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Global Fuchsian Cauchy Problem

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Abstract. We show the unique solvability of a Fuchsian Cauchy problem in the space of entire functions. As immediate corollaries, we give some injectivity and bijectivity results for a class of partial differential operators on the space of entire functions of exponential type.

1. Introduction

First let us review some known results on a noncharacteristic Cauchy problem in which the coefficients of the operator and the data are all entire. The uniqueness of the solution is trivial and one tries to prove the existence of an entire solution.

As [2] showed by using a classical result due to Bieberbach, such a solution does not always exist; a solution exists near the initial surface but in some cases it is impossible to extend it to the whole space.

Several authors have given sufficient conditions on the operator for the existence of an entire solution. For example, [4] proved the following result (see also [5] for a proof based on majorant functions).

In $\mathbf{C}_t \times \mathbf{C}^n_x$, let us consider the Cauchy problem

$$(*)\left\{\begin{array}{rcl} D_t^m u(t,x) & = & \displaystyle\sum_{\ell=1}^{m}\sum_{|\beta|\le\ell} a_{\ell\beta}(t,x) D_t^{m-\ell} D_x^{\beta} u(t,x) + v(t,x), \\ D_t^{\lambda} u(0,x) & = & w_\lambda(x) \quad (0 \le \lambda \le m-1), \end{array}\right.$$

where $a_{\ell\beta}(t,x)$ is entire. Moreover, if $|\beta| = \ell$, it has the form $a_{\ell\beta}(t,x) = \sum_{|\gamma|\le\ell} a_{\ell\beta\gamma}(t)x^\gamma$ with entire functions $a_{\ell\beta\gamma}(t)$.

Theorem (Persson). *For any entire functions $v(t,x)$ and $w_\lambda(x)$, the Cauchy problem $(*)$ has a (unique) entire solution $u(t,x)$.*

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On the other hand, [1] studied a characteristic Cauchy problem for local holomorphic functions.

In the present paper we treat a characteristic Cauchy problem for entire functions. Moreover, in §5. we prove some injectivity and bijectivity results for a class of partial differential operators on the space of entire functions of exponential type.

2. Statement of Main Result

We will consider a Fuchsian Cauchy problem for entire functions in $\mathbf{C}_t \times \mathbf{C}^n_x$, $x = (x_1, \ldots, x_n)$.

Let $P = P(t, x, D_t, D_x)$ be a linear partial differential operator of order m with entire coefficients. Here $D_t = \partial/\partial t$ and $D_x = (D_1, \ldots, D_n) = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$. We assume that P has the following form:

$$\begin{aligned} P(t, x, D_t, D_x) \quad = \quad & t^{m'} D_t^m + a_{m-1} t^{m'-1} D_t^{m-1} + \cdots + a_{m-m'} D_t^{m-m'} \\ & + \sum_{j=0}^{m-1} \sum_{|\beta|=m-j} \sum_{k=\alpha(j)}^{\text{finite}} t^k a_{jk\beta}(x) D_t^j D_x^\beta \\ & + \sum_{j=0}^{m-1} \sum_{|\beta| \le m-j-1} t^{\alpha(j)} a_{j\beta}(t, x) D_t^j D_x^\beta, \end{aligned}$$

where $0 \le m' \le m$, $\alpha(j) = \max\{0, j - (m - m') + 1\}$, $D_x^\beta = D_1^{\beta_1} \cdots D_n^{\beta_n}$, and $|\beta| = \beta_1 + \cdots + \beta_n$. Here we assume that

- a_{m-j} $(j = 1, \ldots, m')$ is a constant,
- $a_{jk\beta}(x)$ is a polynomial in x of degree $\le m - j - 1$ (if $j = m - 1$, then $a_{jk\beta}$ is a constant),
- $a_{j\beta}(t, x)$ is entire in $\mathbf{C}_t \times \mathbf{C}^n_x$.

Obviously P is a Fuchsian operator of weight $m - m'$ in the sense of [1].

We call the ordinary differential operator

$$P_m(t, D_t) = t^{m'} D_t^m + a_{m-1} t^{m'-1} D_t^{m-1} + \cdots + a_{m-m'} D_t^{m-m'}$$

the Fuchsian principal part of P.

The characteristic polynomial associated with P is

$$\begin{aligned} C(\lambda) &= t^{-\lambda}t^{m-m'}P_m t^{\lambda} \\ &= \lambda(\lambda-1)\cdots(\lambda-m+1)+a_{m-1}\lambda(\lambda-1)\cdots(\lambda-m+2) \\ &\qquad +\cdots+a_{m-m'}\lambda(\lambda-1)\cdots(\lambda-m+m'+1) \end{aligned}$$

and its roots, called the characteristic roots, are denoted by

$$\lambda_1,\ldots,\lambda_{m'},\lambda_{m'+1}=0,\lambda_{m'+2}=1,\ldots,\lambda_m=m-m'-1.$$

Our main result is

THEOREM 1. *If $C(\lambda)\neq 0$ for any integer $\lambda\geq m-m'$, then for any entire function $f(t,x)$ in $\mathbf{C}_t\times\mathbf{C}_x^n$ and entire functions $f_\lambda(x)$ $(0\leq\lambda\leq m-m'-1)$ in $\mathbf{C}_x^n$, the following Cauchy problem has a unique entire solution $u(t,x)$:*

$$Pu=f,\qquad D_t^\lambda u(0,x)=f_\lambda(x)\quad(0\leq\lambda\leq m-m'-1). \tag{1}$$

REMARK. If $m=m'$, we impose no initial condition: the equation $Pu=f$ has a unique solution u.

3. Integral Operators

In this section we follow [1]. See also [9].

Assume that $m=m'$ and that $\operatorname{Re}\lambda_r<0$ for all r. If $g(t)$ is entire, then

$$\begin{aligned} v(t) &= H[g](t) \\ &= \int_{[0,1]^m} s_1^{-\lambda_1-1}\cdots s_m^{-\lambda_m-1}g(s_1\cdots s_m t)ds_1\cdots ds_m \end{aligned}$$

gives the unique entire solution to $P_m v=g$.

We see that there exists a positive constant C independent of g and t such that

$$|H[g](t)|\leq C\sup_{|\tau|\leq|t|}|g(\tau)|.$$

We will need other estimates on $H[g]$. Set $H_0[g]=g$ and

$$H_k[g](t)=\int_{[0,1]^k} s_1^{-\lambda_1-1}\cdots s_k^{-\lambda_k-1}g(s_1\cdots s_k t)ds_1\cdots ds_k$$

for $k = 1, 2, \ldots, m$. It is trivial that $H_m = H$ and that $|H_k[g](t)| \leq C_k \sup_{|\tau| \leq |t|} |g(\tau)|$, where C_k is a positive constant independent of g and t. By the repeated application of the formula $tD_t H_k[g] = \lambda_k H_k[g] + H_{k-1}[g]$ $(k = 1, 2, \ldots, m)$, we find that

$$|(tD_t)^j H[g](t)| \leq C' \sup_{|\tau| \leq |t|} |g(\tau)| \quad (j = 0, 1, \ldots, m)\,,$$

where C' is a positive constant independent of g and t.

Since $(D_t t)^j$ is a linear combination of $1, tD_t, \ldots, (tD_t)^j$, we obtain

PROPOSITION 1. *There exists a positive constant A independent of g and t such that*

$$|(D_t t)^j H[g](t)| \leq A \sup_{|\tau| \leq |t|} |g(\tau)| \quad (j = 0, 1, \ldots, m).$$

Set for $\ell \geq 1$,

$$G_\ell[g](t) = \int_{[0,1]^\ell} g(s_1 \cdots s_\ell t) ds_1 \cdots ds_\ell \quad \text{for} \quad g = g(t).$$

Then it is easy to see that

$$G_\ell = \underbrace{G_1 \circ \cdots \circ G_1}_{\ell\,\text{factors}}, \quad G_1 \circ D_t t = \text{identity}.$$

Hence $G_\ell \circ (D_t t)^\ell = \text{identity}, \quad (D_t t)^j = G_{m-j} \circ (D_t t)^m$. Moreover we have

LEMMA 1. *If $|g(t)| \leq \sum_{p=0}^{\bar{p}} C_p |t|^p$, where $\bar{p} \in \mathbf{N} = \{0, 1, 2, \ldots\}$ and $C_p \geq 0$ for all p, then we have*

$$|G_\ell[g](t)| \leq \sum_{p=0}^{\bar{p}} C_p \frac{|t|^p}{(p+1)^\ell}$$

PROOF. $|G_\ell[g](t)| \leq \sum_{p=0}^{\bar{p}} C_p \int_0^1 ds_1 \cdots \int_0^1 ds_\ell \left\{ s_1^p \cdots s_\ell^p |t|^p \right\}$. $\square$

4. Proof of Theorem 1

First we will prove Theorem 1 in the weight 0 case. In other words we assume that $m = m'$. In this case we impose no initial condition.

For $r > 0$ and $R > 1$, set

$$B(r) = \{t \in \mathbf{C}; |t| < r\},\ D(R) = \{x \in \mathbf{C}^n; \max_{j=1,\dots,n} |x_j| < R\}.$$

The distance from $x \in D(R)$ to the boundary of $D(R)$ is denoted by $d_R(x)$ and we have $d_R(x) = \min_{j=1,\dots,n}(R - |x_j|)$. It is easy to see that

$$d_R(x) \sim R \quad \text{as} \quad R \to \infty \tag{2}$$

uniformly on any bounded subset of $\mathbf{C}^n_x$.

When f and u are expressed in the form $f(t,x) = \sum_{\lambda=0}^{\infty} f_\lambda(x)t^\lambda$, $u(t,x) = \sum_{\lambda=0}^{\infty} u_\lambda(x)t^\lambda$, we have the following recurrence relation:

$$\begin{aligned} C(0)u_0(x) &= f_0(x) \\ C(\lambda)u_\lambda(x) &= f_\lambda(x) + \sum_{\nu=0}^{\lambda-1} P_\lambda^\nu(x, D_x)u_\nu(x), \quad \lambda = 1, 2, \dots, \end{aligned}$$

for some differential operator $P_\lambda^\nu(x, D_x)$. By the assumption $C(\lambda) \neq 0$, we can find a unique holomorphic function $u_\lambda(x)$ for all λ.

Next assume that $\mathrm{Re}\lambda_r < h \in \mathbf{Z}_+ = \{1, 2, \dots\}$ for $r = 1, \dots, m$. Obviously $Q = t^{-h}Pt^h$ is an m-th order Fuchsian operator of weight 0. We denote its characteristic roots by μ_r. Then we have $\mu_r = \lambda_r - h$ and $\mathrm{Re}\,\mu_r < 0$ for all r. Put $u(t,x) = \sum_{\lambda=0}^{h-1} u_\lambda(x)t^\lambda + t^h v(t,x)$. Then we have

$$P(t^h v) = f - P\left(\sum_{\lambda=0}^{h-1} u_\lambda(x)t^\lambda\right),$$

which is equivalent to the equation

$$Qv = t^{-h}\left\{f - P\left(\sum_{\lambda=0}^{h-1} u_\lambda(x)t^\lambda\right)\right\}.$$

The right hand side, which we denote by $g(t,x)$, is a known entire function. We have only to solve $Qv = g$. So we may assume from the beginning that all the characteristic roots of P have a negative real part.

We have

$$P_m - P = \sum_{j=0}^{m-1} \{\sum_{k=1}^{\bar{k}} t^k P_{jk}(x, D_x) + tS_j(t, x, D_x)\}(D_t t)^j.$$

Here $\bar{k}$ is a positive integer and

$$P_{jk}(x, D_x) = \sum_{|\alpha|=m-j} P_{jk\alpha}(x) D_x^\alpha,$$

where $P_{jk\alpha}(x)$ is a polynomial in x of degree $\leq m-j-1$. Notice that P_{jk} has no lower order terms. The operator S_j has entire coefficients, commutes with t and is of order $\leq m-j-1$.

Set $\max_{|\alpha|=m-j} \deg P_{jk\alpha}(x) = d_{jk} \leq m-j-1$ and $S = \{(j,k); P_{jk} \neq 0\}$. Obviously S is a finite set.

We are going to prove

THEOREM 2. *There exists a positive constant C independent of R such that $Pu = f$ has a unique holomorphic solution in the open set*

$$\Omega_R = \left\{ (t, x) \in B(R) \times D(R);\ t \in \bigcap_{(j,k)\in S} B(CR^{-d_{jk}/k} d_R(x)^{(m-j)/k}) \right\}.$$

By (2), any bounded subset of $\mathbf{C}_t \times \mathbf{C}_x^n$ is contained in Ω_R for a sufficiently large R. Therefore Theorem 1 (weight 0 case) is a direct consequence of Theorem 2.

Our equation $Pu = f$ is equivalent to the following differential-integral equation:

$$u = H\left[f + \sum_{j=0}^{m-1} \{\sum_{k=1}^{\bar{k}} t^k P_{jk}(x, D_x) + tS_j(t, x, D_x)\}(D_t t)^j u \right].$$

Here H is the integral operator introduced in §3. Define a sequence of functions $\{u_p(t,x)\}_p$ by

$$\begin{aligned} u_0 &= 0, \\ u_{p+1} &= H\left[f + \sum_{j=0}^{m-1} \{\sum_{k=1}^{\bar{k}} t^k P_{jk} + tS_j\}(D_t t)^j u_p \right] \quad \text{for} \quad p \geq 0. \end{aligned}$$

The solution u will be obtained as $u = \lim_p u_p$. We set $v_p = (D_t t)^m (u_{p+1} - u_p)$, which satisfies for $p \geq 0$,

$$v_{p+1} = (D_t t)^m H \left[\sum_{j=0}^{m-1} \{ \sum_{k=1}^{\bar{k}} t^k P_{jk}(x, D_x) + tS_j(t, x, D_x)\} G_{m-j}[v_p] \right].$$

To show the existence of the limit $u = \lim_p u_p$, it is enough to prove the convergence of $\sum_p v_p$.

Now we fix $R > 1$. Then we have

$$|f(t,x)| \leq \frac{C_{f,R}}{d_R(x)}, \quad |v_0(t,x)| \leq \frac{C'_{f,R}}{d_R(x)}$$

in $B(R) \times D(R)$ for some positive constants $C_{f,R}$ and $C'_{f,R}$. As a matter of fact, the functions f and v_0 are bounded in $B(R) \times D(R)$. We have introduced a negative power of $d_R(x)$ in order to employ Lemmas 3 and 4 below.

Lemma 2. *Assume that a holomorphic function $g(x)$ in $D(R)$ satisfies $|g(x)| \leq \sum_{\ell=1}^{\bar{\ell}} C_\ell d_R(x)^{-\ell}$, where $\bar{\ell} \in \mathbf{Z}_+$ and $C_\ell \geq 0$ for $\ell = 1, \ldots, \bar{\ell}$. Then we have*

$$|D_j g(x)| \leq e(\bar{\ell}+1) \sum_{\ell=1}^{\bar{\ell}} C_\ell d_R(x)^{-(\ell+1)}$$

for $x \in D(R), j = 1, 2, \ldots, n$.

Proof. We may assume that $j = 1$. We have

$$D_1 g(x) = \frac{1}{2\pi i} \oint_{\Gamma_x} \frac{g(y,x')}{(y-x_1)^2} dy, \tag{3}$$

where $x' = (x_2, \ldots, x_n)$ and $\Gamma_x \subset \mathbf{C}_y$ is a circle defined by $\Gamma_x = \{y \in \mathbf{C}_y; |y - x_1| = d_R(x)/(\bar{\ell}+1)\}$. If $y \in \Gamma_x$, then $(y, x') \in D(R)$ and

$$d_R(y,x') \geq d_R(x) - \frac{1}{\bar{\ell}+1} d_R(x) = \frac{\bar{\ell}}{\bar{\ell}+1} d_R(x),$$

from which we deduce

$$\begin{aligned} |g(y,x')| &\leq \sum_{\ell=1}^{\bar{\ell}} C_\ell \left(\frac{\bar{\ell}}{\bar{\ell}+1}\right)^{-\ell} d_R(x)^{-\ell} \\ &\leq e \sum_{\ell=1}^{\bar{\ell}} C_\ell d_R(x)^{-\ell}. \end{aligned} \tag{4}$$

By (3) and (4), we obtain the desired estimate. □

In the situation of Lemma 2, we have the following two lemmas:

LEMMA 3. *If* $|\alpha| = m - j$, *then*

$$|D^\alpha g(x)| \le e^{m-j}(\bar{\ell}+1)_{m-j} \sum_{\ell=1}^{\bar{\ell}} C_\ell d_R(x)^{-(\ell+m-j)}.$$

Here we employ the Pochhammer symbol: $(\lambda)_n = \lambda(\lambda+1)\cdots(\lambda+n-1)$.

PROOF. Use Lemma 2 repeatedly. □

LEMMA 4. *If* $|\alpha| \le m - j - 1$, *then*

$$|D^\alpha g(x)| \le (eR)^m (\bar{\ell}+1)_{m-j-1} d_R(x)^{-(m-j)} \sum_{\ell=1}^{\bar{\ell}} C_\ell d_R(x)^{-\ell}, \tag{5}$$

$$|D^\alpha g(x)| \le (eR)^{m-1} (\bar{\ell}+1)_{m-j-1} d_R(x)^{-(m-1)} \sum_{\ell=1}^{\bar{\ell}} C_\ell d_R(x)^{-\ell}. \tag{6}$$

PROOF. By using Lemma 2 and the inequality $d_R(x) \le R$, we have

$$\begin{aligned} |D^\alpha g(x)| &\le e^{|\alpha|}(\bar{\ell}+1)_{|\alpha|} d_R(x)^{-|\alpha|} \sum_{\ell=1}^{\bar{\ell}} C_\ell d_R(x)^{-\ell} \\ &\le e^{m-j-1}(\bar{\ell}+1)_{m-j-1} d_R(x)^{-|\alpha|+m-j} d_R(x)^{-(m-j)} \\ &\qquad \times \sum_{\ell=1}^{\bar{\ell}} C_\ell d_R(x)^{-\ell} \\ &\le e^m (\bar{\ell}+1)_{m-j-1} R^{-|\alpha|+m-j} d_R(x)^{-(m-j)} \sum_{\ell=1}^{\bar{\ell}} C_\ell d_R(x)^{-\ell}, \end{aligned}$$

from which we obtain (5). Similarly we can prove (6) in the following way:

$$|D^\alpha g(x)| \le e^{|\alpha|}(\bar{\ell}+1)_{|\alpha|} d_R(x)^{-|\alpha|} \sum_{\ell=1}^{\bar{\ell}} C_\ell d_R(x)^{-\ell}$$

$$\begin{aligned}
&\le e^{m-j-1}(\bar{\ell}+1)_{m-j-1}d_R(x)^{-|\alpha|+m-1}d_R(x)^{-(m-1)} \\
&\quad \times \sum_{\ell=1}^{\bar{\ell}} C_\ell d_R(x)^{-\ell} \\
&\le e^{m-1}(\bar{\ell}+1)_{m-j-1}R^{-|\alpha|+m-1}d_R(x)^{-(m-1)}\sum_{\ell=1}^{\bar{\ell}} C_\ell d_R(x)^{-\ell}. \ \square
\end{aligned}$$

Proposition 2. *There exists a constant $C_R > 0$ such that*

$$|v_p(t,x)| \le C_R^{p+1}\frac{|t|^p}{d_R(x)^{mp+1}} \tag{7}$$

for $(t,x) \in B(R) \times D(R)$ and $p \ge 0$.

Proof. The case $p = 0$ has already been proved. Assume that (7) is true for p. Then by Lemma 1 we have

$$|G_{m-j}[v_p](t,x)| \le \frac{C_R^{p+1}}{(p+1)^{m-j}}\frac{|t|^p}{d_R(x)^{mp+1}} \tag{8}$$

for $(t,x) \in B(R) \times D(R)$.

We can write

$$\sum_{k=1}^{\bar{k}} t^k P_{jk} + tS_j = t\sum_{|\alpha|\le m-j} a_{j\alpha}(t,x)D_x^\alpha,$$

where $a_{j\alpha}(t,x)$ is entire. Put $C_{Rj\alpha} = \sup_{B(R)\times D(R)} |a_{j\alpha}(t,x)|$, then Lemma 3, (8) and (5) imply that

$$|a_{j\alpha}D_x^\alpha G_{m-j}[v_p]| \le C_{Rj\alpha}e^{m-j}(mp+2)_{m-j}\frac{C_R^{p+1}}{(p+1)^{m-j}}\frac{|t|^p}{d_R(x)^{mp+m-j+1}}$$

if $|\alpha| = m-j$, and that

$$|a_{j\alpha}D_x^\alpha G_{m-j}[v_p]| \le C_{Rj\alpha}(eR)^m(mp+2)_{m-j}\frac{C_R^{p+1}}{(p+1)^{m-j}}\frac{|t|^p}{d_R(x)^{mp+m-j+1}}$$

if $|\alpha| \le m-j-1$.

Set

$$C'_R = \max_j \left\{ e^{m-j} \sum_{|\alpha|=m-j} C_{Rj\alpha} + (eR)^m \sum_{|\alpha|\le m-j-1} C_{Rj\alpha} \right\}.$$

It is independent of p.

We have for all j,

$$\begin{aligned}\left|(\sum_{k=1}^{\bar{k}} t^k P_{jk} + tS_j)G_{m-j}[v_p]\right| &\le C'_R C_R^{p+1} \frac{(mp+2)_{m-j}}{(p+1)^{m-j}} \frac{|t|^{p+1}}{d_R(x)^{mp+m-j+1}} \\ &\le C'_R C_R^{p+1}(m+1)^m R^{m-1} \frac{|t|^{p+1}}{d_R(x)^{m(p+1)+1}}.\end{aligned}$$

Induction proceeds because of Proposition 1, if C_R is sufficiently large. The proof of Proposition 2 is now complete. □

The proposition above is proved in [1] in a slightly different formulation. It is good enough to prove local existence, but it is not sufficient to show our global result because of the dependence of C_R on R. So we give the following Lemma 5 and Proposition 3.

LEMMA 5. *Let I be a finite subset of $\{(a,b) \in \mathbf{Z}_+^2; a \ge p,\ b \le mp\}$ for $p \in \mathbf{Z}_+$. Assume that*

$$|w(t,x)| \le \sum_{(a,b)\in I} C_{ab} \frac{|t|^a}{d_R(x)^{b+1}}$$

holds for $(t,x) \in B(R) \times D(R)$. Then there exist positive constants $C_{jk}^{(1)}$, independent of R and p, and $C_{j,R}^{(2)}$, independent of p (but dependent on R), such that

$$\tag{9} \begin{aligned}&|t^k P_{jk}(x, D_x) G_{m-j}[w](t,x)| \\ &\le C_{jk}^{(1)} \frac{R^{d_{jk}}|t|^k}{d_R(x)^{m-j}} \sum_{(a,b)\in I} C_{ab} \frac{|t|^a}{d_R(x)^{b+1}},\end{aligned}$$

$$\tag{10} \begin{aligned}&|tS_j(t,x,D_x) G_{m-j}[w](t,x)| \\ &\le C_{j,R}^{(2)} \frac{1}{p} \frac{|t|}{d_R(x)^{m-1}} \sum_{(a,b)\in I} C_{ab} \frac{|t|^a}{d_R(x)^{b+1}},\end{aligned}$$

for all $(t,x) \in B(R) \times D(R)$.

PROOF. By Lemmas 1 and 3, we have, if $|\alpha| = m-j$,

$$\begin{aligned} |D_x^\alpha G_{m-j}[w](t,x)| &\le \sum_{(a,b)\in I} e^{m-j} \frac{(mp+2)_{m-j}}{(a+1)^{m-j}} C_{ab} \frac{|t|^a}{d_R(x)^{m-j+b+1}} \\ &\le e^{m-j} \frac{(mp+2)_{m-j}}{(p+1)^{m-j}} \sum_{(a,b)\in I} C_{ab} \frac{|t|^a}{d_R(x)^{m-j+b+1}}. \end{aligned}$$

Recall that $P_{jk}(x,D_x) = \sum_{|\alpha|=m-j} P_{jk\alpha}(x) D_x^\alpha$ and that $\deg P_{jk\alpha} \le d_{jk}$. There exists a constant $C(P_{jk}) > 0$ such that $\sum_{|\alpha|=m-j} \sup_{D(R)} |P_{jk\alpha}(x)| \le C(P_{jk}) R^{d_{jk}}$ for all $R > 1$. Therefore we have

$$\begin{aligned} &|P_{jk}(x,D_x) G_{m-j}[w](t,x)| \\ \le\ & C(P_{jk}) R^{d_{jk}} e^{m-j} \frac{(mp+2)_{m-j}}{(p+1)^{m-j}} \sum_{(a,b)\in I} C_{ab} \frac{|t|^a}{d_R(x)^{m-j+b+1}}. \end{aligned}$$

Then (9) follows from the fact that $(mp+2)_{m-j}/(p+1)^{m-j} \le (m+1)^m$. Next, by Lemma 1 and (6), we have, if $|\alpha| \le m-j-1$,

$$\begin{aligned} &|D_x^\alpha G_{m-j}[w](t,x)| \\ \le\ & \sum_{(a,b)\in I} (eR)^{m-1} \frac{(mp+2)_{m-j-1}}{(a+1)^{m-j}} C_{ab} \frac{|t|^a}{d_R(x)^{(m-1)+(b+1)}} \\ \le\ & (eR)^{m-1} \frac{(mp+2)_{m-j-1}}{(p+1)^{m-j}} \sum_{(a,b)\in I} C_{ab} \frac{|t|^a}{d_R(x)^{(m-1)+(b+1)}}. \end{aligned}$$

The operator $S_j(t,x,D_x)$ can be expressed as

$$S_j(t,x,D_x) = \sum_{|\alpha|\le m-j-1} S_{j\alpha}(t,x) D_x^\alpha,$$

where $S_{j\alpha}(t,x)$ is entire. Set $C(S_j,R) = \sum_{|\alpha|\le m-j-1} \sup_{B(R)\times D(R)} |S_{j\alpha}(t,x)|$. Then

$$\begin{aligned} &|S_j(t,x,D_x) G_{m-j}[w](t,x)| \\ \le\ & C(S_j,R)(eR)^{m-1} \frac{(mp+2)_{m-j-1}}{(p+1)^{m-j}} \sum_{(a,b)\in I} C_{ab} \frac{|t|^a}{d_R(x)^{(m-1)+(b+1)}}. \end{aligned}$$

The estimate (10) is a consequence of the inequality

$$\frac{(mp+2)_{m-j-1}}{(p+1)^{m-j}} \le \frac{1}{p+1} \cdot \frac{(mp+2)_{m-j-1}}{(p+1)^{m-j-1}} \le \frac{1}{p} \cdot (m+1)^{m-1}. \ \square$$

Put $C_S = A \max_{(j,k)\in S} C^{(1)}_{jk}$, $C^{(2)}_R = A \sum_{j=0}^{m-1} C^{(2)}_{j,R}$, where A is the one in Proposition 1. Notice that C_S is independent of R.

Fix temporarily a positive integer $q \ge 1$. We have

PROPOSITION 3. *If $p \ge q$, we have*

$$|v_p(t,x)| \le \left(C_S \sum_{(j,k)\in S} \frac{R^{d_{jk}}|t|^k}{d_R(x)^{m-j}} + \frac{C^{(2)}_R}{q} \frac{|t|}{d_R(x)^{m-1}} \right)^{p-q} \cdot C_R^{q+1} \frac{|t|^q}{d_R(x)^{mq+1}}$$

for $(t,x) \in B(R) \times D(R)$.

PROOF. The case $p = q$ is included in (7).

Assume that the estimate holds for p. It can be written in the following form:

$$|v_p(t,x)| \le \sum_{(a,b)\in I_p} C^{\{p\}}_{ab} \frac{|t|^a}{d_R(x)^{b+1}},$$

where I_p is a finite subset of $\{(a,b) \in \mathbf{Z}^2_+; a \ge p,\, b \le mp\}$ and $C^{\{p\}}_{ab}$ is a positive constant. By using Lemma 5 and Proposition 1, we obtain

$$\begin{aligned} &|v_{p+1}(t,x)| \\ \le\ & A \left(\sum_{(j,k)\in S} C^{(1)}_{jk} \frac{R^{d_{jk}}|t|^k}{d_R(x)^{m-j}} + \sum_{j=0}^{m-1} C^{(2)}_{j,R} \frac{1}{p} \frac{|t|}{d_R(x)^{m-1}} \right) \\ & \times \sum_{(a,b)\in I_p} C^{\{p\}}_{ab} \frac{|t|^a}{d_R(x)^{b+1}} \\ \le\ & \left(C_S \sum_{(j,k)\in S} \frac{R^{d_{jk}}|t|^k}{d_R(x)^{m-j}} + \frac{C^{(2)}_R}{q} \frac{|t|}{d_R(x)^{m-1}} \right) \sum_{(a,b)\in I_p} C^{\{p\}}_{ab} \frac{|t|^a}{d_R(x)^{b+1}}. \end{aligned}$$

Hence induction proceeds. $\square$

Set $C = \min\{1, (3\,C_S\,\mathrm{card}S)^{-1}\}$. It is independent of q and R.

If $t \in \bigcap_{(j,k)\in S} B(CR^{-d_{jk}/k} d_R(x)^{(m-j)/k})$, then $R^{d_{jk}}|t|^k/d_R(x)^{m-j} < C$ for all $(j,k) \in S$ and

$$C_S \sum_{(j,k)\in S} \frac{R^{d_{jk}}|t|^k}{d_R(x)^{m-j}} < \frac{1}{3}. \tag{11}$$

For $q = 1, 2, \dots$, let $\Omega_{R,q} \subset \mathbf{C}_t \times \mathbf{C}_x^n$ be the intersection of

$$\Omega_R = \left\{ (t,x) \in B(R) \times D(R);\, t \in \bigcap_{(j,k)\in S} B(CR^{-d_{jk}/k} d_R(x)^{(m-j)/k}) \right\}$$

and

$$\omega_{R,q} = \left\{ (t,x) \in B(R) \times D(R);\, |t| < \frac{q}{3C_R^{(2)}} d_R(x)^{m-1} \right\}.$$

Then we have

Proposition 4. *The series $\sum_{p\geq 0} v_p$ is convergent in $\Omega_{R,q}$.*

Proof. We have only to prove the convergence of $\sum_{p\geq q} v_p$. In $\Omega_{R,q}$, we have

$$C_S \sum_{(j,k)\in S} \frac{R^{d_{jk}}|t|^k}{d_R(x)^{m-j}} + \frac{C_R^{(2)}}{q}\frac{|t|}{d_R(x)^{m-1}} < \frac{2}{3}.$$

By using Proposition 3, we obtain

$$|v_p| \leq \left(\frac{2}{3}\right)^{p-q} C_R^{q+1} \frac{|t|^q}{d_R(x)^{mq+1}}, \qquad p \geq q.$$

The convergence of $\sum_{p\geq q} v_p$ follows immediately. $\square$

Since $\bigcup_{q=1}^{\infty} \omega_{R,q} = B(R)\times D(R)$, we have $\bigcup_{q=1}^{\infty} \Omega_{R,q} = \Omega_R$. So Theorem 2 is a consequence of Proposition 4.

Last of all, let us prove Theorem 1 in the case of positive weight. We look for u in the form

$$u(t,x) = \sum_{\lambda=0}^{m-m'-1} \frac{1}{\lambda!} f_\lambda(x) t^\lambda + t^{m-m'} v(t,x).$$

The initial condition is satisfied. The equation $Pu = f$ is equivalent to

$$Pt^{m-m'}v = f - P\left\{\sum_{\lambda=0}^{m-m'-1} \frac{1}{\lambda!} f_\lambda(x)t^\lambda\right\},$$

of which the right hand side is a known entire function. It is easy to see that v can be obtained by applying the result of the weight 0 case to $Pt^{m-m'}$.

5. Entire Functions of Exponential Type

In this section we investigate the action of a class of partial differential operators on entire functions of exponential type. The operators to be studied are the Laplace transform of some Fuchsian operators.

Let $\mathcal{H}(\mathbf{C}^{n+1})$ be the space of entire functions in $\mathbf{C}^{n+1} = \mathbf{C}_t \times \mathbf{C}^n_x$. Its dual, the space of analytic functionals, is denoted by $\mathcal{H}'(\mathbf{C}^{n+1})$. An element $f(t,x)$ of $\mathcal{H}(\mathbf{C}^{n+1})$ is said to be of exponential type if and only if there exists a positive constant C such that for all $(t,x) \in \mathbf{C}^{n+1}$,

$$|f(t,x)| \le C \exp(C|(t,x)|), \quad |(t,x)| = (|t|^2 + \textstyle\sum_{j=1}^n |x_j|^2)^{1/2}.$$

We denote the space of such functions by $\mathrm{Exp}(\mathbf{C}^{n+1})$. It is known (see [3] for example) that the Laplace transformation maps $\mathcal{H}'(\mathbf{C}^{n+1})$ bijectively onto $\mathrm{Exp}(\mathbf{C}^{n+1})$.

Consider the following partial differential operator $Q(t,x,D_t,D_x)$ with polynomial coefficients:

$$\begin{aligned} Q(t,x,D_t,D_x) \quad = \quad & t^m D_t^{m'} + a_{m-1}t^{m-1}D_t^{m'-1} + \cdots + a_{m-m'}t^{m-m'} \\ & + \sum_{j=0}^{m-1} \sum_{|\beta|=m-j} \sum_{k=\alpha(j)}^{\text{finite}} t^j x^\beta D_t^k a_{jk\beta}(D_x) \\ & + \sum_{j=0}^{m-1} \sum_{|\beta|\le m-j-1} t^j x^\beta D_t^{\alpha(j)} a_{j\beta}(D_t,D_x). \end{aligned}$$

Here $0 \le m' \le m$, $\alpha(j) = \max\{0, j-(m-m')+1\}$ and

- $a_{m-j}\,(j = 1,2,\ldots,m')$ is a constant,
- $a_{jk\beta}(\xi)$ is a polynomial in $\xi = (\xi_1,\ldots,\xi_n)$ of degree $\le m-j-1$ (if $j = m-1$, then $a_{jk\beta}$ is a constant),

- $a_{j\beta}(\tau, \xi)$ is a polynomial in τ and ξ of arbitrary degree.

Moreover we assume that for any integer $\lambda \geq m - m'$,

$$\lambda(\lambda-1)\cdots(\lambda-m+1) + a_{m-1}\lambda(\lambda-1)\cdots(\lambda-m+2) \\ +\cdots + a_{m-m'}\lambda(\lambda-1)\cdots(\lambda-m+m'+1) \neq 0.$$

In this situation we have

THEOREM 3. (i) *The operator Q is injective on* $\mathrm{Exp}(\mathbf{C}^{n+1})$.
(ii) *If $m = m'$, then Q maps* $\mathrm{Exp}(\mathbf{C}^{n+1})$ *bijectively onto itself.*

PROOF. Consider the following Fuchsian partial differential operator P of weight $m - m'$ with polynomial coefficients:

$$\begin{aligned} P(t, x, D_t, D_x) \quad &= \quad t^{m'} D_t^m + a_{m-1} t^{m'-1} D_t^{m-1} + \cdots + a_{m-m'} D_t^{m-m'} \\ &\quad + \sum_{j=0}^{m-1} \sum_{|\beta|=m-j} \sum_{k=\alpha(j)}^{\text{finite}} t^k a_{jk\beta}(x) D_t^j D_x^\beta \\ &\quad + \sum_{j=0}^{m-1} \sum_{|\beta|\leq m-j-1} t^{\alpha(j)} a_{j\beta}(t,x) D_t^j D_x^\beta. \end{aligned}$$

By Theorem 1 it induces a continuous surjection from $\mathcal{H}(\mathbf{C}^{n+1})$ onto itself and its transpose ${}^tP : \mathcal{H}'(\mathbf{C}^{n+1}) \to \mathcal{H}'(\mathbf{C}^{n+1})$ is an injection. By using the Laplace transformation we get (i).

If $m = m'$, then P induces a topological automorphism of $\mathcal{H}(\mathbf{C}^{n+1})$ and ${}^tP$ is a bijection. □

Example. Let us consider the operator $Q = D_t^j + D_t^k D_x^\ell + D_t^{m'} t^m$ in $\mathbf{C}^2$. (It is the Laplace transform of the transpose of $t^j + t^k x^\ell + D_t^m t^{m'}$.) We assume that $0 \leq m' \leq m$, $k \geq 1$.

By Theorem 3, the operator Q is injective on $\mathrm{Exp}(\mathbf{C}^2)$. If $m = m'$, it maps $\mathrm{Exp}(\mathbf{C}^2)$ bijectively onto itself.

On the other hand, it is known that Q is *not* injective on $\mathcal{H}(\mathbf{C}^2)$ if $j > \max(k + \ell, m')$. More precisely, the Cauchy problem for Q with the noncharacteristic initial hypersurface $t = 0$ has a unique solution in $\mathcal{H}(\mathbf{C}^2)$; see [4] Theorem 3 or [5] Théorème 3.1 for the precise statement.

The injectivity of Q on $\mathrm{Exp}(\mathbf{C}^2)$ implies that the (noncharacteristic) Cauchy problem for Q does not always have a solution in that space. However, it has a unique solution in the space of functions of exponential type of higher order. This fact is included in [5] Théorème 2.5. See also [6] Corollary 4 for a result on operators with constant coefficients.

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