J. Math. Sci. Univ. Tokyo
7 (2000), 221–240.

The L^p-continuity of Wave Operators for One Dimensional Schrödinger Operators

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Abstract. We consider the wave operators $W_\pm u = \text{s}-\lim_{t\to\pm\infty} e^{itH}e^{-itH_0}u$ for a pair of Schrödinger operators $H_0 = -\dfrac{d^2}{dx^2}$ and $H = H_0 + V(x)$ on the line $\mathbf{R}$. We show that $W_\pm$ is bounded in $L^p(\mathbf{R})$ for any $1 < p < \infty$ under the condition that $\int_{\mathbf{R}^1}(1+|x|)|V(x)|^\gamma dx < \infty$, where $\gamma = 3$ if V is of generic type and $\gamma = 4$ if V is of exceptional type.

1. Introduction

We consider the Schrödinger operator $H = H_0 + V(x)$, $H_0 = -\dfrac{d^2}{dx^2}$ on the line $\mathbf{R}^1$. We assume that V is real valued and $\int_{\mathbf{R}^1}(1+|x|)|V(x)|dx < \infty$. Then the quadratic form in the Hilbert space $L^2(\mathbf{R}^1)$ defined by

$$Q(u) = \int_{\mathbf{R}^1}(|u'(x)|^2 + V(x)|u(x)|^2)dx, \quad u \in D(Q) = H^1(\mathbf{R}^1),$$

$H^1(\mathbf{R}^1)$ being the first order Sobolev space, is closed and bounded from below, and Q defines a unique selfadjoint operator H. It is well-known that the spectrum of H consists of the absolutely continuous spectrum $[0,\infty)$ and the finite number of simple negative eigenvalues; the singular continuous spectrum is absent; and the wave operators defined by the limits $W_\pm u = \text{s}-\lim_{t\to\pm\infty} e^{itH}e^{-itH_0}u$ exist. The wave operators $W_\pm$ are unitary from $L^2(\mathbf{R}^2)$ onto the absolutely continuous spectral subspace $L^2_{\text{ac}}(H)$ for H and

1991 *Mathematics Subject Classification*. Primary 81U05, 34L25; Secondary 42B20, 34L40.

*Supported by scholarship of the Ministry of Education, Sports and Culture of Japan.

†Partly supported by the grant from the Ministry of Education, Sports and Culture of Japan, grant number #11304006.

intertwine H_0 and the absolutely continuous part $H_{\rm ac}$ of H: $W_\pm H_0 W_\pm^* = HP_{\rm ac}$, where $P_{\rm ac}$ is the projection onto $L^2_{\rm ac}(H)$, hence

$$f(H)P_{\rm ac} = W_\pm f(H_0) W_\pm^* \tag{1.1}$$

for any Borel functions on $\mathbf{R}^1$. It follows that various mapping properties of $f(H)P_{\rm ac}$ may be derived from those of $f(H_0)$ if corresponding properties are established for $W_\pm$ and $W_\pm^*$. Note that $f(H_0)$ is the convolution operator by the Fourier transform of the function $f(\xi^2)$.

When the spatial dimensions $m \geq 3$, one of the authors has shown in [11] and [12] that the wave operators $W_\pm$ are bounded in $L^p(\mathbf{R}^m)$ for all $1 \leq p \leq \infty$ under suitable conditions on the smoothness and the decay at infinity of $V(x)$ and the additional spectral condition for H at zero energy, viz. $\lambda = 0$ is not an eigenvalue nor resonance of H. In lower dimensions, however, because of the high singularities at $z = 0$ of the free resolvent $R_0(z) = (H_0 - z)^{-1}$, the methods in [11] and [12] do not apply at least directly and it has been an open question whether or not the wave operators are bounded in L^p. The purpose of this paper is to give a positive answer to this question for the one dimensional case and show that the wave operators are bounded in L^p for any $1 < p < \infty$ under suitable decay conditions at infinity on the potential V. The two dimensional case is treated in the accompanying paper ([13]) by employing techniques slightly different from here.

For stating our main result, we introduce some notation. We denote by $f_\pm(x,k)$ the solution of the Schrödinger equation $-f'' + Vf = k^2 f$ which satisfies $|f_\pm(x,k) - e^{\pm ikx}| \to 0$ as $x \to \pm\infty$. We write $[u(x), v(x)] = u'(x)v(x) - u(x)v'(x)$ for the Wronskian of u and v. For $\gamma \in \mathbf{R}$, $L^1_\gamma(\mathbf{R}^1)$ is the weighted L^1-space:

$$L^1_\gamma(\mathbf{R}^1) = \{u(x) : \int_{\mathbf{R}^1} (1+|x|)^\gamma |u(x)| dx < \infty\}.$$

DEFINITION 1.1. We say that the potential V is of generic type if $[f_+(x,0), f_-(x,0)] \neq 0$. We say V is of exceptional type otherwise.

Thus, the potential V is of exceptional type if and only if $f_+(x,0)$ and $f_-(x,0)$ are linearly dependent, or the equation $-f'' + Vf = 0$ admits a solution $f(x)$ which converges to non-vanishing constants as $x \to \pm\infty$. Recall that 0 is not an eigenvalue of H (cf. e.g. [2]).

THEOREM 1.2. *Assume either $V \in L^1_3(\mathbf{R}^1)$ and V is of generic type, or $V \in L^1_4(\mathbf{R}^1)$ and V is of exceptional type. Assume further that $V' \in L^1_2(\mathbf{R})$. Then $W_\pm$ is a bounded operator in $L^p(\mathbf{R}^1)$ for any $1 < p < \infty$. More precisely, the operator $W_\pm$ defined on $L^p(\mathbf{R}^1) \cap L^2(\mathbf{R}^1)$ has an (unique) extension which is bounded in $L^p(\mathbf{R}^1)$.*

The rest of the paper is devoted to the proof of the Theorem 1.2. We shall prove the Theorem for W_- only. The proof for W_+ is similar. As is well known and is easy to show, the free resolvent $R_0(k^2) = (H_0 - k^2)^{-1}$, Im $k > 0$, has the integral kernel $\dfrac{e^{ik|x-y|}}{-2ik}$ and $R_0(k^2)$ has a very strong singularity at $k = 0$. This singularity prevents us from applying to one dimension the method used for the higher dimensional cases, which is based on the the resolvent estimates and the representation formulae for the terms appearing in the perturbation expansion of $W_\pm$ as superpositions of compositions of essentially one dimensional convolution operators. However, because of the simplicity of structures in one dimension, we are able to circumvent this difficulty by representing the wave operator W_- in the form

$$W_- f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \phi(x,k) e^{-iyk} dk \right) f(y) dy \tag{1.2}$$

using the generalized eigenfunctions $\phi(x,k)$ of H characterized by the Lippmann-Schwinger equation, and by estimating the integral kernel $\int_{-\infty}^{\infty} \phi(x,k) e^{-iyk} dk$ by exploiting the detailed properties of $\phi(x,k)$.

In Section 2, we recall the derivation of the representation formula (1.2) starting from Kato-Kuroda's abstract theory of scattering [4] and prove that the high energy part of the wave operator is bounded in L^p for any $1 < p < \infty$. In the high energy region, the singularity at 0 is irrelevant and the properties of $\phi(x,k)$ may be studied by the perturbation expansion, viz. the iteration method for the Lippmann-Schwinger equation. We prove that the low energy part is also bounded in L^p, $1 < p < \infty$, in Section 4. For studying $\phi(x,k)$ for small $|k|$, it is more convenient to use the solution $f_\pm(x,k)$ mentioned above or the Jost functions $m_\pm(x,k)$ which are defined by $m_\pm(x,k) = e^{\mp ikx} f_\pm(x,k)$. These functions are related by

$$\phi(x, \pm k) = T(k) f_\pm(x,k) = e^{\pm ixk} T(k) m_\pm(x,k), \quad k > 0,$$

where $T(k)$ is the transmission coefficient, and in Section 3, we study some properties of $m_\pm(x,k)$, $T(k)$ and the reflection coefficients $R_j(k)$ which will be used in Section 4. After submission of the paper, we are informed of a work by Weder[10] where a similar result is obtained under slightly weaker assumptions.

We shall use the following notation and convention. We write $\langle x \rangle = (1+|x|^2)^{1/2}$. We denote the norm of u in $L^p(\mathbf{R})$ by $\|u\|_p$. $\hat{u}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-ikx} u(x) dx$ is the Fourier transform of u. For Banach spaces X and Y, $B(X,Y)$ is the Banach space of all bounded operators from X to Y. $B(X) = B(X,X)$. Various constants whose specific values are not important are denoted by the same letter C and it may differ from one place to another. The letter k stands for real numbers. Complex momentum will not be used in this paper. When the region of integration is not specified, the integral is understood to be taken on the whole line.

2. High Energy Estimate

We set $\phi_{0\pm}(x,\lambda) = e^{\pm i\sqrt{\lambda}x}$ and $c(\lambda) = 2^{-1/2}\lambda^{-1/4}(2\pi)^{-1/2}$ for $\lambda > 0$. Then, $\{\phi_{0\pm}(x,\lambda), \lambda > 0\}$ is a complete set of generalized eigenfunctions of H_0, $-\phi''_{0\pm}(x,\lambda) = \lambda \phi_{0\pm}(x,\lambda)$, and, if we define $F_0(\lambda)f = (F_{0+}(\lambda)f, F_{0-}(\lambda)f) \in \mathbf{C}^2$ for $f \in L^1 \cap L^2$ by

$$F_{0\pm}(\lambda)f = c(\lambda)\langle f, \phi_{0\pm}(\cdot,\lambda)\rangle = c(\lambda) \int_{\mathbf{R}} f(x) \overline{\phi_{0\pm}(x,\lambda)} dx \tag{2.1}$$

and $(F_0 f)(\lambda) = F_0(\lambda) f$, then F_0 is a spectral representation of H_0, viz. F_0 extends to a unitary operator from $L^2(\mathbf{R})$ onto $L^2([0,\infty), \mathbf{C}^2)$ and $F_0 H_0 F_0^*$ is the multiplication operator by the variable λ. Moreover, in virtue of Riemann-Lebesgue theorem, $F_0(\lambda)$, $\lambda \geq 0$, is a strongly continuous family of bounded operators from $L^1(\mathbf{R})$ to $\mathbf{C}^2$. A spectral representation for H is constructed by a perturbation of F_0. Under the assumption, the limiting absorption principle holds and the resolvent $R(z) = (H-z)^{-1}$ defined on $L^1 \cap L^2$ extends to a bounded operator from L^1 to L^∞ and, as a $B(L^1, L^\infty)$-valued function of $z \in \mathbf{C}^\pm = \{z : \pm \mathrm{Im}\, z > 0\}$, admits a strongly continuous extensions to $\mathbf{C}^{\pm} \cup (0,\infty)$. We denote the boundary values on $(0,\infty)$ from the upper and the lower half planes by $R^{\pm}(\lambda) = \lim_{\epsilon\to+0} R(\lambda \pm i\epsilon)$ and define for $f \in L^1 \cap L^2$,

$$\begin{aligned} F(\lambda)f &= (F_{0+}(\lambda)(1 - VR^{-}(\lambda))f, F_{0-}(\lambda)(1 - VR^{-}(\lambda))f) \\ &= (F_{+}(\lambda)f, F_{-}(\lambda)f) \in \mathbf{C}^2, \end{aligned} \tag{2.2}$$

and $(Ff)(\lambda) = F(\lambda)f$. Then F is a spectral representation of the absolutely continuous part H_{ac} of H, viz. F extends to a unitary operator from the absolutely continuous subspace $L^2_{\mathrm{ac}}(H)$ for H onto $L^2([0,\infty),\mathbf{C}^2)$ and $FH_{\mathrm{ac}}F^*$ is the multiplication operator by the variable λ. In terms of the spectral representations F_0 and F, the wave operator W_- is represented by by

$$W_- f = F^* F_0 f = \sum_{\pm} \int_0^{\infty} F_{\pm}(\lambda)^* F_{0,\pm}(\lambda) f d\lambda. \tag{2.3}$$

The equation (2.3) can be translated to the representation formula via the generalized eigenfunctions of H as follows. Define

$$\phi_{\pm}(\cdot,\lambda) = (1 - VR^{-}(\lambda))^* \phi_{0\pm}(\cdot,\lambda) \tag{2.4}$$

Then, $\phi_{\pm}(x,\lambda)$ satisfies $-\phi_{\pm}'' + V(x)\phi_{\pm} = \lambda\phi_{\pm}$ and we have from (2.1) and (2.2) that

$$F_{\pm}(\lambda)f = c(\lambda)\langle f, \phi_{\pm}(\cdot,\lambda)\rangle = c(\lambda)\int_{\mathbf{R}} f(x)\overline{\phi_{\pm}(x,\lambda)}dx, \tag{2.5}$$

and from (2.1), (2.3) and (2.5) that

$$W_- f(x) = \sum_{\pm} \frac{1}{(2\pi)2\sqrt{\lambda}} \int_0^{\infty} \phi_{\pm}(x,\lambda) \left(\int_{\mathbf{R}} \overline{\phi_{0\pm}(y,\lambda)} f(y) dy\right) d\lambda. \tag{2.6}$$

Define now

$$\begin{aligned} \phi_0(x,k) &= \begin{cases} \phi_{0+}(x,k^2), & k > 0, \\ \phi_{0-}(x,k^2), & k < 0, \end{cases} \\ \phi(x,k) &= \begin{cases} \phi_{+}(x,k^2), & k > 0, \\ \phi_{-}(x,k^2), & k < 0, \end{cases} \end{aligned} \tag{2.7}$$

Then, $\phi_0(x,k) = e^{ikx}$, $\phi(x,k) = (1 - VR^{-}(k^2))^* e^{ikx}$ and (2.6) may be rewritten as

$$W_- f(x) = \frac{1}{2\pi}\int_{\mathbf{R}} \phi(x,k) \left(\int_{\mathbf{R}} e^{-iky} f(y) dy\right) dk. \tag{2.8}$$

Because $(1-VR^-(k^2))^* = (1+R_0^+(k^2)V)^{-1}$, the defining equation (2.4) of $\phi(x,k)$ may be written as $\phi(x,k) = e^{ikx} - R_0^+(k^2)V\phi(x,k)$, which, when the integral kernel is written explicitly, is nothing but the celebrated Lippmann-Schwinger equation:

$$\phi(x,k) = e^{ixk} + \frac{1}{2i|k|}\int e^{i|k||x-y|}V(y)\phi(y,k)dy \tag{2.9}$$

It is obvious that $\|R_0^+(k^2)V\|_{B(L^\infty)} \le \dfrac{\|V\|_{L^1}}{2|k|}$ and (2.9) may be solved by a uniformly convergent series

$$\phi(x,k) = \sum_{n=0}^{+\infty}(-1)^n\left(R_0^+(k^2)V\right)^n e^{ikx} \tag{2.10}$$

for $|k| > \|V\|_{L^1}/2$. We set $l_1 = 4\|V\|_{L^1_2} + 1$ and $l_2 = 2l_1$.

Lemma 2.3. *For $|k| > l_1$ the generalized eigenfunction $\phi(x,k)$ can be written in the form*

$$\begin{aligned}\phi(x,k) &= e^{ikx} + \frac{e^{ikx}}{2ik}\Phi_+(x,k) + \frac{e^{-ikx}}{2ik}\Phi_-(x,k)\\ &\quad + \frac{e^{ixk}}{(2ik)^2}\Psi_+(x,k) + \frac{e^{-ixk}}{(2ik)^2}\Psi_-(x,k),\end{aligned} \tag{2.11}$$

where, for $k > 0$, $\Phi_+(x,\pm k) = \pm\int_{\mp\infty}^x V(y)dy$ and $\Phi_-(x,\pm k) = \pm\int_x^{\pm\infty} e^{2iky}V(y)dy$, and $\Psi_\pm$ satisfy the estimate for $\alpha = 0,1,2$:

$$|(\partial/\partial k)^\alpha \Psi_\pm(x,k)| \le C, \quad x \in \mathbf{R},\ l_1 < |k|. \tag{2.12}$$

Proof. We prove the lemma for $k > l_1$. The proof for $k < -l_1$ is similar. Decomposing $-R_0^+(k^2)Vf$ into two parts, one with positive and the other with negative phases:

$$\begin{aligned}\frac{1}{2ik}\int e^{ik|x-y|}V(y)f(y)dy &= \frac{e^{ixk}}{2ik}\int_{-\infty}^x e^{-iky}V(y)f(y)dy\\ &\quad + \frac{e^{-ixk}}{2ik}\int_x^\infty e^{iky}V(y)f(y)dy,\end{aligned}$$

we see that $(-R_0^+(k^2)V)^n e^{ikx}$ can be written as $(2ik)^{-n}(e^{ixk}\Psi_{+,n}(x,k) + e^{-ixk}\Psi_{-,n}(x,k))$ where $\Psi_{+,n}$ and $\Psi_{-,n}$ are determined by the recurrent formulae: $\Psi_{+,0} = 1$, $\Psi_{-,0} = 0$ and for $n \geq 0$,

$$\Psi_{+,n+1}(x,k) = \int_{-\infty}^{x} V(z)\left(\Psi_{+,n}(z,k) + \Psi_{-,n}(z,k)e^{-2ikz}\right)dz, \tag{2.13}$$

$$\Psi_{-,n+1}(x,k) = \int_{x}^{\infty} V(z)\left(\Psi_{+,n}(z,k)e^{2ikz} + \Psi_{-,n}(z,k)\right)dz. \tag{2.14}$$

We obtain (2.11) from (2.10) by setting $\Psi_\pm(x,k) = \sum_{n=2}^{\infty}(2ik)^{2-n}\Psi_{\pm,n}(x,k)$. From the equations (2.13) and (2.14), we have $\|\Psi_{+,n+1}\|_\infty + \|\Psi_{-,n+1}\|_\infty \leq 2\|V\|_{L^1}(\|\Psi_{+,n}\|_\infty + \|\Psi_{-,n}\|_\infty)$, and hence $\|\Psi_{+,n}\|_\infty + \|\Psi_{-,n}\|_\infty \leq 2^n\|V\|_{L^1}^n$. Thus, (2.12) is satisfied for $\alpha = 0$. We next show

$$|\dot\Psi_{\pm,n}(x,k)| \leq (2\|V\|_{L^1_1})^n, \quad |\ddot\Psi_{\pm,n}(x,k)| \leq (4\|V\|_{L^1_2})^n, \quad n = 0, 1, \ldots, \tag{2.15}$$

by induction. Here and hereafter we denote $\dot\Psi_{\pm,n} = \partial\Psi_{\pm,n}/\partial k$, $\ddot\Psi_{\pm,n} = \partial^2\Psi_{\pm,n}/\partial k^2$ and etc.

When $n = 0$, (2.15) is obvious and we suppose that (2.15) is satisfied upto n. Differentiating (2.13) and using the induction hypothesis, we estimate

$$\begin{aligned}\left|\dot\Psi_{+,n+1}(x,k)\right| &= \left|\int_x^\infty V(y)\left(\dot\Psi_{+,n} + \dot\Psi_{-,n}e^{-2iky} - 2iy\Psi_{-,n}e^{-2iky}\right)dy\right| \\ &\leq 2\|V\|_{L^1}\left(2\|V\|_{L^1_1}\right)^n + 2\|yV\|_{L^1}\left(2\|V\|_{L^1}\right)^n \\ &\leq \left(2\|V\|_{L^1_1}\right)^{n+1};\end{aligned}$$

$$\begin{aligned}&\left|\ddot\Psi_{+,n+1}(x,k)\right| \\ &= \left|\int_x^{+\infty} V(y)\left(\ddot\Psi_{+,n} + \left(\ddot\Psi_{-,n} - 4iy\dot\Psi_{-,n} - 4y^2\Psi_{-,n}\right)e^{-2iky}\right)dk\right| \\ &\leq 2\|V\|_{L^1}\left(4\|V\|_{L^1_2}\right)^n + 4\|yV\|_{L^1}\left(2\|V\|_{L^1_1}\right)^n \\ &\quad + 4\|y^2V\|_{L^1}\left(2\|V\|_{L^1}\right)^n \leq \left(4\|V\|_{L^1_2}\right)^{n+1}.\end{aligned}$$

Likewise, we obtain $\left|\dot{\Psi}_{-,n+1}(x,k)\right| \le \left(2\|V\|_{L^1_1}\right)^{n+1}$ and $\left|\ddot{\Psi}_{-,n+1}(x,k)\right| \le \left(4\|V\|_{L^1_2}\right)^{n+1}$ by estimating the derivatives of (2.14). Hence, (2.15) holds for all $n = 0, 1, \dots$ and the Lemma follows. □

We now state and prove the main result of this section. We let $\varphi_2(k) \in C^\infty(\mathbf{R})$ be such that $\varphi_2(k) = 1$ for $|k| \ge l_2$, $\varphi_2(k) = 0$ if $|k| \le l_1$ and $0 \le \varphi_2(k) \le 1$ if $l_1 \le |k| \le l_2$, and consider the operator $W_-\varphi_2(H_0)$. It follows from (2.8) that

$$W_-\varphi_2(H_0)f(x) = \frac{1}{2\pi}\int_{\mathbf{R}} \phi(x,k)\varphi_2(k^2)\left(\int_{\mathbf{R}} e^{-iky}f(y)dy\right)dk. \tag{2.16}$$

THEOREM 2.4. *Let V and $V' \in L^1_2$ and let $l_1 > 4\|V\|_{L^1_2}$. Then for any $1 \le p \le \infty$, $W_-\varphi_2(H_0)$ is bounded in L^p:*

$$\|W_-\varphi_2(H_0)f\|_p \le C_p\|f\|_p, \quad f \in L^2 \cap L^p, \tag{2.17}$$

where $C_p > 0$ is a constant.

PROOF. By inserting (2.11) into (2.16) we decompose $W_-\varphi_2(H_0)$ into five pieces:

$$\begin{aligned} W_-\varphi_2(H_0)f(x) = &\frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} e^{ikx}\varphi_2(k^2)\hat{f}(k)dk \\ &+ \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} \frac{e^{ikx}}{2ik}\Phi_+(x,k)\varphi_2(k^2)\hat{f}(k)dk \\ &+ \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} \frac{e^{-ikx}}{2ik}\Phi_-(x,k)\varphi_2(k^2)\hat{f}(k)dk \\ &+ \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} \frac{e^{ixk}}{(2ik)^2}\Psi_+(x,k)\varphi_2(k^2)\hat{f}(k)dk \\ &+ \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} \frac{e^{-ixk}}{(2ik)^2}\Psi_-(x,k)\varphi_2(k^2)\hat{f}(k)dk. \end{aligned} \tag{2.18}$$

We denote the operators in the right of (2.18) by $T_1, \dots, T_5$, respectively, and we show that they are bounded in L^p for any $1 < p < \infty$ separately. Because $\varphi_1(k^2) = 1 - \varphi_2(k^2) \in C_0^\infty(\mathbf{R})$, $\varphi_1(H_0)$ is a convolution with a rapidly decreasing function and $T_1 = I - \varphi_1(H_0)$ is bounded in $L^p(\mathbf{R})$

for any $1 \le p \le \infty$. We next estimate T_4. Let $f \in L^1 \cap L^2$. Since $(2ik)^{-2}\Psi_+(x,k)\varphi_2(k^2)$ is integrable with respect to the variable k, we may write by using Fubini's theorem

$$\begin{aligned} T_4 f(x) &= \int_{\mathbf{R}} dy \left(\frac{1}{2\pi} \int \frac{e^{i(x-y)k}}{(2ik)^2} \varphi_2(k^2)\Psi_+(x,k)dk \right) f(y)dy \\ &\equiv \int_{\mathbf{R}} K_4(x,y)f(y)dy. \end{aligned}$$

We apply integration by part to the integral in the parenthesis, which we wrote $K_4(x,y)$, by using the identity

$$\frac{1}{\langle x-y\rangle^2}\left(1-\frac{\partial^2}{\partial k^2}\right)e^{i(x-y)k} = e^{i(x-y)k}.$$

In virtue of the decaying factor $(2ik)^{-2}$ and (2.12) the boundary term does not appear and

$$\begin{aligned} 2\pi|K_4(x,y)| &= \left| \frac{1}{\langle x-y\rangle^2} \int e^{i(x-y)k} \left(1-\frac{\partial^2}{\partial k^2}\right)\left(\frac{\varphi_2(k^2)\Psi_+(x,k)}{(2ik)^2}\right)dk \right| \\ &\le C\langle x-y\rangle^{-2}. \end{aligned}$$

If follows that T_4 is bounded in L^p for any $1 \le p \le \infty$. Entirely similarly, we can show that T_5 may be written as an integral operator $T_5 f(x) = \int_{\mathbf{R}} K_5(x,y)f(y)$ with the kernel $K_5(x,y)$ satisfying $|K_5(x,y)| \le C(1+|x+y|)^{-2}$, and T_5 is also bounded in L^p for any $1 \le p \le \infty$.

Recalling $\Phi_+(x,\pm k) = \pm\int_{\mp\infty}^{x} V(y)dy$, $k>0$, are independent of k, we write

$$T_2 f(x) = V_1(x)\int_{\mathbf{R}} G(x-y)f(y)dy + V_2(x)\int_{\mathbf{R}} G(y-x)f(y)dy.$$

Here $V_1(x) = \int_{-\infty}^{x} V(z)dz$ and $V_2(x) = \int_{x}^{\infty} V(z)dz$ are bounded and $G(t) = \frac{1}{2\pi}\int_0^{+\infty} \frac{\varphi_2(k^2)}{2ik} e^{ikt}dk$ is integrable. Indeed, $G(t) \in L^2(\mathbf{R})$ because $\varphi_2(k^2)/(2ik) \in L^2(\mathbf{R})$ and integration by parts shows that $G(t)$ is rapidly decreasing as $t \to \pm\infty$. It follows by Young's inequality that T_2 is bounded in $L^p(\mathbf{R})$ for any $1 \le p \le \infty$. T_3 may be written as a sum:

$$T_3 f(x) = \int_{\mathbf{R}} S_+(x,y) f(y) dy + \int_{\mathbf{R}} S_-(x,y) f(y) dy$$

where $S_\pm$ are given by $S_\pm(x,y) = \dfrac{\pm 1}{2\pi} \displaystyle\int_0^{\pm\infty} \frac{\varphi_2(k^2)}{2ik} \left(\int_x^{\pm\infty} V(z) e^{\pm 2ikz} dz \right) \cdot e^{\mp ik(x+y)} dk$. Here integration by parts shows that

$$\int_x^{\pm\infty} V(z) e^{\pm 2ikz} dz = \frac{\pm i}{2k} \left(V(x) e^{\pm 2ikx} + \int_x^{\pm\infty} V'(x) e^{\pm 2ikz} dz \right)$$

and $S_\pm(x,y)$ may be written in the form

$$\frac{1}{2\pi} \int_0^\infty \frac{\varphi_2(k^2)}{(2k)^2} V(x) e^{\pm ik(x-y)} dy \\ + \frac{1}{2\pi} \int_0^\infty \frac{\varphi_2(k^2)}{(2k)^2} \left(\int_x^{\pm\infty} V'(z) e^{\pm 2ikz} dz \right) e^{\pm ik(x+y)} dy.$$

The first term produces an operator of the form T_2; $\displaystyle\int_x^{\pm\infty} V'(z) e^{\pm 2ikz} dz$ are bounded with first two derivatives with respect to k by the assumption and the second term produces an operator of the form T_4. Hence T_3 is also bounded in $L^p(\mathbf{R})$ for all $1 \le p \le \infty$. This completes the proof of the Theorem 2.4. □

3. Jost Function

From the Lippmann-Schwinger equation (2.9), we can read off the asymptotic behavior of the generalized eigenfunction $\phi(x,k)$ as $x \to \pm\infty$: For $k > 0$,

$$\phi(x,k) \sim \begin{cases} e^{ikx} + e^{-ikx} \left(\dfrac{1}{2ik} \displaystyle\int e^{iky} V(y) \phi(y,k) dy \right), & x \to -\infty, \\ e^{ikx} \left(1 + \dfrac{1}{2ik} \displaystyle\int e^{-iky} V(y) \phi(y,k) dy \right), & x \to \infty, \end{cases} \tag{3.1}$$

and for $k < 0$,

$$\phi(x,k) \sim \begin{cases} e^{ikx} \left(1 - \dfrac{1}{2ik} \displaystyle\int e^{-iky} V(y) \phi(y,k) dy \right), & x \to -\infty, \\ e^{ikx} - e^{-ikx} \left(\dfrac{1}{2ik} \displaystyle\int e^{iky} V(y) \phi(y,k) dy \right), & x \to \infty. \end{cases} \tag{3.2}$$

For relating the solutions $\phi(x,k)$ to $f_\pm(x,k)$, we recall some definitions ([2], [1], [5]). We do so in an ad hoc way as systematic derivation is not our purpose here. Pairs of functions $\{f_+(x,k), f_+(x,-k)\}$ and $\{f_-(x,k), f_-(x,-k)\}$ are both basis of the solution space of $-f''+Vf=k^2 f$. The transmission $T_j(k)$ and the reflection coefficients $R_j(k)$, $j=1,2$, are defined through the coefficients of the linear transformation relating these two bases as follows:

$$f_-(x,k) = \frac{R_1(k)}{T_1(k)} f_+(x,k) + \frac{1}{T_1(k)} f_+(x,-k), \tag{3.3}$$

$$f_+(x,k) = \frac{R_2(k)}{T_2(k)} f_-(x,k) + \frac{1}{T_2(k)} f_-(x,-k), \tag{3.4}$$

By computing the Wronskians and by employing the relations $\overline{f_+(x,k)} = f_+(x,-k)$ and etc. we easily see that $T_1(k) = T_2(k) \equiv T(k)$, $\overline{T(k)} = T(-k)$, $\overline{R_j(k)} = R_j(-k)$, $j=1,2$, and the matrix $S(k)$ defined by $S(k) = \begin{pmatrix} T(k) & R_2(k) \\ R_1(k) & T(k) \end{pmatrix}$ is unitary. $S(k)$ is called the scattering matrix.

By comparing (3.1) and (3.2) with (3.3) and (3.4), we conclude

$$\phi(x,k) = \begin{cases} T(k) f_+(x,k), & \text{for } k>0; \\ T(-k) f_-(x,-k), & \text{for } k<0, \end{cases} \tag{3.5}$$

and, defining the Jost functions by $m_\pm(x,k) = e^{\mp ikx} f_\pm(x,k)$, we obtain

$$W_- u(x) = \frac{1}{2\pi} \int_{\mathbf{R}} \left(\int_0^\infty T(k) \{ e^{ik(x-y)} m_+(x,k) + e^{-ik(x-y)} m_-(x,k) \} dk \right) u(y) dy. \tag{3.6}$$

The function $m_\pm$ satisfies $m_\pm'' \pm 2ik m_\pm' = V m_\pm$ and $m_\pm(x,k) \to 1$ as $x \to \pm\infty$ and is the unique solution of the Volterra equations

$$m_\pm(x,k) = 1 \pm \int_x^{\pm\infty} D_k(\pm(t-x)) V(t) m_\pm(t,k) dt, \tag{3.7}$$

where $D_k(x) = \int_0^x e^{2ikt} dt = \dfrac{e^{2ikx}-1}{2ik}$. The following Lemma 3.5 is well known and may be proved by solving (3.7) by iteration (see [2] and [3] for the details).

LEMMA 3.5. *Suppose $V \in L^1_1$. Then $m_\pm(x,k)$ are of C^1 -class for $k \neq 0$ and $k\dot{m}_\pm(x,k)$ are continuous on $\mathbf{R}^2$. $m_\pm$ satisfy*

$$|m_\pm(x,k)-1| \le \frac{Ce^{C/|k|}}{|k|}, \qquad k \neq 0 \tag{3.8}$$

$$|m_\pm(x,k)-1| \le C\frac{1+\max(\mp x,0)}{1+|k|}, \quad (x,k) \in \mathbf{R}^2. \tag{3.9}$$

If $V \in L^1_2$, then $m_\pm(x,k)$ are of C^1-class on $\mathbf{R}^2$ and satisfy

$$|\dot{m}_\pm(x,k)| \le C(1+\max(\mp x,0)|x|), \quad (x,k) \in \mathbf{R}^2. \tag{3.10}$$

The iteration method used for the proof of Lemma 3.5 in [2] applies to the estimation of higher derivatives $(\partial/\partial k)^\alpha m_\pm(x,k)$ and we obtain the following lemma.

LEMMA 3.6. *Let $V \in L^1_\gamma$, $\gamma \ge 2$. Then, $(\partial/\partial k)^n m_\pm(x,k)$ exist for $0 \le n \le \gamma - 1$, are continuous in $(x,k) \in \mathbf{R}^2$ and satisfy*

$$|(\partial/\partial k)^n m_\pm(x,k)| \le C(1+\max(\mp x,0))^{n+1}, \quad (x,k) \in \mathbf{R}^2. \tag{3.11}$$

PROOF. We give the proof for m_+ by induction. The proofs for m_- is similar. We prove the estimate (3.11) only assuming the existence of the derivatives. The latter may be proved by replacing one of $\partial/\partial k$ by the difference quotient $\triangle_h$ in the formulae to appear and taking the limit $h \to 0$. Estimate (3.10) proves (3.11) for the case $n = 1$. Differentiating (3.7) by using Leibniz' formula, we have

$$\begin{aligned}\partial_k^n m_+(x,k) = &\int_x^\infty \sum_{j=0}^{n-1} {}_nC_j \partial_k^{n-j} D_k(t-x)V(t)\partial_k^j m_+(t,k)dt \\ &+ \int_x^\infty D_k(t-x)V(t)\partial_k^n m_+(t,k)dt.\end{aligned}$$

By the induction hypothesis and by the obvious estimate $|(\partial/\partial k)^j D_k(x)| \le C_j|x|^{j+1}$, we have

$$\begin{aligned}&\int_x^\infty |\partial_k^{n-j} D_k(t-x)V(t)\partial_k^j m_+(t,k)|dt \\ &\qquad \le C\int_x^\infty |(t-x)^{n-j+1}V(t)(1+\max(-t,0))^{j+1}|dt.\end{aligned} \tag{3.12}$$

When $x > 0$, it obvious that (3.12) $\le C\|V\|_{L^1_\gamma}$. If $x < 0$, then (3.12) is bounded by

$$C\int_0^\infty |(t-x)^{n-j+1}V(t)|dt + \int_x^0 |(t-x)^{n-j+1}V(t)(1-t)^{j+1}|dt \\ \le C\|V\|_{L^1_\gamma}(1-x)^{n+1}$$

for $j = 0, \dots, n-1$. It follows that

$$|\partial_k^n m_+(x,k)| \le C(1+\max(-x,0))^{n+1} + \int_x^\infty (t-x)|V(t)\partial_k^n m_+(t,k)|dt.$$

Dividing through by $(1+\max(-x,0))^{n+1}$ and noticing that

$$|t-x|\left\{\frac{1+\max(-t,0)}{1+\max(-x,0))}\right\}^{n+1} \le (|t|+1)^{n+1}, \quad x < t < \infty,$$

we see $h(x,k) = \dfrac{|\partial_k^n m_\pm(x,k)|}{(1+\max(0,-x))^{n+1}}$ satisfies the integral inequality

$$|h(x,k)| \le C + \int_x^\infty (1+|t|)^{n+1}|V(t)h(t,k)|dt. \tag{3.13}$$

Here $(1+|t|)^{n+1}V(t)$ is integrable on $\mathbf{R}$ by the assumption and Gronwall's inequality implies $|h(x,k)| \le C < \infty$. The lemma follows. □

We set $n_\pm(x,k) = \dfrac{m_\pm(x,k) - m_\pm(x,0)}{k}$. The following lemma is obvious from the previous Lemma 3.6.

Lemma 3.7. *Suppose $V \in L^1_4$. Then for each fixed x, $(d/dk)^j n_\pm(x,k)$, $j = 0,1,2$, exist, are continuous in (x,k) and obey the following estimates:*

$$|(d/dk)^j n_\pm(x,k)| \le C\langle x\rangle^{2+j}, \quad (x,k) \in \mathbf{R}^2, \quad j = 0,1,2. \tag{3.14}$$

On the scattering coefficients the followings is well known ([2], [1] and [5]).

Lemma 3.8. *Assume that $V \in L^1_2$. Then:*
1). *$T(k)$ and $R_j(k)$ are bounded and continuous and as $k \to \pm\infty$, $T(k) =$*

$1+O(1/|k|)$ *and* $R_j(k)=O(1/|k|)$, $j=1,2$.
2). *If* V *is of generic type, then, as* $k\to 0$, $T(k)=\alpha k+o(k)$, $\alpha\neq 0$ *and* $R_j(k)=-1+O(k)$ *for* $j=1,2$.
3). *If* V *is of exceptional type, then, as* $k\to 0$, *we have, with* $a=\lim_{x\to-\infty} f_+(x,0)$,

$$\begin{aligned}T(k)&=\frac{2a}{1+a^2}+o(1),\\ R_1(k)&=\frac{1-a^2}{1+a^2}+o(1),\quad R_2(k)=\frac{a^2-1}{1+a^2}+o(1).\end{aligned}\tag{3.15}$$

We need estimate the derivatives of the scattering coefficients. For this we use the following integral representation:

$$\frac{1}{T(k)}=\frac{1}{2ik}[f_+(x,k),f_-(x,k)]=1-\frac{1}{2ik}\int_{-\infty}^{+\infty}V(t)m_+(t,k)dk,\tag{3.16}$$

$$\begin{aligned}\frac{R_1(k)}{T(k)}&=\frac{1}{2ik}[f_-(x,k),f_+(x,-k)]\\ &=\frac{1}{2ik}\int_{-\infty}^{+\infty}e^{-2ikt}V(t)m_-(t,k)dk,\end{aligned}\tag{3.17}$$

$$\begin{aligned}\frac{R_2(k)}{T(k)}&=\frac{1}{2ik}[f_-(x,-k),f_+(x,k)]\\ &=\frac{1}{2ik}\int_{-\infty}^{+\infty}e^{2ikt}V(t)m_+(t,k)dk.\end{aligned}\tag{3.18}$$

Thus, V is of generic type if and only if $\int_{-\infty}^{+\infty}V(t)m_+(t,0)dt\neq 0$.

LEMMA 3.9 (the generic case). *Assume* $V\in L^1_3$ *and generic. Then* $T(k)$ *and* $R_1(k)$, $R_2(k)$ *are of* C^2*-class and*

$$\sum_{i=1}^{2}\left(|(\partial/\partial k)^i T(k)|+\sum_{j=1}^{2}|(\partial/\partial k)^i R_j(k)|\right)\leq\frac{C}{1+|k|}.\quad k\in\mathbf{R}.\tag{3.19}$$

PROOF. Set $g(k) = k - \frac{1}{2i}\int_{-\infty}^{+\infty} V(t)m_+(t,k)dt$ so that $T(k) = \frac{k}{g(k)}$. When V is generic, $g(k) \neq 0$ for any $k \in \mathbf{R}$. In virtue of (3.11), we may differentiate $g(k)$ under the integral sign and obtain

$$(3.20)\quad |\dot{g}(k)| = \left|1 - \frac{1}{2i}\int V(t)\dot{m}_+(t,k)dt\right| \leq 1 + C\int |V(t)|(1+t^2)dt \leq C$$

$$(3.21)\quad |\ddot{g}(k)| = \left|\frac{1}{2i}\int V(t)\ddot{m}_+(t,k)dt\right| \leq C\int |V(t)|(1+|t|)^3dt \leq C.$$

Then, it is clear that $T(k) \in C^2(\mathbf{R})$. From Lemma 3.8, 2) and $|T(k)| \leq 1$ we have

$$(3.22)\quad \left|\frac{T(k)}{k}\right| = \left|\frac{1}{g(k)}\right| \leq \frac{M}{1+|k|}, \quad k \in R,$$

where $M > 0$ is independent of k. Hence, by (3.20), (3.21), (3.22) and $|T(k)| \leq 1$:

$$(3.23)\quad |\dot{T}(k)| = \left|\frac{1}{g(k)} - \frac{k\dot{g}(k)}{g^2(k)}\right| \leq \left|\frac{1}{g(k)}\right|\left(1 + \left|\frac{k}{g(k)}\dot{g(k)}\right|\right) \leq \frac{C_1}{1+|k|}.$$

$$(3.24)\quad |\ddot{T}(k)| = \left|\frac{1}{g(k)}\right|\left(\left|\frac{\dot{g}(k)}{g(k)}\right| + \left|\frac{k(\ddot{g}g - 2\dot{g}^2)}{g^2(k)}\right|\right) \leq C\left|\frac{1}{g(k)}\right| \leq \frac{C_1}{1+|k|}.$$

Using (3.18) and (3.16), we write $R_2(k)$ in the form

$$R_2(k) = T(k)\left(1 + \int_{-\infty}^{+\infty} D_k(t)V(t)m_+(t,k)dt\right) - 1, \quad k \in R.$$

Recall that $D_k(x)$ satisfies $|(\partial/\partial k)^j D_k(x)| \leq C_j|x|^{j+1}$. It is easy to see that $D_k(x)$ also satisfies the estimate $|k(\partial/\partial k)^j D_k(x)| \leq C_j|x|^j$, $j = 0,1,2$ as well. Then, we estimate

$$\begin{aligned}
|\dot{R}_2(k)| &\leq |\dot{T}(k)| \cdot \left|1 + \int D_k(t)V(t)m_+(t,k)dt\right| \\
&\quad + \left|\frac{T(k)}{k}\right| \int \left|\left(k\dot{D}_k(t)m_+(t,k) + kD_k(t)\dot{m}_+(t,k)\right)V(t)\right|dt \\
&\leq \frac{C}{1+|k|}\left(1 + \int (1+|t|)^2|V(t)|dt\right) \leq \frac{C}{1+|k|}\|V\|_{L^1_2}.
\end{aligned}$$

Likewise we have $|\ddot{R}_2(k)| \leq \frac{C_3}{1+|k|}\|V\|_{L^1_3}$. The proof for $R_1(k)$ is similar. □

LEMMA 3.10 (the exceptional case). *Suppose V is of exceptional type and $V \in L^1_4$. Then, $T(k)$, $R_1(k)$ and $R_2(k)$ are of C^2-class and are bounded with their first two derivatives.*

PROOF. If V is exceptional, $\int_{-\infty}^{+\infty} V(t)m_+(t,0)dt = 0$ and in virtue of (3.16)

$$\frac{1}{T(k)} = 1 - \frac{1}{2i}\int V(t)\frac{m_+(t,k)-m_+(t,0)}{k}dt = 1 - \frac{1}{2i}\int V(t)n_+(t,k)dt.$$

Write $p(k) \equiv 1 - \frac{1}{2i}\int V(t)n_+(t,k)dt$. It follows, in virtue of Lemma 3.7 and the assumption $V \in L^1_4$, that $p(k)$ is of C^2-class and $p, \dot{p}$ and $\ddot{p}$ are bounded. Since $T(k)$ is continuous on the line $\mathbf{R}$, $p(k) \neq 0$ for $k \in \mathbf{R}$ and, taking into account of the behavior at infinity of $T(k)$ in Lemma 3.8, we see that $C \leq |T(k)|$ for some $C > 0$. Hence, $T(k) = p^{-1}(k)$ is of C^2-class and is bounded with derivatives.

Using (3.18) and the identity $\int V(t)m_+(t,0)dt = 0$, we write $R_2(k)$ as follows.

$$\begin{aligned} R_2(k) &= T(k)\int \frac{e^{2ikt}}{2ik}V(t)m_+(t,k)dt - \frac{T(k)}{2ik}\int V(t)m_+(t,0)dt \\ &= T(k)\int D_k(t)V(t)m_+(t,k)dt + \frac{T(k)}{2i}\int V(t)n_+(t,k)dt. \end{aligned}$$

The integrals $\frac{1}{2i}\int V(t)n_+(t,k)dt$ and $\int D_k(t)V(t)m_+(t,k)dt$ and their derivatives have been estimated above and are bounded. Hence, the result on $T(k)$ which has just been proved above implies that $\dot{R}_2(k)$ and $\ddot{R}_2(k)$ are bounded. The proof for $R_1(k)$ is similar. □

4. The Low Energy Estimate

In this section we prove the following

THEOREM 4.11. *Assume either $V \in L^1_3(\mathbf{R})$ and V is of generic type, or $V \in L^1_4(\mathbf{R})$ and V is of exceptional type. Then, $W_-\varphi_1(H_0)$ is a bounded operator in L^p for any $1 < p < \infty$.*

$$\|W_-\varphi_1(H_0)f\|_p \leq C_p\|f\|_p, \quad f \in L^2 \cap L^p, \tag{4.1}$$

where $C_p > 0$ is a constant.

PROOF. We recall the representation formula (3.6) and write $W_-\varphi_1(H_0)$ as an integral operator

$$W_-\varphi_1(H_0)u(x) = \frac{1}{2\pi}\int_{\mathbf{R}} K(x,y)u(y)dy \tag{4.2}$$

with the integral kernel

$$\begin{aligned} K(x,y) = \int_0^\infty \varphi_1(k^2)\{&e^{ik(x-y)}T(k)m_+(x,k) \\ &+ e^{-ik(x-y)}T(k)m_-(x,k)\}dk. \end{aligned} \tag{4.3}$$

We have proven in Section 3 that $m_\pm(x,k)$ and its derivatives upto second order are uniformly bounded for $\pm x > 0$. The following equations which may be derived from (3.3)

$$T(k)m_-(x,k) = R_1(k)e^{2ikx}m_+(x,k) + m_+(x,-k), \tag{4.4}$$

$$T(k)m_+(x,k) = R_2(k)e^{-2ikx}m_-(x,k) + m_-(x,-k). \tag{4.5}$$

make it possible to estimate $T(k)m_\pm(x,k)$ in the complementary set $\mp x > 0$. Taking this into account, for $x > 0$, we replace $T(k)m_-(x,k)$ by (4.4) in(4.3) and write

$$\begin{aligned} K(x,y) = &\int_0^\infty e^{ik(x-y)}\varphi_1(k^2)T(k)m_+(x,k)dk \\ &+ \int_0^\infty e^{-ik(x-y)}\varphi_1(k^2)m_+(x,-k)dk \\ &+ \int_0^\infty e^{ik(x+y)}\varphi_1(k^2)R_1(k)m_+(x,k)dk. \end{aligned} \tag{4.6}$$

Likewise, for $x < 0$, we replace $T(k)m_+(x,k)$ by (4.5) in (4.3) and write

$$\begin{aligned} K(x,y) = &\int_0^\infty e^{-ik(x-y)}\varphi_1(k^2)T(k)m_-(x,k)dk \\ &+ \int_0^\infty e^{ik(x-y)}\varphi_1(k^2)m_-(x,-k)dk \\ &+ \int_0^\infty e^{-ik(x+y)}\varphi_1(k^2)R_2(k)m_-(x,k)dk. \end{aligned} \tag{4.7}$$

It is clear from the boundedness of $T(k)$ and $R_j(k)$ and Lemma 3.5 that $K(x,y)$ is bounded. Note that K is the integral kernel of $W_-\varphi_1(H_0)$ which is a fortiori bounded in $L^2(\mathbf{R})$. For proving the Theorem 4.11, therefore, in virtue of Marcinkiewicz' interpolation theorem, it suffices to show that the integral operator K with the integral kernel $K(x,y)$ and its adjoint operator K^* are weak-type $(1,1)$, viz. maps $L^1(\mathbf{R})$ into weak-$L^1(\mathbf{R})$ continuously:

$$\sup_{\lambda>0}\lambda|\{x : |Kf(x)| \geq \lambda\}| \leq C\|f\|_1, \qquad \sup_{\lambda>0}\lambda|\{x : |K^*f(x)| \geq \lambda\}| \leq C\|f\|_1.$$

For this purpose we show that the functions given by the integrals in the right hand sides (4.6) (or (4.7)) may be written as linear combinations of functions of the following forms:

$$\frac{G_+(x,y)}{1+(x+y)^2}, \quad \frac{G_-(x,y)}{1+(x-y)^2}, \quad \frac{H_+(x)(x+y)}{1+(x+y)^2}, \quad \frac{H_-(x)(x-y)}{1+(x-y)^2}, \tag{4.8}$$

and those with $H_\pm(x)$ being replaced by $H_\pm(y)$, where $G_\pm(x,y)$ and $H_\pm(x)$ are bounded in $x>0$ (resp. $x<0$). Once this is shown, then we are done because:

- If $M \in L^\infty(\mathbf{R})$, then integral operator $\displaystyle\int \frac{M(x,y)}{1+|x\pm y|^2} f(y)dy$ is bounded in $L^p(\mathbf{R}^1)$ for all $1 \leq p \leq \infty$.
- The multiplication operator by a bounded function and the reflection operator $f(x) \to f(-x)$ are continuous both in L^1 and in weak-L^1 spaces.
- The integral operator with integral kernel $(x-y)/(1+(x-y)^2)$ is of weak type $(1,1)$ because $(x-y)/(1+(x-y)^2)$ satisfies Hörmander condition:

$$\int_{|x-y|\geq 2\delta} \left| \frac{x-y}{(1+(x-y)^2)} - \frac{x-y'}{1+(x-y')^2} \right| dx \leq C, \quad |y-y'|<\delta,\ \delta>0.$$

As a prototype we show that the integral $K_3(x,y) = \int_0^\infty e^{ik(x+y)}\varphi_1(k^2)R_1(k)m_+(x,k)dk$ may be written as a linear combination of the functions of the forms in (4.8). The proof for other integrals in (4.6) and (4.7) are similar and are left to the readers. By integration by parts, we have

$$
\begin{aligned}
K_3(x,y) &= \int_0^\infty \left(\frac{1 - i(x+y)(\partial/\partial k)}{1+(x+y)^2} e^{ik(x+y)} \right) \varphi_1(k^2) R_1(k) m_+(x,k) dk \\
&= \frac{K_3(x,y)}{1+(x+y)^2} + \frac{i(x+y)}{1+(x+y)^2} \varphi_1(0) R_1(0) m_+(x,0) \\
&\quad + \frac{i(x+y)}{1+(x+y)^2} \int_0^\infty e^{ik(x+y)} (\partial/\partial k)(\varphi_1(k^2) R_1(k) m_+(x,k)) dk
\end{aligned}
$$

The first term $\dfrac{K_3(x,y)}{1+(x+y)^2}$ is of the type of the first function in (4.8) because $K_3(x,y)$ is bounded as noticed above; the second summand $\dfrac{i(x+y)}{1+(x+y)^2}\varphi_1(0)R_1(0)m_+(x,0)$ is of the type of third function of (4.8) because $m_+(x,0)$ is bounded in $x > 0$. We apply the integration by parts again to the last term in the right hand side and write it in the form

$$
\begin{aligned}
&\frac{-1}{1+(x+y)^2} \{(\partial/\partial k)(\varphi_1(k^2) R_1(k) m_+(x,k))\}|_{k=0} \\
&\qquad + \frac{-1}{1+(x+y)^2} \int_0^\infty e^{ik(x+y)} (\partial/\partial k)^2 \{\varphi_1(k^2) R_1(k) m_+(x,k)\} dk.
\end{aligned}
$$

It is easy to check by using Lemma 3.5 and Lemma 3.7 $\sim$ Lemma 3.10 that this is again of the form $G_+(x,y)(1+(x+y)^2)^{-1}$ with $G_+(x,y)$ bounded for $x > 0$. This completes the proof of Theorem 4.11 and hence of the main Theorem 1.2. □

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(Received May 10, 1999)

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