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Exact Hausdorff Dimension of Self-avoiding Processes on the Multi-dimensional Sierpinski Gasket

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Abstract. We determine the 'exact Hausdorff dimension' for a class of multi-type random constructions. As an application, we consider a model of self-avoiding walk called the 'branching model' on the multi-dimensional Sierpinski gasket. We take its continuum limit and determine the exact Hausdorff dimension of the path of the limit process.

1. Introduction

We consider a model of self-avoiding walk on the d-dimensional pre-Sierpinski gasket (the Sierpinski gasket lattice). In our model we assign weights to self-avoiding paths in such a way that each step of a path on a coarser lattice splits into finer structures according to a given probability law to yield a path on a finer lattice. In view of this construction, we call our model the 'branching model.' We take the continuum limit of our self-avoiding walk, that is, the limit as the lattice spacing tends to zero. With an appropriate time-scale transformation, we obtain a non-trivial stochastic process on the d-dimensional Sierpinski gasket in the limit. The limit path K of the branching model is a random closed subset of $\mathbf{R}^d$. We are concerned with determining the 'exact Hausdorff dimension' of the limit path.

Let $h : (0,1) \to \mathbf{R}_+$ be a non-decreasing, continuous function. We call $h(t)$ with such properties, a dimension function. The Hausdorff measure $\mathcal{H}^h$ with regard to a dimension function h is defined as follows. Let E be a subset of $\mathbf{R}^d$. A countable family of subsets of $\mathbf{R}^d$, $\{U_i\}$ is called a δ-cover of E if $E \subset \bigcup_{i=1}^{\infty} U_i$ and $|U_i| \leq \delta$, $i = 1, 2, \cdots$, where $|U_i|$ denotes the diameter

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of U_i. For $\delta > 0$, set

$$\mathcal{H}^h_\delta(E) = \inf\{\sum_{i=1}^{\infty} h(|U_i|) \,:\, \{U_i\} \text{ is a } \delta\text{-cover of } E\},$$

and

$$\mathcal{H}^h(E) = \sup_{\delta>0} \mathcal{H}^h_\delta(E).$$

$\mathcal{H}^h$ is an outer meausure such that all Borel sets of $\mathbf{R}^d$ are $\mathcal{H}^h$-measurable. If we find a dimension function h satisfying

$$0 < \mathcal{H}^h(E) < \infty,$$

we say that we have determined the exact Hausdorff dimension.

We will show that for the limit path K of the branching model,

$$0 < \mathcal{H}^h(K) < \infty \quad \text{a.s.},$$

where

$$h(t) = t^\alpha |\log|\log t||^\theta,$$

α is almost surely the Hausdorff dimension of K, and θ is a constant determined by the dimension d and some parameters.

$\mathcal{H}^h$ is a generalization of the usual s-dimensional Hausdorff measure $\mathcal{H}^s$, which corresponds to the dimension function $h(t) = t^s$ for $s \geq 0$. The Hausdorff dimension of E, $\dim_H(E)$ is defined by

$$\dim_H(E) = \inf\{s \geq 0 \,:\, \mathcal{H}^s(E) = 0\} \; (= \sup\{s \geq 0 \,:\, \mathcal{H}^s(E) = \infty\}).$$

It was shown that if $E \subset \mathbf{R}^d$ is a deterministic self-similar set (for example, the Sierpinski gaskets) (see [9]), or a deterministic set defined by a strongly connected graph directed construction (multi-type) (see [8], [10]),

$$0 < \mathcal{H}^D(E) < \infty,$$

where $D = \dim_H(E)$. In contrast, for the limit path K of the branching model considered here, $\dim_H(K) = \alpha$ a.s., and

$$\mathcal{H}^\alpha(K) = 0 \quad \text{a.s.}$$

(The Hausdorff dimensions for $d = 2$ and 3 were determined in [4].) The occurence of zero Hausdorff measure is caused by stochastic fluctuations and is often the case with random sets. (See [2], [3], [7], [10].) We seek for a measure that gives a finite and positive value for K.

When we study the exact Hausdorff dimension, we regard the limit path as a **multi-type** random construction. In [3], Graf, Mauldin and Williams have shown for a wide class of **single-type** random constructions such that $\mathcal{H}^\alpha(K) = 0$ a.s., that

$$0 < \mathcal{H}^h(K) < \infty, \quad \text{if } K(\omega) \neq \emptyset, \quad \text{a.s.},$$

where

$$h(t) = t^\alpha |\log|\log t||^\theta,$$

and θ is determined by the construction.

On the other hand, in [10], Tsujii studied a certain class of N-type random constructions with stochastic geometric self-similarity. (He calls them the random Markov-self-similar sets.) He showed that under some conditions,

$$0 < \mathcal{H}^\alpha(K^{(r)}) < \infty \text{ for all } \ r \in \{1, 2, \cdots, N\} \ \text{ a.s.},$$

where α is almost surely the Hausdorff dimension of all $K^{(r)}$'s, if and only if

$$\sum_{j=1}^{N} T_{rj}^\alpha x_j = x_r \text{ for all } \ r \in \{1, 2, \cdots, N\} \ \text{ a.s.}, \tag{1.1}$$

where T_{ij}'s are the random ratios that govern the construction of the random sets and $\{x_i\}$ is the Frobenius' eigenvector of the ratio matrix $\mathbf{R}(\alpha)$. T_{ij}'s and $\mathbf{R}(\alpha)$ are defined in Section 2. (1.1) is a quite restrictive condition.

In this paper, we will concern ourselves with determining the exact Hausdorff dimension for multi-type random constructions which do not satisfy (1.1). Our limit path of the self-avoiding walk belongs to this case. In Section 2, we introduce definitions, notations and basic tools that were prepared in previous works. In Section 3, we extend the result of [3] to multi-type random constructions and obtain a general theorem that determines the exact Hausdorff dimension. In Section 4, we apply the theorem to the branching model on the d-dimensional Sierpinski gasket.

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2. Notation and Definitions

A multi-type random construction is defined as follows: Let J be a non-empty compact subset of $\mathbf{R}^d$ such that $J = cl(int(J))$, where $cl(A)$ and $int(A)$ denote the closure and the interior of a set A, respectively. Let $C = \{1, 2, \cdots, N\}$, $C_n = \{1, \cdots, N\}^n$, and $C^* = \bigcup_{n=0}^{\infty} C_n$, where C_0 stands for $\{\emptyset\}$. For σ, $\tau \in C^*$, $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n)$, $\tau = (\tau_1, \tau_2, \cdots, \tau_m)$, denote $\sigma * \tau = (\sigma_1, \cdots, \sigma_n, \tau_1, \cdots, \tau_m)$, $|\sigma| = n$ and $t(\sigma) = \sigma_n$. For $m \in \mathbf{N}$ and $\sigma = (\sigma_1, \cdots, \sigma_k) \in \cup_{n=m}^{\infty} C_n$, denote $\sigma|m = (\sigma_1, \cdots, \sigma_m)$. Set $\sigma|0 = \emptyset$.

On a probability space $(\Omega, \mathcal{F}, P)$, consider N families of random subsets of $\mathbf{R}^d$,

$$\mathbf{J}^{(k)}(\omega) = \{J_\sigma^{(k)}(\omega) \,:\, \sigma \in C^*\}, \quad k \in C.$$

Assume each $\mathbf{J}^{(k)}$ satisfies the following conditions:

(J1) $J_\emptyset^{(k)}(\omega) = J$ for a.a. $\omega \in \Omega$. For all $\sigma \in C$ and for a.a. $\omega \in \Omega$, if $J_\sigma^{(k)}(\omega) \neq \emptyset$, then $J_\sigma^{(k)}(\omega)$ is geometrically similar to J, and the map $\omega \mapsto J_\sigma^{(k)}(\omega)$ is measurable with respect to the Hausdorff metric on the space of compact subsets of $\mathbf{R}^d$.

(J2) For all $\sigma \in C^*$ and for all $i \in C$,

$$J_{\sigma*i}^{(k)}(\omega) \subset J_\sigma^{(k)}(\omega), \quad \text{for a.a. } \omega \in \Omega.$$

For all $\sigma \in C^*$ and for $i, j \in C$ with $i \neq j$,

$$int(J_{\sigma*i}^{(k)}(\omega)) \cap int(J_{\sigma*j}^{(k)}(\omega)) = \emptyset, \quad \text{for a.a. } \omega \in \Omega.$$

(J3) There are mutually independent random vectors $\mathbf{T}_\sigma^{(k)}(\omega) = (T_{\sigma*1}^{(k)}(\omega), \cdots, T_{\sigma*N}^{(k)}(\omega))$, $\sigma \in C^*$, such that for all $\sigma \in C^*$ and for all $i \in C$,

$$\mathrm{diam}(J_{\sigma*i}^{(k)}(\omega)) = T_{\sigma*i}^{(k)}(\omega)\,\mathrm{diam}(J_\sigma^{(k)}(\omega)) \quad \text{for a.a. } \omega \in \Omega,$$

and $\mathbf{T}_\sigma^{(k)}$ is equal in law to $\mathbf{T}_\emptyset^{(r)}$ if $t(\sigma) = r$, $r \in C$.

Throughout this paper, we denote

$$\mathbf{T}_\emptyset^{(r)}(\omega) = (T_1^{(r)}(\omega), \cdots, T_N^{(r)}(\omega)) = (T_{r1}(\omega), \cdots, T_{rN}(\omega)). \tag{2.1}$$

We call the system $\{\mathbf{J}^{(k)}\}$ a multi-type random construction. We define the random sets $K^{(k)}$, $k \in C$ by

$$K^{(k)}(\omega) = \bigcap_{n=1}^{\infty} \bigcup_{\sigma \in C_n} J_\sigma^{(k)}(\omega).$$

For $\beta \geq 0$, define an $N \times N$ matrix $\mathbf{R}(\beta) = \{R(\beta)_{ij}\}_{i,j=1,\cdots,N}$ by,

$$R(\beta)_{ij} = E[T_{ij}^\beta], \tag{2.2}$$

where $0^0 = 0$.

An $M \times M$ matrix A is called irreducible if for every $(i,j) \in \{1, \cdots, M\}^2$, there is an $m \in \mathbf{N}$ such that $(A^m)_{ij} > 0$.

If $R(\beta)$ is irreducible, Frobenius' theorem implies that there exists a positive eigenvalue that is simple and greater in absolute value than any other eigenvalues (the Frobenius' root), and a positive eigenvector associated with it. We denote by $\lambda(\beta)$ the Frobenius' root of $R(\beta)$. Assume $\lambda(0) > 1$. From the definition of $J_\sigma^{(k)}$'s, considering the d-dimensional volumes of $J_j^{(k)}$'s and J, we have $\sum_{j=1}^{N} T_{kj}^d \leq 1$ almost surely, where d is the dimension of the Euclidean space. Thus $\lambda(d) \leq 1$. This combined with $\lambda(0) > 1$ and that $\lambda(\beta)$ is continuous and strictly decreasing with respect to β leads to the unique existence of an $\alpha > 0$ such that $\lambda(\alpha) = 1$. Let $\{x_i\}$ be the positive right eigenvector associated with $\lambda(\alpha) = 1$.

Suppose $\mathbf{R}(0)$ is irreducible and $\lambda(0) > 1$. It was proved that for each k, $K^{(k)}$ is non-empty with positive probability, and if $K^{(k)}$ is non-empty, almost surely $K^{(k)}$ has Hausdorff dimension α independent of k. (See [10].)

If random vectors $\mathbf{T}_\emptyset^{(r)} = (T_{r1}, \cdots, T_{rN})$, $r = 1, \cdots, N$ have the same distribution, it corresponds to a (single-type) random construction studied in [2], [3] and [7].

We give a simple example of multi-type random constructions. Let $C = \{1, 2\}$. Let $\mathbf{T}_\sigma^{(k)}$, $\sigma \in C^*$, be mutually independent random vectors such that

$$P[\ \mathbf{T}_\emptyset^{(1)} = (\frac{1}{2}, \frac{1}{4})\ ] = p,$$

$$P[\ \mathbf{T}_\emptyset^{(1)} = (\frac{1}{2}, 0)\ ] = 1 - p,$$

$$P[\ \mathbf{T}_\emptyset^{(2)} = (\frac{1}{2}, \frac{1}{4})\ ] = q,$$

$$P[\ \mathbf{T}_\emptyset^{(2)} = (0, \frac{1}{4})\ ] = 1 - q,$$

where $0 < p, q < 1$, and for each $\sigma \neq \emptyset$, $\mathbf{T}_\sigma^{(k)}$ is equal in law to $\mathbf{T}_\emptyset^{(\tau(\sigma))}$. We define a family of random sets $J_\sigma^{(k)}$ inductively as follows. For each $k \in C$, set

$$J_\emptyset^{(k)}(\omega) = [0, 1],$$

for all $\omega \in \Omega$. Set

$$J_1^{(k)}(\omega) = [0, \frac{1}{2}], \quad \text{if } \ T_1^{(k)}(\omega) = \frac{1}{2},$$

$$J_2^{(k)}(\omega) = [\frac{3}{4}, 1], \quad \text{if } \ T_2^{(k)}(\omega) = \frac{1}{4},$$

and

$$J_i^{(k)}(\omega) = \emptyset, \quad \text{otherwise .}$$

Suppose $J_\sigma^{(k)}(\omega)$, $\sigma \in C^*$, is defined and is either $\emptyset$ or a closed interval $[a, b]$ with $0 \leq a < b \leq 1$. $J_{\sigma * i}^{(k)}(\omega)$, $i \in C$, is defined in the following way. Set

$$J_{\sigma * i}^{(k)}(\omega) = \emptyset, \quad \text{if } J_\sigma^{(k)}(\omega) = \emptyset.$$

If $J_\sigma^{(k)}(\omega) = [a, b]$, $0 \leq a < b \leq 1$, set

$$J_{\sigma * 1}^{(k)}(\omega) = [a, a + \frac{1}{2}(b - a)], \quad \text{if } \ T_{\sigma * 1}^{(k)}(\omega) = \frac{1}{2},$$

$$J_{\sigma * 2}^{(k)}(\omega) = [b - \frac{1}{4}(b - a), b], \quad \text{if } \ T_{\sigma * 2}^{(k)}(\omega) = \frac{1}{4},$$

$$J_{\sigma * i}^{(k)}(\omega) = \emptyset, \quad \text{otherwise .}$$

This two-type random construction defines Cantor-like random subsets $K^{(1)}$ and $K^{(2)}$ of $\mathbf{R}$. It is obvious that for any $\alpha > 0$ and any positive vector ${}^t(x_1, x_2)$ the condition (1.1) is not satisfied.

Let $\{\mathbf{J}^{(k)}\}$ be a multi-type random construction such that $\mathbf{R}(0)$ is irreducible and $\lambda(0) > 1$. Set

$$\ell_\sigma^{(k)} = \mathrm{diam} J_\sigma^{(k)}, \quad \sigma \in C^*.$$

Then

$$\ell_\sigma^{(k)} = \prod_{m=0}^{|\sigma|} T_{\sigma|m}^{(k)}, \tag{2.3}$$

where $T_{\sigma|0}^{(k)} = \mathrm{diam} J$. In [10] it was shown that for each $k \in C$,

$$\limsup_{n\to\infty} \{\ell_\sigma^{(k)} \,:\, \sigma \in C_n\} = 0 \quad \text{a.s.} \tag{2.4}$$

Define

$$f_n^{(k)} = \sum_{\sigma\in C_n} \prod_{m=0}^{n} (T_{\sigma|m}^{(k)})^\alpha x_{t(\sigma)},$$

$$f_0^{(k)} = x_k (\mathrm{diam} J)^\alpha,$$

where $\{x_i\}$ is the positive right eigenvector corresponding to $\lambda(\alpha)$. Let $\mathcal{F}_\sigma$ be the σ-field genereted by $(T_{(\sigma|i)*1}^{(k)}, \cdots, T_{(\sigma|i)*N}^{(k)}),\ i = 0, \cdots, |\sigma|-1,\ k \in C$, and $\mathcal{F}_n = \vee_{\sigma\in C_n} \mathcal{F}_\sigma$.

It was shown that for every $p \in \mathbf{N}$ and for every $k \in C$, $\{f_n^{(k)}\}_{n\in\mathbf{N}}$ is a L^p-bounded martingale with respect to $\{\mathcal{F}_n\}$ and converges a.s. and in L^p to a random variable $X^{(k)}$. It satisfies

$$E[X^{(k)}] = x_k (\mathrm{diam} J)^\alpha. \tag{2.5}$$

(See [10].) For $\sigma \in C^* \cup \{\emptyset\}$ and $k \in C$, define random variables $X_\sigma^{(k)}$ by

$$X_\sigma^{(k)} = \lim_{n\to\infty} \sum_{\tau\in C_n} \prod_{m=1}^{n} (T_{\sigma*(\tau|m)}^{(k)})^\alpha x_{t(\tau)}.$$

This limit exists a.s. and in L^p. For $\sigma \in C^*$, $X_\sigma^{(k)}$ has the same distribution as $X^{(t(\sigma))}/(\mathrm{diam} J)^\alpha$, thus its distribution does not depend on k. $X_\emptyset^{(k)}$ is distributed as $X^{(k)}/(\mathrm{diam} J)^\alpha$. From the definition, it is clear that $X_\sigma^{(k)}$ is independent of $\mathcal{F}_\sigma$. For any $\sigma \in C^* \cup \{\emptyset\}$, we have

$$X_\sigma^{(k)} = \sum_{i=1}^{N} (T_{\sigma * i}^{(k)})^\alpha X_{\sigma * i}^{(k)}, \quad \text{a.s.} \tag{2.6}$$

Set

$$D = \{1, \cdots, N\}^{\mathbf{N}},$$

For $m \in \mathbf{N}$ and $\sigma = (\sigma_1, \sigma_2, \cdots) \in D$, denote $\sigma|m = (\sigma_1, \cdots, \sigma_m)$. For $\eta \in D \cup C^*$ and $\sigma \in C^*$, write $\eta \succ \sigma$ if $\eta|m = \sigma$ for some $m \in \mathbf{N}$.

The following definitions are the extension of those in [3] to multi-type constructions. We define a family of probability measures $\{Q^{(k)}\}$, $k \in C$ on the product space $(D \times \Omega, \mathcal{B}(D) \times \mathcal{F})$, where $\mathcal{B}(D)$ is the Borel field of D.

For $\sigma \in C^*$, let $A(\sigma) = \{\eta \in D : \eta \succ \sigma\}$. Let $\mu_\omega^{(k)}$ be the random Borel measure on Ω uniquely determined by

$$\mu_\omega^{(k)}(A(\sigma)) = (\ell_\sigma^{(k)}(\omega))^\alpha X_\sigma^{(k)}(\omega).$$

For $B \in \mathcal{B}(D) \times \mathcal{F}$, let

$$B_\omega = \{\eta \in D : (\eta, \omega) \in B\}.$$

For each $k \in C$, $Q^{(k)}$ is defined by

$$Q^{(k)}(B) = \frac{1}{x_k\,(\mathrm{diam} J)^\alpha}\{\int \mu_\omega^{(k)}(B_\omega) dP(\omega)\}.$$

From the definition, we have the following property. If $m \in \mathbf{N}$ and a random variable Y satisfies $Y(\eta, \omega) = Y(\eta', \omega)$ for any η, $\eta' \in D$ with $\eta|m = \eta'|m$, then

$$E_Q^{(k)}[Y] = \frac{1}{x_k\,(\mathrm{diam} J)^\alpha} E[\sum_{\sigma \in C_m} (\ell_\sigma^{(k)})^\alpha X_\sigma^{(k)} Y(\sigma, \cdot)], \tag{2.7}$$

where $E_Q^{(k)}$ denotes the expectation with regard to $Q^{(k)}$. Define random variables $\{\ell_m\}_{m=1,2,\cdots}$ and $\{T_m\}_{m=1,2,\cdots}$ on $(D \times \Omega, \mathcal{B}(D) \times \mathcal{F}, Q^{(k)})$ by

$$\ell_m(\eta, \omega) = \ell_{\eta|m}^{(k)}(\omega), \tag{2.8}$$

$$T_m(\eta,\omega) = T^{(k)}_{\eta|m}(\omega). \tag{2.9}$$

A maximal antichain is a subset Γ of C^* such that if $\sigma, \tau \in \Gamma$ then neither $\tau \succ \sigma$ or $\sigma \succ \tau$ holds, and for each $\eta \in D$ there is a unique k such that $\eta|k \in \Gamma$. From (2.6), the following holds for every maximal antichain $\Gamma \subset C^*$,

$$X^{(k)}_\sigma(\omega) = \sum_{\tau\in\Gamma} \prod_{m=1}^{|\tau|} (T^{(k)}_{\sigma*(\tau|m)})^\alpha(\omega) X^{(k)}_{\sigma*\tau}(\omega), \quad P\text{-a.s.} \tag{2.10}$$

(See [3].)

3. Exact Hausdorff Dimension for Multi-type Random Constructions

In this section, we extend the results in [3] and obtain a multi-type version of theorems for the Hausdorff measures.

Throughout this section, assume that $\mathbf{R}(0)$ is irreducible and $\lambda(0) > 1$. α denotes the positive number such that $\lambda(\alpha) = 1$.

Let $\beta > 0$. For $k \in C$, let $r^{(k)}_\beta \in [0,\infty]$ be the radius of convergence of $E[\exp\{t(X^{(k)})^\beta\}]$, the moment generating function of $(X^{(k)})^\beta$. As in [3], we use $r^{(k)}_\beta$ to obtain the estimate of the Hausdorff measure.

Theorem 3.1 through Theorem 3.3 below are obtained as extensions of Theorems 2.5, 2.7, and 2.11 in [3] to the multi-type case. Since these extensions are straightforward and the proofs are lengthy, we omit the proofs and just show the multi-type version of the recursion formula essential for their proofs in Appendix A.

Let T_{ij} be as in (2.1).

Theorem 3.1. *Let $\beta > 1$.*

(1) If $P[\sum_{i=1}^N T_{ki}^{\alpha/(1-\frac{1}{\beta})} > 1] > 0$ for all $k \in C$, then $r^{(k)}_\beta = 0$ for all $k \in C$.

(2) If $P[\sum_{i=1}^N T_{ki}^{\alpha/(1-\frac{1}{\beta})} \le 1] = 1$ for all $k \in C$, then $r^{(k)}_\beta > 0$, for all $k \in C$.

Theorem 3.2. *Assume*

$$\sup\{\beta > 1 \,:\, \sum_{i=1}^N T_{ki}^{\alpha/(1-\frac{1}{\beta})} \le 1,\ P\text{-a.s.}\ \}$$

has the same finite value for all $k \in C$. Denote this value by $1/\theta$. Then $0 < \theta < 1$ and for every $0 < \beta < 1/\theta$, $r_\beta^{(k)} = \infty$ for all $k \in C$, and for every $\beta > 1/\theta$, $r_\beta^{(k)} = 0$ for all $k \in C$. Moreover, $r_{1/\theta}^{(k)} > 0$ for all $k \in C$.

As a sufficient condition for $r_{1/\theta}^{(k)} < \infty$, we have the following theorem.

THEOREM 3.3. *Let $\gamma = \alpha/(1-\theta)$. Assume there exists an $a \in (\frac{1}{N}, 1) \setminus \{1/\nu : \nu = 1, \cdots, N-1\}$ such that*

$$\prod_{\nu=0}^{\infty} \min_{\ell \in C} (E[(\sum_{j=1}^{N} T_{\ell j}^{\gamma})^{a^{-\nu}} \prod_{i=1}^{N} \mathbf{1}\{T_{\ell i}^{\gamma} / \sum_{j=1}^{N} T_{\ell j}^{\gamma} \leq a\}])^{a^{\nu}} > 0.$$

Then $r_{1/\theta}^{(k)} < \infty$, for all $k \in C$.

Let $\delta > 0$ and $L \in \{1, \cdots, N-1\}$. In the following, for $i \in C$, we denote by $i + L$ the integer $m \in C$ such that $i + L \equiv m$ (mod N). Define $N \times N$ matrices, $\mathbf{R}^A = \mathbf{R}^A(\delta, L)$ and $\mathbf{R}^B = \mathbf{R}^B(\delta, L)$ by

$$R_{i,j}^{A} = \mathrm{E}[\, T_{ij}^{\alpha}, \quad T_{i,j+L} \geq \delta \,] \, \frac{x_j}{x_i},$$

$$R_{i,j}^{B} = \mathrm{E}[\, T_{ij}^{\alpha}, \quad T_{i,j+L} < \delta \,] \, \frac{x_j}{x_i}, \quad i, j \in C.$$

Let $\mathbf{R}(\delta, L)$ be a $2N \times 2N$ matrix defined by

$$\mathbf{R}(\delta, L) = \begin{pmatrix} \mathbf{R}^A & \mathbf{R}^B \\ \mathbf{R}^A & \mathbf{R}^B \end{pmatrix}.$$

THEOREM 3.4. *(Upper bound)*
Assume $\lambda(0) > 1$. Suppose there are a $\delta > 0$ and an $L \in \{1, \cdots, N-1\}$ such that $\mathbf{R}(\delta, L)$ is irreducible. Suppose $\beta > 1$ is such that $r_\beta^{(i)} < \infty$ for some $i \in C$. Let $h : (0, 1) \longrightarrow \mathbf{R}_+$ be defined by

$$h(t) = t^{\alpha} |\log \log(1/t)|^{1/\beta}.$$

Then there is a constant $c > 0$ such that for all $r \in C$,

$$\mathcal{H}^h(K^{(r)}(\omega)) \leq c X^{(r)}(\omega) < \infty, \quad P\text{-a.s.}$$

The following lemma is the key to the proof of Theorem 3.4, and its proof requires considerable modifications to obtain our multi-type version. The proofs of Lemma 3.5 and Theorem 3.4 are found in Appendix B.

LEMMA 3.5. *Let $\beta > 1$, $i \in C$, $\delta > 0$ and $L \in \{1, 2, \cdots, N-1\}$ be as in Theorem 3.4. Choose $t > 0$ such that*

$$E[\exp(t\delta^{\alpha\beta} X^{(i)\beta}/\{3(\mathrm{diam} J)^{\alpha}\})] = \infty. \tag{3.1}$$

For $k \in \mathbf{N}$, $r \in C$ and $\omega \in \Omega$, define

$$\begin{aligned} B_k^{(r)}(\omega) \quad = \quad \{\sigma \in C_k \,:\, X_{\sigma|\nu}^{(r)}(\omega) < (\frac{1}{t}|\log|\log \ell_{\sigma|\nu}^{(r)}(\omega)||)^{\frac{1}{\beta}} \\ \textit{for } \nu = [\log k], \cdots, k\}. \end{aligned}$$

Then for each $r \in C$, there exists a sequence $\{k_j^{(r)}\}_{j \in \mathbf{N}}$ satisfying

$$\lim_{j \to \infty} \sum_{\sigma \in B_{k_j}^{(r)}} (\ell_\sigma^{(r)})^{\alpha} |\log|\log \ell_\sigma^{(r)}||^{\frac{1}{\beta}} = 0, \quad P\text{-}a.s. \tag{3.2}$$

For $\epsilon > 0$, $k \in C$ and $\omega \in \Omega$, define

$$\mathcal{C}_\epsilon^{(k)} = \mathcal{C}_\epsilon^{(k)}(\omega) = \{\sigma \in C^* \,:\, \ell_{\sigma||\sigma|-1}^{(k)}(\omega) \geq \epsilon,\ \ell_\sigma^{(k)}(\omega) < \epsilon\}.$$

Then $\mathcal{C}_\epsilon^{(k)}$ is a maximal antichain defined in Section 2. For $\tau \in \mathcal{C}_\epsilon^{(k)}$, let

$$\mathcal{C}_{\epsilon,\tau}^{(k)} = \{\sigma \in \mathcal{C}_\epsilon^{(k)} \,:\, \mathrm{dist}(J_\sigma^{(k)}, J_\tau^{(k)}) < \epsilon\},$$

where $\mathrm{dist}(\emptyset, E) = \infty$, for any set E. For $\eta \in D$, let $\eta_\epsilon = \eta^{(k)}(\epsilon, \omega)$ be the unique $\sigma \in \mathcal{C}_\epsilon^{(k)}$ with $\eta \succ \sigma$. Let $\mathcal{G}_\epsilon^{(k)}(\eta, \omega) = \mathcal{C}_{\epsilon,\eta_\epsilon}^{(k)}$.

The following theorem gives a sufficient condition for $\mathcal{H}^h$ to be positive.

THEOREM 3.6. *(Lower bound)*

Suppose a value θ as in Theorem 3.2 exists. Let $\gamma = \alpha/(1-\theta)$. Assume further

(1) For some $r \in C$,

$$\sup_{\epsilon>0} \| \frac{1}{\epsilon^\gamma} \sum_{\sigma\in\mathcal{G}_\epsilon} (\ell_\sigma^{(r)})^\gamma \|_\infty < \infty,$$

where $\|\cdot\|_\infty$ *denotes the* $L^\infty(Q^{(r)})$*-norm.*

(2) For the same r *as in (1), there exist* $c > 0$ *and* $0 < b < 1$ *such that*

$$Q^{(r)}[\sharp\mathcal{G}_\epsilon^{(r)} = m] \le cb^m \text{ for all } \epsilon > 0 \text{ and all } m \in \mathbf{N}.$$

Then

$$\mathcal{H}^h(K^{(r)}(\omega)) > 0, \quad \text{if } K^{(r)}(\omega) \ne \emptyset, \quad P\text{-a.s.},$$

where $h(t) = t^\alpha |\log|\log t||^\theta$.

Since Theorem 3.6 is obtained as an extension of the results in Section 4 of [3], we omit the proof here.

Combining Theorem 3.4 and Theorem 3.6, we summerize our main theorem on the exact Hausdorff dimension for multi-type random constructions in a general setting as follows.

THEOREM 3.7. *Let* $(\mathbf{J}^{(1)}, \cdots, \mathbf{J}^{(N)})$ *be a random construction satisfying* $\lambda(0) > 1$. *Let* α *be the solution to* $\lambda(\alpha) = 1$. *Suppose*

(1) There exists a $\theta > 0$ *such that*

$$\sup\{\beta > 1 : \sum_{i=1}^N (T_{ki})^{\alpha/(1-\frac{1}{\beta})} \le 1, \ P\text{-a.s.}\} = \frac{1}{\theta}, \quad \text{for all } k \in C.$$

(2) $r_{1/\theta}^{(i)}$ *is finite for some* $i \in C$.

(3) There exist a $\delta > 0$ *and an* $L \in \{1, 2, \cdots, N-1\}$ *such that* $\mathbf{R}(\delta, L)$ *is irreducible.*

(4) For some $r \in C$,

$$\sup_{\epsilon>0} \| \frac{1}{\epsilon^\gamma} \sum_{\sigma\in\mathcal{G}_\epsilon} (\mathrm{diam} J_\sigma^{(r)})^\gamma \|_\infty < \infty.$$

(5) For the same r as in (4), there exist $c > 0$ and $0 < b < 1$ such that

$$Q^{(r)}[\sharp \mathcal{G}_\epsilon^{(r)} = m] \le cb^m, \text{ for all } \epsilon > 0 \text{ and all } m \in \mathbf{N}.$$

Then

$$0 < \mathcal{H}^h(K^{(r)}(\omega)) < \infty, \quad \text{if } K^{(r)}(\omega) \neq \emptyset, \quad \text{for } P\text{-a.a. } \omega,$$

where

$$h(t) = t^\alpha |\log|\log t||^\theta.$$

In the case that each T_{ij} takes only finitely many values almost surely, the theorem is considerably simplified as follows.

THEOREM 3.8. *Let $(\mathbf{J}^{(1)}, \cdots, \mathbf{J}^{(N)})$ be a random construction satisfying $\lambda(0) > 1$ such that each T_{ij} takes only finitely many values strictly less than 1 almost surely. Let α be the solution to $\lambda(\alpha) = 1$. Suppose*

(1) There exists a $\theta > 0$ such that

$$\sup\{\beta > 1 \,:\, \sum_{i=1}^N (T_{ki})^{\alpha/(1-\frac{1}{\beta})} \le 1, \ P\text{-a.s.}\} = \frac{1}{\theta}, \quad \text{for all } k \in C.$$

(3) There exist a $\delta > 0$ and an $L \in \{1, 2, \cdots, N-1\}$ such that $\mathbf{R}(\delta, L)$ is irreducible.

Then for all $k \in C$,

$$0 < \mathcal{H}^h(K^{(k)}(\omega)) < \infty, \ \text{if } K^{(k)}(\omega) \neq \emptyset, \quad \text{for } P\text{-a.a. } \omega,$$

where

$$h(t) = t^\alpha |\log|\log t||^\theta.$$

The derivation of Theorem 3.8 from Theorem 3.7 is given in Appendix C. In the next section, we apply Theorem 3.8 to the branching model.

Here let us apply Theorem 3.8 to the example given in Section 2. By explicit calculation, it is easily seen that both $\mathbf{R}(0)$ and $\mathbf{R}(\frac{1}{4}, 1)$ are irreducible and $\lambda(0) > 1$.

The common Hausdorff dimension α of $K^{(1)}$ and $K^{(2)}$ is given almost surely as the solution to

$$(\frac{1}{2})^{\alpha+1}\{1+(\frac{1}{2})^{\alpha}+\sqrt{1+(\frac{1}{2})^{2\alpha}+(4pq-2)(\frac{1}{2})^{\alpha}}\}=1,$$

and satisfies

$$0<\alpha<\frac{\log\frac{1+\sqrt{5}}{2}}{\log 2}.$$

θ in Condition (1) exists as the solution to

$$(\frac{1}{2})^{\alpha/(1-\theta)}+(\frac{1}{4})^{\alpha/(1-\theta)}=1,$$

thus,

$$\theta=1-\frac{\alpha\log 2}{\log\frac{1+\sqrt{5}}{2}}.$$

For any $k\in C$ and a.a. $\omega\in\Omega$, $\bigcup_{\sigma\in C_n} J_{\sigma}^{(k)}(\omega)$, $n=1,2,\cdots$, is a nested sequence of non-empty closed sets. This implies that $K^{(k)}(\omega)$ is non-empty for a.a. $\omega\in\Omega$. From Theorem 3.8 we have for a.a. ω,

$$0<\mathcal{H}^h(K^{(k)}(\omega))<\infty,$$

where

$$h(t)=t^{\alpha}|\log|\log t||^{\theta}.$$

So far we assumed only stochastic *ratio* self-similarity, that is, $\mathbf{T}_{\sigma}^{(k)}$ has the same distribution as $\mathbf{T}_{\emptyset}^{(t(\sigma))}$. In that case, $\bigcup_{i=1}^{N} J_{\sigma*i}^{(k)}$ does not necessarily have the same distribution as $\bigcup_{i=1}^{N} J_{i}^{(t(\sigma))}$. If we further assume stochastic geometrical self-similarity, $\mathcal{H}^h(K^{(k)}(\omega))$ is shown to be almost surely equal to $X^{(k)}(\omega)$ up to a constant.

Let $\mathcal{J}_n$ be the σ-algebra on Ω generated by $\{J_{\sigma}^{(k)}:|\sigma|\le n\}$ and $\mathcal{B}(J)$ be the Borel field of J. Suppose for all $\sigma\in C_n$ and $k\in C$, $P[J_{\sigma}^{(k)}\neq\emptyset]>0$, and $F_{\sigma}^{(k)}:\Omega\times J\longrightarrow\mathbf{R}^d$ satisfies

(1) $F_{\sigma}^{(k)}$ is $\mathcal{J}_n\times\mathcal{B}(J)$ measurable.

(2) $F_\sigma^{(k)}(\omega, J) = J_\sigma^{(k)}(\omega)$, if $J_\sigma^{(k)}(\omega) \neq \emptyset$, for P-a.a. ω.

(3) $F_\sigma^{(k)}(\omega, \cdot)$ is a geometric similarity map with domain J for P-a.a. ω, that is, there exists a postive constant $c = c(k, \sigma, \omega)$ such that for any $x, y \in J$,

$$\| F_\sigma^{(k)}(\omega, x) - F_\sigma^{(k)}(\omega, y) \| = c \| x - y \|, \quad P\text{-a.s.}$$

Define

$$\tilde{\mathbf{J}}_\sigma^{(k)} = \{\tilde{J}_{\sigma;\eta}^{(k)} : \eta \in C^*\},$$

by

$$\tilde{J}_{\sigma;\eta}^{(k)} = [F_\sigma^{(k)}(\omega, \cdot)]^{-1}(J_{\sigma*\eta}^{(k)}(\omega)).$$

Then given $J_\sigma^{(k)}(\omega) \neq \emptyset$, $\tilde{\mathbf{J}}_\sigma^{(k)}$ is a random construction. We say a multi-type construction is 'stochastically *geometrically* self-similar' if under $J_\sigma^{(k)} \neq \emptyset$, $\tilde{\mathbf{J}}_\sigma^{(k)}$ has the same distribution as $\mathbf{J}_\sigma^{(k)}$, i.e., if B is a Borel subset of $(2^J)^{C^*}$ and $\sigma \in C_n$, then

$$P[\tilde{\mathbf{J}}_\sigma^{(k)} \in B \mid \mathcal{J}_n, J_\sigma^{(k)} \neq \emptyset] = P[\mathbf{J}^{(t(\sigma))} \in B].$$

Theorem 3.9. *Suppose our multi-type construction is stochastically geometrically self-similar and has the property that for all $k \in C$,*

$$\sharp(J_i^{(k)} \cap J_j^{(k)}) < \infty \quad \textit{for} \quad i, j \in C, \quad i \neq j, \ P\text{-a.s.}$$

Let $h(t) = t^\alpha |\log\log(1/t)|^\theta$, $\alpha > 0$, and assume that for all $k \in C$,

$$0 < \mathcal{H}^h(K^{(k)}(\omega)) < \infty, \ P\text{-a.s.}$$

Then there exists a constant $a > 0$ such that

$$\mathcal{H}^h(K^{(k)}(\omega)) = aX^{(k)}(\omega), \ \textit{for all} \ \ k \in C, \quad P\text{-a.s.}$$

Since it is straightforward to extend Theorem 5.5 in [3] to obtain Theorem 3.9, we omit the proof here.

4. Self-avoiding Paths on the d-dimensional Sierpinski Gasket

We start with the definition of the finite d-dimensional Sierpinski gasket for $d \geq 2$. Consider a unit d-dimensional simplex, that is, a polyhedron made up of $(d+1)$ vertices, each of which is connected to the other d vertices by edges of unit length. Embed it in $\mathbf{R}^d$ so that one vertex lies on the origin O. Let $a_1 = O$. Name the other vertices $a_2, a_3, \cdots, a_{d+1}$. Let $G_0 = \{a_1, \cdots, a_{d+1}\}$ and let F_0 be the set of all the points on the edges of the unit d-dimensional simplex whose vertices belong to G_0. Let us define two sequences of sets inductively by,

$$G_{n+1} = \frac{1}{2}\{ \bigcup_{y \in G_0} (G_n + y)\},$$

$$F_{n+1} = \frac{1}{2}\{ \bigcup_{y \in G_0} (F_n + y)\}\,,\ n \in \mathbf{Z}_+\,,$$

where, $A + y = \{x + y \in \mathbf{R}^d \ :\ x \in A\}$, $y \in \mathbf{R}^d$, and $k\,A = \{k\,x \in \mathbf{R}^d \ :\ x \in A\}$, $k \in \mathbf{R}$. F_n's are called (finite) pre-Sierpinski gaskets. Let $F = cl(\bigcup_{n=0}^{\infty} F_n)$. F is the (finite) d-dimensional Sierpinski gasket. We define $\mathcal{T}_n$ to be the set of closed d-simplices in $\mathbf{R}^d$ which are the translations of the d-simplex with vertices $2^{-n}a_1, \cdots, 2^{-n}a_{d+1}$, and whose edges lie in F_n.

Next, for $a, b \in G_0$, we define $W^n(a,b)$ to be the set of sequences $w = (w(0), w(1), \cdots, w(L(w))) \in \cup_{m=1}^{\infty}(G_n)^m$, $L(w) \in \mathbf{N}$, satisfying the following conditions.

(W1) $w(0) = a$.

(W2) $w(L(w)) = b$.

(W3) $|w(i) - w(i+1)| = 2^{-n}$, $\overline{w(i)w(i+1)} \subset F_n\,,\ i = 0, 1, 2, \cdots, L(w)-1$, where $\overline{w(i)w(i+1)}$ denotes the line segment connecting $w(i)$ and $w(i+1)$.

(W4) $w(i) \neq w(j)\,,\ i, j = 0, 1, 2 \cdots, L(w)\,,\ i \neq j$.

(W5) There is no $\Delta \in \mathcal{T}_n$ and no k, $0 \leq k \leq L(w)-2$, satisfying $w(k), w(k+1), w(k+2) \in \Delta$.

$W^n(a,b)$ is a set of paths on G_n starting from a and ending at b, and $L(w)$ is the arrival time at b. (W4) means that $w \in W_1^n$ is self-avoiding. We will comment on the assumption (W5) in Section 5.

Let $W_1^n = W^n(a_1,a_2)$. To describe the large-scale (coarse) behavior of each path $w \in W_1^n$, we define a sequence of 'hitting times' $\{T_i^m(w)\}$, for $m \le n$. Let $T_0^m(w) = 0$, and by induction,

$$T_i^m(w) = \inf\{t > T_{i-1}^m(w) \,:\, w(t) \in G_m\} \ , \quad i \ge 1 \, ,$$

if the right hand side is finite, otherwise, $T_i^m(w) = \infty$. $T_i^m(w)$ is the time when w hits the points in G_m for the i-th time. Noting that $w(L(w)) = a_2$, we obtain an integer $M = M(w)$ and a finite sequence $\{T_i^m(w)\}_{i=1,\cdots,M}$ such that $w(T_M^m(w)) = a_2$, $w(T_i^m(w)) \ne a_2$, $i = 0,\cdots,M-1$.

Let $\{T_i^m(w)\}$, $i = 0,1,2,\cdots,M$, be the sequence obtained above. For $m \in \mathbf{Z}_+$, we define a 'decimation' map $Q_m : \bigcup_{k=m}^{\infty} W_1^k \to W_1^m$, by

$$(Q_m w)(i) = w\left(T_i^m(w)\right) \ , \quad i = 0,1,\cdots,M \, ,$$

and

$$M = L(Q_m w).$$

$Q_m w$ shows the behavior of w on the scale of 2^{-m}. Note that if $k \le m$, we have $Q_k \circ Q_m = Q_k$.

Since a d-simplex has $(d+1)$ vertices, a self-avoiding path in W_1^n is permitted to go through any d-simplex in $\mathcal{T}_n$ at most $[\frac{d+1}{2}]$ times, where $[\frac{d+1}{2}]$ is the largest integer that does not exceed $\frac{d+1}{2}$. Here we meant by "w goes through a d-simplex Δ" that w visits two vertices of Δ in succession. Throughout the following, we write $K = [\frac{d+1}{2}]$.

We define also W_k^n, $k = 2,\cdots,K$ as the set of k-tuples of mutually avoiding, self-avoiding paths, $w = (w^1,\cdots,w^k)$ such that

$$w^j \in W^n(a_{2j-1},a_{2j}), \quad j = 1,\cdots,k.$$

$$\{w^i(t) \,:\, t = 1,\cdots,L(w^i)\} \cap \{w^j(t) \,:\, t = 1,\cdots,L(w^j)\} = \emptyset, \quad \text{if } i \ne j.$$

For $w \in W_k^n$, $k = 1,\cdots,K$, define

$$M_{n,\ell}(w) = \sharp\{\Delta \in \mathcal{T}_n \,:\, \sharp(\{w^j(i) \,:\, i = 1,\cdots,L(w^j), j = 1,\cdots,k\} \cap \Delta) = 2\ell\}.$$

$M_{n,\ell}$ is the number of $\Delta \in \mathcal{T}_n$ through which w goes just ℓ times. We say a $\Delta \in \mathcal{T}_n$ is of Type ℓ with regard to w if w goes through it just ℓ times.

For $\mathbf{x} = (x_1, \cdots, x_K)$ with $x_i > 0$, $i = 1, \cdots, K$, we define polynomials

$$\phi_k(\mathbf{x}) = \sum_{w \in W_k^1} \prod_{i=1}^{K} x_i^{M_{1,i}(w)}, \ k = 1, \cdots, K.$$

Define a probability measure $\Lambda_k(\mathbf{x})$ on W_k^1 $k = 1, \cdots, K$ by

$$\Lambda_k(\mathbf{x})[w] = \phi_k(\mathbf{x})^{-1} \prod_{i=1}^{K} x_i^{M_{1,i}(w)}, \text{ for each } w \in W_k^1.$$

Set $P_1(\mathbf{x}) = \Lambda_1(\mathbf{x})$. For $n = 2, 3, \cdots$, we define $P_n(\mathbf{x})$ on W_1^n inductively by

$$P_n(\mathbf{x})[w] = P_{n-1}(\mathbf{x})[Q_{n-1}w] \prod_{i=1}^{K} \{\phi_i(\mathbf{x})^{-M_{n-1,i}(Q_{n-1}w)} \ x_i^{M_{n,i}(w)}\},$$

for each $w \in W_1^n$.

The inductive definition of $P_n(\mathbf{x})$ on W_1^n can be interpreted as follows: Choose $w \in W_1^{n-1}$ randomly according to $P_{n-1}(\mathbf{x})$. Then add finer structures to each step of w independently. Add them in such a way that if $\Delta \in \mathcal{T}_{n-1}$ is of Type k with regard to w, then the part of the finer path within Δ is similar to some $w' \in W_k^1$ or its reflection. We get a part similar to w' (or a reflection of w') with probability $\Lambda_k[w']$. $P_{n-1}(\mathbf{x})[Q_{n-1}w]$ is the weight of the "parent" path in W_1^{n-1} and the rest corresponds to those of children born in the elements of $\mathcal{T}_{n-1}$ through which the parent goes.

From the definition of $P_n(\mathbf{x})$'s, we see

$$Q_m P_n(\mathbf{x}) = P_m(\mathbf{x}), \tag{4.1}$$

for $m < n$, where $Q_m P_n$ denotes the image measure of P_n induced by Q_m.

By virtue of (4.1) and Kolmogorov's extension theorem, for each $\mathbf{x}$, there is a probability measure $P(\mathbf{x})$ on $(\Omega, \mathcal{F})$, where $\Omega = \prod_{n=0}^{\infty} W_1^n$ and $\mathcal{F}$ is the product Borel field, such that

$$P(\mathbf{x})[\ \omega = \{w_k\}_{k=1}^{\infty} \ : \ Q_m w_n = w_m, \ n \geq m\ ] \ = \ 1,$$

and

$$\pi_n P(\mathbf{x}) = P_n(\mathbf{x}),$$

where π_n denotes the natural projection from Ω to W_1^n, and $\pi_n\omega = w_n$.

Now we construct a multi-type random construction from our self-avoiding paths in such a way that the picture of self-avoiding paths is kept apparent. Let J be the closed d-simplex $a_1 a_2 \cdots a_{d+1}$, $N = (d+1)K$ and $C = \{1, 2, \cdots, N\}$. On $(\Omega, \mathcal{F}, P)$, let us define $\mathbf{J}_{SAP} = \{J_\sigma(\omega) : \sigma \in C^*\}$ as follows. First set $J_\emptyset(\omega) = J$, for all $\omega \in \Omega$, and then define J_σ, $\sigma \in C_n$, $n = 1, 2, \cdots$, inductively as follows. Assume for each $\sigma \in C_n$, J_σ is defined and satisfies for P-a.e. ω,

(1) $J_\sigma(\omega) \in \mathcal{T}_n \cup \{\emptyset\}$.

(2) If $J_\sigma(\omega) \in \mathcal{T}_n$, then w_n passes through $J_\sigma(\omega)$.

(3) $int(J_\sigma(\omega)) \cap int(J_\tau(\omega)) = \emptyset$, for any $\sigma, \tau \in C_n,\ \ \tau \neq \sigma$.

We define $J_{\sigma * i}(\omega)$, $i \in C$ as follows. First, consider the case that $J_\sigma(\omega) \in \mathcal{T}_n$. If it is of Type ℓ with regard to w_n, then there are integers $0 \le i_1 < i_1 + 1 < i_2 < \cdots < i_\ell$ and distinct points $u_1, \cdots, u_{2\ell} \in J_\sigma(\omega) \cap G_n$ such that $u_{2j-1} = w_n(i_j)$, $u_{2j} = w_n(i_j + 1)$, $j = 1, \cdots, \ell$. Name the other $d + 1 - 2\ell$ vertices of $J_\sigma(\omega)$ as $u_{2\ell+1}, \cdots, u_{d+1}$ (the assignment is arbitrary: for example, assign them in the lexicographical order with regard to the coordinates). Denote the $d+1$ elements of $\mathcal{T}_{n+1}$ in $J_\sigma(\omega)$ by $\Delta_1, \cdots, \Delta_{d+1}$, so that $u_i \in \Delta_i$, $i = 1, \cdots, d+1$. We classify Δ_i's into types with regard to w_{n+1}. For each $i \in \{1, \cdots, d+1\}$, if Δ_i is of Type ℓ with regard to w_{n+1}, set $J_{\sigma * \{(\ell-1)(d+1)+i\}}(\omega) = \Delta_i$, and for $k \in \{1, \cdots, K\}$, $k \neq \ell$, set $J_{\sigma * \{(k-1)(d+1)+i\}}(\omega) = \emptyset$. If w_{n+1} does not go through Δ_i, set $J_{\sigma * \{(k-1)(d+1)+i\}}(\omega) = \emptyset$, for all $k \in \{1, \cdots, K\}$. Next, in the case that $J_\sigma(\omega) \notin \mathcal{T}_n$, set $J_{\sigma * j}(\omega) = \emptyset$ for all $j \in C$. $J_{\sigma * i}$'s defined in this way satisfy for P-a.e. ω,

(0) $J_{\sigma * i}(\omega) \subset J_\sigma(\omega)$.

(1) $J_{\sigma * i}(\omega) \in \mathcal{T}_{n+1} \cup \{\emptyset\}$.

(2) If $J_{\sigma * i}(\omega) \in \mathcal{T}_{n+1}$, w_{n+1} passes through $J_{\sigma * i}(\omega)$.

(3) $int(J_{\sigma * i}(\omega)) \cap int(J_{\tau * j}(\omega)) = \emptyset$, for any $\sigma, \tau \in C_n$, $i, j \in C$, with $\tau \neq \sigma$, or $\ i \neq j$.

Now we construct independent random vectors $\mathbf{T}_\sigma$'s in (J3) from J_σ's. Let $\Delta_1, \cdots, \Delta_{d+1}$ be the elements of $\mathcal{T}_1$ such that $a_i \in \Delta_i$. For each $k \in \{1, \cdots, K\}$, let $\{\tilde{T}_{k,(\ell-1)(d+1)+j}\}$, $\ell = 1, \cdots, K$, $j = 1, \cdots, d+1$, be a set of random variables on $(W_k^1, \Lambda_k(\mathbf{x}))$ defined by

$$\tag{4.2} \tilde{T}_{k,(\ell-1)(d+1)+j}(w) = \begin{cases} \frac{1}{2}, & \text{if } \Delta_j \text{ is of Type } \ \ell \text{ with regard to } w, \\ 0, & \text{otherwise .} \end{cases}$$

For $\sigma \in C^*$ and for $\omega \in \Omega$ such that $J_\sigma(\omega) \neq \emptyset$, set

$$\mathbf{T}_\sigma(\omega) = (T_{\sigma*1}(\omega), \cdots, T_{\sigma*N}(\omega)),$$

$$T_{\sigma*i} = \frac{\mathrm{diam}(J_{\sigma*i}(\omega))}{\mathrm{diam}(J_\sigma(\omega))}.$$

The following facts follow from the definitions of $P_n(\mathbf{x})$ and J_σ's.

(1) $(T_{\sigma_1}, \cdots, T_{\sigma*N})$ under $J_\sigma \neq \emptyset$ is distributed as $(\tilde{T}_{k1}, \cdots, \tilde{T}_{kN})$, if $t(\sigma) = (k-1)(d+1) + r$ for some $r = 1, \cdots, d+1$.

(2) Let $\sigma \in C^*$ and let $\mathcal{F}_\sigma^*$ be the σ-field generated by $\{\mathrm{diam}(J_\tau),\ \sigma \not< \tau\}$. For any Borel set A of $\mathbf{R}^N$, and for any $B \in \mathcal{F}_\sigma^*$ such that $\{J_\sigma \neq \emptyset\} \cap B \neq \emptyset$,

$$P[\ \mathbf{T}_\sigma \in A \mid J_\sigma \neq \emptyset, B\ ] = P[\ \mathbf{T}_\sigma \in A \mid J_\sigma \neq \emptyset\ ].$$

These facts allow us to extend $\mathbf{T}_\sigma$'s to $\{\omega \in \Omega\ :\ J_\sigma(\omega) = \emptyset\}$ without changing the distribution so that $\mathbf{T}_\sigma$'s are independent random vectors defined on the whole sample space (if necessary, by enlarging the sample space). Actually, what is relevant in determining the exact Hausdorff dimension is only the distribution of $\mathbf{T}_\sigma$'s under $J_\sigma \neq \emptyset$, but this extension simplifies the description considerably. Thus in the following we regard $\mathbf{T}_\sigma$'s as independent random vectors defined on the whole Ω.

$\mathbf{J}_{SAP}$ constructed this way forms a random construction corresponding to $\mathbf{J}^{(1)}$ defined in Section 2. (We could define also $\mathbf{J}^{(2)}, \cdots, \mathbf{J}^{(N)}$, but we are interested only in paths that go through the whole d-simplex J once.)

Let

$$K(\omega) = \bigcap_{n=1}^{\infty} \bigcup_{\sigma \in C_n} J_\sigma(\omega).$$

$K(\omega)$ corresponds to the 'continuum limit path' in the following sense. For $\omega = (w_1, w_2, \cdots) \in \Omega$, and for each n let w_n^* be the continuous path obtained by connecting $w_n(0), w_n(1), \cdots w_n(L(w_n))$ in this order by line segments of length 2^{-n}. For P-a.e. $\omega \in \Omega$, as n tends to infinity, the limit of w_n^* exists with regard to the Hausdorff metric and coincides with $K(\omega)$.

We turn to the $(d+1)K \times (d+1)K$ matrix $\mathbf{R}(\beta) = \mathbf{R}(\beta, \mathbf{x}) = \{R(\beta)_{ij}\} = \{E[T_{ij}^{\beta}]\}$, where $\mathbf{T}_i = (T_{i1}, \cdots, T_{iN})$, $i \in C$. We have

LEMMA 4.1.

$$R(\beta)_{ij} = (\frac{1}{2})^{\beta} R(0)_{ij} = (\frac{1}{2})^{\beta} P[T_{ij} > 0].$$

$\mathbf{R}(0)$ *is irreducible and its Frobenius' root,* $\rho(\mathbf{x})$ *satisfies* $2 < \rho(\mathbf{x}) < d+1$.

PROOF. The first statement is straightforward because T_{ij} is either $\frac{1}{2}$ or 0.

Assume $w_1 \in W_1^1$ goes through $\Delta \in \mathcal{T}_1$. Let $\Delta_1, \cdots, \Delta_{d+1} \subset \Delta$ be the elements of $\mathcal{T}_2$ named as in the inductive definition of $J_{\sigma * i}$ from J_σ. Since every $\Delta' \cap F_2$, $\Delta' \in \mathcal{T}_1$ has the same substructure, we have for $i \in C$, $j = 1, \cdots, d+1$ and $k = 1, \cdots, K$,

$$T_{(k-1)(d+1)+j,i} \overset{\mathrm{d}}{=} T_{(k-1)(d+1)+1,i}. \tag{4.3}$$

Taking expectation, it follows that

$$R(0)_{(k-1)(d+1)+j,i} = R(0)_{(k-1)(d+1)+1,i}. \tag{4.4}$$

Suppose $\Delta \in \mathcal{T}_1$ is of Type k with regard to $w_1 \in W_1^1$. From the assignment of $\Delta_1, \cdots, \Delta_{d+1} \subset \Delta$, we see for $r = 1, \cdots, 2k$,

$$\sharp(w_1 \cap \Delta_r \cap G_1) = \sharp(\{w_1(t) \,:\, t = 0, \cdots, L(w_1)\} \cap \Delta_r \cap G_1) = 1, \text{ a.s.},$$

thus

$$\sharp(w_2 \cap \Delta_r \cap G_1) = 1, \text{ a.s.}$$

On the other hand, for $r = 2k+1, \cdots, d+1$, (W5) prohibits w_2 to visit $\Delta_r \cap G_1$. Thus, these two groups behave differently, and within each group $w_2 \cap \Delta_r$ has the same distribution up to translations, rotations and reflections.

To express these in terms of $\mathbf{R}(0)$, for $k, \ell = 1, \cdots, K$, $j = 1, \cdots, d+1$, $r = 1, \cdots, 2k$, and $s = 2k+1, \cdots, d+1$,

$$R(0)_{(k-1)(d+1)+j,(\ell-1)(d+1)+r} = R(0)_{(k-1)(d+1)+1,(\ell-1)(d+1)+1}. \tag{4.5}$$

$$R(0)_{(k-1)(d+1)+j,(\ell-1)(d+1)+s} = R(0)_{(k-1)(d+1)+1,\ell(d+1)}. \tag{4.6}$$

(4.4) through (4.6) implies that to show the irreducibility of $\mathbf{R}(0)$, it suffices to show

$$R(0)_{(k-1)(d+1)+1,1} > 0, \text{ for } k = 1, \cdots, K. \tag{4.7}$$

$$R(0)_{1,(\ell-1)(d+1)+1} > 0, \text{ for } \ell = 1, \cdots, K. \tag{4.8}$$

$$R(0)_{1,(\ell-1)(d+1)+3} > 0, \text{ for } 1 \le \ell \le \frac{d}{2}. \tag{4.9}$$

$$R(0)_{(K-1)(d+1)+1,(K-1)(d+1)+1} > 0 \text{ if } K = \frac{d+1}{2}. \tag{4.10}$$

In fact, from (4.4) and (4.7), we have

$$R(0)_{i1} > 0, \quad i = 1, \cdots, K(d+1). \tag{4.11}$$

(4.5) and (4.8) implies

$$R(0)_{1,(\ell-1)(d+1)+1} = R(0)_{1,(\ell-1)(d+1)+2} > 0 \quad \ell = 1, \cdots, K. \tag{4.12}$$

(4.6) and (4.9) gives

$$R(0)_{1,(\ell-1)(d+1)+s} > 0 \quad s = 3, \cdots, d+1, \quad \ell \le \frac{d}{2}. \tag{4.13}$$

Combining (4.11) through (4.13), we have

$$R(0)_{i1} R(0)_{1j} > 0,$$

for all $i, j \in C$ if $K = \frac{d}{2}$ (d is even), and for all $i \in C$ and $j \in \{1, \cdots, (K-1)(d+1)\}$ if $K = \frac{d+1}{2}$ (d is odd). In the case that d is odd, we need also (4.10) to show that for $j > (K-1)(d+1)$

$$R(0)_{i1} R(0)_{1,(K-1)(d+1)+1} R(0)_{(K-1)(d+1)+1,j} > 0.$$

Now we will turn to the derivation of (4.7) through (4.10). Since for any $\Delta' \in \mathcal{T}_1$, $\Delta' \cap F_2$ is similar to F_1, in the following let us consider in terms of paths in W_k^1 instead of the part of w_2 inside Δ', using the fact that $T_{(k-1)(d+1)+r,(\ell-1)(d+1)+j}$ is distributed as $\tilde{T}_{k,(\ell-1)(d+1)+j}$ defined in (4.2). Let $a_1, \cdots, a_{d+1}$ be the vertices of G_0 defined as above and let Δ_i's be the elements of $\mathcal{T}_1$ such that $a_i \in \Delta_i,\ i = 1, \cdots, d+1$. Denote $b_{ij} = b_{ji} = \Delta_i \cap \Delta_j$, for $i \neq j$. Note that

$$\begin{aligned}
&R(0)_{(k-1)(d+1)+i,(\ell-1)(d+1)+j} \\
&= P[T_{(k-1)(d+1)+i,(\ell-1)(d+1)+j} > 0] \\
&= \Lambda_k[w \in W_k^1 \ :\ \Delta_j \text{ is of Type } \ell \text{ with regard to } w\ ].
\end{aligned}$$

We will show (4.7) through (4.10) by explicitly constructing paths on G_1.

For (4.7), consider an k-tuple of paths, $w = (w^1, \cdots, w^k)$, with $w^1 = (a_1, b_{12}, a_2)$, $w^2 = (a_3, b_{34}, a_4)$, ..., $w^k = (a_{2k-1}, b_{2k-1,2k}, a_{2k})$. Obviously, $w \in W_k^1$, and Δ_1 is of Type 1 with regard to w and $\Lambda_k[w] > 0$, thus we have (4.7).

To show (4.8), consider

$$w = (a_1, b_{13}, b_{34}, b_{41}, b_{15}, b_{56}, \cdots, b_{r,1}, b_{1,r+1}, b_{r+1,r+2}, \cdots, b_{2\ell,1}, b_{12}, a_2).$$

$w \in W_1^1$ and w visits $\mathcal{T}_1$-simplices in the order: $\Delta_1 \to \Delta_3 \to \Delta_4 \to \Delta_1 \to \Delta_5 \to \Delta_6 \to \Delta_1 \to \cdots \to \Delta_{2\ell} \to \Delta_1 \to \Delta_2$. Hence Δ_1 is of Type ℓ with regard to w. In the following we will express long paths in terms of the sequences of $\mathcal{T}_1$-simplices they visit.

We can show (4.9) by constructing a path $w \in W_1^1$ such that Δ_3 is of Type ℓ with regard to it. We can realize it by: $\Delta_1 \to \Delta_3 \to \Delta_4 \to \Delta_5 \to \Delta_3 \to \Delta_6 \to \Delta_7 \to \cdots \to \Delta_3 \to \Delta_{2\ell} \to \Delta_{2\ell+1} \to \Delta_3 \to \Delta_2$. This path exists if and only if $\ell \leq \frac{d}{2}$. In fact, for $d = 3$, we get $R_{ij}(0) = 0$ for $i = 1, \cdots, 4$, $j = 7, 8$.

To show (4.10), let $w = (w^1, \cdots, w^K) \in W_K^1$ be w^1: $\Delta_1 \to \Delta_{i_1} \to \Delta_{i_2} \to \Delta_1 \to \Delta_{i_3} \to \Delta_{i_4} \to \Delta_1 \to \cdots \to \Delta_{i_{2K-3}} \to \Delta_{i_{2K-2}} \to \Delta_1 \to \Delta_2$, where $(i_1, \cdots, i_{2K-2})$ is a permutation of $(3, 4, \cdots, 2K)$ and $(i_{2n-1}, i_{2n}) \neq (3,4), (5,6), \cdots, (2K-1, 2K)$, for $n = 1, \cdots, K-1$,

$$w^2 = (a_3, b_{34}, a_4),$$

$$w^3 = (a_5, b_{56}, a_6),$$
$$\cdots,$$
$$w^K = (a_{2K-1}, b_{2K-1,2K}, a_{2K}).$$

Here Δ_1 is of Type K with regard to w.

Thus we have shown that $\mathbf{R}(0)$ is irreducible. Combining this with the non-negativity that is obvious from the definition, we have, by virtue of Frobenius' theorem, the existence of the Frobenius' root $\rho(\mathbf{x})$.

Finally, we will show $2 < \rho(\mathbf{x}) < d+1$. Note

$$2 < \sum_{j=1}^{K(d+1)} R(0)_{ij} \leq d+1, \tag{4.14}$$

$$\sum_{j=1}^{K(d+1)} R(0)_{1j} < d+1. \tag{4.15}$$

The lower bound in (4.14) can be seen from the fact that for any $k \in \{1, \cdots, K\}$, every $w \in W_k^1$ goes through at least two simplices in $\mathcal{T}_1$ and more than two with positive probability. (4.15) is seen from

$$P[\sum_{j=1}^{K(d+1)} T_{1j}^0 = 2] \geq P_1(\mathbf{x})[w = (a_1, b_{12}, a_2)] = \phi_1(\mathbf{x})^{-1} x_1^2 > 0.$$

From (4.14) and (4.15), we have $2 < \rho(\mathbf{x}) < d+1$. This completes the proof. □

Next, we consider $\mathbf{R}(\delta, L)$ defined in Section 3.

LEMMA 4.2. *$\mathbf{R}(\frac{1}{2}, 1)$ is irreducible.*

PROOF. Recall $N = K(d+1)$. It suffices to show for any $i, j \in \{1, 2, \cdots, 2N\}$,

$$R(\frac{1}{2}, 1)_{i,(K-1)(d+1)+1+N} R(\frac{1}{2}, 1)_{(K-1)(d+1)+1+N,j} > 0.$$

We will show this by constructing appropriate paths in $\bigcup_{\ell=1}^{K} W_\ell^1$, as in the proof of Lemma 4.1. Let Δ_i's be the elements of $\mathcal{T}_1$ such that $a_i \in \Delta_i$,

$i = 1, \cdots, d+1$. In the following, we interpret $j+1$ as in $T_{i,j+1}$ as the integer $j_0 \in \{1, \cdots, N\}$ such that $j+1 \equiv j_0$ (mod N). From the definition of $\mathbf{R}(\delta, L)$, it suffices to show for any $i \in \{1, \cdots, N\}$,

$$P[T_{i,(K-1)(d+1)+1} > 0,\ T_{i,(K-1)(d+1)+2} = 0] > 0, \tag{4.16}$$

$$P[T_{(K-1)(d+1)+1,i} > 0,\ T_{(K-1)(d+1)+1,i+1} > 0] > 0, \tag{4.17}$$

$$P[T_{(K-1)(d+1)+1,i} > 0,\ T_{(K-1)(d+1)+1,i+1} = 0] > 0. \tag{4.18}$$

First we will show (4.16). (4.3) implies that it suffices to show (4.16) for $i = (\ell-1)(d+1)+1$, $\ell = 1, \cdots, K$. We show it by constructing a path $w = (w^1, \cdots, w^\ell) \in W_\ell^1$ regard to which Δ_1 is of Type K and Δ_2 is of Type 1. Let w^1 be the path segment we constructed to show (4.10), and if $\ell \geq 2$, let w^i's be the path segments $(a_{2i-1}, b_{2i-1,2i}, a_{2i})$, $i = 2, \cdots, \ell$. This ℓ-tuple satisfies the condition.

Now we will go on to (4.17). Let $i = (\ell-1)(d+1)+j$, $\ell = 1, \cdots, K$ and $j = 1, \cdots, d+1$. In the case that $j \neq d+1$ and $j < 2K$, since all $\Delta_i \cap F_2$'s have the same structure, it is enough to show for $j = 1$ and $j = 2$. For $\ell = 1$, the path $w^i = (a_{2i-1}, b_{2i-1,2i}, a_{2i})$, $i = 1, \cdots, K$ will do. For $\ell \geq 2$, Δ_1, Δ_2 and Δ_3 are all of Type ℓ with regard to the following path, which proves both cases of $j = 1$ and $j = 2$. w^1: $\Delta_1 \to \Delta_{i_1} \to \Delta_2 \to \Delta_{i_2} \to \Delta_1 \to \Delta_{i_3} \to \Delta_2 \to \Delta_{i_4} \to \Delta_1 \to \cdots \to \Delta_{i_{2\ell-3}} \to \Delta_2 \to \Delta_{i_{2\ell-2}} \to \Delta_1 \to \Delta_2$, where $(i_1, \cdots, i_{2\ell-2})$ is a permutation of $(3, 4, \cdots, 2\ell)$ and $(i_{2n-1}, i_{2n}) \neq (3,4), (5,6), \cdots, (2\ell-1, 2\ell)$. w^2: $\Delta_3 \to \Delta_{k_1} \to \Delta_{k_2} \to \Delta_3 \to \Delta_{k_3} \to \Delta_{k_4} \to \Delta_3 \to \cdots \to \Delta_{k_{2\ell-5}} \to \Delta_{k_{2\ell-4}} \to \Delta_3 \to \Delta_4$, where $(k_1, \cdots, k_{2\ell-4})$ is a permutation of $(5, 6, \cdots, 2\ell)$ and $(k_{2n-1}, k_{2n}) \neq (3,4), (5,6), \cdots, (2\ell-1, 2\ell)$. $w^i : (a_{2i-1}, b_{2i-1,2i}, a_{2i})$, $i = 3, \cdots, \ell$.

We deal with the case that $j = d = 2K$ separately, because the path does not visit a_{d+1}. Let w^K be $\Delta_{d-1} \to \Delta_d \to \Delta_{d+1} \to \Delta_{i_2} \to \Delta_d \to \Delta_{i_3} \to \Delta_{i_4} \to \Delta_d \to \cdots \to \Delta_{i_{2\ell-3}} \to \Delta_{i_{2\ell-2}} \to \Delta_d$, where $(i_2, \cdots, i_{2\ell-2})$ is a permutation of $(1, 2, \cdots, d-2)$ and $(i_{2n-1}, i_{2n}) \neq (3,4), (5,6), \cdots, (d-3, d-2)$. Let w^1 be $\Delta_1 \to \Delta_{d+1} \to \Delta_{j_1} \to \Delta_{j_2} \to \Delta_{d+1} \to \Delta_{j_3} \to \Delta_{j_4} \to \Delta_{d+1} \to \cdots \to \Delta_{i_{2\ell-5}} \to \Delta_{i_{2\ell-4}} \to \Delta_{d+1} \to \Delta_2$, where $(j_1, \cdots, j_{2\ell-4})$ is a permutation of $(3, 4, \cdots, d-2)$ and $(j_{2n-1}, j_{2n}) \notin \{(3,4), (5,6), \cdots, (d-3, d-2), (i_{2m-1}, i_{2m}),\ m = 2, \cdots, \ell-1\}$. For $i = 2, \cdots, K-1$, set $w^i = (a_{2i-1}, b_{2i-1,2i}, a_{2i})$. Δ_d and Δ_{d+1} are of Type ℓ with regard to $w = (w^1, \cdots, w^K)$.

(4.17) for $j = d+1$ is shown from the existence of a K-tuple $w \in W_K^1$ with regard to which Δ_{d+1} is of Type ℓ and Δ_1 is of Type ℓ+1 (Type 1 if $\ell = K$). (4.18) is shown in a similar way by constructing paths explicitly. We omit further details here. This completes the proof. □

Since $\lambda(\alpha) = (\frac{1}{2})^\alpha \rho(\mathbf{x})$, we have

$$\alpha = \alpha(\mathbf{x}) = \frac{\log \rho(\mathbf{x})}{\log 2}.$$

Now we apply Theorem 3.8 to $\mathbf{J}_{SAP}$ defined above. As we saw in Lemma 4.1, $\mathbf{R}(0)$ is irreducible and has the Frobenius' root greater than 1. Next, we show that (1) holds. We see by an explicit construction as in the proof of Lemma 4.1, that $P[\sum_{j=1}^N T_{ij}^0 = d+1] > 0$ for every $i \in C$. This combined with the fact that $P[\sum_{j=1}^N T_{ij}^0 \leq d+1] = 1$ for every $i \in C$, we see that θ as in Theorem 3.8 exists and obtained as the solution to

$$(d+1)(\frac{1}{2})^{\alpha/(1-\theta)} = 1,$$

hence,

$$\theta = \theta(\mathbf{x}) = 1 - \frac{\log \rho(\mathbf{x})}{\log(d+1)}.$$

From Lemma 4.2, $\mathbf{R}(\frac{1}{2}, 1)$ is irreducible, thus (3) is satisfied. Thus we have shown that the conditions in Theorem 3.8 are all satisfied. Furthermore, $K(\omega) \neq \emptyset$ P-a.s.. In fact, for P-a.e. ω, $\{\bigcup_{\sigma \in C_n} J_\sigma(\omega)\}$, $n = 1, 2, \cdots$ is a nested sequence of decreasing non-empty compact sets, therefore $\bigcap_{n=1}^{\infty} \bigcup_{\sigma \in C_n} J_\sigma(\omega) \neq \emptyset$, P-a.s.

Thus we have

THEOREM 4.3. *Let $K(\omega) = K_d(\omega, \mathbf{x})$ be the limit path of the branching model of self-avoiding paths on the d-dimensional Sierpinski gasket. Then*

$$K(\omega) \neq \emptyset, \quad P\text{-}a.s.,$$

$$\dim_H(K(\omega)) = \alpha, \quad P\text{-}a.s.,$$

and

$$0 < \mathcal{H}^h(K(\omega)) < \infty, \quad P\text{-}a.s.,$$

where

$$h(t) = t^{\alpha}|\log|\log t||^{\theta},$$

$$\alpha = \alpha(\mathbf{x}) = \frac{\log\rho(\mathbf{x})}{\log 2}, \quad 1 < \alpha < \frac{\log(d+1)}{\log 2},$$

$$\theta = \theta(\mathbf{x}) = 1 - \frac{\log\rho(\mathbf{x})}{\log(d+1)},$$

and $\rho(\mathbf{x})$ *is the Frobenius' root of* $\mathbf{R}(0)$.

REMARK. $\mathbf{J}_{SAP}$ satisfies the conditions for stochastically geometrically self-similar construction, so Theorem 3.9 holds.

5. Concluding Remarks

In [4], we studied the branching models on the 2- and 3-dimensional Sierpinski gaskets as stochastic processes. We have proved that the existence of the continuum limit, that is, the convergence of the process in law to a continuous process on F, under an appropriate time-scale transformation. Also for $d \geq 4$, we can prove the existence of the continuum limit. The proof is omitted here because it goes parallelly as in [4]. What is given in Theorem 4.3 of this paper is the exact Hausdorff dimension of the paths of this continuum limit.

Condition (W5) in the definintion of W_1^n is not essential in obtaining the exact Hausdorff dimension. It simplifies the problem, while the continuum limit of our model is large enough to accommodate important processes. In [5] and [6], Hattori, Hattori and Kusuoka studied 'canonical' models of self-avoiding paths on the 2- and 3-dimensional Sierpinski gaskets. In these models, we assigned to each self-avoiding path w (we do not impose (W5)) from a_1 to a_2, a weight propotional to $e^{-\beta L(w)}$, where $\beta > 0$ is a parameter and $L(w)$ is as in Condition (W2). It was shown that the unique non-trivial continuum limits for $d = 2, 3$ coincide up to time transformations with those of the branching models with $\mathbf{x}$ satisfying

$$\phi_k(\mathbf{x}) = x_k, \quad k = 1, \cdots, K, \tag{5.1}$$

and, for $d = 3$, in addition,

$$x_1^2 > x_2.$$

It was also proved that the limit processes are self-avoiding. (We note for emphasis that the continuum limit of self-avoiding paths is not always self-avoiding. See [5] and [4].) We have good reason to expect that also for $d \geq 4$, the limilt processes of the $e^{-\beta L(w)}$-model coincide with the branching models with $\mathbf{x}$ satisfying (5.1) up to time transformations, and we conjecture also that this limit processes are self-avoiding.

Appendix

A. Recursion Formula

Here we show the multi-type version of the recursion formula corresponding to (2.10) in [3]. It is essential for the proofs of Theorem 3.1 through Theorem 3.3. Let $\beta > 0$. For each $k \in C$, let

$$s_0^{(k)}(\beta) = 1,$$

and

$$s_n^{(k)}(\beta) = E[(X^{(k)})^n]/\Gamma(\frac{n}{\beta} + 1) \quad \text{for } n \geq 1.$$

Let

$$\Delta_N = \{(y_1, \cdots, y_N) \in [0,1]^N \, : \, \sum_{i=1}^{N} y_i = 1\}.$$

Let O_N denote the measure on Δ_N which is the image of Lebesgue measure on $\{(y_1, \cdots, y_{N-1}) \in [0,1]^{N-1} \, : \, \sum_{i=1}^{N-1} y_i \leq 1\}$ induced by the transformation $(y_1, \cdots, y_{N-1}) \longrightarrow (y_1, \cdots, y_{N-1}, 1 - \sum_{i=1}^{N-1} y_i)$. In a similar way to the proof of Theorem 2.1 in [3], it is proved that

$$(r_\beta^{(k)})^{-1} = \{\limsup_{n\to\infty}(s_n^{(k)}(\beta))^{1/n}\}^\beta.$$

$\{s_n^{(k)}(\beta)\}$, $k \in C$, satisfies the following recursion formula.

$$\begin{aligned} s_n^{(k)}(\beta) &= \prod_{i=1}^{N-1}(\frac{n}{\beta}+1) \sum_{j_1+\cdots+j_N=n, 0\le j_1,\cdots,j_N<n} [\frac{n!}{j_1!\cdots j_N!}\int_{\Delta_N}(\prod_{i=1}^{N} y_i^{j_i/\beta})\, dO_N \\ &\qquad (\prod_{i=1}^{N} s_{j_i}^{(i)}(\beta))\{\sum_{\ell=1}^{N}(I-R(n\alpha))_{k\ell}^{-1} E[\prod_{i=1}^{N} T_{\ell i}^{\alpha j_i}]\}], \quad n \ge 2, \end{aligned}$$

where I is the $N \times N$ unit matrix and $R(n\alpha)$ the ratio matrix defined in Section 2.

B. Proofs of Lemma 3.5 and Theorem 3.4

Throughout the following, let $\text{diam} J = 1/e$ so that we can simply replace $|\log|\log \ell_\sigma^{(k)}||$ by $\log|\log \ell_\sigma^{(k)}|$. The general case follows by scaling. To prove Lemma 3.5 we need some preparations. Let S be a set of $2N$ symbols, $\{1_A, 1_B, \cdots, N_A, N_B\}$. Fix $\delta > 0$ and $L \in \{1, 2, \cdots, N-1\}$ as in Theorem 3.4.

For each k, we will define a sequence of S-valued random variables $\{\Sigma_n\}_{n=0,1,2,\cdots}$ on $(D \times \Omega, \mathcal{B}(D) \times \mathcal{F}, Q^{(k)})$. For $\sigma \in D$, denote $\sigma|n* = (\sigma_1, \cdots, \sigma_{n-1}, \sigma_n + L)$. Here $n + L$ denotes the integer $m \in C$ such that $n + L \equiv m$ $(mod\ N)$. For each $(\sigma, \omega) = ((\sigma_1, \sigma_2, \cdots), \omega) \in D \times \Omega$, define random variables T_n^*, $n = 1, 2, \cdots$ by

$$T_n^*(\sigma, \omega) = T_{\sigma|n*}^{(k)}(\omega). \tag{B.1}$$

Define $\{\Sigma_n\}$ by

$$\Sigma_0(\sigma, \omega) = k_A,$$

$$\Sigma_n(\sigma, \omega) = \begin{cases} i_A, & \text{if } \sigma_n = i \text{ and } T_n^*(\sigma, \omega) \ge \delta, \\ i_B, & \text{if } \sigma_n = i \text{ and } T_n^*(\sigma, \omega) < \delta, \end{cases}$$

$$i \in C, \quad n \ge 1.$$

LEMMA B.1. *For each $k \in C$, $\{\Sigma_n\}$ defined as above on $(D \times \Omega, \mathcal{B}(D) \times \mathcal{F}, Q^{(k)})$ is a Markov chain with the transition matrix $\mathbf{P} = \{P_{IJ}\}_{I,J \in S}$ given by*

$$P_{i_A, j_A} = P_{i_B, j_A} = R_{ij}^A,$$

$$P_{i_A,j_B} = P_{i_B,j_B} = R^B_{ij}, \quad i,j \in C,$$

where $\mathbf{R}^A$ *and* $\mathbf{R}^B$ *are defined in Section 3.*

PROOF. Fix $k \in C$. For $I_r \in S, \quad r = 1, \cdots, m$, we will show

$$\begin{aligned} &Q^{(k)}[\Sigma_m = I_m \mid \Sigma_r = I_r,\ r = 1, \cdots, m-1] \\ &= Q^{(k)}[\Sigma_m = I_m \mid \Sigma_{m-1} = I_{m-1}] \\ &= \begin{cases} R^A_{ij}, & \text{if } I_{m-1} = i_A \text{ or } i_B\,, \text{ and } I_m = j_A, \\ R^B_{ij}, & \text{if } I_{m-1} = i_A \text{ or } i_B\,, \text{ and } I_m = j_B\,. \end{cases} \end{aligned}$$

In the following, we will consider the case that all the I_r's are A-states. Generalization to the case that some of the I_r's are B-states is straightforward.

For $i_r \in C$, $r = 1, \cdots, m$,

$$\begin{aligned} &Q^{(k)}[\Sigma_m = i_{mA} \mid \Sigma_r = i_{rA},\ r = 1, \cdots, m-1] \\ &= \frac{Q^{(k)}[(\sigma,\omega) \in D \times \Omega\,:\, \sigma_r = i_r, T^*_r \geq \delta, r = 1, \cdots, m]}{Q^{(k)}[(\sigma,\omega) \in D \times \Omega\,:\, \sigma_r = i_r, T^*_r \geq \delta, r = 1, \cdots, m-1]}\,. \end{aligned}$$

Set $\tau = (i_1, \cdots, i_m) \in C_m$ and $i_0 = k$. From (2.7), (2.3), (B.1), the mutual independence of $\mathbf{T}_\eta$'s, $\eta \in C^*$, and the fact that $X^{(k)}_\eta$ is independent of $\ell^{(k)}_\eta$ and $T^{(k)}_{\eta|r*}$'s, $r = 1, \cdots, m$,

$$\begin{aligned} &Q^{(k)}[\sigma_r = i_r, T^*_r \geq \delta, r = 1, \cdots, m] \\ &= e^\alpha \int (\ell^{(k)}_\tau(\omega))^\alpha X^{(k)}_\tau(\omega)\ \mathbf{1}\{T^{(k)}_{\tau|r*} \geq \delta, r = 1, \cdots, m\} dP(\omega)/x_k \\ &= E[\prod_{r=1}^m (T^{(k)}_{\tau|r})^\alpha\ X^{(k)}_\tau\ \mathbf{1}\{T^{(k)}_{\tau|r*} \geq \delta, r = 1, \cdots, m\}]/x_k \\ &= \prod_{r=1}^m E[(T^{(k)}_{\tau|r})^\alpha,\ T^{(k)}_{(\tau|r)*} \geq \delta] E[X^{(k)}_\tau]/x_k \\ &= \prod_{r=1}^m E[T^\alpha_{i_{r-1},i_r}, T_{i_{r-1},i_r+L} \geq \delta] x_{i_m}/x_k. \end{aligned}$$

Similarly, we have

$$\begin{aligned} &Q^{(k)}[\sigma_r = i_r, T^*_r \geq \delta, r = 1, \cdots, m-1] \\ &= \prod_{r=1}^{m-1} E[T^\alpha_{i_{r-1},i_r}, T_{i_{r-1},i_r+L} \geq \delta] x_{i_{m-1}}/x_k. \end{aligned}$$

Thus

$$\begin{aligned}&Q^{(k)}[\Sigma_m = i_{mA} \mid \Sigma_r = i_{rA},\ r = 1, \cdots, m-1]\\&= \frac{x_{i_m}}{x_{i_{m-1}}} E[T^{\alpha}_{i_{m-1},i_m}, T_{i_{m-1},i_{m+L}} \geq \delta].\end{aligned}$$

Similarly we can show

$$Q^{(k)}[\Sigma_m = I_{mA} \mid \Sigma_{m-1} = i_{m-1A}] = \frac{x_{i_m}}{x_{i_{m-1}}} E[T^{\alpha}_{i_{m-1},i_m}, T_{i_{m-1},i_{m+L}} \geq \delta].$$

Thus we have

$$\begin{aligned}&Q^{(k)}[\Sigma_m = i_{mA} \mid \Sigma_r = i_{rA},\ r = 1, \cdots, m-1]\\&= Q^{(k)}[\Sigma_m = I_{mA} \mid \Sigma_{m-1} = i_{m-1A}]\\&= R^A_{i_{m-1},i_m}.\end{aligned}$$

This completes the proof. □

We will use some basic results about general Markov chains. Let S' be a countable set. Let $\{\Sigma'_n\}_{n=0,1,\cdots}$ be an S'-valued Marcov chain on some $(\Omega, \mathcal{F}, P_x), x \in S'$ with $P_x[\Sigma'_0 = x] = 1$. Let

$$N_n(x) = \sum_{m=1}^{n} \mathbf{1}\{\Sigma'_m = x\}$$

be the number of visits to x by time n. Denote the time of the i-th visit to x by

$$\xi^{(1)}_x = \xi_x = \inf\{n > 0 \,:\, \Sigma'_n = x\},$$
$$\xi^{(i)}_x = \inf\{n > \xi^{(i-1)}_x \,:\, \Sigma'_n = x\}, \quad \text{for} \quad i \geq 2.$$

We say $x \in S'$ is recurrent if $P_x[\xi_x < \infty] = 1$.

Theorem B.2 below is found as Theorem (5.1) in Chapter 5 of [1]. Lemma B.3 is an application of Lemma (9.4) in Chapter 1 of [1].

Theorem B.2. *Suppose $y \in S'$ is recurrent. For any $x \in S'$, as $n \to \infty$,*

$$\frac{N_n(y)}{n} \longrightarrow \frac{1}{E_y[\xi_y]} \mathbf{1}\{\xi_y < \infty\}, \quad P_x - a.s.,$$

where $\frac{1}{\infty} = 0$.

LEMMA B.3. *Suppose* $E_x[\exp(u\xi_x)] < \infty$ *for some* $u > 0$ *and some* $x \in S'$. *If* $a > E_x[\xi_x]$, *there is a constant* c, $0 < c < 1$ *such that*

$$P_x[\frac{\xi_x^{(1)} + \cdots + \xi_x^{(n)}}{n} > a] \le c^n.$$

Now we go back to our case. From our assumption on δ and L, our Markov chain $\{\Sigma_n\}$ on $(D \times \Omega, \mathcal{B}(D) \times \mathcal{F}, Q^{(k)})$ is irreducible, that is, the transition matrix $\mathbf{P}$ is irreducible.

LEMMA B.4. *Let* $0 < \mu < 1$. *For each* $I \in S$, $r_I = \lim_{n\to\infty} \dfrac{N_n(I)}{n}$ *exists* $Q^{(k)}$*-a.s. and there is a constant* c, $0 < c < 1$, *such that*

$$Q^{(k)}[\frac{N_n(I)}{n} < \mu r_I] \le c^n.$$

PROOF. Since our Markov chain is finite-state and irreducible, any $I \in S$ is recurrent and it is easily seen that there is a $u > 0$ satisfying

$$E_Q^{(k)}[\exp(u\xi_I)] < \infty.$$

Moreover, $Q^{(k)}[\xi_I < \infty] = 1$. Theorem B.2 implies that $r_I = \lim_{n\to\infty} \dfrac{N_n(I)}{n}$ exists, and combined with the strong law of large numbers further implies

$$\lim_{n\to\infty} \frac{\xi_I^{(1)} + \cdots + \xi_I^{(n)}}{n} = \frac{1}{r_I}, \quad Q^{(k)}\text{-a.s.}$$

It follows from Lemma B.3

$$\begin{aligned}
&Q^{(k)}[\frac{N_n(I)}{n} < \mu r_I] \\
&= Q^{(k)}[N_n(I) < n\mu r_I] \\
&= Q^{(k)}[\xi_I^{(1)} + \cdots + \xi_I^{[n\mu r_I]+1} > n] \\
&= Q^{(k)}[\frac{\xi_I^{(1)} + \cdots + \xi_I^{[n\mu r_I]+1}}{n\mu r_I} > \frac{1}{\mu r_I}] \\
&\le c^n,
\end{aligned}$$

for some $0 < c < 1$. □

Lemma B.5. *(Lemma 3.2 of [3]).*

Let $g : \mathbf{R}_+ \longrightarrow [0,1]$ *be a non-increasing function with* $\int_0^\infty g(x)dx = \infty$. *Then, for every* $\lambda > 0$ *and for every sequence* $\{j(k)\}_{k=1}^\infty$ *of postive integers with* $\limsup_{k\to\infty} \frac{j(k^3)}{k^3} < 1$, *we have*

$$\limsup_{k\to\infty}\{\int_{j(k)^{1/3}}^{k^{1/3}} g(x)x^2dx - \lambda \log k\} = \infty.$$

Proof of Lemma 3.5. In the following fix $r \in C$. Set

$$I_k^{(r)} = \int_\Omega \sum_{\sigma\in B_k^{(r)}} (\ell_\sigma^{(r)})^\alpha (\log|\log \ell_\sigma^{(r)}|)^{\frac{1}{\beta}} dP.$$

Since L^1- convergence implies almost sure convergence of a subsequence, it suffices to show

$$\liminf_{k\to\infty} I_k^{(r)} = 0.$$

From (2.6),

$$X_{\sigma|\nu}^{(r)}(\omega) \geq (T_{(\sigma|\nu+1)*}^{(r)}(\omega))^\alpha X_{(\sigma|\nu+1)*}^{(r)}(\omega) \quad P\text{-a.s.}$$

Notice that the family $\{X_{(\sigma|\nu+1)*}^{(r)}\}$, $\nu = [\log k], \cdots, k$, is mutually independent and also independent of $\mathcal{F}_\sigma$. This combined with the definition of $B_k^{(r)}$

yields

$$\begin{aligned}
&I_k^{(r)}\\
&\le \sum_{\sigma\in C_k}\int_\Omega \prod_{\nu=[\log k]}^{k-1} \mathbf{1}\{(T^{(r)}_{(\sigma|\nu+1)*}(\omega))^\alpha X^{(r)}_{(\sigma|\nu+1)*}(\omega)\\
&\qquad\qquad < (\frac{1}{t}\log|\log \ell^{(r)}_{\sigma|\nu}(\omega)|)^{1/\beta}\}\\
&\qquad\qquad \times(\ell^{(r)}_\sigma(\omega))^\alpha(\log|\log \ell^{(r)}_\sigma(\omega)|)^{1/\beta} dP(\omega)\\
&= \sum_{\sigma\in C_k}\int_\Omega \prod_{\nu=[\log k]}^{k-1} P[\ \omega' \ :\ (T^{(r)}_{(\sigma|\nu+1)*}(\omega))^\alpha X^{(r)}_{(\sigma|\nu+1)*}(\omega')\\
&\qquad\qquad < (\frac{1}{t}\log|\log \ell^{(r)}_{\sigma|\nu}(\omega)|)^{1/\beta}]\\
&\qquad\qquad \times(\ell^{(r)}_\sigma(\omega))^\alpha(\log|\log \ell^{(r)}_\sigma(\omega)|)^{1/\beta} dP(\omega)\\
&\le \sum_{\sigma\in C_k}\int_\Omega (\ell^{(r)}_\sigma(\omega))^\alpha \exp\{-\sum_{\nu=[\log k]}^{k-1}\\
&\qquad\qquad P[\ \omega' \ :\ \exp\{t(T^{(r)}_{(\sigma|\nu+1)*}(\omega))^{\alpha\beta}(X^{(r)}_{(\sigma|\nu+1)*}(\omega'))^\beta\}\\
&\qquad\qquad \ge |\log \ell^{(r)}_{\sigma|\nu}(\omega)|]\}\\
&\qquad\qquad \times(\log|\log \ell^{(r)}_\sigma(\omega)|)^{1/\beta} dP(\omega).
\end{aligned}$$

Here we conditioned on $\mathcal{F}_\sigma$ to obtain the equality part. From the fact that $X^{(r)}_\sigma$ is independent of the integrand and that $E[X^{(r)}_\sigma] = x_{t(\sigma)}$,

$$\begin{aligned}
&I_k^{(r)}\\
&\le \sum_{\sigma\in C_k}\int_\Omega (\ell^{(r)}_\sigma(\omega))^\alpha X^{(r)}_\sigma(\omega) \exp\{-\sum_{\nu=[\log k]}^{k-1}\\
&\qquad\qquad P[\ \omega' \ :\ \exp\{t(T^{\alpha}_{(\sigma|\nu+1)*}(\omega))^{\alpha\beta}(X^{(r)}_{(\sigma|\nu+1)*}(\omega'))^\beta\}\\
&\qquad\qquad \ge |\log \ell^{(r)}_{\sigma|\nu}(\omega)|]\}\\
&\qquad\qquad \times(\log|\log \ell^{(r)}_\sigma(\omega)|)^{1/\beta} dP(\omega)/(\min_{1\le j\le N} x_j)\\
&\le e^{-\alpha} x_r(\min_{1\le j\le N} x_j)^{-1}\int_{D\times\Omega}(\log|\log \ell_k(\eta,\omega)|)^{1/\beta}\exp\{-\sum_{\nu=[\log k]}^{k-1}\\
&\qquad P[\ \omega' \ :\ \exp\{t(T^*_{\nu+1}(\eta,\omega))^{\alpha\beta} e^{\alpha\beta}(X^{(\eta_{\nu+1}+L)}(\omega'))^\beta\}\\
&\qquad\quad \ge |\log \ell_\nu(\eta,\omega)|]\} dQ^{(r)},
\end{aligned}$$

where we used (2.7), (2.8), the fact that $X^{(r)}_\tau$ is distributed as $X^{t(\tau)}/(\mathrm{diam}J)^\alpha$, $\tau\in C^*$ and the assumption that $\mathrm{diam}J = 1/e$.

Let i be as in Theorem 3.4. For $(\eta, \omega) \in D \times \Omega$, define

$$\Delta_k(\eta, \omega) = \{\nu \in \{[\log k], \cdots, k\} : \eta_{\nu+1} + L = i, \quad T^*_{\nu+1} \geq \delta\}.$$

Suppose a is a constant such that

$$0 < a < \lim_{n \to \infty} \frac{N_n(i_A)}{n}, \ Q^{(r)}\text{-a.s.},$$

for all $r \in \{1, \cdots, N\}$. Fix b, $0 < b < \alpha$, arbitrarily. Let $c > 0$ be such that

$$\lambda(\alpha - b)e^{-bc} < 1. \tag{B.2}$$

Define subsets $E_k^{(r)}$, $F_k^{(r)}$, $G_k^{(r)} \subset D \times \Omega$ by

$$E_k^{(r)} = \{|\log \ell_\nu| \leq c\nu, \nu = [\log k], \cdots, k\} \cap \{\sharp\Delta_k > a(k - [\log k])\},$$

$$F_k^{(r)} = \{|\log \ell_\nu| \leq c\nu, \nu = [\log k], \cdots, k\} \cap \{\sharp\Delta_k \leq a(k - [\log k])\},$$

$$G_k^{(r)} = \{ \text{ There exists } \nu \in \{[\log k], \cdots, k\} \text{ such that } |\log \ell_\nu| > c\nu\},$$

where $\sharp\Delta_k$ denotes the cardinality of Δ_k. Then we have

$$I_k^{(r)} \leq e^{-\alpha} x_r (\min_{1 \leq j \leq N} x_j)^{-1} (I_{k,1}^{(r)} + I_{k,2}^{(r)} + I_{k,3}^{(r)}),$$

where

$$I_{k,1}^{(r)} = \int_{E_k^{(r)}} Y(\eta, \omega) dQ^{(r)},$$

$$I_{k,2}^{(r)} = \int_{F_k^{(r)}} Y(\eta, \omega) dQ^{(r)},$$

$$I_{k,3}^{(r)} = \int_{G_k^{(r)}} Y(\eta, \omega) dQ^{(r)},$$

and

$$\begin{aligned} & Y(\eta, \omega) \\ & = (\log |\log \ell_k(\eta, \omega)|)^{\frac{1}{\beta}} \\ & \exp\{- \sum_{\nu=[\log k]}^{k-1} P[\ \omega' : \exp\{t(T^*_{\nu+1}(\eta, \omega))^{\alpha\beta} e^{\alpha\beta} (X^{(\eta_{\nu+1}+L)}(\omega'))^\beta\} \\ & \qquad \geq |\log \ell_\nu(\eta, \omega)|]\}. \end{aligned}$$

First we will show

$$\liminf_{k\to\infty} I_{k,1}^{(r)} = 0.$$

From the definitions of $E_k^{(r)}$ and Δ_k and the choice of i, for a large enough k with $ck > 1$,

$$I_{k,1}^{(r)} \le (\log ck)^{\frac{1}{\beta}} \int_{\{\sharp\Delta_k > a(k-[\log k])\}} \exp\{-\sum_{\nu\in\Delta_k\setminus\{k\}} P[\ \omega' \ :\ \exp\{t\delta^{\alpha\beta}e^{\alpha\beta}(X^{(i)}(\omega'))^{\beta}\} \ge c\nu]\} dQ^{(r)}.$$

Set

$$j(k) = k - [a(k - [\log k])].$$

Then using the definition of $E_k^{(r)}$ again, we have

$$\begin{aligned}
&\sum_{\nu\in\Delta_k\setminus\{k\}} P[\ \omega' \ :\ \exp\{t\delta^{\alpha\beta}e^{\alpha\beta}(X^{(i)}(\omega'))^{\beta}\} \ge c\nu] \\
&\ge \sum_{\nu=j(k)}^{k-1} P[\ \omega' \ :\ \exp\{t\delta^{\alpha\beta}e^{\alpha\beta}(X^{(i)}(\omega'))^{\beta}\} \ge c\nu] \\
&\ge \sum_{\nu=j(k)}^{k-1} \int_{\nu}^{\nu+1} P[\ \omega' \ :\ \frac{1}{c}\exp\{t\delta^{\alpha\beta}e^{\alpha\beta}(X^{(i)}(\omega'))^{\beta}\} \ge x] dx \\
&= \int_{j(k)}^{k} P[\ \omega' \ :\ \frac{1}{c}\exp\{t\delta^{\alpha\beta}e^{\alpha\beta}(X^{(i)}(\omega'))^{\beta}\} \ge x] dx.
\end{aligned}$$

Note that the last expression is independent of (η, ω). We have

$$\begin{aligned}
&I_{k,1}^{(r)} \\
&\le (\log ck)^{\frac{1}{\beta}} \exp\{-\int_{j(k)}^{k} P[\ \omega' \ :\ \frac{1}{c}\exp\{t\delta^{\alpha\beta}e^{\alpha\beta}(X^{(i)}(\omega'))^{\beta}\} \ge x] dx\} \\
&\le \exp\{\frac{1}{\beta}\log ck - \int_{j(k)}^{k} P[\ \omega' \ :\ \frac{1}{c}\exp\{t\delta^{\alpha\beta}e^{\alpha\beta}(X^{(i)}(\omega'))^{\beta}\} \ge x] dx\}.
\end{aligned}$$

Set

$$g(x) = P[\ \omega' \ :\ (\frac{1}{c})^{1/3}\exp\{\frac{1}{3}t\delta^{\alpha\beta}e^{\alpha\beta}(X^{(i)}(\omega'))^{\beta}\} \ge x].$$

From (3.1) and the definition of $j(k)$, we see that $g(x)$ and $\{j(k)\}_{k=1}^{\infty}$ satisfy the assumptions in Lemma B.5. Thus we have

$$\liminf_{k\to\infty} I_{k,1}^{(r)} = 0.$$

Next we will show

$$\lim_{k\to\infty} I_{k,2}^{(r)} = 0.$$

$$\begin{aligned}
&I_{k,2}^{(r)} \\
&\le (\log ck)^{\frac{1}{\beta}} \int_{\{\sharp\Delta_k \le a(k-[\log k])\}} \exp\{-\sum_{\nu\in\Delta_k\setminus\{k\}} \\
&\qquad P[\ \omega' : \exp\{t\delta^{\alpha\beta}e^{\alpha\beta}(X^{(i)}(\omega'))^{\beta}\} \ge c\nu]\}dQ^{(r)} \\
&\le (\log ck)^{\frac{1}{\beta}} Q^{(r)}[\sharp\Delta_k \le a(k-[\log k])] \\
&\le (\log ck)^{\frac{1}{\beta}} c_1^{k-[\log k]},
\end{aligned}$$

for some c_1 with $0 < c_1 < 1$.

Lemma B.4 was used for the last inequality. Thus we have

$$\lim_{k\to\infty} I_{k,2}^{(r)} = 0.$$

Finally we will show

$$\lim_{k\to\infty} I_{k,3}^{(r)} = 0.$$

From the definition of $G_k^{(r)}$ and Hölder's inequality,

$$\begin{aligned}
&I_{k,3}^{(r)} \\
&\le \sum_{\nu=[\log k]}^{k} \int \mathbf{1}\{|\log \ell_\nu| > c\nu\}(\log|\log \ell_k(\eta,\omega)|)^{\frac{1}{\beta}} dQ^{(r)} \\
&\le \sum_{\nu=[\log k]}^{k} (Q^{(r)}[|\log \ell_\nu| > c\nu])^{1-\frac{1}{\beta}} (\int \log|\log \ell_k(\eta,\omega)| dQ^{(r)})^{\frac{1}{\beta}}.
\end{aligned}$$

We have

$$
\begin{aligned}
&Q^{(r)}[|\log \ell_\nu| > c\nu\}] \\
&\le e^{-bc\nu} E_Q^{(r)}[\exp(b|\log \ell_\nu|)] \quad \text{(Chebyshev's inequality)} \\
&\le e^{-bc\nu} E_Q^{(r)}[(\ell_\nu)^{-b}] \\
&= e^{-bc\nu+\alpha} \sum_{\sigma\in C_\nu} E[(\ell_\sigma^{(r)})^\alpha X_\sigma (\ell_\sigma^{(r)})^{-b}]/x_r \quad (\text{ from } (2.7)) \\
&= e^{-bc\nu+b} \sum_{\sigma\in C_\nu} x_{\sigma_\nu} E[(T_{r\sigma_1})^{-b+\alpha}] \prod_{m=2}^{\nu} E[(T_{\sigma_{m-1}\sigma_m})^{-b+\alpha}]/x_r \\
&\qquad\qquad (\text{from } (2.3) \text{ and } T_\emptyset^{(r)} = \frac{1}{e}) \\
&\le e^{-bc\nu+b} \sum_{j=1}^{N} (R(\alpha-b)^\nu)_{rj} x_j / x_r \quad (\text{from } (2.2)) \\
&\le e^{-bc\nu+b} \lambda(\alpha-b)^\nu.
\end{aligned}
$$

From (B.2), $\lambda(\alpha - b)e^{-bc} = c_2 < 1$. Thus

$$
\begin{aligned}
&I_{k,3}^{(r)} \\
&\le \sum_{\nu=[\log k]}^{k} e^{b(1-\frac{1}{\beta})} c_2^{(1-\frac{1}{\beta})\nu} \left(\int \log|\log \ell_k(\eta,\omega)| dQ^{(r)}\right)^{\frac{1}{\beta}} \\
&\le e^{b(1-\frac{1}{\beta})} \frac{c_2^{(1-\frac{1}{\beta})[\log k]}}{1 - c_2^{(1-\frac{1}{\beta})}} \left(\int \log|\log \ell_k(\eta,\omega)| dQ^{(r)}\right)^{\frac{1}{\beta}}.
\end{aligned}
$$

From Jensen's inequatlity,

$$
\begin{aligned}
&\int \log|\log \ell_k| dQ^{(r)} \\
&= \log k + \int \log(|\log \ell_k|/k) dQ^{(r)}) \\
&\le \log k + \log(\frac{1}{k} \int |\log \ell_k| dQ^{(r)}).
\end{aligned}
$$

From (2.7), (2.8) and the fact that $\{T_{\sigma|\nu}\}_\nu$ is a family of independent random variables, we have

$$
\begin{aligned}
&\int |\log \ell_k| dQ^{(r)} \\
&= 1 + \sum_{m=1}^{k} \int |\log T_m| dQ^{(r)} \\
&= 1 + \sum_{m=1}^{k} \frac{1}{x_r} E[\sum_{\sigma \in C_m} (\ell_\sigma^{(r)})^\alpha X_\sigma |\log T_\sigma|] e^\alpha \\
&= 1 + \sum_{m=1}^{k} \sum_{\sigma \in C_m} \frac{x_{\sigma_m}}{x_r} E[\prod_{\nu=0}^{m} (T_{\sigma|\nu}^{(r)})^\alpha |\log T_\sigma|] \\
&= 1 + \sum_{m=1}^{k} c_3 N \frac{\max_{1\le s\le N} x_s}{x_r} \sum_{j=1}^{N} (R(\alpha)^{m-1})_{rj} \\
&\le 1 + c_4 k,
\end{aligned}
$$

where c_3 and c_4 are positive constants independent of k. Here we used the boundedness of the function $x^\alpha|\log x|$ in $0 < x \le 1$. Thus we have

$$I_{k,3}^{(r)} \le A c_5^{[\log k]} \{\log k + \log(\frac{1}{k} + c_4)\},$$

where A, c_4 and c_5 are postitive constants and $0 < c_5 < 1$. Therefore, $\lim_{k\to\infty} I_{k,3}^{(r)} = 0$. This completes the proof. □

Proof of Theorem 3.4. Once Lemma 3.5 is proved, the rest of the proof proceeds in the same way as that of Theorem 3.1 in [3], so we just give the sketch here.

Fix $r \in C$. Note that (2.4), (2.10) and (3.2) hold for P-a.e.ω. Take such an ω. Then for any $\delta_0 > 0$ and for any $\epsilon > 0$, we can find a maximal antichain $\Gamma = \Gamma(\omega)$ such that

(1) $\{J_\sigma\}_{\sigma\in\Gamma}$ forms a δ_0-cover of $K^{(r)}(\omega)$.

(2) The sum of $(\ell_\sigma^{(r)}(\omega))^\alpha (\log|\log \ell_\sigma^{(r)}(\omega)|)^{1/\beta}$ over $\sigma \in \Gamma$ satisfying

$$X_\sigma^{(r)}(\omega) < (\frac{1}{t} \log|\log \ell_\sigma^{(r)}(\omega)|)^{1/\beta}$$

is less than ϵ.

(1) is guaranteed by (2.4), and (2) by (2.10) and (3.2). It follows that

$$\begin{aligned}&\mathcal{H}^h_{\delta_0}(K^{(r)}(\omega))\\ &\le \sum_{\sigma\in\Gamma} h(\mathrm{diam}(J^{(r)}_\sigma(\omega)))\\ &= \sum_{\sigma\in\Gamma}(\ell^{(r)}_\sigma(\omega))^\alpha(\log|\log \ell^{(r)}_\sigma(\omega)|)^{1/\beta}\\ &\le \sum_{\sigma\in\Gamma}(\ell^{(r)}_\sigma(\omega))^\alpha t^{1/\beta}X^{(r)}_\sigma(\omega)+\epsilon\\ &= t^{1/\beta}X^{(r)}(\omega)+\epsilon,\end{aligned}$$

In the last equality we used (2). Thus we have

$$\mathcal{H}^h(K^{(r)}(\omega)) \le t^{1/\beta}X^{(r)}(\omega) < \infty \quad P\text{-a.s.}$$

This completes the proof. □

C. Proof of Theorem 3.8

It suffices to show that Conditions (2), (4) and (5) in Theorem 3.7 hold.

Since T_{ij} takes only finitely many values almost surely, there is a $\delta > 0$ such that for all $i, j \in C$,

$$P[\,T_{ij} > 0\,] = P[\,T_{ij} > \delta\,].$$

Let $k \in C$. From the definition of $\mathcal{G}^{(k)}_\epsilon(\eta,\omega)$, all $J^{(k)}_\sigma(\omega)$'s with $\sigma \in \mathcal{G}^{(k)}_\epsilon(\eta,\omega)$ are included in a ball of radius 3ϵ centered in $J^{(k)}_{\eta_\epsilon}(\omega)$ and $\mathrm{diam}J^{(k)}_\sigma(\omega) > \delta\epsilon$. This combined with the similarity of all $J^{(k)}_\sigma(\omega)$'s with $\sigma \in \mathcal{G}^{(k)}_\epsilon(\eta,\omega)$ implies that there is an $M > 0$ such that for any $\epsilon > 0$,

$$\sharp\mathcal{G}^{(k)}_\epsilon \le M, \quad Q^{(k)}\text{-a.s.}$$

From this, Conditions (4) and (5) follow immediately.

To show that Condition (2) is satisfied, we use Theorem 3.3. Let

$$G_i = \{\,\omega\,:\,\sum_{j=1}^N T^\gamma_{ij}(\omega) = 1,\ \sum_{j=1}^N T^0_{ij}(\omega) \ge 2\,\}.$$

From the assumptions that T_{ij} takes finitely many values almost surely and $P[\,T_{ij} < 1\,] = 1$, for all $i, j \in C$, we have

$$P[\,G_i\,] > 0, \quad \text{for all} \ \ i \in C.$$

Set

$$p = \min_{i \in C} P[\, G_i \,].$$

Notice that for P-a.a. $\omega \in G_i$,

$$\frac{T_{ik}^{\gamma}}{\sum_{j=1}^{N} T_{ij}^{\gamma}} \leq 1 - \delta^{\gamma} \quad \text{a.s.,} \qquad \text{for all } k \in C.$$

Choose an a from $(\max\{\frac{1}{N}, 1 - \delta^{\gamma}\}, 1) \setminus \{\frac{1}{\nu} \,:\, \nu = 1, \cdots, N-1\}$. Then for any $t > 0$,

$$\begin{aligned}
&E[(\sum_{j=1}^{N} T_{ij}^{\gamma})^t \ \prod_{k=1}^{N} \mathbf{1}\{T_{ik}^{\gamma} / \sum_{j=1}^{N} T_{ij}^{\gamma} \leq a\}] \\
&\geq E[(\sum_{j=1}^{N} T_{ij}^{\gamma})^t \ \prod_{k=1}^{N} \mathbf{1}\{T_{ik}^{\gamma} / \sum_{j=1}^{N} T_{ij}^{\gamma} \leq a\} \, \mathbf{1}_{G_i}] \\
&\geq P[\, G_i \,] \\
&\geq p > 0.
\end{aligned}$$

From this, we see that the left-hand side of the condition in Theorem 3.3 is bounded from below by

$$\prod_{\nu=0}^{\infty} p^{a^{\nu}} = p^{1/(1-a)} > 0.$$

Thus Condition (2) holds. This completes the proof. □

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