, r$ are positive numbers.

Definition 3.6. We call a point b_j a *branch point of order* B_j. We call a point p_i a *puncture point of order* P_i.

In the following, we will denote by $M(\mathfrak{g}, \eta)$ the Riemann surface $M = \bar{M} - \{p_1, \dots, p_r\}$, where $\{p_1, \dots, p_r\}$ is the set of puncture points. From the above discussion, we can assume that

$$(\mathfrak{g})_\infty = \sum_{i=1}^{r_1+r_2} p_i + \sum_{k=1}^{a} q_k, \tag{3.7}$$

$$\begin{aligned} &\operatorname{Res}(p_i; \mathfrak{g}) = u_i/2\pi + m_i, && i = 1, \dots, r_1, \\ &\operatorname{Res}(p_i; \mathfrak{g}) = n_i, && i = r_1 + 1, \dots, r_2, \\ &\operatorname{Res}(q_k; \mathfrak{g}) = Q_k, && k = 1, \dots, a, \end{aligned} \tag{3.8}$$

$$\begin{aligned} (\eta) = &\sum_{j=1}^{s} B_j \cdot b_j + \sum_{k=1}^{a} |Q_k| \cdot q_k - \sum_{i=1}^{r_1} (P_i - |u_i/2\pi + m_i|) \cdot p_i \\ &- \sum_{i=r_1+1}^{r_1+r_2} (P_i - |n_i|) \cdot p_i - \sum_{i=r_1+r_2+1}^{r_1+r_2+r_3} P_i \cdot p_i, \end{aligned} \tag{3.9}$$

where $r_1 + r_2 + r_3 = r$, $n_i, m_i, Q_k \in \mathbb{Z}$, $u_i = \pm u$, $\{b_j; p_i\}$ are $s + r$ distinct points, $\{p_i; q_k\}$ are $r + a$ distinct points, and $z_0 \notin \{p_i; q_k\}$.

Definition 3.7. We will call the conditions (3.6), (3.7), (3.8), and (3.9) the *divisor conditions*.

Let the genus of $\bar{M}$ be equal to e. Then we may see the relations

$$\sum_{i=1}^{r_1}(u_i/2\pi + m_i) + \sum_{i=r_1+1}^{r_1+r_2} n_i + \sum_{k=1}^{a} Q_k = 0, \tag{3.10}$$

$$\begin{aligned}\sum_{j=1}^{s} B_j + \sum_{k=1}^{a} |Q_k| - \sum_{i=1}^{r_1}(P_i - |u_i/2\pi + m_i|) \\ - \sum_{i=r_1+1}^{r_1+r_2}(P_i - |n_i|) - \sum_{i=r_1+r_2+1}^{r_1+r_2+r_3} P_i = 2e - 2\end{aligned} \tag{3.11}$$

hold since the sum of all the residues of a meromorphic 1-form on a compact Riemann surface is 0 and the degree of a meromorphic 1-form on a compact Riemann surface of genus e is $2e - 2$.

We will denote by $\mathcal{C} = \mathcal{C}(\bar{M})$ the set of all simple closed curves in $\bar{M}$ where the orientations are counterclockwise and $\mathcal{M} = \mathcal{M}(\bar{M})$ the set of all meromorphic 1-forms on $\bar{M}$. Let us define a map $\tau\colon \mathcal{C} \times \mathcal{M} \to \mathbb{C}$ by

$$\tau(\gamma, \eta) = \int_\gamma \eta. \tag{3.12}$$

We may see that τ is well-defined on a generic subset of $\mathcal{C} \times \mathcal{M}$.

Let $\delta\colon [0,1] \to M(\mathfrak{g}, \eta)$ be a simple closed curve. Then we may see that $\operatorname{Re}\tau(\delta, \eta) = \tilde{f}_3(\tilde{\delta}(1)) - \tilde{f}_3(\tilde{\delta}(0))$, where $\tilde{\delta}$ is a lift of δ to $\tilde{M}$. Let α_i be a simple closed curve around p_i, $i = 1, \dots, r$ whose orientation is counterclockwise. Then

$$\operatorname{Re}\tau(\alpha_i, \eta) = \begin{cases} v_i, & i = 1, \dots, r_1, \\ 0, & i = r_1 + 1, \dots, r_3, \end{cases} \tag{3.13}$$

where $|v_i| = v$.

DEFINITION 3.8. We will call the condition (3.13) the *period condition*.

Conversely, if $\bar{M} = \mathbb{C}P^1$ and if the pair $(\mathfrak{g}, \eta, [c])$ satisfies the conditions (3.7), (3.8), (3.9), and (3.13), then we can obtain a branched multi-valued complete conformal minimal surface $\check{f}\colon M(\mathfrak{g}, \eta) \to \mathbb{R}^3$ by integration:

$$\check{f}(z) = \mathrm{Re} \int_{z_1}^{z} \check{\Psi}, \tag{3.14}$$

$$\check{\Psi} = \left(\frac{1}{\check{g}} - \check{g}, \sqrt{-1} \left(\frac{1}{\check{g}} + \check{g} \right), 2 \right) \frac{\eta}{2}, \tag{3.15}$$

$$\check{g}(z) = c \exp \int_{z_0}^{z} \mathfrak{g}, \tag{3.16}$$

where z is a local coordinate of $M(\mathfrak{g}, \eta)$ and $c \in [c]$. If we choose another base point z_1 of integral, then the image of the immersion shifts by a translation in $\mathbb{R}^3$.

When $u = 0$, that is $S = T = T(0, 0, v)$, the following relation also holds:

$$\mathrm{Re} \int_{\alpha_i} \Psi_1 = \mathrm{Re} \int_{\alpha_i} \Psi_2 = 0, \quad i = 1, \dots, r. \tag{3.17}$$

If the pair $(\mathfrak{g}, \eta, [c])$ satisfies the conditions (3.7), (3.8), (3.9), (3.13), and (3.17), then we obtain a branched complete minimal surface $(f, \mathbb{C}P^1, \mathbb{R}^3/T)$ by integration (3.14), (3.15), and (3.16).

We will denote by $I = I(s, r_1, r_2, r_3, a)$ the pair $(B_j; P_i; u_i; m_i; n_i; Q_k)$ which appeared in the divisor conditions. Fix $\bar{M} = \mathbb{C}P^1$, I, and $z_0 \in \mathbb{C}P^1$. Let $\mathcal{U} = \mathcal{U}(I, z_0)$ be the set $\{(\mathfrak{g}, \eta, [c])\}$ of pairs satisfying the conditions (3.7), (3.8), (3.9), and (3.13). Let $\mathcal{W} = \mathcal{W}(I, z_0)$ be the set $\{(\mathfrak{g}, \eta, c)\}$ of pairs satisfying the conditions (3.7), (3.8), (3.9), (3.13), and (3.17). Let $\mathcal{A} = \mathcal{A}(I, z_0)$ the set of minimal surfaces $(f, \mathbb{C}P^1, \mathbb{R}^3/T)$ corresponding to elements of $\mathcal{W}$ and $\tilde{\mathcal{A}} = \tilde{\mathcal{A}}(I, z_0)$ the set of multi-valued minimal surfaces $\check{f}\colon M \to \mathbb{R}^3$ corresponding to elements of $\mathcal{U}$. For F and $G \in \mathcal{A}$, let $F \sim G$ means $F = G + b$ for some $b \in \mathbb{R}^3$. From the discussion above, the following holds:

LEMMA 3.9. *There exists a bijective correspondence between* (1) $\mathcal{A}/\sim$ *and* $\mathcal{W}$, *and* (2) $\tilde{\mathcal{A}}/\sim$ *and* $\mathcal{U}$.

4. A Variety of Weierstrass Data

In this section, we will show the set $\mathcal{U}$ and $\mathcal{W}$ become varieties.

THEOREM 4.1. *If $\mathcal{U}$ is nonempty, then it becomes a real analytic variety. If u is a rational number, then $\mathcal{U}$ contains a complex analytic subvariety of dimension not less than $s + a + 3$.*

PROOF. Let $\mathcal{R} = \mathcal{R}(I)$ be the set $\{(\mathfrak{g}, \eta)\}$ of pairs of meromorphic 1-forms satisfying the conditions (3.7), (3.8), and (3.9). Then we can see that the set $\mathcal{U}$ is a subset of $\mathcal{R} \times (\mathbb{C}^*/\sim)$. Let $\mathcal{D} = \mathcal{D}(I)$ be the set $\{(D_1, D_2)\}$ of pairs of divisors on $\mathbb{C}P^1$ satisfying the following conditions:

$$D_1 = \sum_{i=1}^{r_1+r_2} p_i + \sum_{k=1}^{a} q_k, \tag{4.1}$$

$$\begin{aligned} D_2 = &\sum_{j=1}^{s} B_j \cdot b_j + \sum_{k=1}^{a} |Q_k| \cdot q_k - \sum_{i=1}^{r_1} (P_i - |u_i/2\pi + m_i|) \cdot p_i \\ &- \sum_{i=r_1+1}^{r_1+r_2} (P_i - |n_i|) \cdot p_i - \sum_{i=r_1+r_2+1}^{r_1+r_2+r_3} P_i \cdot p_i, \end{aligned} \tag{4.2}$$

where $z_0 \notin \{p_i; q_k\}$, $\{b_j; p_i\}$ are $s+r$ distinct points, and $\{p_i; q_k\}$ are $r+a$ distinct points. These are the conditions (3.7) and (3.9) with $\mathfrak{g}$ and η replaced by D_1 and D_2 respectively. Then there exists a bijective correspondence between $\mathcal{R}$ and $\mathcal{D} \times \mathbb{C}^*$. The bijective correspondence is given as follows:

$$(\mathfrak{g}, \eta) \mapsto ((\mathfrak{g})_\infty, (\eta), \eta/\eta_0)), \tag{4.3}$$

$$(D_1, D_2, c_1) \mapsto (\mathfrak{g}_0, c_1\eta_0), \tag{4.4}$$

where

$$\mathfrak{g}_0 = \left(\sum_{i=1}^{r_1} \frac{u_i/2\pi + m_i}{z - p_i} + \sum_{i=r_1+1}^{r_1+r_2} \frac{n_i}{z - p_i} + \sum_{k=1}^{a} \frac{Q_k}{z - q_k} \right) dz, \tag{4.5}$$

$$\eta_0 = \prod_{j=1}^{s}(z-b_j)^{B_j}\prod_{k=1}^{a}(z-q_k)^{|Q_k|}\prod_{i=1}^{r_1}(z-p_i)^{-P_i+|n_i|} \times \prod_{i=r_1+1}^{r_1+r_2}(z-p_i)^{-P_i+|u_i/2\pi+m_i|}\prod_{i=r_2+1}^{r_1+r_2+r_3}(z-p_i)^{-P_i}dz, \tag{4.6}$$

and z is the standard holomorphic coordinate on $\mathbb{C}$.

The set $\mathcal{D}$ is considered as the set $\mathcal{D}'$ of pairs (E_1, E_2, E_3) of divisors on $\mathbb{C}P^1$ satisfying the following conditions:

$$E_1 = \sum_{j=1}^{s} b_j,\ E_2 = \sum_{i=1}^{r_1+r_2+r_3} p_i,\ E_3 = \sum_{k=1}^{a} q_k, \tag{4.7}$$

where $z_0 \notin \{p_i; q_k\}$, $\{b_j; p_i\}$ are $s+r$ distinct points, and $\{p_i; q_k\}$ are $r+a$ distinct points. Hence, we can consider $\mathcal{D}' \times \mathbb{C}^*$, or $\mathcal{R}$ as a complex analytic variety of dimension $s+r+a+1$.

The set $\mathcal{U}$ consists of all the elements in $\mathcal{R} \times (\mathbb{C}^*/\sim)$ satisfying the condition (3.13). We will assume that $(\hat{\mathfrak{g}}, \hat{\eta}, [\hat{c}]) \in \mathcal{U}$ and that $\hat{p}_i$, $i = 1, \dots, r$ are corresponding puncture points. Fix a simple closed curves α_i around $\hat{p}_i$ such that the orientation of each curve is counterclockwise, that $\operatorname{Re}\tau(\alpha_i, \hat{\eta}) = v_i$, $i = 1, \dots, r_1$, and that $\operatorname{Re}\tau(\alpha_i, \hat{\eta}) = 0$, $i = r_1+1, \dots, r$. We can see that $\tau(\alpha_i, \cdot)$, $i = 1, \dots, r$, are local holomorphic functions on $\mathcal{R}$. We may see that the set of solutions to the system of equations $\operatorname{Re}\tau(\alpha_i, \cdot) = v_i$, $i = 1, \dots, r_1$ and $\operatorname{Re}\tau(\alpha_i, \cdot) = 0$, $i = r_1+1, \dots, r$ on $\mathcal{R} \times (\mathbb{C}^*/\sim)$ is a subset of $\mathcal{U}$. Thus if $\mathcal{U}$ is nonempty, then it is a real analytic variety.

Let u be a rational number. We will assume that $\tau(\alpha_i, \hat{\eta}) = R_i \in \mathbb{C}$, $i = 1, \dots, r$. Then, the set $\mathcal{V}$ of solutions to the system of equations $\tau(\alpha_i, \cdot) = R_i$, $i = 1, \dots, r$ on $\mathcal{R} \times (\mathbb{C}^*/\sim)$ is a subset of $\mathcal{U}$. Since $\tau(\alpha_i, \cdot)$ is holomorphic on a generic subset of $\mathcal{R} \times (\mathbb{C}^*/\sim)$, $\mathcal{V}$ is a complex analytic variety of $\mathcal{U}$. Thus if u is a rational number and if $\mathcal{U}$ is nonempty, then it contains a complex analytic variety. Since $\tau(\cdot, \eta)\colon H_1(M(\mathfrak{g}, \eta), \mathbb{Z}) \to \mathbb{C}$ is homomorphism and α_i, $i = 1, \dots, r-1$ become a basis of $H_1(M(\mathfrak{g}, \eta), \mathbb{Z})$, we can see that there exist integers $e_i \in \mathbb{Z}$, $i = 1, \dots, r$ such that

$$\sum_{i=1}^{r} e_i \tau(\alpha_i, \cdot) = 0. \tag{4.8}$$

Hence the dimension of $\mathcal{V}$ is not less than $s+a+3$. $\square$

THEOREM 4.2. *If $\mathcal{W}$ is nonempty, then it becomes a real analytic variety and contains a complex analytic subvariety of dimension not less than $s+a+5-2r$.*

PROOF. The set $\mathcal{W}$ is considered as a subset of $\mathcal{R}\times\mathbb{C}^*$ consisted of the elements satisfying the conditions (3.13) and (3.17). The functions $\int_{\alpha_i}\Psi_k$, $i=1,\dots,r$, $k=1,2$ are local holomorphic functions on $\mathcal{R}$. Hence if $\mathcal{W}$ is nonempty, then it is a real analytic variety of $\mathcal{R}\times\mathbb{C}^*$. In a similar fashion as above, we can prove that if $\mathcal{W}$ is nonempty, it contains a complex analytic subvariety of dimension not less than $s+a+5-2r$. $\square$

5. Examples

In this section, we apply the discussion of the previous section and compare with the possibility of deformations of the Scherk's saddle tower and helicoidal saddle tower.

Example 5.1. The Weierstrass data $(\mathfrak{g}_0, \eta_0, [c_0])$ for a helicoidal saddle tower in $\mathbb{R}^3/S(u,1)$ which appeared in [4] is given as follows:

$$\mathfrak{g}_0 = \frac{dz}{z} + \frac{udz}{z+R} + \frac{udz}{z-R} - \frac{udz}{z+\sqrt{-1}/R} - \frac{udz}{z-\sqrt{-1}/R}, \tag{5.1}$$

$$\eta_0 = \frac{1}{2}\left(\frac{z\sqrt{-1}}{(z+R)(z-R)(z+\sqrt{-1}/R)(z-\sqrt{-1}/R)}\right)dz, \tag{5.2}$$

$$[c_0] = [\exp[\pi\sqrt{-1}/4]], \tag{5.3}$$

where R is a real number depending on u. Hence, $(\mathfrak{g}_0, h_0, [c_0]) \in \mathcal{U}(I, z_0)$, where $I = I(0,0,4,0,2)$. Thus, $\mathcal{U}(I, z_0)$ is nonempty and contains a complex analytic variety of dimension not less than 5.

Example 5.2. If $u = 0$ in (5.1), (5.2), and (5.3), then the pair $(\mathfrak{g}_0, \eta_0, [c_0])$ becomes a Weierstrass data for a Scherk's saddle tower. Hence, $\mathcal{W}(I, z_0)$ is nonempty and contains a complex variety with dimension not less than -1, which is not useful.

REMARK 5.3. In [3], we can see a family of one real parameter which contains a Scherk's saddle tower.

REMARK 5.4. I take this occasion to correct errors in my paper [7]. I would like to thank R. Miyaoka for her comment about this correction.

The statement of Theorem 1.2 in p. 122 should be modified by "If $FD(M_g, \Omega, B_{k,r}, \alpha, \beta)$ is nonempty, then it has the structure of a real analytic variety. If the nullity of the Jacobian of the map $(\mathrm{Re}\, \lambda_i^a)$ defined in p. 132 at a point in FD is 0, then the dimension of FD is at least $2[(k + 2\alpha + 2\beta + 5) - \{(7 - l)g + r\}]$".

The first line in p. 133 of the proof of Theorem 1.2 should be replaced by "Hence, FD is a real analytic subvariety of AD. If the nullity of the Jacobian of the map $(\mathrm{Re}\, \lambda_i^a)$ at a point in FD is 0, then the dimension of FD is at least $2[(k + 2\alpha + 2\beta + 5) - \{(7 - l)g + r\}]$".

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(Received March 23, 1999)
(Revised June 10, 1999)

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