

A Limit Theorem for Weyl Transformation in Infinite-Dimensional Torus and Central Limit Theorem for Correlated Multiple Wiener Integrals

By Hiroshi SUGITA and Satoshi TAKANOBU

Abstract. We show that under many of the probabilities on \mathbb{T}^∞ , infinite-dimensional torus, a random system $(1/\sqrt{N} \sum_{i=1}^N f(x_i + p\alpha_i))$ converges to a centered Gaussian system whose covariance is determined only by the distribution of $(\alpha_i)_{i=1}^\infty$ over \mathbb{T} . Moreover we show the convergence of a system of symmetric statistics to that of correlated multiple Wiener integrals defined by the Gaussian system.

Also we study the central limit theorem for a sequence of the correlated multiple Wiener integrals.

0. Introduction and Notation

We study a limit theorem for a random system of Weyl transformation in infinite-dimensional torus \mathbb{T}^∞ .

In the previous paper [16], taking infinite-dimensional Lebesgue probability \mathbf{P}^∞ as underlying probability on \mathbb{T}^∞ , we showed that on $(\mathbb{T}^\infty, \mathbf{P}^\infty)$

$$(0.1) \quad \left(\frac{1}{\sqrt{N}} \sum_{i=1}^N f(x_i + p\alpha_i) \right)_{p \in \mathbb{Z}, f \in \mathcal{CL}_2} \xrightarrow{\text{f.d.}} \left(I^{(p)}(f) \right)_{p \in \mathbb{Z}, f \in \mathcal{CL}_2} \quad \text{as } N \rightarrow \infty,$$

if and only if $\alpha = (\alpha_1, \alpha_2, \dots) \in \mathbb{T}_{dx}^\infty$, i.e., α is uniformly distributed over \mathbb{T} . (We expressed this phenomenon by saying that the disappearance of dependency happens.) Here \mathcal{CL}_2 is the totality of square integrable real functions on \mathbb{T} with $\int_{\mathbb{T}} f(x) dx = 0$, “ $\xrightarrow{\text{f.d.}}$ ” means the convergence in finite dimensional distribution, $I^{(p)}(f)$ is the Wiener integral of f with respect to $B^{(p)}$ and $\{B^{(p)} = (B^{(p)}(t))_{0 \leq t \leq 1}\}_{p \in \mathbb{Z}}$ is a sequence of independent 1-dimensional Brownian motions starting at 0. Moreover this was generalized

1991 *Mathematics Subject Classification.* Primary 60F05.

to the convergence of a system of symmetric statistics to that of multiple Wiener integrals, i.e.,

$$(0.2) \quad \left(\left(\frac{1}{\sqrt{N}} \right)^n \sum_{1 \leq i_1 < \dots < i_n \leq N} h(x_{i_1} + p\alpha_{i_1}, \dots, x_{i_n} + p\alpha_{i_n}) \right)_{p \in \mathbb{Z}, h \in \mathcal{CSL}_2^n} \\ \xrightarrow{\text{f.d.}} \left(\frac{1}{n!} I_n^{(p)}(h) \right)_{p \in \mathbb{Z}, h \in \mathcal{CSL}_2^n} \quad \text{as } N \rightarrow \infty.$$

Here $I_n^{(p)}(h)$ is the n -ple Wiener integral of $h \in \mathcal{CSL}_2^n$ (for which see (0.7) below) with respect to $B^{(p)}$.

In the present paper we take, as underlying probability, product probability and generally absolutely continuous one relative to it. When a product probability μ is mostly dominated by \mathbf{P}^∞ , i.e., the singular part of μ with respect to \mathbf{P}^∞ is small in a certain sense, we show that on any absolutely continuous probability space relative to μ ,

$$(0.3) \quad \left(\frac{1}{\sqrt{N}} \sum_{i=1}^N f(x_i + p\alpha_i) \right)_{p \in \mathbb{Z}, f \in C(\mathbb{T}) \cap \mathcal{CL}_2} \\ \xrightarrow{\text{f.d.}} \left(I^{(p)}(f) \right)_{p \in \mathbb{Z}, f \in C(\mathbb{T}) \cap \mathcal{CL}_2} \quad \text{as } N \rightarrow \infty,$$

if and only if $\alpha \in \mathbb{T}_{dx}^\infty$, and that for general $\alpha \in \mathbb{T}^\infty$, chosen a subsequence $\{N_m\}_{m=1}^\infty$ of $\{1, 2, 3, \dots\}$ and a probability μ on \mathbb{T} so that $\alpha \in \mathbb{T}_{\mu, \{N_m\}}^\infty$ (for which see (0.8) below), the convergence (0.3) holds by replacing $\{N\}$ and $(I^{(p)}(f))$ by $\{N_m\}$ and $(I^{(p)}(f; \mu))$, respectively, where $(I^{(p)}(f; \mu))$ is a centered, correlated Gaussian system (cf. (1.1)). For details, see Theorem 1.2. For this reason one may say that under many of the probabilities on \mathbb{T}^∞ a random system $(\frac{1}{\sqrt{N}} \sum_{i=1}^N f(x_i + p\alpha_i))_{p \in \mathbb{Z}, f \in C(\mathbb{T}) \cap \mathcal{CL}_2}$ converges to a centered Gaussian system whose covariance is determined only by the distribution of $(\alpha_1, \alpha_2, \dots)$ over \mathbb{T} .

We further improve the convergence (0.2) in the following way: In case the μ above satisfies some additional conditions and $\alpha \in \mathbb{T}_{\mu, \{N_m\}}^\infty$, on any absolutely continuous probability space relative to μ

$$(0.4) \quad \left(\left(\frac{1}{\sqrt{N_m}} \right)^n \sum_{1 \leq i_1 < \dots < i_n \leq N_m} h(x_{i_1} + p\alpha_{i_1}, \dots, x_{i_n} + p\alpha_{i_n}) \right)_{p \in \mathbb{Z}, h \in \mathcal{CSL}_2^n} \\ \xrightarrow{\text{f.d.}} \left(\frac{1}{n!} I_n^{(p)}(h; \mu) \right)_{p \in \mathbb{Z}, h \in \mathcal{CSL}_2^n} \quad \text{as } m \rightarrow \infty,$$

where $I_n^{(p)}(\cdot; \mu)$ is the n -ple multiple Wiener integral with respect to $I^{(p)}(\cdot; \mu)$. See Theorem 2.1.

Next we study the central limit theorem (abbr. CLT) for a sequence $\{I_n^{(p)}(\cdot; \mu)\}_{p=1}^\infty$. For general $\mu(dx)$, the sequence is correlated in p , but when $\mu(dx)$ is the Lebesgue measure dx , it is independent. So it is expected that if $\mu(dx)$ is not very far from dx in some sense, then the dependence of $\{I_n^{(p)}(\cdot; \mu)\}_{p=1}^\infty$ will be not very large, so that the CLT for $\{I_n^{(p)}(\cdot; \mu)\}_{p=1}^\infty$ must hold. Indeed, when $\mu(dx)$ is absolutely continuous relative to dx , and its density is continuous and positive, we show that for any non-zero $h \in \mathcal{CSL}_2^n$, $\frac{1}{\sqrt{P}} \sum_{p=1}^P I_n^{(p)}(h; \mu)$ converges to a nondegenerate Gaussian as $P \rightarrow \infty$. See Theorem 3.3.

In Section 4, combining this with the convergence (0.4), we present a justification of the claim of Sobol' et al, by which our study in [16] was inspired. In the last section, related topics on the disappearance of dependency are introduced.

Let us here explain the notation used in this paper.

Let $\mathbb{T} \cong [0, 1)$ be the 1-dimensional torus and $\mathbf{P}(dx) = dx$ the Lebesgue measure on it. For $m = 1, 2, \dots, \infty$ let \mathbb{T}^m be the m -dimensional torus, i.e., $\mathbb{T}^m = \underbrace{\mathbb{T} \times \dots \times \mathbb{T}}_m$. Let \mathcal{F} be a σ -algebra on \mathbb{T}^∞ generated by cylindrical sets and \mathbf{P}^∞ the Lebesgue measure on $(\mathbb{T}^\infty, \mathcal{F})$, i.e., $\mathbf{P}^\infty(dx_1 dx_2 \dots) = \prod_{i=1}^\infty dx_i$. The addition and scalar multiplication on \mathbb{T}^m are always considered coordinatewise in the sense of modulo 1.

Let

$$(0.5) \quad \mathcal{L}_2^n = L_2(\mathbb{T}^n; dx_1 \cdots dx_n),$$

$$(0.6) \quad \mathcal{SL}_2^n = \left\{ h \in \mathcal{L}_2^n; \begin{array}{l} h \text{ is symmetric in each two variables, i.e.,} \\ h(\dots, \overset{i}{x}, \dots, \overset{j}{y}, \dots) = h(\dots, \overset{i}{y}, \dots, \overset{j}{x}, \dots) \end{array} \right\},$$

$$(0.7) \quad \mathcal{CSL}_2^n = \left\{ h \in \mathcal{SL}_2^n; \int_{\mathbb{T}} h(x_1, \dots, x_{n-1}, y) dy = 0 \right. \\ \left. \text{a.a. } (x_1, \dots, x_{n-1}) \in \mathbb{T}^{n-1} \right\}.$$

When $n = 1$, $\mathcal{L}_2^1 = \mathcal{SL}_2^1$ and \mathcal{CSL}_2^1 are simply written as \mathcal{L}_2 and \mathcal{CL}_2 ,

respectively. The norm and inner product on \mathcal{L}_2^n are denoted by $\|\cdot\|_{\mathcal{L}_2^n}$ and $(\cdot, *)_{\mathcal{L}_2^n}$, respectively. In some cases, they may be written in dropping the subscript \mathcal{L}_2^n . For $f_1, \dots, f_n \in \mathcal{L}_2$, $f_1 \otimes \dots \otimes f_n \in \mathcal{L}_2^n$ is defined by

$$f_1 \otimes \dots \otimes f_n(x_1, \dots, x_n) := f_1(x_1) \times \dots \times f_n(x_n).$$

If $f_1 = \dots = f_n = f$, this is denoted by $f^{\otimes n}$. Note that $f^{\otimes n} \in \mathcal{CSL}_2^n$ if $f \in \mathcal{CL}_2$.

Functions on \mathbb{T} are identified with 1-periodic functions on \mathbb{R} in an obvious way. By this identification a continuous function on \mathbb{T} is regarded as a continuous 1-periodic function on \mathbb{R} . Let $C(\mathbb{T})$ be the space of all such functions with the supremum norm $\|\cdot\|_\infty$.

For $f \in L_1(\mathbb{T}; dx)$ and generally a finite measure $\nu(dx)$ on \mathbb{T} , $\widehat{f}(n)$ and $\widehat{\nu}(n)$ are the n -th Fourier coefficients of f and ν , respectively, i.e., $\widehat{f}(n) = \int_{\mathbb{T}} f(x) e^{-\sqrt{-1} 2n\pi x} dx$ and $\widehat{\nu}(n) = \int_{\mathbb{T}} e^{-\sqrt{-1} 2n\pi x} \nu(dx)$. For a probability measure μ on \mathbb{T} and a subsequence $\{N_m\}_{m=1}^\infty$ of $\{1, 2, 3, \dots\}$, let

$$(0.8) \quad \mathbb{T}_{\mu, \{N_m\}}^\infty = \left\{ \begin{array}{l} \alpha = (\alpha_1, \alpha_2, \dots) \in \mathbb{T}^\infty; \\ \lim_{m \rightarrow \infty} \frac{1}{N_m} \sum_{i=1}^{N_m} \delta_{\alpha_i}(dx) = \mu(dx) \text{ weakly} \end{array} \right\}.$$

If $N_m = m$, this is simply written as \mathbb{T}_μ^∞ .

1. Convergence to a Centered Gaussian System

In this section underlying probability space $(\mathbb{T}^\infty, \mathcal{F}, \mu)$ is a product probability space, i.e., for a sequence $\{\mathbf{P}_i\}_{i=1}^\infty$ of probability measures on \mathbb{T} , $\mu = \prod_{i=1}^\infty \mathbf{P}_i$. Set $\mathcal{F}_n := \sigma(X_1, \dots, X_n)$, where X_i is the coordinate function, i.e., $X_i(x) = x_i, x = (x_j)_{j=1}^\infty \in \mathbb{T}^\infty$. Note that $\mathcal{F} = \sigma(\bigcup_n \mathcal{F}_n)$. For a probability measure μ on \mathbb{T} let $(I^{(p)}(f; \mu))_{p \in \mathbb{Z}, f \in \mathcal{CL}_2}$ be a centered Gaussian system on a probability space (Ω, \mathcal{F}, P) such that

$$(1.1) \quad E \left[I^{(p)}(f; \mu) I^{(q)}(g; \mu) \right] = \sum_{|i| \geq 1} \widehat{f}(i) \overline{\widehat{g}(i)} \widehat{\mu}(i(q-p)),$$

$$\forall f, g \in \mathcal{CL}_2, \forall p, q \in \mathbb{Z}.$$

THEOREM 1.1. Let $\alpha \in \mathbb{T}^\infty$ and let $\{N_m\}_{m=1}^\infty$ be a subsequence of $\{1, 2, 3, \dots\}$. The following (A) and (B) are equivalent:

- (A) $\left(\frac{1}{\sqrt{N_m}} \sum_{i=1}^{N_m} f(x_i + p\alpha_i) \right)_{p \in \mathbb{Z}, f \in C(\mathbb{T}) \cap \mathcal{CL}_2}$ under μ
 $\xrightarrow{\text{f.d.}} \left(I^{(p)}(f; \mu) \right)_{p \in \mathbb{Z}, f \in C(\mathbb{T}) \cap \mathcal{CL}_2}$ as $m \rightarrow \infty$.
- (B) (B.1) $\lim_{m \rightarrow \infty} \frac{1}{N_m} \sum_{i=1}^{N_m} \left(\int_{\mathbb{T}} f(x + p\alpha_i) \mathbf{P}_i(dx) \right)^2 = 0,$
 $\forall f \in C(\mathbb{T}) \cap \mathcal{CL}_2, \forall p \in \mathbb{Z}$
- (B.2) $\lim_{m \rightarrow \infty} \frac{1}{\sqrt{N_m}} \sum_{i=1}^{N_m} \int_{\mathbb{T}} f(x + p\alpha_i) \mathbf{P}_i(dx) = 0,$
 $\forall f \in C(\mathbb{T}) \cap \mathcal{CL}_2, \forall p \in \mathbb{Z}$
- (B.3) $\lim_{m \rightarrow \infty} \frac{1}{N_m} \sum_{i=1}^{N_m} \int_{\mathbb{T}} f(x + p\alpha_i) g(x + q\alpha_i) \mathbf{P}_i(dx)$
 $= E \left[I^{(p)}(f; \mu) I^{(q)}(g; \mu) \right],$
 $\forall f, g \in C(\mathbb{T}) \cap \mathcal{CL}_2, \forall p, q \in \mathbb{Z}; p \neq q.$

To prove the theorem we present a lemma:

LEMMA 1.1. Let $\alpha \in \mathbb{T}^\infty$. Let $p_0 \in \mathbb{Z}$, $K \in \mathbb{N}$ and $f_1, \dots, f_L \in C(\mathbb{T}) \cap \mathcal{CL}_2$. For $(a_{pk})_{\substack{1 \leq p \leq K \\ 1 \leq k \leq L}} \in S^{KL-1}$, i.e., $(a_{pk})_{\substack{1 \leq p \leq K \\ 1 \leq k \leq L}} \in \mathbb{R}^{KL}$ with $\sum_{\substack{1 \leq p \leq K \\ 1 \leq k \leq L}} a_{pk}^2 =$

1, set

$$\zeta_{Ni} := \frac{1}{\sqrt{N}} \sum_{\substack{1 \leq p \leq K \\ 1 \leq k \leq L}} a_{pk} f_k(x_i + (p + p_0)\alpha_i), \quad 1 \leq i \leq N,$$

$$T_N(t) := \prod_{i=1}^N (1 + \sqrt{-1} t \zeta_{Ni}), \quad t \in \mathbb{R}.$$

Let λ be a probability measure on $(\mathbb{T}^\infty, \mathfrak{F})$. Then, for $\forall v \geq 0$ and \forall sub-

sequence $\{N_j\}_{j=1}^\infty$ of $\{1, 2, 3, \dots\}$

$$\begin{aligned} & \limsup_{j \rightarrow \infty} \mathbf{E}^\lambda \left[\left| e^{\sqrt{-1}t \sum_{i=1}^{N_j} \zeta_{N_j i}} - T_{N_j}(t) e^{-\frac{t^2}{2}v} \right| \right] \\ & \leq \left(1 + e^{\frac{t^2}{2}K \sum_{k=1}^L \|f_k\|_\infty^2} \right) \lim_{\varepsilon \downarrow 0} \limsup_{j \rightarrow \infty} \lambda \left(\left| \sum_{i=1}^{N_j} \zeta_{N_j i}^2 - v \right| > \varepsilon \right). \end{aligned}$$

PROOF. Using an expression

$$(1.2) \quad e^z = (1+z)e^{\frac{z^2}{2}-r(z)}, \quad \operatorname{Re} z > -1$$

where

$$(1.3) \quad r(z) = z^3 \int_0^1 \frac{s^2}{1+zs} ds,$$

we write for $0 < \varepsilon \leq 1$ and $v \geq 0$

$$\begin{aligned} & e^{\sqrt{-1}t \sum_{i=1}^N \zeta_{Ni}} \\ & = e^{\sqrt{-1}t \sum_{i=1}^N \zeta_{Ni}} \mathbf{1}_{\{\max_{1 \leq i \leq N} |\zeta_{Ni}| > \varepsilon\}} \\ & + e^{\sqrt{-1}t \sum_{i=1}^N \zeta_{Ni}} \mathbf{1}_{\{\max_{1 \leq i \leq N} |\zeta_{Ni}| \leq \varepsilon, |\sum_{i=1}^N \zeta_{Ni}^2 - v| > \varepsilon\}} \\ & - T_N(t) e^{-\frac{t^2}{2}v} \mathbf{1}_{\{\max_{1 \leq i \leq N} |\zeta_{Ni}| > \varepsilon \text{ or } |\sum_{i=1}^N \zeta_{Ni}^2 - v| > \varepsilon\}} \\ & + T_N(t) \left(e^{-\frac{t^2}{2} \sum_{i=1}^N \zeta_{Ni}^2 - \sum_{i=1}^N r(\sqrt{-1}t \zeta_{Ni})} - e^{-\frac{t^2}{2}v} \right) \\ & \quad \times \mathbf{1}_{\{\max_{1 \leq i \leq N} |\zeta_{Ni}| \leq \varepsilon, |\sum_{i=1}^N \zeta_{Ni}^2 - v| \leq \varepsilon\}} \\ & + T_N(t) e^{-\frac{t^2}{2}v}. \end{aligned}$$

Then

$$\begin{aligned} & |e^{\sqrt{-1}t \sum_{i=1}^N \zeta_{Ni}} - T_N(t) e^{-\frac{t^2}{2}v}| \\ & \leq \mathbf{1}_{\{\max_{1 \leq i \leq N} |\zeta_{Ni}| > \varepsilon\}} + \mathbf{1}_{\{|\sum_{i=1}^N \zeta_{Ni}^2 - v| > \varepsilon\}} \\ & \quad + |T_N(t)| \mathbf{1}_{\{\max_{1 \leq i \leq N} |\zeta_{Ni}| > \varepsilon \text{ or } |\sum_{i=1}^N \zeta_{Ni}^2 - v| > \varepsilon\}} \\ & \quad + |T_N(t)| \sup \left\{ |e^z - e^w|; \begin{array}{l} |z|, |w| \leq (\frac{t^2}{2} + \frac{|t|^3}{3})(1+v), \\ |z-w| \leq \varepsilon(\frac{t^2}{2} + \frac{|t|^3}{3})(1+v) \end{array} \right\}. \end{aligned}$$

Hence, noting that

$$(1.4) \quad |T_N(t)| \leq e^{\frac{t^2}{2} K \sum_{k=1}^L \|f_k\|_\infty^2},$$

$$(1.5) \quad \max_{1 \leq i \leq N} |\zeta_{Ni}| \leq \frac{1}{\sqrt{N}} \sqrt{K \sum_{k=1}^L \|f_k\|_\infty^2},$$

we have the conclusion. \square

PROOF OF “ (B) \Rightarrow (A) ”. We suppose (B). Let $K, L \in \mathbb{N}$ be fixed arbitrarily. Let $p_0 \in \mathbb{Z}$, $f_1, \dots, f_L \in C(\mathbb{T}) \cap \mathcal{CL}_2$ and $(a_{pk})_{\substack{1 \leq p \leq K \\ 1 \leq k \leq L}} \in S^{KL-1}$.

By the Cramér-Wold device (cf. [1]) it suffices to show that for $\forall t \in \mathbb{R}$

$$\begin{aligned} & \lim_{m \rightarrow \infty} \mathbf{E}^\mu \left[e^{\sqrt{-1} t \sum_{p=1}^K \sum_{k=1}^L a_{pk} \frac{1}{\sqrt{N_m}} \sum_{i=1}^{N_m} f_k(x_i + (p+p_0)\alpha_i)} \right] \\ &= e^{-\frac{t^2}{2} E[(\sum_{p=1}^K \sum_{k=1}^L a_{pk} I^{(p+p_0)}(f_k; \mu))^2]}. \end{aligned}$$

Let ζ_{Ni} and $T_N(t)$ be as in Lemma 1.1. From (B.2) note that

$$\lim_{m \rightarrow \infty} \frac{1}{N_m} \sum_{i=1}^{N_m} \mathbf{E}_i[f(\cdot + p\alpha_i)] = \mathbf{E}[f], \quad \forall f \in C(\mathbb{T}), \forall p \in \mathbb{Z}.$$

We write

$$\begin{aligned} \sum_{i=1}^N \zeta_{Ni}^2 &= \sum_{\substack{1 \leq p \leq K \\ 1 \leq k \leq L}} \sum_{\substack{1 \leq q \leq K \\ 1 \leq l \leq L}} a_{pk} a_{ql} \\ &\times \left(\frac{1}{N} \sum_{i=1}^N (f_k(x_i + (p+p_0)\alpha_i) f_l(x_i + (q+p_0)\alpha_i) \right. \\ &\quad \left. - \mathbf{E}_i[f_k(\cdot + (p+p_0)\alpha_i) f_l(\cdot + (q+p_0)\alpha_i)]) \right. \\ &\quad \left. + \frac{1}{N} \sum_{i=1}^N \mathbf{E}_i[f_k(\cdot + (p+p_0)\alpha_i) f_l(\cdot + (q+p_0)\alpha_i)] \right). \end{aligned}$$

By the note above and (B.3)

$$\lim_{m \rightarrow \infty} \frac{1}{N_m} \sum_{i=1}^{N_m} \mathbf{E}_i[f_k(\cdot + p\alpha_i) f_l(\cdot + q\alpha_i)] = E[I^{(p)}(f_k; \mu) I^{(q)}(f_l; \mu)],$$

and also by the strong law of large numbers (abbr. SLLN)

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N (f_k(x_i + p\alpha_i) f_l(x_i + q\alpha_i) - \mathbf{E}_i[f_k(\cdot + p\alpha_i) f_l(\cdot + q\alpha_i)]) = 0 \quad \mu\text{-a.s.}$$

Hence substituting these into the expression of $\sum_{i=1}^{N_m} \zeta_{N_m i}^2$, we have

$$(1.6) \quad \lim_{m \rightarrow \infty} \sum_{i=1}^{N_m} \zeta_{N_m i}^2 = E \left[\left(\sum_{\substack{1 \leq p \leq K \\ 1 \leq k \leq L}} a_{pk} I^{(p+p_0)}(f_k; \mu) \right)^2 \right] \quad \mu\text{-a.s.}$$

Next let us check

$$(1.7) \quad \lim_{m \rightarrow \infty} \mathbf{E}^\mu [T_{N_m}(t); A] = \mu(A), \quad \forall t \in \mathbb{R}, \forall A \in \mathcal{F}.$$

To do so, for $m > n \geq 1$ we divide $\mathbf{E}^\mu [T_{N_m}(t); A]$ as

$$\begin{aligned} \mathbf{E}^\mu [T_{N_m}(t); A] &= \mathbf{E}^\mu \left[T_{N_m}(t) (\mu(A | \mathcal{F}_{N_m}) - \mu(A | \mathcal{F}_{N_n})) \right] \\ &\quad + \mathbf{E}^\mu \left[T_{N_m}(t) \mu(A | \mathcal{F}_{N_n}) \right]. \end{aligned}$$

By (1.4)

$$|\text{the first term}| \leq e^{\frac{t^2}{2} K \sum_{k=1}^L \|f_k\|^2} \mathbf{E}^\mu \left[|\mu(A | \mathcal{F}_{N_m}) - \mu(A | \mathcal{F}_{N_n})| \right].$$

Since $\mu(A | \mathcal{F}_N) \rightarrow 1_A$ in L_1 as $N \rightarrow \infty$,

$$\begin{aligned} \limsup_{m \rightarrow \infty} |\text{the first term}| &\leq e^{\frac{t^2}{2} K \sum_{k=1}^L \|f_k\|^2} \mathbf{E}^\mu \left[|1_A - \mu(A | \mathcal{F}_{N_n})| \right] \\ &\longrightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

By the independence of \mathcal{F}_{N_n} and $\sigma(\zeta_{N_m i}; N_n + 1 \leq i \leq N_m)$ and by (1.2),

the second term

$$= \mathbf{E}^\mu \left[\prod_{i=1}^{N_n} \left(1 + \frac{\sqrt{-1}t}{\sqrt{N_m}} \sum_{\substack{1 \leq p \leq K \\ 1 \leq k \leq L}} a_{pk} f_k(x_i + (p+p_0)\alpha_i) \right) \mu(A | \mathcal{F}_{N_n}) \right]$$

$$\begin{aligned} & \times e^{\sqrt{-1}t \sum_{p=1}^K \sum_{k=1}^L a_{pk} \frac{1}{\sqrt{N_m}} \sum_{i=N_n+1}^{N_m} \mathbf{E}_i[f_k(\cdot + (p+p_0)\alpha_i)]} \\ & \times e^{\frac{t^2}{2} \frac{1}{N_m} \sum_{i=N_n+1}^{N_m} \left(\sum_{p=1}^K \sum_{k=1}^L a_{pk} \mathbf{E}_i[f_k(\cdot + (p+p_0)\alpha_i)] \right)^2} \\ & \times e^{\sum_{i=N_n+1}^{N_m} r \left(\frac{\sqrt{-1}t}{\sqrt{N_m}} \sum_{p=1}^K \sum_{k=1}^L a_{pk} \mathbf{E}_i[f_k(\cdot + (p+p_0)\alpha_i)] \right)}. \end{aligned}$$

Clearly

$$\text{the first product} \longrightarrow \mathbf{E}^\mu \left[\boldsymbol{\mu}(A | \mathcal{F}_{N_n}) \right] = \boldsymbol{\mu}(A) \quad \text{as } m \rightarrow \infty.$$

By (B.2) and (B.1), the second and third products $\rightarrow 1$. By the fact $|r(\sqrt{-1}x)| \leq \frac{|x|^3}{3}$ (cf. (1.3)), the fourth product $\rightarrow 1$. Hence, putting the above we have (1.7).

By (1.6) and (1.7), Lemma 1.1 gives us the desired conclusion. \square

PROOF OF “(A) \Rightarrow (B)”. We suppose (A). The proof is done in three steps.

1° For $\forall f \in C(\mathbb{T})$ and $\forall p \in \mathbb{Z}$

$$\lim_{m \rightarrow \infty} \frac{1}{N_m} \sum_{i=1}^{N_m} f(x_i + p\alpha_i) = \mathbf{E}[f] \quad \boldsymbol{\mu}\text{-a.s.}$$

PROOF. Let $p \in \mathbb{Z}$ and $f \in C(\mathbb{T}) \cap \mathcal{CL}_2$. By the assumption

$$\frac{1}{\sqrt{N_m}} \sum_{i=1}^{N_m} f(x_i + p\alpha_i) \implies I^{(p)}(f; \boldsymbol{\mu}) \quad \text{as } m \rightarrow \infty,$$

and so

$$\lim_{m \rightarrow \infty} \frac{1}{N_m} \sum_{i=1}^{N_m} f(x_i + p\alpha_i) = 0 \quad \text{in } \boldsymbol{\mu}.$$

But, since $\left\{ \frac{1}{N} \sum_{i=1}^N f(x_i + p\alpha_i) \right\}_{N=1}^\infty$ is uniformly bounded, and hence uniformly integrable, this convergence also holds in $L_1(\boldsymbol{\mu})$. Therefore, by taking expectation

$$\lim_{m \rightarrow \infty} \frac{1}{N_m} \sum_{i=1}^{N_m} \mathbf{E}_i[f(\cdot + p\alpha_i)] = 0.$$

On the other hand, by the SLLN

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \left(f(x_i + p\alpha_i) - \mathbf{E}_i[f(\cdot + p\alpha_i)] \right) = 0 \quad \boldsymbol{\mu}\text{-a.s.}$$

Combining these we have

$$\lim_{m \rightarrow \infty} \frac{1}{N_m} \sum_{i=1}^{N_m} f(x_i + p\alpha_i) = 0 \quad \boldsymbol{\mu}\text{-a.s.}$$

From this the assertion for general $f \in C(\mathbb{T})$ will be clear. \square

2° (B.1) and (B.2) hold.

PROOF. Let $p \in \mathbb{Z}$ and $f \in C(\mathbb{T}) \cap \mathcal{CL}_2$. We may suppose that $\mathbf{E}[f^2] = \int_{\mathbb{T}} f(x)^2 dx = 1$. For simplicity set $\zeta_{Ni} := \frac{1}{\sqrt{N}} f(x_i + p\alpha_i)$, $1 \leq i \leq N$. By the assumption $\sum_{i=1}^{N_m} \zeta_{N_m i} \Rightarrow I^{(p)}(f; \mu) \sim \mathfrak{N}(0, 1)$, and by 1° $\sum_{i=1}^{N_m} \zeta_{N_m i}^2 \rightarrow 1$ $\boldsymbol{\mu}$ -a.s. Therefore we apply Lemma 1.1 to have

$$\lim_{m \rightarrow \infty} \mathbf{E}^\mu \left[\prod_{i=1}^{N_m} (1 + \sqrt{-1} t \zeta_{N_m i}) \right] = 1, \quad \forall t \in \mathbb{R}.$$

Since, by (1.2)

$$\begin{aligned} & e^{\sqrt{-1} t \frac{1}{\sqrt{N_m}} \sum_{i=1}^{N_m} \mathbf{E}_i[f(\cdot + p\alpha_i)]} \\ &= \mathbf{E}^\mu \left[\prod_{i=1}^{N_m} (1 + \sqrt{-1} t \zeta_{N_m i}) \right] \\ & \quad \times e^{-\frac{t^2}{2} \frac{1}{N_m} \sum_{i=1}^{N_m} (\mathbf{E}_i[f(\cdot + p\alpha_i)])^2} e^{-\sum_{i=1}^{N_m} r(\frac{\sqrt{-1} t}{\sqrt{N_m}} \mathbf{E}_i[f(\cdot + p\alpha_i)])}, \end{aligned}$$

and hence

$$\begin{aligned} & e^{\frac{t^2}{2} \frac{1}{N_m} \sum_{i=1}^{N_m} (\mathbf{E}_i[f(\cdot + p\alpha_i)])^2} \\ &= \left| \mathbf{E}^\mu \left[\prod_{i=1}^{N_m} (1 + \sqrt{-1} t \zeta_{N_m i}) \right] \right| \left| e^{-\sum_{i=1}^{N_m} r(\frac{\sqrt{-1} t}{\sqrt{N_m}} \mathbf{E}_i[f(\cdot + p\alpha_i)])} \right|, \end{aligned}$$

letting $m \rightarrow \infty$ in the second expression yields

$$\lim_{m \rightarrow \infty} \frac{1}{N_m} \sum_{i=1}^{N_m} \left(\mathbf{E}_i[f(\cdot + p\alpha_i)] \right)^2 = 0,$$

which says (B.1), and next putting this into the first expression yields

$$\lim_{m \rightarrow \infty} e^{\sqrt{-1}t \frac{1}{\sqrt{N_m}} \sum_{i=1}^{N_m} \mathbf{E}_i[f(\cdot + p\alpha_i)]} = 1, \quad \forall t \in \mathbb{R},$$

i.e.,

$$\lim_{m \rightarrow \infty} \frac{1}{\sqrt{N_m}} \sum_{i=1}^{N_m} \mathbf{E}_i[f(\cdot + p\alpha_i)] = 0,$$

which says (B.2). \square

3° (B.3) holds.

PROOF. Let $f, g \in C(\mathbb{T}) \cap \mathcal{C}\mathcal{L}_2$ and $p \neq q$. We may suppose that $\int_{\mathbb{T}} f(x)^2 dx = \int_{\mathbb{T}} g(x)^2 dx = 1$. Set

$$\zeta_{Ni} := \frac{1}{\sqrt{N}} \left(\frac{1}{\sqrt{2}} f(x_i + p\alpha_i) + \frac{1}{\sqrt{2}} g(x_i + q\alpha_i) \right), \quad 1 \leq i \leq N.$$

By the assumption

$$(1.8) \quad \sum_{i=1}^{N_m} \zeta_{N_m i} \implies \frac{1}{\sqrt{2}} \left(I^{(p)}(f; \mu) + I^{(q)}(g; \mu) \right) \quad \text{as } m \rightarrow \infty.$$

Since we have (B.1) and (B.2), in the same way as in the proof of (1.7)

$$\lim_{m \rightarrow \infty} \mathbf{E}^\mu \left[\prod_{i=1}^{N_m} (1 + \sqrt{-1} t \zeta_{N_m i}) \right] = 1, \quad \forall t \in \mathbb{R}.$$

To view the convergence of $\sum_{i=1}^{N_m} \zeta_{N_m i}^2$, we write

$$\begin{aligned} \sum_{i=1}^{N_m} \zeta_{N_m i}^2 &= \frac{1}{2} \frac{1}{N_m} \sum_{i=1}^{N_m} f(x_i + p\alpha_i)^2 + \frac{1}{2} \frac{1}{N_m} \sum_{i=1}^{N_m} g(x_i + q\alpha_i)^2 \\ &\quad + \frac{1}{N_m} \sum_{i=1}^{N_m} \left(f(x_i + p\alpha_i)g(x_i + q\alpha_i) \right. \\ &\quad \quad \left. - \mathbf{E}_i[f(\cdot + p\alpha_i)g(\cdot + q\alpha_i)] \right) \\ &\quad + \frac{1}{N_m} \sum_{i=1}^{N_m} \mathbf{E}_i[f(\cdot + p\alpha_i)g(\cdot + q\alpha_i)]. \end{aligned}$$

By 1°, the first and second terms $\rightarrow \frac{1}{2} \mu$ -a.s. By the SLLN, the third term $\rightarrow 0$ μ -a.s. The fourth term is bounded in m , and so for \forall subsequence $\{m'\}$ we can take a further subsequence $\{m''\}$ of $\{m'\}$ and $c \in \mathbb{R}$ such that

$$\frac{1}{N_{m''}} \sum_{i=1}^{N_{m''}} \mathbf{E}_i[f(\cdot + p\alpha_i)g(\cdot + q\alpha_i)] \longrightarrow c.$$

Then putting these into the expression of $\sum_{i=1}^{N_m} \zeta_{N_m i}^2$ yields that $\sum_{i=1}^{N_{m''}} \zeta_{N_{m''} i}^2 \rightarrow 1 + c$ μ -a.s.

Now Lemma 1.1 tells us that $\sum_{i=1}^{N_{m''}} \zeta_{N_{m''} i} \Rightarrow \mathfrak{N}(0, 1 + c)$. Combined with (1.8) this implies that

$$c = E \left[I^{(p)}(f; \mu) I^{(q)}(g; \mu) \right].$$

Therefore, since the convergence above holds with this value for \forall subsequence $\{m'\}$, we must have

$$\lim_{m \rightarrow \infty} \frac{1}{N_m} \sum_{i=1}^{N_m} \mathbf{E}_i[f(\cdot + p\alpha_i)g(\cdot + q\alpha_i)] = E \left[I^{(p)}(f; \mu) I^{(q)}(g; \mu) \right],$$

which says (B.3). \square

CLAIM 1.1. *Let $\alpha \in \mathbb{T}^\infty$ and $\{N_m\}_{m=1}^\infty$ be as in Theorem 1.1. Suppose the conditions (B.1) and (B.2). The convergence of (A) holds under some absolutely continuous probability relative to $\boldsymbol{\mu}$, if and only if the condition (B.3) holds. In this case the convergence of (A) remains valid under any absolutely continuous one.*

PROOF. Suppose the convergence of (A) under an absolutely continuous probability $\boldsymbol{\nu}$ relative to $\boldsymbol{\mu}$. Let ζ_{N_i} be as in 3° above. By the assumption

$$(1.9) \quad \sum_{i=1}^{N_m} \zeta_{N_m i} \quad \text{under } \boldsymbol{\nu} \implies \frac{1}{\sqrt{2}} \left(I^{(p)}(f; \boldsymbol{\mu}) + I^{(q)}(g; \boldsymbol{\mu}) \right) \quad \text{as } m \rightarrow \infty.$$

By virtue of the conditions (B.1) and (B.2)

$$\lim_{m \rightarrow \infty} \mathbf{E}^\nu \left[\prod_{i=1}^{N_m} (1 + \sqrt{-1} t \zeta_{N_m i}) \right] = 1, \quad \forall t \in \mathbb{R},$$

which is easily seen from (1.7). In the same way as in 3°, for any subsequence $\{m'\}$ we take a further subsequence $\{m''\}$ such that $\lim_{m''} \sum_{i=1}^{N_{m''}} \zeta_{N_{m''} i}^2 = 1 + c$ in $\boldsymbol{\nu}$, where

$$c = \lim_{m''} \frac{1}{N_{m''}} \sum_{i=1}^{N_{m''}} \mathbf{E}_i [f(\cdot + p\alpha_i)g(\cdot + q\alpha_i)].$$

Then, by Lemma 1.1 and (1.9), $c = E[I^{(p)}(f; \boldsymbol{\mu})I^{(q)}(g; \boldsymbol{\mu})]$. This implies the condition (B.3).

Conversely suppose the condition (B.3). This time let ζ_{N_i} be as in Lemma 1.1. It suffices to show that for $\forall Z \in L_1(\mathbb{T}^\infty, \mathcal{F}, \boldsymbol{\mu})$ and $\forall t \in \mathbb{R}$

$$\lim_{m \rightarrow \infty} \mathbf{E}^\mu [Z e^{\sqrt{-1} t \sum_{i=1}^{N_m} \zeta_{N_m i}}] = \mathbf{E}^\mu [Z] e^{-\frac{t^2}{2} E[(\sum_{p=1}^K \sum_{k=1}^L a_{pk} I^{(p+p_0)}(f_k; \boldsymbol{\mu}))^2]}.$$

Since elements in $L_1(\mathbb{T}^\infty, \mathcal{F}, \boldsymbol{\mu})$ are approximated by \mathcal{F} -simple functions, we may assume the Z above to be a defining function.

Let $A \in \mathcal{F}$ and $t \in \mathbb{R}$ be fixed arbitrarily. Then by (1.6) and (1.7), Lemma 1.1 implies that

$$\lim_{m \rightarrow \infty} \mathbf{E}^\mu \left[e^{\sqrt{-1} t \sum_{i=1}^{N_m} \zeta_{N_m i}; A} \right] = \boldsymbol{\mu}(A) e^{-\frac{t^2}{2} E[(\sum_{p=1}^K \sum_{k=1}^L a_{pk} I^{(p+p_0)}(f_k; \boldsymbol{\mu}))^2]},$$

which is just the desired conclusion. \square

CLAIM 1.2. *The condition (B.1) + (B.3) is equivalent to that (C)_k ($k = 1$ or 2) + (M):*

$$(C)_k \quad \lim_{m \rightarrow \infty} \frac{1}{N_m} \sum_{i=1}^{N_m} |\widehat{\mathbf{P}}_i(n)|^k = 0, \quad \forall n \in \mathbb{Z} \setminus \{0\},$$

$$(M) \quad \alpha \in \mathbb{T}_{\mu, \{N_m\}}^\infty.$$

PROOF. The implications (B.1) \Rightarrow (C)₂ \Rightarrow (C)₁ \Rightarrow (B.1) are easily seen. Indeed, the first implication follows from an equality

$$\begin{aligned} & \left(\int_{\mathbb{T}} \cos 2n\pi(x + p\alpha_i) \mathbf{P}_i(dx) \right)^2 + \left(\int_{\mathbb{T}} \sin 2n\pi(x + p\alpha_i) \mathbf{P}_i(dx) \right)^2 \\ &= |\widehat{\mathbf{P}}_i(\mp n)|^2; \end{aligned}$$

the second does from an inequality $|a| \leq \frac{1}{2}(\varepsilon^2 + \frac{|a|^2}{\varepsilon^2})$, $\varepsilon > 0$; the third does from the Weierstrass approximation theorem.

Next we show the implication (C)₁ + (M) \Rightarrow (B.3). Let $f, g \in C(\mathbb{T}) \cap \mathcal{CL}_2$ and let $p, q \in \mathbb{Z}$ be such that $p \neq q$. By virtue of the Weierstrass approximation theorem we may assume the f and g to be real trigonometric polynomials. Letting

$$f(x) = \sum_{1 \leq |k| \leq N} c_k e^{\sqrt{-1} 2\pi k x}, \quad g(x) = \sum_{1 \leq |l| \leq N} d_l e^{\sqrt{-1} 2\pi l x},$$

where $\overline{c}_k = c_{-k}$ and $\overline{d}_l = d_{-l}$, we write

$$\begin{aligned} & \frac{1}{N_m} \sum_{i=1}^{N_m} \mathbf{E}_i[f(\cdot + p\alpha_i)g(\cdot + q\alpha_i)] \\ &= \sum_{1 \leq |k| \leq N} c_k \overline{d}_k \frac{1}{N_m} \sum_{i=1}^{N_m} e^{\sqrt{-1} 2\pi k(p-q)\alpha_i} \\ & \quad + \sum_{\substack{1 \leq |k|, |l| \leq N; \\ k \neq l}} c_k \overline{d}_l \frac{1}{N_m} \sum_{i=1}^{N_m} e^{\sqrt{-1} 2\pi(kp-lq)\alpha_i} \widehat{\mathbf{P}}_i(l-k). \end{aligned}$$

By (M), the first term $\rightarrow \sum_{1 \leq |k| \leq N} c_k \overline{d_k} \widehat{\mu}(k(q-p)) = E[I^{(p)}(f; \mu) I^{(q)}(g; \mu)]$, and by (C)₁, the second term $\rightarrow 0$. Hence we have (B.3).

Finally we show the implication (B.3) + (C)₁ \Rightarrow (M). Let $\{m'\}$ be any subsequence. Because of the compactness of \mathbb{T} there exist a further subsequence $\{m''\}$ and a probability measure ν on \mathbb{T} so that $\alpha \in \mathbb{T}_{\nu, \{N_{m''}\}}^\infty$. From what we have shown above and our assumption

$$\begin{aligned} & \sum_{|i| \geq 1} \widehat{f}(i) \overline{\widehat{g}(i)} \widehat{\nu}(i(q-p)) \\ &= \sum_{|i| \geq 1} \widehat{f}(i) \overline{\widehat{g}(i)} \widehat{\mu}(i(q-p)), \quad \forall f, g \in C(\mathbb{T}) \cap \mathcal{CL}_2, \forall p, q \in \mathbb{Z}. \end{aligned}$$

This easily implies $\nu = \mu$, so that we have $\alpha \in \mathbb{T}_{\mu, \{N_m\}}^\infty$, which is just (M). \square

We present a sufficient condition for $\{\mathbf{P}_i\}_{i=1}^\infty$ to satisfy the conditions (B.1) and (B.2).

THEOREM 1.2. *Suppose that a sequence $\{\mathbf{P}_i\}_{i=1}^\infty$ of probability measures on \mathbb{T} satisfies the following: For some $\mathbb{N}_0 \subset \mathbb{N}$,*

(i) $\mathbf{P}_i(dx) \ll dx$ ($\forall i \in \mathbb{N}_0$) and $\prod_{i \in \mathbb{N}_0} \int_{\mathbb{T}} \sqrt{\frac{d\mathbf{P}_i}{dx}(x)} dx > 0$,

(ii) $\#(\mathbb{N} \setminus \mathbb{N}_0) \cap \{1, \dots, N\} = O(\sqrt{N})$ as $N \rightarrow \infty$ and

$$\lim_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \sum_{i \in (\mathbb{N} \setminus \mathbb{N}_0) \cap \{1, \dots, N\}} |\widehat{\mathbf{P}}_i(n)| = 0, \quad \forall n \neq 0.$$

Then, for $\forall \alpha \in \mathbb{T}^\infty$, (B.1) and (B.2) hold with the whole sequence $\{1, 2, 3, \dots\}$. Thus, the convergence of (A) holds under some / any absolutely continuous probability relative to $\boldsymbol{\mu} = \prod_{i=1}^\infty \mathbf{P}_i$, if and only if $\alpha \in \mathbb{T}_{\boldsymbol{\mu}, \{N_m\}}^\infty$.

In particular the disappearance of dependency happens as $N \rightarrow \infty$ in the sense that on some / any absolutely continuous probability space relative to

$$\boldsymbol{\mu} = \prod_{i=1}^\infty \mathbf{P}_i$$

$$\left(\frac{1}{\sqrt{N}} \sum_{i=1}^N f(x_i + p\alpha_i) \right)_{p \in \mathbb{Z}, f \in C(\mathbb{T}) \cap \mathcal{CL}_2}$$

$$\xrightarrow{\text{f.d.}} \left(I^{(p)}(f) \right)_{p \in \mathbb{Z}, f \in C(\mathbb{T}) \cap \mathcal{L}_2} \quad \text{as } N \rightarrow \infty,$$

if and only if $\alpha \in \mathbb{T}_{dx}^\infty$. Here $\left\{ (B^{(p)}(t))_{0 \leq t \leq 1} \right\}_{p \in \mathbb{Z}}$ is a sequence of independent 1-dimensional Brownian motions starting at 0, and $I^{(p)}(f)$ is the Wiener integral of $f \in \mathcal{L}_2$ with respect to $B^{(p)}$, i.e., $I^{(p)}(f) = \int_0^1 f(s) dB^{(p)}(s)$.

PROOF. It suffices to check (C)₂ and (B.2) with $N_m = m$. First of all note that for a sequence $\{a_k\}_{k=1}^\infty$ with $0 < a_k \leq 1$ ($\forall k$)

$$\prod_{k=1}^{\infty} a_k > 0 \quad \stackrel{\text{iff}}{\iff} \quad \sum_{k=1}^{\infty} (1 - a_k) < \infty.$$

Set $\rho_i(x) := \frac{d\mathbf{P}_i}{dx}(x)$, $i \in \mathbb{N}_0$. By the assumption (i) and the note above

$$\sum_{i \in \mathbb{N}_0} \left(1 - \int_{\mathbb{T}} \sqrt{\rho_i(x)} dx \right) < \infty.$$

Also, by the fact $\int_{\mathbb{T}} \rho_i(x) dx = 1$

$$\begin{aligned} \int_{\mathbb{T}} |1 - \sqrt{\rho_i(x)}|^2 dx &= 2 \left(1 - \int_{\mathbb{T}} \sqrt{\rho_i(x)} dx \right), \\ \int_{\mathbb{T}} |1 - \rho_i(x)| dx &\leq 2 \sqrt{\int_{\mathbb{T}} |1 - \sqrt{\rho_i(x)}|^2 dx}. \end{aligned}$$

Therefore

$$(1.10) \quad \sum_{i \in \mathbb{N}_0} \int_{\mathbb{T}} |1 - \sqrt{\rho_i(x)}|^2 dx < \infty,$$

$$(1.11) \quad \sum_{i \in \mathbb{N}_0} \left(\int_{\mathbb{T}} |1 - \rho_i(x)| dx \right)^2 < \infty.$$

Now we show (C)₂. Let $n \in \mathbb{Z} \setminus \{0\}$. Then, by (1.11) and the assumption (ii) we have

$$\frac{1}{N} \sum_{i=1}^N |\widehat{\mathbf{P}}_i(n)|^2 = \frac{1}{N} \sum_{i \in \mathbb{N}_0 \cap \{1, \dots, N\}} \left| \int_{\mathbb{T}} e^{-\sqrt{-1} 2n\pi x} \rho_i(x) dx \right|^2$$

$$\begin{aligned}
 & + \frac{1}{N} \sum_{i \in (\mathbb{N} \setminus \mathbb{N}_0) \cap \{1, \dots, N\}} \left| \int_{\mathbb{T}} e^{-\sqrt{-1} 2n\pi x} \mathbf{P}_i(dx) \right|^2 \\
 & \leq \frac{1}{N} \sum_{i \in \mathbb{N}_0 \cap \{1, \dots, N\}} \left(\int_{\mathbb{T}} |1 - \rho_i(x)| dx \right)^2 \\
 & \quad + \frac{1}{\sqrt{N}} \frac{1}{\sqrt{N}} \# (\mathbb{N} \setminus \mathbb{N}_0) \cap \{1, \dots, N\} \\
 & \longrightarrow 0 \quad \text{as } N \rightarrow \infty,
 \end{aligned}$$

which is just (C)₂.

Next we show (B.2). Let $f \in C(\mathbb{T}) \cap \mathcal{CL}_2$ and $p \in \mathbb{Z}$. We divide $\frac{1}{\sqrt{N}} \sum_{i=1}^N \mathbf{E}_i[f(\cdot + p\alpha_i)]$ into two terms as

$$\begin{aligned}
 \frac{1}{\sqrt{N}} \sum_{i=1}^N \mathbf{E}_i[f(\cdot + p\alpha_i)] & = \frac{1}{\sqrt{N}} \sum_{i \in \mathbb{N}_0 \cap \{1, \dots, N\}} \mathbf{E}_i[f(\cdot + p\alpha_i)] \\
 & \quad + \frac{1}{\sqrt{N}} \sum_{i \in (\mathbb{N} \setminus \mathbb{N}_0) \cap \{1, \dots, N\}} \mathbf{E}_i[f(\cdot + p\alpha_i)].
 \end{aligned}$$

As for the first term, noting that

$$\mathbf{E}_i[f(\cdot + p\alpha_i)] = \int_{\mathbb{T}} f(x + p\alpha_i) (\sqrt{\rho_i(x)} + 1) (\sqrt{\rho_i(x)} - 1) dx,$$

we have by (1.10)

[the first term]

$$\begin{aligned}
 & = \left| \frac{1}{\sqrt{N}} \sum_{i \in \mathbb{N}_0 \cap \{1, \dots, M\}} \mathbf{E}_i[f(\cdot + p\alpha_i)] \right. \\
 & \quad \left. + \frac{1}{\sqrt{N}} \sum_{i \in \mathbb{N}_0 \cap \{M+1, \dots, N\}} \int_{\mathbb{T}} f(x + p\alpha_i) (\sqrt{\rho_i(x)} + 1) (\sqrt{\rho_i(x)} - 1) dx \right| \\
 & \leq \frac{M}{\sqrt{N}} \|f\|_{\infty} + \frac{1}{\sqrt{N}} \left(\sum_{i \in \mathbb{N}_0 \cap \{M+1, \dots, N\}} \int_{\mathbb{T}} f(x + p\alpha_i)^2 (\sqrt{\rho_i(x)} + 1)^2 dx \right)^{\frac{1}{2}} \\
 & \quad \times \left(\sum_{i \in \mathbb{N}_0 \cap \{M+1, \dots, N\}} \int_{\mathbb{T}} |1 - \sqrt{\rho_i(x)}|^2 dx \right)^{\frac{1}{2}}
 \end{aligned}$$

$$\begin{array}{l}
 \xrightarrow{\quad} 0. \\
 \text{first } N \rightarrow \infty \\
 \text{second } M \rightarrow \infty
 \end{array}$$

As for the second term, for $\forall \varepsilon > 0$ we take a trigonometric polynomial $P_\varepsilon(x) = \sum_{1 \leq |k| \leq n} c_k e^{\sqrt{-1}2k\pi x}$ such that $\|f - P_\varepsilon\|_\infty < \varepsilon$. Then we have by the assumption (ii)

$$\begin{aligned}
& |\text{the second term}| \\
&= \left| \frac{1}{\sqrt{N}} \sum_{i \in (\mathbb{N} \setminus \mathbb{N}_0) \cap \{1, \dots, N\}} \int_{\mathbb{T}} (f - P_\varepsilon)(x + p\alpha_i) \mathbf{P}_i(dx) \right. \\
&\quad \left. + \sum_{1 \leq |k| \leq n} c_k \frac{1}{\sqrt{N}} \sum_{i \in (\mathbb{N} \setminus \mathbb{N}_0) \cap \{1, \dots, N\}} e^{\sqrt{-1}2k\pi p\alpha_i} \int_{\mathbb{T}} e^{\sqrt{-1}2k\pi x} \mathbf{P}_i(dx) \right| \\
&\leq \varepsilon \frac{1}{\sqrt{N}} \# (\mathbb{N} \setminus \mathbb{N}_0) \cap \{1, \dots, N\} \\
&\quad + \sum_{1 \leq |k| \leq n} |c_k| \frac{1}{\sqrt{N}} \sum_{i \in (\mathbb{N} \setminus \mathbb{N}_0) \cap \{1, \dots, N\}} |\widehat{\mathbf{P}}_i(-k)| \\
&\quad \xrightarrow[\text{second } \varepsilon \rightarrow 0]{\text{first } N \rightarrow \infty} 0.
\end{aligned}$$

These convergences clearly imply (B.2). \square

2. Convergence to a System of Correlated Multiple Wiener Integrals

In this section we study the convergence of a system of symmetric statistics to that of correlated multiple Wiener integrals. As in the previous section let $(I^{(p)}(f; \mu))_{p \in \mathbb{Z}, f \in \mathcal{CL}_2}$ be a centered Gaussian system with covariance (1.1), and let us denote by $\boldsymbol{\mu}$ a product probability $\prod_{i=1}^{\infty} \mathbf{P}_i$.

We begin with the following definition:

DEFINITION 2.1. For $\kappa \in [0, \infty)$, we define a real Hilbert space $(H_\kappa, (\cdot, *)_{H_\kappa})$ by

$$\begin{aligned}
H_\kappa &:= \left\{ (h_n)_{n=1}^\infty; \sum_{n=1}^\infty \left(\frac{1}{n!} + \sum_{r=1}^n \frac{\kappa^r}{r!} \frac{1}{\sqrt{(n-r)!}} \right) \|h_n\|_{\mathcal{L}_2^n}^2 < \infty \right\}, \\
(h^{(1)}, h^{(2)})_{H_\kappa} &:= \sum_{n=1}^\infty \left(\frac{1}{n!} + \sum_{r=1}^n \frac{\kappa^r}{r!} \frac{1}{\sqrt{(n-r)!}} \right) (h_n^{(1)}, h_n^{(2)})_{\mathcal{L}_2^n}, \quad h^{(1)}, h^{(2)} \in H_\kappa.
\end{aligned}$$

Clearly

$$\{h^f := (f^{\otimes n})_{n=1}^{\infty}; f \in \mathcal{CL}_2\} \subset H_{\kappa},$$

and moreover the following holds (cf. Lemma 1 of [16]):

$$(2.1) \quad \text{c.l.s.}\{h^f; f \in \mathcal{CL}_2\} = H_{\kappa}.$$

Also it holds that (cf. Lemma 3 of [16])

$$(2.2) \quad \|h^{\phi} - h^{\psi}\|_{H_{\kappa}} \leq \text{const}(\kappa, \|\phi\| \vee \|\psi\|)\|\phi - \psi\|$$

for $\forall \phi, \psi \in \mathcal{CL}_2$, where

$$\text{const}(\kappa, t) = \sqrt{\sum_{n=1}^{\infty} \left(\frac{1}{n!} + \sum_{r=1}^n \frac{\kappa^r}{r!} \frac{1}{\sqrt{(n-r)!}} \right) n^2 t^{2(n-1)}}.$$

DEFINITION 2.2. We define symmetric statistics $\sigma_n^N(\cdot; h)$ ($h \in \mathcal{SL}_2^n$) and $Y_N(\cdot; h)$ ($h = (h_n) \in H_0$) by

$$\sigma_n^N(y; h) := \begin{cases} \sum_{1 \leq i_1 < \dots < i_n \leq N} h(y_{i_1}, \dots, y_{i_n}) & n \leq N \\ 0 & n > N, \end{cases}$$

$$Y_N(y; h) := \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{N}} \right)^n \sigma_n^N(y; h_n).$$

As a CONS of \mathcal{CL}_2 , we take

$$\{\phi_k\}_{k=1}^{\infty} = \left\{ \sqrt{2} \cos 2\pi n x, \sqrt{2} \sin 2\pi m x \right\}_{n,m \in \mathbb{N}}.$$

CLAIM 2.1. For $\psi \in \mathcal{CL}_2$ and $R \in \mathbb{N}$, set $\psi^{(R)} := \sum_{k=1}^R (\psi, \phi_k) \phi_k$. Suppose that a sequence $\{\mathbf{P}_i\}_{i=1}^{\infty}$ satisfies the conditions (i) and (ii) in Theorem

1.2. Then, for $\forall \alpha \in \mathbb{T}_{\mu, \{N_m\}}^\infty$ and $\forall \nu \ll \mu$

$$\begin{aligned} & \left(1 + Y_{N_m}(\cdot + p\alpha; h^{\psi^{(R)}}) \right)_{p \in \mathbb{Z}, \psi \in \mathcal{CL}_2} \quad \text{under } \nu \\ & \xrightarrow{\text{f.d.}} \left(e^{I^{(p)}(\psi^{(R)}; \mu) - \frac{1}{2} \|\psi^{(R)}\|^2} \right)_{p \in \mathbb{Z}, \psi \in \mathcal{CL}_2} \quad \text{as } m \rightarrow \infty. \end{aligned}$$

PROOF. Let $\alpha \in \mathbb{T}_{\mu, \{N_m\}}^\infty$ and $\nu \ll \mu$. By Theorem 1.2

$$\begin{aligned} & \left(\frac{1}{\sqrt{N_m}} \sum_{i=1}^{N_m} \phi_k(x_i + p\alpha_i) \right)_{p \in \mathbb{Z}, k \in \mathbb{N}} \quad \text{under } \nu \\ (2.3) \quad & \xrightarrow{\text{f.d.}} \left(I^{(p)}(\phi_k; \mu) \right)_{p \in \mathbb{Z}, k \in \mathbb{N}} \quad \text{as } m \rightarrow \infty. \end{aligned}$$

Also, by 1° in the proof of the implication “(A) \Rightarrow (B)”

$$(2.4) \quad \lim_{m \rightarrow \infty} \frac{1}{N_m} \sum_{i=1}^{N_m} \phi_k(x_i + p\alpha_i) \phi_l(x_i + p\alpha_i) = \delta_{kl} \quad \text{in } \nu.$$

For fixed $(c_1, \dots, c_R) \in \mathbb{R}^R$, set $z_{Ni} := \frac{1}{\sqrt{N}} \sum_{k=1}^R c_k \phi_k(x_i + p\alpha_i)$. Let $N \geq (2\sqrt{2} \sum_{k=1}^R |c_k|)^2$. Then $|z_{Ni}| \leq \frac{1}{2}$, and so, by (1.2)

$$\begin{aligned} 1 + z_{Ni} &= e^{z_{Ni} - \frac{1}{2} z_{Ni}^2 + r_{Ni}}, \\ |r_{Ni}| &= |r(z_{Ni})| \leq \frac{2}{3} |z_{Ni}|^3. \end{aligned}$$

This tells us that

$$\begin{aligned} & \left| \prod_{i=1}^N (1 + z_{Ni}) - e^{\sum_{i=1}^N z_{Ni} - \frac{1}{2} \sum_{i=1}^N z_{Ni}^2} \right|^2 \\ & \leq e^{\sum_{i=1}^N z_{Ni} - \frac{1}{2} \sum_{i=1}^N z_{Ni}^2} \left(e^{\frac{2\sqrt{2}}{3} \frac{1}{\sqrt{N}} \sum_{k=1}^R |c_k| \sum_{i=1}^N z_{Ni}^2} - 1 \right). \end{aligned}$$

By (2.3) and (2.4), the inequality above yields that

$$\left(\prod_{i=1}^{N_m} \left(1 + \frac{1}{\sqrt{N_m}} \sum_{k=1}^R c_k \phi_k(x_i + p\alpha_i) \right) \right)_{p \in \mathbb{Z}, (c_1, \dots, c_R) \in \mathbb{R}^R} \quad \text{under } \nu$$

$$\xrightarrow{\text{f.d.}} \left(e^{I^{(p)}(\sum_{k=1}^R c_k \phi_k; \mu) - \frac{1}{2} \|\sum_{k=1}^R c_k \phi_k\|^2} \right)_{p \in \mathbb{Z}, (c_1, \dots, c_R) \in \mathbb{R}^R} \quad \text{as } m \rightarrow \infty.$$

But

$$1 + Y_N(y; h^{\psi^{(R)}}) = \prod_{i=1}^N \left(1 + \frac{1}{\sqrt{N}} \sum_{k=1}^R (\psi, \phi_k) \phi_k(y_i) \right).$$

From this together with the above the assertion follows immediately. \square

We state a main theorem in this section.

THEOREM 2.1. *Suppose that a sequence $\{\mathbf{P}_i\}_{i=1}^\infty$ satisfies the following:*

- (a) $\mathbf{P}_i(dx) \ll dx$ ($\forall i \in \mathbb{N}$).
- (b) *There exists a subset \mathbb{N}_0 of $\{1, 2, 3, \dots\}$ such that*
 - (b.i) $\prod_{i \in \mathbb{N}_0} \int_{\mathbb{T}} \sqrt{\frac{d\mathbf{P}_i}{dx}}(x) dx > 0$,
 - (b.ii) $\sup_{N \in \mathbb{N}} \frac{1}{\sqrt{N}} \#(\mathbb{N} \setminus \mathbb{N}_0) \cap \{1, \dots, N\} =: C < \infty$,
 - (b.iii) $\lim_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \sum_{i \in (\mathbb{N} \setminus \mathbb{N}_0) \cap \{1, \dots, N\}} \left| \widehat{\frac{d\mathbf{P}_i}{dx}}(n) \right| = 0$ for $\forall n \in \mathbb{Z} \setminus \{0\}$,
 - (b.iv) $\sup_{i \in \mathbb{N} \setminus \mathbb{N}_0} \sqrt{\int_{\mathbb{T}} \left(\frac{d\mathbf{P}_i}{dx}(x) \right)^2 dx} =: M < \infty$.

Then, for $\forall \alpha \in \mathbb{T}_{\mu, \{N_m\}}^\infty$ and $\forall \nu \ll \mu$

$$\left(Y_{N_m}(\cdot + p\alpha; h) \right)_{p \in \mathbb{Z}, h=(h_n) \in H_{CM}} \quad \text{under } \nu$$

$$\xrightarrow{\text{f.d.}} \left(\sum_{n=1}^\infty \frac{1}{n!} I_n^{(p)}(h_n; \mu) \right)_{p \in \mathbb{Z}, h=(h_n) \in H_{CM}} \quad \text{as } m \rightarrow \infty.$$

Here $I_n^{(p)}(h_n; \mu)$ is an n -ple Wiener integral of $h_n \in \mathcal{CSL}_2^n$ with respect to $I^{(p)}(\cdot; \mu)$.

COROLLARY 1. *For $\forall (h_n)_{n=1}^\infty$ such that $h_n \in \mathcal{CSL}_2^n$ ($n \geq 1$)*

$$\left(N_m^{-\frac{n}{2}} \sigma_n^{N_m}(\cdot + p\alpha; h_n) \right)_{p \in \mathbb{Z}, n \in \mathbb{N}} \xrightarrow{\text{f.d.}} \left(\frac{1}{n!} I_n^{(p)}(h_n; \mu) \right)_{p \in \mathbb{Z}, n \in \mathbb{N}} \quad \text{as } m \rightarrow \infty.$$

REMARK 2.1. (i) The conditions on $\{\mathbf{P}_i\}_{i=1}^\infty$ in Theorem 2.1 are stronger than those in Theorem 1.2.

(ii) When $\mathbb{N}_0 = \mathbb{N}$, we can regard the conditions (b.ii), (b.iii) and (b.iv) as satisfied, and we understand as $C = M = 0$. In this case, by Kakutani dichotomy theorem, the condition (b.i) is equivalent to that $\prod_{i=1}^\infty \mathbf{P}_i \ll \mathbf{P}^\infty$.

For the proof of Theorem 2.1, we present two propositions:

PROPOSITION 1. (i) For $\forall f \in \mathcal{CL}_2$

$$I_n^{(p)}(f^{\otimes n}; \mu) = n! \|f\|^n H_n \left(I_n^{(p)} \left(\frac{f}{\|f\|}; \mu \right) \right).$$

Here $(H_n)_{n=0}^\infty$ are Hermite polynomials, i.e., $H_n(\xi) = \frac{(-1)^n}{n!} e^{\frac{\xi^2}{2}} \frac{d^n}{d\xi^n} \left(e^{-\frac{\xi^2}{2}} \right)$.

(ii) For $\forall h \in \mathcal{CSL}_2^n$ and $\forall k \in \mathcal{CSL}_2^m$

$$E \left[I_n^{(p)}(h; \mu) \right] = 0,$$

$$E \left[I_n^{(p)}(h; \mu) I_m^{(p)}(k; \mu) \right] = n! \delta_{nm} (h, k)_{\mathcal{L}_2^n}.$$

For the proof see [8].

PROPOSITION 2. Suppose the conditions (a) and (b) in Theorem 2.1. Then there exists an increasing function $\xi : [0, \infty) \rightarrow [0, \infty)$ with $\xi(0+) = \xi(0) = 0$ such that

$$\mu \left(|Y_N(\cdot + p\alpha; h)| > \eta \right) \leq \xi \left(\frac{\|h\|_{H_{CM}}}{\eta} \right)$$

for $\forall N \in \mathbb{N}$, $\forall p \in \mathbb{Z}$, $\forall \alpha \in \mathbb{T}^\infty$, $\forall h \in H_{CM}$ and $\forall \eta > 0$. Here C and M are constants in (b.ii) and (b.iv), respectively.

PROOF. For simplicity set $\mathbb{N}_1 := \mathbb{N} \setminus \mathbb{N}_0$ and $\rho_i := \frac{d\mathbf{P}_i}{dx}$ ($i \in \mathbb{N}$). Let $N \in \mathbb{N}$ and $h = (h_n)_{n=1}^\infty \in H_{CM}$. We rewrite

$$Y_N(y; h) = \sum_{n=1}^N \left(\frac{1}{\sqrt{N}} \right)^n \frac{1}{n!} \sum_{\substack{i_1, \dots, i_n \in \mathbb{N}_0 \cap \{1, \dots, N\}; \\ i_1, \dots, i_n \text{ are distinct}}} h_n(y_{i_1}, \dots, y_{i_n})$$

$$\begin{aligned}
 & + \sum_{n=1}^N \left(\frac{1}{\sqrt{N}} \right)^n \sum_{r=1}^n \frac{1}{r!(n-r)!} \\
 & \quad \times \sum_{\substack{i_1, \dots, i_{n-r} \in \mathbb{N}_0 \cap \{1, \dots, N\}, \\ j_1, \dots, j_r \in \mathbb{N}_1 \cap \{1, \dots, N\}; \\ i_1, \dots, i_{n-r} \text{ are distinct,} \\ j_1, \dots, j_r \text{ are distinct}}} h_n(y_{i_1}, \dots, y_{i_{n-r}}, y_{j_1}, \dots, y_{j_r}).
 \end{aligned}$$

Let $\alpha \in \mathbb{T}^\infty$, $p \in \mathbb{N}$ and $\eta > 0$. Then, by Fubini's theorem

$$\begin{aligned}
 & \mu \left(|Y_N(\cdot + p\alpha; h)| > \eta \right) \\
 & \leq \left(\prod_{k \in \mathbb{N}_0} \mathbf{P}_k \right) \left(\left| \sum_{n=1}^N \left(\frac{1}{\sqrt{N}} \right)^n \frac{1}{n!} \sum_{\substack{i_1, \dots, i_n \in \mathbb{N}_0 \cap \{1, \dots, N\}; \\ i_1, \dots, i_n \text{ are distinct}}} \right. \right. \\
 & \quad \left. \left. \times h_n(x_{i_1} + p\alpha_{i_1}, \dots, x_{i_n} + p\alpha_{i_n}) \right| > \frac{\eta}{2} \right) \\
 & + \mathbf{E}^{\prod_{k \in \mathbb{N}_0} \mathbf{P}_k} \left[\left(\prod_{k \in \mathbb{N}_1} \mathbf{P}_k \right) \left(\left| \sum_{n=1}^N \left(\frac{1}{\sqrt{N}} \right)^n \sum_{r=1}^n \frac{1}{r!(n-r)!} \sum_{\substack{i_1, \dots, i_{n-r} \in \mathbb{N}_0 \cap \{1, \dots, N\}, \\ j_1, \dots, j_r \in \mathbb{N}_1 \cap \{1, \dots, N\}; \\ i_1, \dots, i_{n-r} \text{ are distinct,} \\ j_1, \dots, j_r \text{ are distinct}}} \right. \right. \right. \\
 & \quad \left. \left. \times h_n(x_{i_1} + p\alpha_{i_1}, \dots, x_{i_{n-r}} + p\alpha_{i_{n-r}}, \right. \right. \\
 & \quad \left. \left. \left. x_{j_1} + p\alpha_{j_1}, \dots, x_{j_r} + p\alpha_{j_r} \right) \right| > \frac{\eta}{2} \right) \right]
 \end{aligned}$$

=: the first term + the second term.

First we estimate the first term. By Chebyshev's inequality

$$\begin{aligned}
 & \left(\prod_{k \in \mathbb{N}_0} dx_k \right) \left(\left| \sum_{n=1}^N \left(\frac{1}{\sqrt{N}} \right)^n \frac{1}{n!} \sum_{\substack{i_1, \dots, i_n \in \mathbb{N}_0 \cap \{1, \dots, N\}; \\ i_1, \dots, i_n \text{ are distinct}}} \right. \right. \\
 & \quad \left. \left. \times h_n(x_{i_1} + p\alpha_{i_1}, \dots, x_{i_n} + p\alpha_{i_n}) \right| > \frac{\eta}{2} \right)
 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{2}{\eta} \left(\mathbf{E}^{\prod_{k \in \mathbb{N}_0} dx_k} \left[\sum_{1 \leq n, m \leq N} \left(\frac{1}{\sqrt{N}} \right)^n \left(\frac{1}{\sqrt{N}} \right)^m \right. \right. \\
&\quad \times \frac{1}{n!} \frac{1}{m!} \sum_{\substack{i_1, \dots, i_n \in \mathbb{N}_0 \cap \{1, \dots, N\}; \\ i_1, \dots, i_n \text{ are distinct}}} \sum_{\substack{j_1, \dots, j_m \in \mathbb{N}_0 \cap \{1, \dots, N\}; \\ j_1, \dots, j_m \text{ are distinct}}} \\
&\quad \times h_n(x_{i_1} + p\alpha_{i_1}, \dots, x_{i_n} + p\alpha_{i_n}) \\
&\quad \left. \left. \times h_m(x_{j_1} + p\alpha_{j_1}, \dots, x_{j_m} + p\alpha_{j_m}) \right] \right)^{\frac{1}{2}}.
\end{aligned}$$

By the fact $\int_{\mathbb{T}} h_n(x_1, \dots, x_{n-1}, y) dy = 0$, the right hand side equals

$$\frac{2}{\eta} \left(\sum_{n=1}^N \left(\frac{1}{\sqrt{N}} \right)^n \sum_{\substack{i_1, \dots, i_n \in \mathbb{N}_0 \cap \{1, \dots, N\}; \\ i_1, \dots, i_n \text{ are distinct}}} \frac{1}{n!} \|h_n\|^2 \right)^{\frac{1}{2}},$$

and this is dominated by $\frac{2}{\eta} \left(\sum_{n=1}^N \frac{1}{n!} \|h_n\|^2 \right)^{\frac{1}{2}} \leq \frac{2}{\eta} \|h\|_{H_{CM}}$. Hence it turns out that

$$(2.5) \quad \text{the first term} \leq \sup \left\{ \left(\prod_{k \in \mathbb{N}_0} \mathbf{P}_k \right) (A); \left(\prod_{k \in \mathbb{N}_0} dx_k \right) (A) \leq \frac{2}{\eta} \|h\|_{H_{CM}} \right\}.$$

Next we estimate the second term. To do so, we temporarily denote by Φ the integrand in expectation $\mathbf{E}^{\prod_{k \in \mathbb{N}_0} \mathbf{P}_k}$. Since $0 \leq \Phi \leq 1$,

$$(2.6) \quad \text{the second term} \leq \delta + \left(\prod_{k \in \mathbb{N}_0} \mathbf{P}_k \right) (\Phi > \delta)$$

for $\forall \delta > 0$. In the following we treat the right hand side.

By Chebyshev's inequality and then by Schwarz inequality

$$\begin{aligned}
\Phi &\leq \frac{2}{\eta} \sum_{n=1}^N \left(\frac{1}{\sqrt{N}} \right)^n \sum_{r=1}^n \frac{1}{r!(n-r)!} \sum_{\substack{j_1, \dots, j_r \in \mathbb{N}_1 \cap \{1, \dots, N\}; \\ j_1, \dots, j_r \text{ are distinct}}} \\
&\quad \times \int_{\mathbb{T}^r} \left| \sum_{\substack{i_1, \dots, i_{n-r} \in \mathbb{N}_0 \cap \{1, \dots, N\}; \\ i_1, \dots, i_{n-r} \text{ are distinct}}} h_n(x_{i_1} + p\alpha_{i_1}, \dots, x_{i_{n-r}} + p\alpha_{i_{n-r}}, \right. \\
&\quad \left. x_{j_1} + p\alpha_{j_1}, \dots, x_{j_r} + p\alpha_{j_r}) \right|
\end{aligned}$$

$$\begin{aligned}
 & \times \rho_{j_1}(x_{j_1}) \times \cdots \times \rho_{j_r}(x_{j_r}) dx_{j_1} \cdots dx_{j_r} \\
 \leq & \frac{2}{\eta} \sum_{n=1}^N \left(\frac{1}{\sqrt{N}}\right)^n \sum_{r=1}^n \frac{1}{r!(n-r)!} \sum_{\substack{j_1, \dots, j_r \in \mathbb{N}_1 \cap \{1, \dots, N\}; \\ j_1, \dots, j_r \text{ are distinct}}} \\
 & \times \left\| \sum_{\substack{i_1, \dots, i_{n-r} \in \mathbb{N}_0 \cap \{1, \dots, N\}; \\ i_1, \dots, i_{n-r} \text{ are distinct}}} h_n(x_{i_1} + p\alpha_{i_1}, \dots, x_{i_{n-r}} + p\alpha_{i_{n-r}}, \cdot) \right\|_{\mathcal{L}_2^r} \\
 & \times \|\rho_{j_1}\| \times \cdots \times \|\rho_{j_r}\| \\
 \leq & \frac{2}{\eta} \sum_{n=1}^N \sum_{r=1}^n \frac{(CM)^r}{r!(n-r)!} \left(\frac{1}{\sqrt{N}}\right)^{n-r} \\
 & \times \left\| \sum_{\substack{i_1, \dots, i_{n-r} \in \mathbb{N}_0 \cap \{1, \dots, N\}; \\ i_1, \dots, i_{n-r} \text{ are distinct}}} h_n(x_{i_1} + p\alpha_{i_1}, \dots, x_{i_{n-r}} + p\alpha_{i_{n-r}}, \cdot) \right\|_{\mathcal{L}_2^r}.
 \end{aligned}$$

Here we have used the conditions (b.ii) and (b.iv) in the last line. By this estimate, and then by Chebyshev's inequality and Schwarz one again

$$\begin{aligned}
 & \left(\prod_{k \in \mathbb{N}_0} dx_k \right) (\Phi > \delta) \\
 \leq & \frac{1}{\delta} \frac{2}{\eta} \sum_{n=1}^N \sum_{r=1}^n \frac{(CM)^r}{r!(n-r)!} \left(\frac{1}{\sqrt{N}}\right)^{n-r} \\
 & \times \left(\mathbf{E}^{\prod_{k \in \mathbb{N}_0} dx_k} \left[\int_{\mathbb{T}^r} \left| \sum_{\substack{i_1, \dots, i_{n-r} \in \mathbb{N}_0 \cap \{1, \dots, N\}; \\ i_1, \dots, i_{n-r} \text{ are distinct}}} \right. \right. \right. \\
 & \quad \times \left. \left. \left. h_n(x_{i_1} + p\alpha_{i_1}, \dots, x_{i_{n-r}} + p\alpha_{i_{n-r}}, \right. \right. \right. \\
 & \quad \left. \left. \left. y_1, \dots, y_r) \right|^2 dy_1 \cdots dy_r \right] \right)^{\frac{1}{2}} \\
 = & \frac{1}{\delta} \frac{2}{\eta} \sum_{n=1}^N \sum_{r=1}^n \frac{(CM)^r}{r!(n-r)!} \left(\frac{1}{\sqrt{N}}\right)^{n-r} \left(\sum_{\substack{i_1, \dots, i_{n-r} \in \mathbb{N}_0 \cap \{1, \dots, N\}; \\ i_1, \dots, i_{n-r} \text{ are distinct}}} (n-r)! \|h_n\|^2 \right)^{\frac{1}{2}}
 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\delta} \frac{2}{\eta} \sum_{n=1}^N \sum_{r=1}^n \frac{(CM)^r}{r!} \frac{\|h_n\|}{\sqrt{(n-r)!}} \\
&\leq \frac{1}{\delta} \frac{2}{\eta} (e^{CM} - 1) \sum_{m=0}^{\infty} \frac{1}{\sqrt{m!}} \|h\|_{H_{CM}}.
\end{aligned}$$

Hence, combining this with (2.6) yields that

the second term

$$\begin{aligned}
(2.7) \quad &\leq \delta + \sup \left\{ \left(\prod_{k \in \mathbb{N}_0} \mathbf{P}_k \right) (A); \left(\prod_{k \in \mathbb{N}_0} dx_k \right) (A) \right. \\
&\quad \left. \leq \frac{1}{\delta} \frac{2}{\eta} (e^{CM} - 1) \sum_{m=0}^{\infty} \frac{1}{\sqrt{m!}} \|h\|_{H_{CM}} \right\}.
\end{aligned}$$

Now, letting $\delta = \left(\frac{2}{\eta} \|h\|_{H_{CM}}\right)^{\frac{1}{2}}$ in (2.7), and then putting (2.5) and (2.7) we have

$$\boldsymbol{\mu} \left(|Y_N(\cdot + p\alpha; h)| > \eta \right) \leq \xi \left(\frac{\|h\|_{H_{CM}}}{\eta} \right),$$

where

$$\begin{aligned}
\xi(t) &= \sup \left\{ \left(\prod_{k \in \mathbb{N}_0} \mathbf{P}_k \right) (A); \left(\prod_{k \in \mathbb{N}_0} dx_k \right) (A) \leq 2t \right\} \\
&\quad + \sqrt{2t} + \sup \left\{ \left(\prod_{k \in \mathbb{N}_0} \mathbf{P}_k \right) (A); \left(\prod_{k \in \mathbb{N}_0} dx_k \right) (A) \right. \\
&\quad \left. \leq (e^{CM} - 1) \sum_{m=0}^{\infty} \frac{1}{\sqrt{m!}} \sqrt{2t} \right\}.
\end{aligned}$$

Since our condition (b.i) is equivalent to that $\prod_{k \in \mathbb{N}_0} \mathbf{P}_k \ll \prod_{k \in \mathbb{N}_0} dx_k$ by Kakutani dichotomy theorem, ξ satisfies $\xi(0+) = \xi(0) = 0$, and thus this is the desired function. The proof is complete. \square

PROOF OF THEOREM 2.1. Let $\alpha \in \mathbb{T}_{\mu, \{N_m\}}^{\infty}$ and $\nu \ll \mu$. Set $Z := \frac{d\nu}{d\mu}$. Note that $Z \geq 0$ and $\mathbf{E}^{\mu}[Z] = 1$. The proof is done in two steps.

$$\begin{aligned}
1^{\circ} \quad &\left(1 + Y_{N_m}(\cdot + p\alpha; h^{\psi}) \right)_{p \in \mathbb{Z}, \psi \in \mathcal{CL}_2} \quad \text{under } \nu \\
&\xrightarrow{\text{f.d.}} \left(e^{I^{(p)}(\psi; \mu) - \frac{1}{2} \|\psi\|^2} \right)_{p \in \mathbb{Z}, \psi \in \mathcal{CL}_2} \quad \text{as } m \rightarrow \infty.
\end{aligned}$$

PROOF. By Proposition 2 and (2.2),

$$(2.8) \quad \begin{aligned} & \boldsymbol{\mu} \left(|Y_N(\cdot + p\alpha; h^\psi) - Y_N(\cdot + p\alpha; h^{\psi^{(R)}})| > \eta \right) \\ & \leq \xi \left(\text{const}(CM, \|\psi\|) \frac{1}{\eta} \sqrt{\sum_{k=R+1}^{\infty} (\psi, \phi_k)^2} \right). \end{aligned}$$

Clearly

$$(2.9) \quad E \left[|I^{(p)}(\psi; \mu) - I^{(p)}(\psi^{(R)}; \mu)|^2 \right] = \sum_{k=R+1}^{\infty} (\psi, \phi_k)^2.$$

Let $p_0 \in \mathbb{Z}$, $\psi_1, \dots, \psi_L \in \mathcal{CL}_2$ and $(a_{pl})_{\substack{1 \leq p \leq K \\ 1 \leq l \leq L}} \in \mathbb{R}^{KL}$. For $\forall \varepsilon > 0$

$$\begin{aligned} & \left| \mathbf{E}^\nu \left[e^{\sqrt{-1} \sum_{p=1}^K \sum_{l=1}^L a_{pl} (1 + Y_{N_m}(\cdot + (p+p_0)\alpha; h^{\psi_l}))} \right] \right. \\ & \quad \left. - E \left[e^{\sqrt{-1} \sum_{p=1}^K \sum_{l=1}^L a_{pl} e^{I^{(p+p_0)}(\psi_l; \mu) - \frac{1}{2} \|\psi_l\|^2}} \right] \right| \\ & \leq \left| \mathbf{E}^\mu \left[Z \left(e^{\sqrt{-1} \sum_{p=1}^K \sum_{l=1}^L a_{pl} (1 + Y_{N_m}(\cdot + (p+p_0)\alpha; h^{\psi_l}))} \right) \right. \right. \\ & \quad \left. \left. - e^{\sqrt{-1} \sum_{p=1}^K \sum_{l=1}^L a_{pl} (1 + Y_{N_m}(\cdot + (p+p_0)\alpha; h^{\psi_l^{(R)}}))} \right) \right] \right| \\ & \quad + \left| \mathbf{E}^\nu \left[e^{\sqrt{-1} \sum_{p=1}^K \sum_{l=1}^L a_{pl} (1 + Y_{N_m}(\cdot + (p+p_0)\alpha; h^{\psi_l^{(R)}}))} \right] \right. \\ & \quad \left. - E \left[e^{\sqrt{-1} \sum_{p=1}^K \sum_{l=1}^L a_{pl} e^{I^{(p+p_0)}(\psi_l^{(R)}; \mu) - \frac{1}{2} \|\psi_l^{(R)}\|^2}} \right] \right| \\ & \quad + \left| E \left[e^{\sqrt{-1} \sum_{p=1}^K \sum_{l=1}^L a_{pl} e^{I^{(p+p_0)}(\psi_l^{(R)}; \mu) - \frac{1}{2} \|\psi_l^{(R)}\|^2}} \right] \right. \\ & \quad \left. - e^{\sqrt{-1} \sum_{p=1}^K \sum_{l=1}^L a_{pl} e^{I^{(p+p_0)}(\psi_l; \mu) - \frac{1}{2} \|\psi_l\|^2}} \right] \right| \\ & \leq 2 \sup_{\boldsymbol{\mu}(B) \leq \delta_R(\varepsilon)} \mathbf{E}^\mu [Z; B] + \varepsilon \\ & \quad + \left| \mathbf{E}^\nu \left[e^{\sqrt{-1} \sum_{p=1}^K \sum_{l=1}^L a_{pl} (1 + Y_{N_m}(\cdot + (p+p_0)\alpha; h^{\psi_l^{(R)}}))} \right] \right. \\ & \quad \left. - E \left[e^{\sqrt{-1} \sum_{p=1}^K \sum_{l=1}^L a_{pl} e^{I^{(p+p_0)}(\psi_l^{(R)}; \mu) - \frac{1}{2} \|\psi_l^{(R)}\|^2}} \right] \right| \end{aligned}$$

$$+ \left| E \left[e^{\sqrt{-1} \sum_{p=1}^K \sum_{l=1}^L a_{pl}} e^{I^{(p+p_0)}(\psi_l^{(R)}; \mu) - \frac{1}{2} \|\psi_l^{(R)}\|^2} - e^{\sqrt{-1} \sum_{p=1}^K \sum_{l=1}^L a_{pl}} e^{I^{(p+p_0)}(\psi_l; \mu) - \frac{1}{2} \|\psi_l\|^2} \right] \right|.$$

Here $\delta_R(\varepsilon)$ in the first term is

$$\sup_{N \in \mathbb{N}} \mu \left(\left| \sum_{\substack{1 \leq p \leq K \\ 1 \leq l \leq L}} a_{pl} (Y_N(\cdot + (p+p_0)\alpha; h^{\psi_l}) - Y_N(\cdot + (p+p_0)\alpha; h^{\psi_l^{(R)}})) \right| > \varepsilon \right).$$

By (2.8) this converges to 0 as $R \rightarrow \infty$. Also, by (2.9) the fourth term $\rightarrow 0$ as $R \rightarrow \infty$, and by Claim 2.1 the third term $\rightarrow 0$ as $m \rightarrow \infty$. Hence we have

$$\limsup_{m \rightarrow \infty} \left| \mathbf{E}^\nu \left[e^{\sqrt{-1} \sum_{p=1}^K \sum_{l=1}^L a_{pl} (1 + Y_{N_m}(\cdot + (p+p_0)\alpha; h^{\psi_l}))} - E \left[e^{\sqrt{-1} \sum_{p=1}^K \sum_{l=1}^L a_{pl}} e^{I^{(p+p_0)}(\psi_l; \mu) - \frac{1}{2} \|\psi_l\|^2} \right] \right] \right| \leq \varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0,$$

which implies the assertion. \square

2° Take $h^{(1)}, \dots, h^{(L)} \in H_{CM}$ arbitrarily. By virtue of (2.1), for $\forall \varepsilon > 0$ and $1 \leq \forall l \leq L$ there exist $\psi_{l1}, \dots, \psi_{lk_l} \in \mathcal{CL}_2$ and $t_{l1}, \dots, t_{lk_l} \in \mathbb{R}$ such that $\|h^{(l)} - \sum_{i=1}^{k_l} t_{li} h^{\psi_{li}}\|_{H_{CM}} < \varepsilon$. By Propositions 2 and 1(ii)

$$\mu \left(\left| Y_N(\cdot + p\alpha; h^{(l)}) - \sum_{i=1}^{k_l} t_{li} Y_N(\cdot + p\alpha; h^{\psi_{li}}) \right| > \sqrt{\varepsilon} \right) \leq \xi(\sqrt{\varepsilon}),$$

$$E \left[\left| \sum_{n=1}^{\infty} \frac{1}{n!} I_n^{(p)} \left(\sum_{i=1}^{k_l} t_{li} \psi_{li}^{\otimes n}; \mu \right) - \sum_{n=1}^{\infty} \frac{1}{n!} I_n^{(p)} (h_n^{(l)}; \mu) \right|^2 \right] < \varepsilon^2.$$

On the other hand, by Proposition 1(i)

$$e^{I^{(p)}(\psi; \mu) - \frac{1}{2} \|\psi\|^2} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} I_n^{(p)}(\psi^{\otimes n}; \mu), \quad \forall \psi \in \mathcal{CL}_2,$$

and hence

$$\sum_{i=1}^{k_l} t_{li} \left(e^{I^{(p)}(\psi_{li}; \mu) - \frac{1}{2} \|\psi_{li}\|^2} - 1 \right) = \sum_{n=1}^{\infty} \frac{1}{n!} I_n^{(p)} \left(\sum_{i=1}^{k_l} t_{li} \psi_{li}^{\otimes n}; \mu \right).$$

Therefore, collecting these we have that for $\forall (a_{pl})_{\substack{1 \leq p \leq K \\ 1 \leq l \leq L}} \in \mathbb{R}^{KL}$

$$\begin{aligned}
 & \left| \mathbf{E}^\nu \left[e^{\sqrt{-1} \sum_{p=1}^K \sum_{l=1}^L a_{pl} Y_{N_m}(\cdot + (p+p_0)\alpha; h^{(l)})} \right] \right. \\
 & \quad \left. - E \left[e^{\sqrt{-1} \sum_{p=1}^K \sum_{l=1}^L a_{pl} \sum_{n=1}^\infty \frac{1}{n!} I_n^{(p+p_0)}(h_n^{(l)}; \mu)} \right] \right| \\
 & \leq \left| \mathbf{E}^\mu \left[Z \left(e^{\sqrt{-1} \sum_{p=1}^K \sum_{l=1}^L a_{pl} Y_{N_m}(\cdot + (p+p_0)\alpha; h^{(l)})} \right. \right. \right. \\
 & \quad \left. \left. - e^{\sqrt{-1} \sum_{p=1}^K \sum_{l=1}^L a_{pl} \sum_{i=1}^{k_l} t_{li} Y_{N_m}(\cdot + (p+p_0)\alpha; h^{\psi_{li}})} \right) \right] \right| \\
 & \quad + \left| \mathbf{E}^\nu \left[e^{\sqrt{-1} \sum_{p=1}^K \sum_{l=1}^L \sum_{i=1}^{k_l} a_{pl} t_{li} Y_{N_m}(\cdot + (p+p_0)\alpha; h^{\psi_{li}})} \right] \right. \\
 & \quad \left. - E \left[e^{\sqrt{-1} \sum_{p=1}^K \sum_{l=1}^L \sum_{i=1}^{k_l} a_{pl} t_{li} (e^{I^{(p+p_0)}(\psi_{li}; \mu) - \frac{1}{2} \|\psi_{li}\|^2} - 1)} \right] \right| \\
 & \quad + \left| E \left[e^{\sqrt{-1} \sum_{p=1}^K \sum_{l=1}^L a_{pl} \sum_{n=1}^\infty \frac{1}{n!} I_n^{(p+p_0)}(\sum_{i=1}^{k_l} t_{li} \psi_{li}^{\otimes n}; \mu)} \right] \right. \\
 & \quad \left. - e^{\sqrt{-1} \sum_{p=1}^K \sum_{l=1}^L a_{pl} \sum_{n=1}^\infty \frac{1}{n!} I_n^{(p+p_0)}(h_n^{(l)}; \mu)} \right] \right| \\
 & \leq 2 \sup_{\mu(B) \leq KL\xi(\sqrt{\varepsilon})} \mathbf{E}^\mu[Z; B] + \max_{\substack{1 \leq p \leq K \\ 1 \leq l \leq L}} |a_{pl}| KL\sqrt{\varepsilon} \\
 & \quad + \left| \mathbf{E}^\nu \left[e^{\sqrt{-1} \sum_{p=1}^K \sum_{l=1}^L \sum_{i=1}^{k_l} a_{pl} t_{li} Y_{N_m}(\cdot + (p+p_0)\alpha; h^{\psi_{li}})} \right] \right. \\
 & \quad \left. - E \left[e^{\sqrt{-1} \sum_{p=1}^K \sum_{l=1}^L \sum_{i=1}^{k_l} a_{pl} t_{li} (e^{I^{(p+p_0)}(\psi_{li}; \mu) - \frac{1}{2} \|\psi_{li}\|^2} - 1)} \right] \right| \\
 & \quad + \sum_{\substack{1 \leq p \leq K \\ 1 \leq l \leq L}} |a_{pl}| \varepsilon \\
 & \xrightarrow{\substack{\text{first } m \rightarrow \infty \\ \text{second } \varepsilon \rightarrow 0}} 0,
 \end{aligned}$$

which implies the conclusion of the theorem. \square

3. Central Limit Theorem for Correlated Multiple Wiener Integrals

In this section we study the CLT for a sequence $\{I_n^{(p)}(\cdot; \mu)\}_{p=1}^\infty$.

We begin with the following. The system $\left(I^{(p)}(f) \right)_{p \in \mathbb{Z}, f \in \mathcal{CL}_2}$ in Theorem 1.2 is nondegenerate in the sense that for \forall linearly independent $f_1, \dots, f_L \in$

\mathcal{CL}_2 , $\forall p_0 \in \mathbb{Z}$ and $\forall K \in \mathbb{N}$, a random vector $\left(I^{(p+p_0)}(f_i) \right)_{1 \leq p \leq K, 1 \leq i \leq L}$ is nondegenerate Gaussian. This nondegeneracy is completely determined for general Gaussian system $\left(I^{(p)}(f; \mu) \right)_{p \in \mathbb{Z}, f \in \mathcal{CL}_2}$:

THEOREM 3.1. *The Gaussian system $\left(I^{(p)}(f; \mu) \right)_{p \in \mathbb{Z}, f \in \mathcal{CL}_2}$ is nondegenerate in the sense above, i.e., it holds that for \forall linearly independent $f_1, \dots, f_L \in \mathcal{CL}_2$, $\forall p_0 \in \mathbb{Z}$ and $\forall K \in \mathbb{N}$, a random vector $\left(I^{(p+p_0)}(f_i; \mu) \right)_{1 \leq p \leq K, 1 \leq i \leq L}$ is nondegenerate Gaussian, if and only if $\# \text{supp } \mu = \infty$.*

PROOF. First as for the “if” part. We suppose $\# \text{supp } \mu = \infty$ and show the nondegeneracy of $\left(I^{(p+p_0)}(f_i; \mu) \right)_{1 \leq p \leq K, 1 \leq i \leq L}$ where $f_1, \dots, f_L \in \mathcal{CL}_2$ are linearly independent, $p_0 \in \mathbb{Z}$ and $K \in \mathbb{N}$. For this let $(\xi_{pi})_{1 \leq p \leq K, 1 \leq i \leq L} \in \mathbb{R}^{KL}$ be such that $\sum_{p=1}^K \sum_{i=1}^L \xi_{pi} I^{(p+p_0)}(f_i; \mu) = 0$. Then

$$\begin{aligned} 0 &= E \left[\left(\sum_{p=1}^K \sum_{i=1}^L \xi_{pi} I^{(p+p_0)}(f_i; \mu) \right)^2 \right] \\ &= \sum_{|n| \geq 1} \int_{\mathbb{T}} \left| \sum_{p=1}^K \sum_{i=1}^L \xi_{pi} \widehat{f}_i(n) e^{\sqrt{-1} 2\pi n p x} \right|^2 \mu(dx), \end{aligned}$$

and hence

$$\mu \left(\{x \in \mathbb{T}; e^{\sqrt{-1} 2\pi n x} \in Z(n)\} \right) = 1, \quad \forall n \geq 1,$$

where

$$Z(n) := \left\{ z \in \mathbb{C}; \sum_{p=1}^K \left(\sum_{i=1}^L \xi_{pi} \widehat{f}_i(n) \right) z^p = 0 \right\}.$$

Since $\{x \in \mathbb{T}; e^{\sqrt{-1} 2\pi n x} \in Z(n)\}$ is closed in \mathbb{T} , this implies

$$\text{supp } \mu \subset \{x \in \mathbb{T}; e^{\sqrt{-1} 2\pi n x} \in Z(n)\},$$

so from the assumption

$$\#\{x \in \mathbb{T}; e^{\sqrt{-1}2\pi nx} \in Z(n)\} = \infty.$$

But, by the fundamental theorem of algebra we observe that $\left(\sum_{i=1}^L \xi_{pi} \widehat{f}_i(n)\right)_{1 \leq p \leq K} \neq 0$ implies $\#Z(n) \leq K$. Consequently $\left(\sum_{i=1}^L \xi_{pi} \widehat{f}_i(n)\right)_{1 \leq p \leq K} = 0$ ($\forall n \geq 1$), i.e., $\sum_{i=1}^L \xi_{pi} f_i = 0$ ($1 \leq \forall p \leq K$). By the linear independence of f_1, \dots, f_L , this implies $(\xi_{pi})_{1 \leq p \leq K, 1 \leq i \leq L} = 0$, and the nondegeneracy of $\left(I^{(p+p_0)}(f_i; \mu)\right)_{1 \leq p \leq K, 1 \leq i \leq L}$ is obtained.

Next as for the “only if” part. We suppose $\#\text{supp } \mu < \infty$. Then there exist $0 \leq \theta_1 < \dots < \theta_m < 1$ and $a_1, \dots, a_m > 0$ such that $\sum_{i=1}^m a_i = 1$ and $\mu = \sum_{i=1}^m a_i \delta_{\theta_i}$. For $N \in \mathbb{N}$ we define $(\beta_0, \dots, \beta_{2Nm}) \in \mathbb{R}^{2Nm+1}$ as

$$\begin{aligned} \sum_{p=0}^{2Nm} \beta_p z^p &= \prod_{n=1}^N \prod_{i=1}^m (z - e^{\sqrt{-1}2\pi n\theta_i})(z - e^{-\sqrt{-1}2\pi n\theta_i}) \\ &= \prod_{n=1}^N \prod_{i=1}^m (z^2 - 2(\cos 2\pi n\theta_i)z + 1). \end{aligned}$$

By definition, for $f \in \mathcal{CL}_2$ such that $\widehat{f}(n) = 0$ ($\forall |n| \geq N+1$)

$$\begin{aligned} E \left[\left(\sum_{p=1}^{2Nm+1} \beta_{p-1} I^{(p+p_0)}(f; \mu) \right)^2 \right] \\ &= \sum_{1 \leq |n| \leq N} |\widehat{f}(n)|^2 \sum_{i=1}^m a_i \left| \sum_{p=0}^{2Nm} \beta_p (e^{\sqrt{-1}2\pi n\theta_i})^p \right|^2 \\ &= 0, \end{aligned}$$

and also $(\beta_0, \dots, \beta_{2Nm}) \neq 0$ because $\beta_0 = \beta_{2Nm} = 1$. These say the degeneracy of $\left(I^{(p+p_0)}(f; \mu)\right)_{1 \leq p \leq 2Nm+1}$, so that $\#\text{supp } \mu = \infty$ is necessary. \square

For $h_k \in \mathcal{CSL}_2^k$ ($1 \leq k \leq L$), $\{(I_1^{(p)}(h_1), \dots, I_L^{(p)}(h_L))\}_{p \in \mathbb{Z}}$ is a sequence of i.i.d. random vectors with mean zero and finite covariance. The Lindeberg CLT says

$$\frac{1}{\sqrt{P}} \sum_{p=1}^P (I_1^{(p)}(h_1), \dots, I_L^{(p)}(h_L)) \xrightarrow{P \rightarrow \infty} \mathfrak{N} \left(0, \begin{bmatrix} 1! \|h_1\|^2 & & 0 \\ & \ddots & \\ 0 & & L! \|h_L\|^2 \end{bmatrix} \right).$$

For a sequence $\{I_n^{(p)}(\cdot; \mu)\}_{p=1}^\infty$ of general multiple Wiener integrals this is also valid when μ is a “good” probability measure on \mathbb{T} . To see this, in the following we suppose that $\mu(dx) \ll dx$, i.e., $\mu(dx)$ is absolutely continuous relative to dx , and let $m := \frac{d\mu}{dx}$.

PROPOSITION 3. For $\forall h_n \in \mathcal{CSL}_2^n$, $\forall k_l \in \mathcal{CSL}_2^l$ and $\forall p, q \in \mathbb{Z}$

$$\begin{aligned} & E \left[I_n^{(p)}(h_n; \mu) I_l^{(q)}(k_l; \mu) \right] \\ &= \begin{cases} n! \sum_{|i_1|, \dots, |i_n| \geq 1} \widehat{h}_n(i_1, \dots, i_n) \overline{\widehat{k}_n(i_1, \dots, i_n)} \\ \quad \times \widehat{\mu}(i_1(q-p)) \cdots \widehat{\mu}(i_n(q-p)) & \text{if } n = l, \\ 0 & \text{if } n \neq l. \end{cases} \end{aligned}$$

Here

$$\widehat{h}_n(i_1, \dots, i_n) := \int_{\mathbb{T}^n} e^{-\sqrt{-1} 2\pi(i_1 x_1 + \cdots + i_n x_n)} h_n(x_1, \dots, x_n) dx_1 \cdots dx_n.$$

PROOF. The proof is done in three steps.

1° For $\forall p, q \in \mathbb{Z}$, $\forall n, l \in \mathbb{N}$ and $\forall f, g \in \mathcal{CL}_2$ such that $\|f\| = \|g\| = 1$

$$E \left[H_n(I^{(p)}(f; \mu)) H_l(I^{(q)}(g; \mu)) \right] = \delta_{nl} \frac{1}{n!} \left(E \left[I^{(p)}(f; \mu) I^{(q)}(g; \mu) \right] \right)^n.$$

PROOF. In case $p = q$ and $|(f, g)| = 1$ (i.e., $f = \pm g$), we have

$$\begin{aligned} \text{the left hand side} &= (\pm 1)^n \int_{\mathbb{R}} H_n(x) H_l(x) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &= (\pm 1)^n \delta_{nl} \frac{1}{n!} = \text{the right hand side.} \end{aligned}$$

In case $p \neq q$ or $|(f, g)| < 1$, by Theorem 3.1 $(I^{(p)}(f; \mu), I^{(q)}(g; \mu))$ is nondegenerate Gaussian and its density is as follows:

$$P\left((I^{(p)}(f; \mu), I^{(q)}(g; \mu)) \in dx dy\right) = \frac{1}{2\pi} \frac{1}{\sqrt{1-c^2}} e^{-\frac{1}{2} \frac{1}{1-c^2} (x^2 - 2cxy + y^2)} dx dy,$$

where $c := E\left[I^{(p)}(f; \mu) I^{(q)}(g; \mu)\right]$. Without loss of generality let $n \leq l$. By the expression above

the left hand side

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi(1-c^2)}} e^{-\frac{x^2}{2(1-c^2)}} dx \int_{\mathbb{R}} H_l(y) H_n(x + cy) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy.$$

To compute the y -integral, note that

$$\begin{aligned} H_j(y) e^{-\frac{y^2}{2}} &= \frac{(-1)^j}{j!} \frac{d^j}{dy^j} \left(e^{-\frac{y^2}{2}} \right), \quad j \geq 0, \\ H'_j(y) &= H_{j-1}(y), \quad j \geq 1. \end{aligned}$$

By integrating by parts n -times it turns out that

$$\int_{\mathbb{R}} H_l(y) H_n(x + cy) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = \delta_{nl} \frac{c^n}{n!}.$$

Hence substituting this into the above we have the desired identity. \square

2° For $\forall n, l \in \mathbb{N}$ and $\forall p, q \in \mathbb{Z}$

$$\begin{aligned} &E \left[I_n^{(p)} \left(\sum_{i:\text{finite sum}} c_i f_i^{\otimes n}; \mu \right) I_l^{(q)} \left(\sum_{j:\text{finite sum}} d_j g_j^{\otimes l}; \mu \right) \right] \\ &= \begin{cases} n! \sum_{|k_1|, \dots, |k_n| \geq 1} \widehat{\sum_{i:\text{finite sum}} c_i f_i^{\otimes n}(k_1, \dots, k_n)} \\ \quad \times \widehat{\sum_{j:\text{finite sum}} d_j g_j^{\otimes n}(k_1, \dots, k_n)} \\ \quad \times \widehat{\mu}(k_1(q-p)) \cdots \widehat{\mu}(k_n(q-p)) & \text{if } n = l, \\ 0 & \text{if } n \neq l. \end{cases} \end{aligned}$$

PROOF. Combining Proposition 1 and 1° we have that for $\forall p, q \in \mathbb{Z}$, $\forall n, l \in \mathbb{N}$ and $\forall f, g \in \mathcal{CL}_2$

$$\begin{aligned} & E \left[I_n^{(p)}(f^{\otimes n}; \mu) I_l^{(q)}(g^{\otimes l}; \mu) \right] \\ &= \delta_{nl} n! \left(E \left[I^{(p)}(f; \mu) I^{(q)}(g; \mu) \right] \right)^n \\ &= \delta_{nl} n! \sum_{|k_1|, \dots, |k_n| \geq 1} \widehat{f^{\otimes n}}(k_1, \dots, k_n) \overline{\widehat{g^{\otimes n}}(k_1, \dots, k_n)} \\ &\quad \times \widehat{\mu}(k_1(q-p)) \cdots \widehat{\mu}(k_n(q-p)). \end{aligned}$$

From this the assertion follows immediately. \square

3° Let $h_n, k_n \in \mathcal{CSL}_2^n$. From (2.1) we take, for $\forall \varepsilon > 0$, $\sum_{i: \text{finite sum}} c_i f_i^{\otimes n} =: h_{n\varepsilon}$ and $\sum_{j: \text{finite sum}} d_j g_j^{\otimes n} =: k_{n\varepsilon}$ such that $\|h_n - h_{n\varepsilon}\|, \|k_n - k_{n\varepsilon}\| < \varepsilon$. By Proposition 1 this implies

$$\begin{aligned} & \left| E \left[I_n^{(p)}(h_n; \mu) I_n^{(q)}(k_n; \mu) \right] - E \left[I_n^{(p)}(h_{n\varepsilon}; \mu) I_n^{(q)}(k_{n\varepsilon}; \mu) \right] \right| \\ & \leq n! (\|h_n\| + \|k_n\| + \varepsilon) \varepsilon. \end{aligned}$$

On the other hand

$$\begin{aligned} & \left| \sum_{|i_1|, \dots, |i_n| \geq 1} \widehat{h_n}(i_1, \dots, i_n) \overline{\widehat{k_n}(i_1, \dots, i_n)} \widehat{\mu}(i_1(q-p)) \cdots \widehat{\mu}(i_n(q-p)) \right. \\ & \quad \left. - \sum_{|i_1|, \dots, |i_n| \geq 1} \widehat{h_{n\varepsilon}}(i_1, \dots, i_n) \overline{\widehat{k_{n\varepsilon}}(i_1, \dots, i_n)} \widehat{\mu}(i_1(q-p)) \cdots \widehat{\mu}(i_n(q-p)) \right| \\ & \leq \varepsilon (\|h_n\| + \|k_n\| + \varepsilon). \end{aligned}$$

Combining these with 2° we have

$$\begin{aligned} & \left| E \left[I_n^{(p)}(h_n; \mu) I_n^{(q)}(k_n; \mu) \right] \right. \\ & \quad \left. - n! \sum_{|i_1|, \dots, |i_n| \geq 1} \widehat{h_n}(i_1, \dots, i_n) \overline{\widehat{k_n}(i_1, \dots, i_n)} \widehat{\mu}(i_1(q-p)) \cdots \widehat{\mu}(i_n(q-p)) \right| \\ & \leq 2n! \varepsilon (\|h_n\| + \|k_n\| + \varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0, \end{aligned}$$

which is the conclusion of the proposition for $n = l$. The conclusion for $n \neq l$ will follow from the first part and 2°. \square

CLAIM 3.1. Suppose $m \in C(\mathbb{T})$ and let $h_n \in \mathcal{CSL}_2^n$.

$$(i) \quad E \left[\left(\frac{1}{\sqrt{P}} \sum_{p=1}^P I_n^{(p)}(h_n; \mu) \right)^2 \right] \leq \frac{\pi^2}{4} n! \|m\|_\infty^n \|h_n\|^2, \quad \forall P \geq 1.$$

$$(ii) \quad \lim_{P \rightarrow \infty} E \left[\left(\frac{1}{\sqrt{P}} \sum_{p=1}^P I_n^{(p)}(h_n; \mu) \right)^2 \right] = \begin{cases} g(0) & \text{if } n = 1, \\ n! \int_{\mathbb{T}^{n-1}} g(-t_1, t_1 - t_2, \dots, \\ \quad t_{n-2} - t_{n-1}, t_{n-1}) dt_1 \cdots dt_{n-1} & \text{if } n \geq 2. \end{cases}$$

Here

$$g(x_1, \dots, x_n) := \sum_{k_1, \dots, k_n \geq 1} \sum_{\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}} |\widehat{h}_n(\varepsilon_1 k_1, \dots, \varepsilon_n k_n)|^2 \\ \times \frac{1}{k_1 \cdots k_n} \sum_{j_1=1}^{k_1} \cdots \sum_{j_n=1}^{k_n} m \left(\frac{\varepsilon_1 x_1 + j_1 - \frac{\varepsilon_1 + 1}{2}}{k_1} \right) \times \cdots \times m \left(\frac{\varepsilon_n x_n + j_n - \frac{\varepsilon_n + 1}{2}}{k_n} \right).$$

PROOF. Let $h_n \in \mathcal{CSL}_2^n$. By Proposition 3

$$E \left[\left(\sum_{p=1}^P I_n^{(p)}(h_n; \mu) \right)^2 \right] \\ = n! \sum_{|i_1|, \dots, |i_n| \geq 1} |\widehat{h}_n(i_1, \dots, i_n)|^2 \\ \times \int_{\mathbb{T}^n} \left| \sum_{p=1}^P e^{\sqrt{-1} 2\pi p(i_1 x_1 + \dots + i_n x_n)} \right|^2 m(x_1) \cdots m(x_n) dx_1 \cdots dx_n \\ = n! \sum_{i_1, \dots, i_n \geq 1} \sum_{\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}} |\widehat{h}_n(\varepsilon_1 i_1, \dots, \varepsilon_n i_n)|^2 \\ \times \int_{\mathbb{T}^n} \left| \sum_{p=1}^P e^{\sqrt{-1} 2\pi p(\varepsilon_1 i_1 x_1 + \dots + \varepsilon_n i_n x_n)} \right|^2 m(x_1) \cdots m(x_n) dx_1 \cdots dx_n.$$

The (x_1, \dots, x_n) -integral in the last line can be further computed as

$$\begin{aligned}
& \int_{\mathbb{T}^n} \left| \sum_{p=1}^P e^{\sqrt{-1} 2\pi p(\varepsilon_1 i_1 x_1 + \dots + \varepsilon_n i_n x_n)} \right|^2 m(x_1) \cdots m(x_n) dx_1 \cdots dx_n \\
&= \sum_{\substack{1 \leq j_1 \leq i_1 \\ \dots \\ 1 \leq j_n \leq i_n}} \int_{\frac{j_1-1}{i_1}}^{\frac{j_1}{i_1}} \cdots \int_{\frac{j_n-1}{i_n}}^{\frac{j_n}{i_n}} \left| \sum_{p=1}^P e^{\sqrt{-1} 2\pi p(\varepsilon_1 i_1 x_1 + \dots + \varepsilon_n i_n x_n)} \right|^2 \\
&\quad \times m(x_1) \cdots m(x_n) dx_1 \cdots dx_n \\
&= \sum_{\substack{1 \leq j_1 \leq i_1 \\ \dots \\ 1 \leq j_n \leq i_n}} \int_0^{\frac{1}{i_1}} \cdots \int_0^{\frac{1}{i_n}} \left| \sum_{p=1}^P e^{\sqrt{-1} 2\pi p(\varepsilon_1 i_1 x_1 + \dots + \varepsilon_n i_n x_n)} \right|^2 \\
&\quad \times m\left(x_1 + \frac{j_1-1}{i_1}\right) \cdots m\left(x_n + \frac{j_n-1}{i_n}\right) dx_1 \cdots dx_n \\
&= \sum_{\substack{1 \leq j_1 \leq i_1 \\ \dots \\ 1 \leq j_n \leq i_n}} \int_{\mathbb{T}^n} \left| \sum_{p=1}^P e^{\sqrt{-1} 2\pi p(x_1 + \dots + x_n)} \right|^2 \\
&\quad \times m\left(\frac{\varepsilon_1 x_1 + j_1 - \frac{\varepsilon_1 + 1}{2}}{i_1}\right) \cdots m\left(\frac{\varepsilon_n x_n + j_n - \frac{\varepsilon_n + 1}{2}}{i_n}\right) \frac{dx_1}{i_1} \cdots \frac{dx_n}{i_n},
\end{aligned}$$

so that

$$\begin{aligned}
& E \left[\left(\sum_{p=1}^P I_n^{(p)}(h_n; \mu) \right)^2 \right] \\
&= n! \int_{\mathbb{T}^n} \left| \sum_{p=1}^P e^{\sqrt{-1} 2\pi p(x_1 + \dots + x_n)} \right|^2 g(x_1, \dots, x_n) dx_1 \cdots dx_n.
\end{aligned}$$

From the definition of g it is easily seen that $g \geq 0$, $\int_{\mathbb{T}^n} g(x_1, \dots, x_n) dx_1 \cdots dx_n = \|h_n\|^2 < \infty$, and

$$(3.1) \quad g(x_1, \dots, x_k + 1, \dots, x_n) = g(x_1, \dots, x_k, \dots, x_n),$$

$$(3.2) \quad g(-x_1, \dots, -x_n) = g(x_1, \dots, x_n).$$

By (3.1) we can check that

$$\int_{\mathbb{T}^n} \left| \sum_{p=1}^P e^{\sqrt{-1} 2\pi p(x_1 + \dots + x_n)} \right|^2 g(x_1, \dots, x_n) dx_1 \cdots dx_n$$

$$= \int_{\mathbb{T}} \left| \frac{\sin \pi P x}{\sin \pi x} \right|^2 \kappa(x) dx,$$

where

$$\kappa(x) := \begin{cases} g(x) & \text{if } n = 1, \\ \int_{\mathbb{T}^{n-1}} g(x - t_1, t_1 - t_2, \dots, \\ \quad t_{n-2} - t_{n-1}, t_{n-1}) dt_1 \cdots dt_{n-1} & \text{if } n \geq 2. \end{cases}$$

Consequently, combining two expressions above we have

$$E \left[\left(\frac{1}{\sqrt{P}} \sum_{p=1}^P I_n^{(p)}(h_n; \mu) \right)^2 \right] = n! \frac{1}{P} \int_{\mathbb{T}} \left| \frac{\sin \pi P x}{\sin \pi x} \right|^2 \kappa(x) dx.$$

Now let $m \in C(\mathbb{T})$. Then $g \in C(\mathbb{T}^n)$ with $\|g\|_\infty \leq \|m\|_\infty^n \|h_n\|^2$, and hence $\kappa \in C(\mathbb{T})$ with $\|\kappa\|_\infty \leq \|m\|_\infty^n \|h_n\|^2$. Also, by (3.1) and (3.2), $\kappa(-x) = \kappa(x)$. From these facts

$$\begin{aligned} \frac{1}{P} \int_{\mathbb{T}} \left| \frac{\sin \pi P x}{\sin \pi x} \right|^2 \kappa(x) dx &= \frac{2}{P} \int_0^{\frac{1}{2}} \left| \frac{\sin \pi P x}{\sin \pi x} \right|^2 \kappa(x) dx \\ &= 2 \int_0^\infty \kappa\left(\frac{y}{P}\right) 1_{y < \frac{P}{2}} \left| \frac{\pi \frac{y}{P}}{\sin \pi \frac{y}{P}} \right|^2 \left| \frac{\sin \pi y}{\pi y} \right|^2 dy \\ &\leq 2 \|\kappa\|_\infty \left(\frac{\pi}{2}\right)^2 \int_0^\infty \left| \frac{\sin \pi y}{\pi y} \right|^2 dy \\ &\leq \frac{\pi^2}{4} \|m\|_\infty^n \|h_n\|^2. \end{aligned}$$

By putting the above the assertion (i) is obtained immediately. Also, by letting $P \rightarrow \infty$ the assertion (ii) follows from the Lebesgue convergence theorem. \square

THEOREM 3.2. *Suppose $m \in C(\mathbb{T})$. Then $\{I^{(p)}(\cdot; \mu)\}_{p=1}^\infty$ satisfies the CLT in the sense that for $\forall f_1, \dots, f_L \in \mathcal{CL}_2$*

$$\frac{1}{\sqrt{P}} \sum_{p=1}^P (I^{(p)}(f_1; \mu), \dots, I^{(p)}(f_L; \mu)) \xrightarrow{P \rightarrow \infty} \mathfrak{N}\left(0, \Sigma(f_1, \dots, f_L)\right).$$

Here $\Sigma(f_1, \dots, f_L)$ is a nonnegative symmetric matrix whose (i, j) -component is

$$\begin{aligned} \Sigma_{ij}(f_1, \dots, f_L) &= \sum_{|n| \geq 1} \widehat{f}_i(n) \overline{\widehat{f}_j(n)} \frac{1}{|n|} \sum_{k=1}^{|n|} m\left(\frac{k-1}{|n|}\right) \\ &= \sum_{n=1}^{\infty} 2\operatorname{Re}(\widehat{f}_i(n) \overline{\widehat{f}_j(n)}) \frac{1}{n} \sum_{k=1}^n m\left(\frac{k-1}{n}\right). \end{aligned}$$

Moreover, if m is positive, i.e., $\min_{x \in \mathbb{T}} m(x) > 0$, then the limiting Gaussian $\mathfrak{N}\left(0, \Sigma(f_1, \dots, f_L)\right)$ is nondegenerate in the sense that a matrix $\Sigma(f_1, \dots, f_L)$ is nonsingular whenever $f_1, \dots, f_L \in \mathcal{CL}_2$ are linearly independent.

PROOF. Since, for $\xi = (\xi_1, \dots, \xi_L) \in \mathbb{R}^L$

$$\begin{aligned} &E\left[e^{\sqrt{-1}\sum_{i=1}^L \xi_i \frac{1}{\sqrt{P}} \sum_{p=1}^P I^{(p)}(f_i; \mu)}\right] \\ &= \exp\left\{-\frac{1}{2}E\left[\left(\frac{1}{\sqrt{P}} \sum_{p=1}^P I^{(p)}\left(\sum_{i=1}^L \xi_i f_i; \mu\right)\right)^2\right]\right\}, \end{aligned}$$

Claim 3.1(ii) says

$$\begin{aligned} &\lim_{P \rightarrow \infty} E\left[e^{\sqrt{-1}\sum_{i=1}^L \xi_i \frac{1}{\sqrt{P}} \sum_{p=1}^P I^{(p)}(f_i; \mu)}\right] \\ &= \exp\left\{-\frac{1}{2} \sum_{|n| \geq 1} \left|\sum_{i=1}^L \xi_i \widehat{f}_i(n)\right|^2 \frac{1}{|n|} \sum_{k=1}^{|n|} m\left(\frac{k-1}{|n|}\right)\right\}, \end{aligned}$$

which implies the first part of the theorem. The second part will be clear from $\frac{1}{n} \sum_{k=1}^n m\left(\frac{k-1}{n}\right) \geq \min_{x \in \mathbb{T}} m(x)$. \square

Let us proceed to the CLT for $\{I_n^{(p)}(\cdot; \mu)\}_{p=1}^{\infty}$. For this we define the following:

DEFINITION 3.1. For a stationary sequence $\left\{ (I^{(p)}(f; \mu))_{f \in \mathcal{CL}_2} \right\}_{p \in \mathbb{Z}}$ we define the α -mixing coefficients $(\alpha(N))_{N \geq 1}$ by

$$\alpha(N) := \sup \left\{ |P(A \cap B) - P(A)P(B)|; \begin{array}{l} A \in \sigma(I^{(p)}(f; \mu); f \in \mathcal{CL}_2, p \leq 0 \\ B \in \sigma(I^{(p)}(f; \mu); f \in \mathcal{CL}_2, p \geq N + 1) \end{array} \right\}.$$

CLAIM 3.2. Suppose $m \in C(\mathbb{T})$ and positive. Then

$$\alpha(N) \leq \frac{E_N(m)}{\min_{x \in \mathbb{T}} m(x)}.$$

Here

$$E_N(m) := \inf_{(a_n)_{|n| \leq N} \in \mathbb{C}^{2N+1}} \max_{x \in \mathbb{T}} \left| m(x) - \sum_{|n| \leq N} a_n e^{\sqrt{-1} 2\pi n x} \right|.$$

PROOF. The proof is done in exactly the same way as in Th.17.3.3 of [7]. We set

$$\mathcal{F}_a^b := \sigma(I^{(p)}(f; \mu); f \in \mathcal{CL}_2, a \leq p \leq b)$$

for $a, b \in \mathbb{Z} \cup \{\pm\infty\}$, $a < b$, and

$$\rho(N) := \sup \left\{ \frac{|\text{cov}(X, Y)|}{\sqrt{\text{var}(X)}\sqrt{\text{var}(Y)}}; X \in L_2(\mathcal{F}_{-\infty}^0), Y \in L_2(\mathcal{F}_{N+1}^\infty) \right\}.$$

From [9] this equals

$$\sup \left\{ \frac{|E[XY]|}{\sqrt{E[X^2]}\sqrt{E[Y^2]}}; \begin{array}{l} X \in \text{l.s.} \{I^{(p)}(f; \mu); f \in \mathcal{CL}_2, p \leq 0\}, \neq 0 \\ Y \in \text{l.s.} \{I^{(p)}(f; \mu); f \in \mathcal{CL}_2, p \geq N + 1\}, \neq 0 \end{array} \right\}$$

and satisfies $\alpha(N) \leq \rho(N) \leq 2\pi\alpha(N)$. In the following we treat $\rho(N)$.

We take arbitrarily X and Y from the linear spans above, respectively.

Let

$$\begin{aligned} X &= \sum_{p \leq 0, k \geq 1: \text{finite sum}} c_{pk} I^{(p)}(fk; \mu), \\ Y &= \sum_{q \geq N+1, l \geq 1: \text{finite sum}} d_{ql} I^{(q)}(gl; \mu). \end{aligned}$$

For \forall trigonometric polynomial $P_N(x) = \sum_{|j| \leq N} a_j e^{\sqrt{-1} 2\pi j x}$ of degree N we observe

$$\begin{aligned}
|E[XY]| &= \left| \sum_{|i| \geq 1} \int_{\mathbb{T}} \left(\sum_{p \leq 0, k \geq 1: \text{finite sum}} c_{pk} \widehat{f}_k(i) e^{\sqrt{-1} 2\pi i p x} \right) \right. \\
&\quad \times \left. \left(\sum_{q \geq N+1, l \geq 1: \text{finite sum}} d_{ql} \widehat{g}_l(i) e^{-\sqrt{-1} 2\pi i q x} \right) m(x) dx \right| \\
&= \left| \sum_{|i| \geq 1} \int_{\mathbb{T}} \left(\sum_{p \leq 0, k \geq 1: \text{finite sum}} c_{pk} \widehat{f}_k(i) e^{\sqrt{-1} 2\pi i p x} \right) \right. \\
&\quad \times \left. \left(\sum_{q \geq N+1, l \geq 1: \text{finite sum}} d_{ql} \widehat{g}_l(i) e^{-\sqrt{-1} 2\pi i q x} \right) \right. \\
&\quad \times \left. (m(x) - P_N(x)) dx \right| \\
&\leq \|m - P_N\|_{\infty} \sum_{|i| \geq 1} \sqrt{\int_{\mathbb{T}} \left| \sum_{p \leq 0, k \geq 1: \text{finite sum}} c_{pk} \widehat{f}_k(i) e^{\sqrt{-1} 2\pi i p x} \right|^2 dx} \\
&\quad \times \sqrt{\int_{\mathbb{T}} \left| \sum_{q \geq N+1, l \geq 1: \text{finite sum}} d_{ql} \widehat{g}_l(i) e^{-\sqrt{-1} 2\pi i q x} \right|^2 dx} \\
&\leq \frac{\|m - P_N\|_{\infty}}{\min_{y \in \mathbb{T}} m(y)} \\
&\quad \times \sqrt{\sum_{|i| \geq 1} \int_{\mathbb{T}} \left| \sum_{p \leq 0, k \geq 1: \text{finite sum}} c_{pk} \widehat{f}_k(i) e^{\sqrt{-1} 2\pi i p x} \right|^2 m(x) dx} \\
&\quad \times \sqrt{\sum_{|i| \geq 1} \int_{\mathbb{T}} \left| \sum_{q \geq N+1, l \geq 1: \text{finite sum}} d_{ql} \widehat{g}_l(i) e^{-\sqrt{-1} 2\pi i q x} \right|^2 m(x) dx} \\
&= \frac{\|m - P_N\|_{\infty}}{\min_{y \in \mathbb{T}} m(y)} \sqrt{E[X^2]} \sqrt{E[Y^2]}.
\end{aligned}$$

This implies

$$\rho(N) \leq \frac{\|m - P_N\|_\infty}{\min_{y \in \mathbb{T}} m(y)},$$

and the assertion follows at once. \square

Now we state the CLT for $\{I_n^{(p)}(\cdot; \mu)\}_{p=1}^\infty$.

THEOREM 3.3. *Suppose $m \in C(\mathbb{T})$ and positive. Then for $\forall (h_1, h_2, \dots, h_L) \in \mathcal{CL}_2 \times \mathcal{CSL}_2^2 \times \dots \times \mathcal{CSL}_2^L$,*

$$\frac{1}{\sqrt{P}} \sum_{p=1}^P (I_1^{(p)}(h_1; \mu), \dots, I_L^{(p)}(h_L; \mu)) \xrightarrow{P \rightarrow \infty} \mathfrak{N}\left(0, \begin{bmatrix} v_1 & & 0 \\ & \ddots & \\ 0 & & v_L \end{bmatrix}\right).$$

Here

$$v_k = v(h_k) := \lim_{P \rightarrow \infty} E \left[\left(\frac{1}{\sqrt{P}} \sum_{p=1}^P I_k^{(p)}(h_k; \mu) \right)^2 \right].$$

REMARK 3.1. Since, by Claim 3.1(ii), $v(h_k) \geq k! \left(\min_{x \in \mathbb{T}} m(x) \right)^k \times \|h_k\|^2$, the limiting Gaussian is nondegenerate when $h_1 \neq 0, \dots, h_L \neq 0$.

PROOF. Let $(h_1, h_2, \dots, h_L) \in \mathcal{CL}_2 \times \mathcal{CSL}_2^2 \times \dots \times \mathcal{CSL}_2^L$. The proof is done in three steps.

1° Let $\varepsilon > 0$ be fixed arbitrarily. By (2.1), for each h_k we take $t_{k1}, \dots, t_{kn_k} \in \mathbb{R}$ and $f_{k1}, \dots, f_{kn_k} \in \mathcal{CL}_2$ with $\|f_{ki}\| = 1$ such that $\|h_k - \sum_{i=1}^{n_k} t_{ki} f_{ki}^{\otimes k}\| < \varepsilon$. For simplicity write $h_k^{(\varepsilon)} := \sum_{i=1}^{n_k} t_{ki} f_{ki}^{\otimes k}$. Claim 3.1(i) tells us that for $\forall (\xi_1, \dots, \xi_L) \in \mathbb{R}^L$

$$\begin{aligned} & \left| E \left[e^{\sqrt{-1} \sum_{k=1}^L \xi_k \frac{1}{\sqrt{P}} \sum_{p=1}^P I_k^{(p)}(h_k; \mu)} \right] - E \left[e^{\sqrt{-1} \sum_{k=1}^L \xi_k \frac{1}{\sqrt{P}} \sum_{p=1}^P I_k^{(p)}(h_k^{(\varepsilon)}; \mu)} \right] \right| \\ & \leq \left(\sum_{k=1}^L |\xi_k| (k!)^{\frac{1}{2}} \|m\|_\infty^{\frac{k}{2}} \right) \frac{\pi}{2} \varepsilon, \end{aligned}$$

$$\begin{aligned} & \left| e^{-\frac{1}{2} \sum_{k=1}^L \xi_k^2 v(h_k)} - e^{-\frac{1}{2} \sum_{k=1}^L \xi_k^2 v(h_k^{(\varepsilon)})} \right| \\ & \leq \frac{1}{2} \left(\sum_{k=1}^L \xi_k^2 k! \|m\|_\infty^k (2\|h_k\| + \varepsilon) \right) \frac{\pi^2}{4} \varepsilon. \end{aligned}$$

2° Let $(\xi_1, \dots, \xi_L) \in \mathbb{R}^L$ be fixed and set

$$X_p := \sum_{k=1}^L \xi_k I_k^{(p)}(h_k^{(\varepsilon)}; \mu), \quad p \in \mathbb{Z}.$$

$\{X_p\}_{p \in \mathbb{Z}}$ is a stationary sequence, and it satisfies

$$\sigma(X_p; a \leq p \leq b) \subset \sigma(I^{(p)}(f; \mu); f \in \mathcal{CL}_2, a \leq p \leq b),$$

because by Proposition 1

$$(3.3) \quad I_k^{(p)}(h_k^{(\varepsilon)}; \mu) = \sum_{i=1}^{n_k} t_{ki} I_k^{(p)}(f_{ki}^{\otimes k}; \mu) = \sum_{i=1}^{n_k} t_{ki} k! H_k(I^{(p)}(f_{ki}; \mu)).$$

By Claim 3.2 the α -mixing coefficients $(\alpha(N))$ of $\{X_p\}_{p \in \mathbb{Z}}$ are estimated as

$$\begin{aligned} \alpha(N) & := \sup \left\{ |P(A \cap B) - P(A)P(B)|; \begin{array}{l} A \in \sigma(X_p; p \leq 0), \\ B \in \sigma(X_p; p \geq N+1) \end{array} \right\} \\ & \leq \frac{E_N(m)}{\min_{x \in \mathbb{T}} m(x)}, \quad N \geq 1. \end{aligned}$$

Consequently $\{X_p\}_{p \in \mathbb{Z}}$ is α -mixing (or strongly mixing in terms of [7]).

Let us apply Th.18.4.2 of [7] for $\{X_p\}_{p \in \mathbb{Z}}$. To do so we need to check that a sequence $\{(\frac{1}{\sqrt{P}} \sum_{p=1}^P X_p)^2\}_{P=1}^\infty$ is uniformly integrable. But, by Lemma 3.4 of [17]

$$\lim_{P \rightarrow \infty} E \left[\left(\frac{1}{\sqrt{P}} \sum_{p=1}^P H_k(I^{(p)}(f; \mu)) \right)^4 \right] = 3v_k^2$$

where $f \in \mathcal{CL}_2$ with $\|f\| = 1$ and $v_k = v(\frac{1}{k!} f^{\otimes k}) = \lim_{P \rightarrow \infty} E[(\frac{1}{\sqrt{P}} \sum_{p=1}^P H_k(I^{(p)}(f; \mu))^2] > 0$. This together with (3.3) clearly implies the L_4 -boundedness of $\frac{1}{\sqrt{P}} \sum_{p=1}^P X_p$, $P \in \mathbb{N}$, so the uniform integrability above is valid.

Now, as mentioned earlier, we can apply Th.18.4.2 of [7] to have

$$\lim_{P \rightarrow \infty} E \left[e^{\sqrt{-1} \sum_{k=1}^L \xi_k \frac{1}{\sqrt{P}} \sum_{p=1}^P I_k^{(p)}(h_k^{(\varepsilon)}; \mu)} \right] = e^{-\frac{1}{2} \sum_{k=1}^L \xi_k^2 v(h_k^{(\varepsilon)})}.$$

3° Collecting 1° and 2° yields

$$\lim_{P \rightarrow \infty} E \left[e^{\sqrt{-1} \sum_{k=1}^L \xi_k \frac{1}{\sqrt{P}} \sum_{p=1}^P I_k^{(p)}(h_k; \mu)} \right] = e^{-\frac{1}{2} \sum_{k=1}^L \xi_k^2 v(h_k)},$$

which is just the conclusion of the theorem. \square

4. A Justification of the Claim of Sobol' et al

First let us state the claim of Sobol' et al ([13], [14]).

There are several kinds of deterministic sequences $\{x_p = (x_{1p}, \dots, x_{Np})\}_{p=1}^\infty$ on \mathbb{T}^N , which are called low discrepancy sequences ([2]), having the following property: For $\forall F : \mathbb{T}^N \rightarrow \mathbb{R}$ of finite variation

$$\frac{1}{P} \sum_{p=1}^P F(x_p) = \int_{\mathbb{T}^N} F(x) dx + O\left(\frac{1}{P^{1-\varepsilon}}\right) \quad \text{as } P \rightarrow \infty \quad (\forall \varepsilon > 0).$$

This convergence can be used for numerical integrations in \mathbb{T}^N , which is called the quasi Monte Carlo method. Since the usual Monte Carlo method converges at the rate of $O(\frac{1}{\sqrt{P}})$, this method is theoretically more effective.

However many authors have reported that practically the quasi Monte Carlo method does not converge so fast as it is expected, if the dimension N is very high. Among others there is the following claim of Sobol' et al:

CLAIM. In high dimensions, the quasi Monte Carlo method seems to converge at the rate of exactly $O(\frac{1}{\sqrt{P}})$, if the integrands F depend equally on each coordinate.

We try to give a probabilistic explanation to this claim. From Theorems 2.1 and 3.3 let us view the following generously.

Let $(h_1, \dots, h_L) \in \mathcal{CL}_2 \times \dots \times \mathcal{CSL}_2^L$ be such that $h_k \neq 0$ ($1 \leq k \leq L$). Corollary to Theorem 2.1 says that for $\alpha = (\alpha_1, \alpha_2, \dots) \in \mathbb{T}_\mu^\infty$

$$\left(\frac{1}{\sqrt{N}} \sum_{1 \leq i_1 \leq N} h_1(x_{i_1} + p\alpha_{i_1}), \dots, L! \left(\frac{1}{\sqrt{N}}\right)^L \sum_{1 \leq i_1 < \dots < i_L \leq N} h_L(x_{i_1} + p\alpha_{i_1}, \dots, x_{i_L} + p\alpha_{i_L}) \right)_{p \in \mathbb{N}}$$

$$\xrightarrow{\text{f.d.}} \left(I_1^{(p)}(h_1; \mu), \dots, I_L^{(p)}(h_L; \mu) \right)_{p \in \mathbb{N}} \quad \text{as } N \rightarrow \infty.$$

Consequently, for sufficiently large N we think that

$$\begin{aligned} & \left(\frac{1}{\sqrt{N}} \sum_{1 \leq i_1 \leq N} h_1(x_{i_1} + p\alpha_{i_1}), \right. \\ & \quad \left. \dots, L! \left(\frac{1}{\sqrt{N}} \right)^L \sum_{1 \leq i_1 < \dots < i_L \leq N} h_L(x_{i_1} + p\alpha_{i_1}, \dots, x_{i_L} + p\alpha_{i_L}) \right)_{p \in \mathbb{N}} \\ & \quad = \left(I_1^{(p)}(h_1; \mu), \dots, I_L^{(p)}(h_L; \mu) \right)_{p \in \mathbb{N}}. \end{aligned}$$

On the other hand, Theorem 3.3 says that

$$\begin{aligned} & \frac{1}{\sqrt{P}} \sum_{p=1}^P \left(I_1^{(p)}(h_1; \mu), \dots, I_L^{(p)}(h_L; \mu) \right) \\ & \Rightarrow \mathfrak{N} \left(0, \begin{bmatrix} v(h_1) & & 0 \\ & \ddots & \\ 0 & & v(h_L) \end{bmatrix} \right) \quad (\text{nondegenerate Gaussian}) \quad \text{as } P \rightarrow \infty \end{aligned}$$

provided $m = \frac{d\mu}{dx}$ is continuous and positive. Hence combining these yields

$$\begin{aligned} & \frac{1}{\sqrt{P}} \sum_{p=1}^P \left(\frac{1}{\sqrt{N}} \sum_{1 \leq i_1 \leq N} h_1(x_{i_1} + p\alpha_{i_1}), \right. \\ & \quad \left. \dots, L! \left(\frac{1}{\sqrt{N}} \right)^L \sum_{1 \leq i_1 < \dots < i_L \leq N} h_L(x_{i_1} + p\alpha_{i_1}, \dots, x_{i_L} + p\alpha_{i_L}) \right) \\ & \Rightarrow \mathfrak{N} \left(0, \begin{bmatrix} v(h_1) & & 0 \\ & \ddots & \\ 0 & & v(h_L) \end{bmatrix} \right) \quad (\text{nondegenerate Gaussian}) \quad \text{as } P \rightarrow \infty, \end{aligned}$$

so that for $k = 1, \dots, L$

$$\begin{aligned} & \frac{1}{P} \sum_{p=1}^P \left(\frac{1}{\sqrt{N}} \right)^k \sum_{1 \leq i_1 < \dots < i_k \leq N} h_k(x_{i_1} + p\alpha_{i_1}, \dots, x_{i_k} + p\alpha_{i_k}) \\ & \begin{cases} = O\left(\frac{1}{\sqrt{P}}\right) \\ \neq O\left(\frac{1}{\sqrt{P}^{1+\varepsilon}}\right) \quad (\forall \varepsilon > 0) \end{cases} \quad \text{as } P \rightarrow \infty. \end{aligned}$$

This observation tells us that when $\alpha = (\alpha_1, \alpha_2, \dots)$ is “regularly” distributed over the whole \mathbb{T} (it need not be uniformly distributed), the claim of Sobol’ et al “holds” for a low discrepancy sequence $\{(x_1 + p\alpha_1, \dots, x_N + p\alpha_N)\}_{p=1}^\infty$ and an integrand $F(y_1, \dots, y_N) = \left(\frac{1}{\sqrt{N}}\right)^k \sum_{1 \leq i_1 < \dots < i_k \leq N} h_k(y_{i_1}, \dots, y_{i_k})$.

5. Concluding Remarks

Our interest in this paper is when the disappearance of dependency happens, in other words, how the distribution of $\alpha = (\alpha_1, \alpha_2, \dots)$ in \mathbb{T} influences the dependency. For this subject there are two works close to ours. We briefly introduce them to conclude this paper.

5.1. Fukuyama’s work ([5], [6])

Let $\theta \in (1, \infty)$. We define $\varphi_\theta : \mathbb{T} \rightarrow \mathbb{T}^\infty$ by

$$\varphi_\theta(x) = \left(\{\theta^{i-1}x\} \right)_{i=1}^\infty,$$

where $\{y\}$ denotes the fractional part of $y \in \mathbb{R}$. Let μ be a probability measure on \mathbb{T}^∞ induced by φ_θ , i.e., $\mu = \mathbf{P} \circ \varphi_\theta^{-1}$. Clearly μ is singular relative to \mathbf{P}^∞ , and not a product probability measure. So it is not in a class of probability measures considered above. Also, for $\forall \alpha \in \mathbb{T}$

$$\mu\left(\{x \in \mathbb{T}^\infty; x + \varphi_\theta(\alpha) \in *\}\right) = \mu(*).$$

Let us state a result of Fukuyama. To do so, for $f \in \mathcal{CL}_2$ we set

$$D(f) := \sum_{k=0}^\infty \sqrt{\sum_{2^k \leq |n| < 2^{k+1}} |\widehat{f}(n)|^2}.$$

By definition $\|f\| \leq D(f)$, but $D(f) \in [0, \infty]$ (i.e., $D(f)$ is not necessarily convergent). Let μ be a probability measure on \mathbb{T} , $\{N_m\}_{m=1}^\infty$ a subsequence of \mathbb{N} and $\alpha \in \mathbb{T}$ for which $\varphi_\theta(\alpha) \in \mathbb{T}_{\mu, \{N_m\}}^\infty$, i.e.,

$$\frac{1}{N_m} \sum_{i=1}^{N_m} \delta_{\{\theta^{i-1}\alpha\}}(dx) \implies \mu(dx) \quad \text{as } m \rightarrow \infty.$$

Then the following holds: For $\forall f \in \mathcal{CL}_2$ with $D(f) < \infty$

$$\left(\frac{1}{\sqrt{N_m}} \sum_{i=1}^{N_m} f(x_i + j\varphi_\theta(\alpha)_i) \right)_{j \in \mathbb{Z}} \implies \mathfrak{N}(0, \Sigma(f)) \quad \text{as } m \rightarrow \infty.$$

Here $\Sigma(f)$ is an infinite nonnegative symmetric matrix whose (i, j) -component is given by the following manner: In case $\theta^r \notin \mathbb{Q}$ for $\forall r \in \mathbb{N}$,

$$\Sigma_{ij}(f) = \int_{\mathbb{T}} \mu(dx) \int_{\mathbb{T}} f(t) f(t + |i - j|x) dt;$$

in case $\theta^r \in \mathbb{Q}$ for some $r \in \mathbb{N}$, by letting $s = \min\{n \in \mathbb{N}; \theta^n \in \mathbb{Q}\}$ and writing $\theta^s = \frac{p}{q}$ to be irreducible

$$\Sigma_{ij}(f) = \sum_{k \in \mathbb{Z}} \int_{\mathbb{T}} \mu(dx) \int_{\mathbb{T}} f(p^{|k|}t) f(q^{|k|}(t + \widetilde{\text{sgn}}(k)|i - j|x)) dt,$$

where $\widetilde{\text{sgn}}(k) := \begin{cases} 1 & k \geq 0 \\ -1 & k < 0 \end{cases}$. Putting $\mu(dx) = dx$ in the expression above, we readily see

$$\Sigma_{ij}(f) = 0, \quad i \neq j,$$

and hence we have the disappearance of dependency whenever $\varphi_\theta(\alpha) \in \mathbb{T}_{dx}^\infty$. This is valid at least for $f \in \mathcal{CL}_2$ with $D(f) < \infty$. For $\theta = 2$, Fukuyama remarks that the condition $D(f) < \infty$ is necessary for the CLT: $\frac{1}{\sqrt{N}} \sum_{i=1}^N f(x_i) \Rightarrow \mathfrak{N}(0, \sigma^2)$ as $N \rightarrow \infty$ (under $\mu = \mathbf{P} \circ \varphi_2^{-1}$). For this reason, when θ is in the second case above, this condition is the best possible for the disappearance of dependency. On the other hand, when θ is in the first case, is this really necessary? For, in this case, $\Sigma_{ij}(f) = \delta_{ij} \|f\|^2$. Anyway this question remains open.

5.2. Sugita's work ([15])

We first note that our and Fukuyama's works originate in [15].

Let $\mu = \mathbf{P} \circ \varphi_2^{-1}$. For $N \in \mathbb{N}$, we define a random variable $X^{(N)}$ on $(\mathbb{T}^\infty, \mathcal{F}, \mu)$ by

$$X^{(N)}(x) := \sum_{i=1}^N 1_{[\frac{1}{2}, 1)}(x_i) \pmod{2}.$$

Clearly for $\forall \alpha \in \mathbb{T}$, $\forall N \geq 1$ and $\forall j \in \mathbb{Z}$

$$\mu\left(X^{(N)}(\cdot + j\varphi_2(\alpha)) = 1\right) = \mu\left(X^{(N)}(\cdot + j\varphi_2(\alpha)) = 0\right) = \frac{1}{2}.$$

We have a question whether the disappearance of dependency happens as $N \rightarrow \infty$ for a stationary sequence $\left(X^{(N)}(\cdot + j\varphi_2(\alpha))\right)_{j \in \mathbb{Z}}$. For this Sugita gives an affirmative answer: If, at least $\varphi_2(\alpha) \in \mathbb{T}_{dx}^\infty$, in other words α is a dyadic normal number, a sequence $\left(X^{(N)}(\cdot + j\varphi_2(\alpha))\right)_{j \in \mathbb{Z}}$ converges to the $\{0, 1\}$ -valued fair Bernoulli random variables in finite dimensional distribution as $N \rightarrow \infty$, i.e., for $\forall j_1 < \dots < j_k$ and $\forall \varepsilon_1, \dots, \varepsilon_k \in \{0, 1\}$

$$\lim_{N \rightarrow \infty} \mu\left(X^{(N)}(\cdot + j_1\varphi_2(\alpha)) = \varepsilon_1, \dots, X^{(N)}(\cdot + j_k\varphi_2(\alpha)) = \varepsilon_k\right) = \left(\frac{1}{2}\right)^k.$$

He says that this assumption on α is technical, and in fact the statement above will hold for every irrational number α . However, this is still open.

References

- [1] Billingsley, P., *Probability and measure*, 3rd edition, John Wiley & Sons, 1995.
- [2] Bouleau, N. and D. Lépingle, *Numerical methods for stochastic processes*, John Wiley & Sons, 1994.
- [3] Dynkin, E. B. and A. Mandelbaum, Symmetric statistics, Poisson point processes, and multiple Wiener integrals, *Ann. Statist.* **11** (1983), 739–745.
- [4] Fukuyama, K., The central limit theorem for Rademacher system, *Proc. Japan Acad.* **70** Ser. A (1994), 243–246.
- [5] Fukuyama, K., The central limit theorem for Riesz-Raikov sums, *Probab. Theory Relat. Fields* **100** (1994), 57–75.
- [6] Fukuyama, K., Riesz-Raikov sums and Weyl transform, *Monte Carlo Methods and Appl.* **2** (1996), 271–293.
- [7] Ibragimov, I. A. and Yu. V. Linnik, *Independent and stationary sequences of random variables*, Wolters-Noordhoff, 1971.
- [8] Itô, K., Multiple Wiener integral, *J. Math. Soc. Japan* **3** (1951), 157–169.
- [9] Kolmogorov, A. N. and Yu. A. Rozanov, On strong mixing conditions for stationary Gaussian processes, *Theory Probab. Appl.* **5** (1960), 204–208.
- [10] Kuipers, L. and H. Niederreiter, *Uniform distribution of sequences*, Interscience, 1974.
- [11] McLeish, D. L., Dependent central limit theorems and invariance principles, *Ann. Probab.* **2** (1974), 620–628.

- [12] Salem, R. and A. Zygmund, On lacunary trigonometric series I, Proc. Nat. Acad. Sci. USA **33** (1947), 333–338.
- [13] Sobol', I. M., Turchaninov, V. I. and Yu. L. Levitan, *Quasirandom sequence generators*, Keldysh Institute of Applied Mathematics, Moscow, 1992.
- [14] Sobol', I. M., *A primer for the Monte Carlo method*, CRC Press, Boca Raton/Ann Arbor/London/Tokyo, 1994.
- [15] Sugita, H., Pseudo-random number generator by means of irrational rotation, Monte Carlo Methods and Appl. **1** (1995), 35–57.
- [16] Sugita, H. and S. Takanobu, Limit theorem for symmetric statistics with respect to Weyl transformation: Disappearance of dependency, J. Math. Kyoto Univ. **38** (1998), 653–671.
- [17] Sun, T.-C., A central limit theorem for non-linear functions of a normal stationary process, J. Math. Mech. **12** (1963), 945–977.
- [18] Sun, T.-C., Some further results on central limit theorems for non-linear functions of a normal stationary process, J. Math. Mech. **14** (1965), 71–85.
- [19] Walter, P., *An introduction to ergodic theory*, Springer, 1981.
- [20] Zygmund, A., *Trigonometric series I*, Cambridge Univ. Press, Cambridge, 1959.

(Received April 20, 1999)

Hiroshi Sugita
Graduate School Mathematics
Kyushu University
Fukuoka, 812-8581, Japan
E-mail: sugita@math.kyushu-u.ac.jp

Satoshi Takanobu
Department of Mathematics
Faculty of Science
Kanazawa University
Kanazawa, 920-1192, Japan
E-mail: takanobu@kappa.s.kanazawa-u.ac.jp