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Existence of a Solution to the Mixed Problem for a Vortex Filament Equation with an External Flow Term

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Abstract. The three-dimensional motion of a segment of a vortex filament embedded in an external flow is considered under the localized induction approximation. The mixed problem for a vector equation which describes this motion is proved to have at least one solution by using the parabolic regularization method of the second order.

1. Introduction

In this paper, we consider the mixed problem of the vector equation

$$\boldsymbol{X}_t = \frac{\boldsymbol{X}_\xi \times \boldsymbol{X}_{\xi\xi}}{|\boldsymbol{X}_\xi|^3} + \boldsymbol{V}(\boldsymbol{X}, t) \tag{1.1}$$

for $\boldsymbol{X} = \boldsymbol{X}(\xi, t) \in \mathbf{R}^3$ with $\xi \in I \equiv (0,1)$ and $t > 0$. Here $\boldsymbol{V} : \mathbf{R}^3 \times \{t \geq 0\} \to \mathbf{R}^3$ is a given vector function. The initial and boundary conditions are

$$\boldsymbol{X}(\xi, 0) = \boldsymbol{X}_0(\xi) \tag{1.2}$$

and

$$\boldsymbol{X}(0, t) = \boldsymbol{B}_0(t), \quad \boldsymbol{X}(1, t) = \boldsymbol{B}_1(t). \tag{1.3}$$

Here $\boldsymbol{B}_j(t)$ $(j = 0, 1)$ are solutions to

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$$\frac{d}{dt}\boldsymbol{B}_j = \boldsymbol{V}(\boldsymbol{B}_j, t), \qquad \boldsymbol{B}_j(0) = \boldsymbol{X}_0(j). \tag{1.4}$$

The problem (1.1)–(1.3) describes the three-dimensional motion of a segment of a vortex filament embedded in an inviscid incompressible flow under the localized induction approximation (see [1, eq. (2.1 a)], [2, § 1]). It also describes the motion of a quantized vortex filament in superfluid helium ([5, eq. (13)]). The vector $\boldsymbol{X}$ denotes the position of the vortex filament as a function of the time t and the Lagrangian coordinate ξ defined by the initial arclength of the filament, while $\boldsymbol{V}$ represents the effect of a background flow. By (1.4), the first term in the right member of (1.1) vanishes at the ends of the vortex filament. This implies that the ends may wander, for example, on the boundary of the fluid (see [5]) but the curvature of the filament is kept zero there. From the definition of ξ,

$$\left|\frac{d}{d\xi}\boldsymbol{X}_0\right| \equiv 1 \tag{1.5}$$

should be satisfied.

If the Jacobian matrix of $\boldsymbol{V}(\cdot, t)$ with t fixed is skew-symmetric, then we obtain $\boldsymbol{X}_\xi \cdot \boldsymbol{X}_{\xi t} = 0$ from (1.1) and it implies $|\boldsymbol{X}_\xi| \equiv 1$ for $t > 0$. Particularly, in the case of $\boldsymbol{V} \equiv \mathbf{0}$, (1.1) is rewritten as $\boldsymbol{X}_t = \boldsymbol{X}_\xi \times \boldsymbol{X}_{\xi\xi}$. This is well known by the name of the localized induction equation (LIE) and has been studied by many authors from various points of view. The unique and classical solvability of initial and initial-boundary value problems for LIE can be deduced from that for $\boldsymbol{X}_{\xi t} = \boldsymbol{X}_\xi \times \boldsymbol{X}_{\xi\xi\xi}$ (see [4], [9], [10]).

For a general $\boldsymbol{V}$, however, the constancy of $|\boldsymbol{X}_\xi|$ is no longer guaranteed at $t > 0$. This means that the filament stretches or shrinks by the term $\boldsymbol{V}$. In [2], the author established the existence of a solution to the initial value problem (1.1) and (1.2) assuming that the vortex filament is closed, that is, ξ is on a circle. He applied the parabolic regularization method of the fourth order.

Let us define function spaces and their norms. The space $W_2^r(A)$ ($r > 0$, $A \subset \mathbf{R}$) means that all its elements are bounded with respect to the norm

$\|\cdot\|_{W_2^r(A)}$ defined by

$$\|\boldsymbol{g}\|^2_{W_2^r(A)} = \begin{cases} \displaystyle\sum_{k=0}^{r} \|(d/d\eta)^k \boldsymbol{g}\|^2_{L^2(A)} \equiv \sum_{k=0}^{r} \int_A |(d/d\eta)^k \boldsymbol{g}(\eta)|^2 d\eta \\ \hfill \text{for } r = [r], \\ \\ \|\boldsymbol{g}\|^2_{W_2^{[r]}(A)} + \langle\!\langle \boldsymbol{g} \rangle\!\rangle^2_{W_2^r(A)} \\ \quad \equiv \|\boldsymbol{g}\|^2_{W_2^{[r]}(A)} \\ \qquad + \displaystyle\int_A\int_A \frac{|(d/d\eta)^{[r]}\boldsymbol{g}(\eta_1) - (d/d\eta)^{[r]}\boldsymbol{g}(\eta_2)|^2}{|\eta_1 - \eta_2|^{1+2(r-[r])}} d\eta_1 d\eta_2 \\ \hfill \text{for } r \neq [r] \end{cases}$$

(see [6]). Here $d/d\eta$ stands for the generalized differentiation. The space $W_2^{2r,r}(A_T)$, $A_T \equiv A \times (0,T)$, means that all its elements are bounded with respect to the norm $\|\cdot\|_{W_2^{2r,r}(A_T)}$ defined by

$$\|\boldsymbol{h}\|^2_{W_2^{2r,r}(A_T)} = \int_0^T \|\boldsymbol{h}(\cdot,t)\|^2_{W_2^{2r}(A)} dt + \int_A \|\boldsymbol{h}(\xi,\cdot)\|^2_{W_2^r(0,T)} d\xi.$$

The set of all continuous functions in a Banach space X on $[0,T]$ is denoted by $C([0,T];X)$. In the same way, the spaces $\mathrm{Lip}([0,T];X)$, $C^\beta([0,T];X)$ $(0 < \beta < 1)$ and $L^p(0,T;X)$ $(1 \leq p \leq \infty)$ are defined, where Lip and C^β represent the continuous class of Lipschitz and Hölder, respectively. In order to define the class of $\boldsymbol{V}(\cdot,t)$, we use $C_b^k(\mathbf{R}^3)$ $(k \in \mathbf{N} \cup \{0\})$, which means that any partial derivative up to order k is continuous and bounded on $\mathbf{R}^3$. Then we denote the supremum of $\left(\sum_{|\mu|\leq k} |(\partial/\partial \boldsymbol{y})^\mu \boldsymbol{V}(\boldsymbol{y},t)|^2\right)^{1/2}$ for $\boldsymbol{y} \in \mathbf{R}^3$ by $|\boldsymbol{V}(\cdot,t)|_k$, where μ is a multi-index.

The aim of this paper is to prove the following theorem.

THEOREM 1.1. *Let* $\boldsymbol{V} \in L^1(0,T;C_b^2(\mathbf{R}^3)) \cap C([0,T];C_b^1(\mathbf{R}^3))$ $(0 < T < \infty)$ *and* $\boldsymbol{X}_0 \in W_2^2(I)$. *Then* (1.1) *with* (1.2) *and* (1.3) *has at least a solution* $\boldsymbol{X}$ *in* $L^\infty(0,T;W_2^2(I)) \cap \mathrm{Lip}([0,T];L^2(I))$ *such that*

$$\sup_{(\xi,t)\in I_T} |\boldsymbol{X}_\xi(\xi,t)| + \sup_{(\xi,t)\in I_T} |\boldsymbol{X}_\xi(\xi,t)|^{-1} < \infty.$$

To prove this, we apply the parabolic regularization method of the second order. The condition $\boldsymbol{V} \in C([0,T]; C_b^1(\mathbf{R}^3))$ is used when we obtain $\boldsymbol{B}_0$ and $\boldsymbol{B}_1$ from (1.4).

Through this paper, the symbols c, c', c'', c_*, c'_*, c_1, c_2, c_3, c_4 represent positive constants. The first five constants $c, \ldots, c'_*$ change from line to line, where c_* and c'_* are independent of the coefficient ϵ of the parabolic regularization.

We discuss linear parabolic systems in § 2. In § 3, the parabolic regularization for (1.1) is investigated. In § 4, we prove the theorem letting the parabolic term tend to zero.

2. Linear Parabolic Systems

In addition to the function spaces defined in the introduction, we also use the following in this section.

Let $T > 0$, $\gamma > 0$, $r > 0$ and $r \notin \mathbf{N}$. The space $H_\gamma^r(0,T)$ means that all its elements are bounded with respect to the norm $\|\cdot\|_{H_\gamma^r(0,T)}$ defined by

$$\begin{aligned}\|\boldsymbol{g}\|^2_{H_\gamma^r(0,T)} \;&=\; \gamma^{2r}\int_0^T e^{-2\gamma t}|\boldsymbol{g}(t)|^2 dt \\ &\quad + \int_0^T e^{-2\gamma t} dt \int_0^\infty \frac{|(\partial/\partial t)^{[r]}\boldsymbol{g}_0(t-\tau) - (\partial/\partial t)^{[r]}\boldsymbol{g}_0(t)|^2}{\tau^{1+2(r-[r])}} d\tau,\end{aligned}$$

where $\boldsymbol{g}_0(t) = \boldsymbol{g}(t)$ for $t > 0$ and $\boldsymbol{g}_0(t) = \mathbf{0}$ for $t < 0$. The space $H_\gamma^{2r,r}(A_T)$ ($A \subset \mathbf{R}$, $2r \notin \mathbf{N}$) means that all its elements are bounded with respect to the norm $\|\cdot\|_{H_\gamma^{2r,r}(A_T)}$ defined by

$$\|\boldsymbol{h}\|^2_{H_\gamma^{2r,r}(A_T)} = \int_0^T e^{-2\gamma t}\langle\!\langle \boldsymbol{h}(\cdot,t)\rangle\!\rangle^2_{W_2^{2r}(A)} dt + \int_A \|\boldsymbol{h}(\xi,\cdot)\|^2_{H_\gamma^r(0,T)} d\xi.$$

The domains of definition for these spaces, $(0,T)$ and A_T, can be extended to $(0,\infty)$ and $\mathbf{R}_\infty$ $(= \mathbf{R} \times (0,\infty))$, respectively, so that

$$\|\boldsymbol{g}\|_{H_\gamma^r(0,\infty)} \le c\|\boldsymbol{g}\|_{H_\gamma^r(0,T)},$$

$$\|\boldsymbol{h}\|_{H_\gamma^{2r,r}(\mathbf{R}_\infty)} \le c\|\boldsymbol{h}\|_{H_\gamma^{2r,r}(A_T)}$$

(see [8, § 2]).

Now, we start with the following propositions.

Proposition 2.1. *For given* $\epsilon > 0$, $r \in (0,1)$, $\gamma > 0$, $\boldsymbol{\alpha} \in \mathbf{R}^3$ *and* $\boldsymbol{f} \in H_\gamma^{1+r,(1+r)/2}(\mathbf{R}_\infty)$, *there exists a unique solution* $\boldsymbol{u} \in H_\gamma^{3+r,(3+r)/2}(\mathbf{R}_\infty)$ *of*

$$\boldsymbol{u}_t = \epsilon \boldsymbol{u}_{\xi\xi} + \boldsymbol{\alpha} \times \boldsymbol{u}_{\xi\xi} + \boldsymbol{f}(\xi, t) \tag{2.1}$$

with $\boldsymbol{u}(\xi, 0) = \mathbf{0}$. *At this time, the estimate*

$$\|\boldsymbol{u}\|_{H_\gamma^{3+r,(3+r)/2}(\mathbf{R}_\infty)} \le c\|\boldsymbol{f}\|_{H_\gamma^{1+r,(1+r)/2}(\mathbf{R}_\infty)}$$

is valid.

Proposition 2.2. *For given* $\epsilon > 0$, $r \in (0,\, 1/2)$, $\gamma > 0$, $\boldsymbol{\alpha} \in \mathbf{R}^3$, $\boldsymbol{f} \in H_\gamma^{1+r,(1+r)/2}((0,\infty)^2)$ *and* $\boldsymbol{b} \in H_\gamma^{5/4+r/2}(0,\infty)$, *there exists a unique solution* $\boldsymbol{u} \in H_\gamma^{3+r,(3+r)/2}((0,\infty)^2)$ *of* (2.1) *with* $\boldsymbol{u}(\xi, 0) = \mathbf{0}$ *and* $\boldsymbol{u}(0, t) = \boldsymbol{b}(t)$. *At this time, the estimate*

$$\|\boldsymbol{u}\|_{H_\gamma^{3+r,(3+r)/2}((0,\infty)^2)} \le c\left(\|\boldsymbol{f}\|_{H_\gamma^{1+r,(1+r)/2}((0,\infty)^2)} + \|\boldsymbol{b}\|_{H_\gamma^{5/4+r/2}(0,\infty)}\right)$$

is valid.

These propositions can be proved by use of the Fourier and the Laplace transformations in the same way as [3, Propositions 3.1, 3.4]. In the proof of [3, Proposition 3.4], the equality on line 4 in p. 443 is mistaken and it should be corrected as $(\|g\|_\gamma^{(3/4+\alpha/2)})^2 \ge c\int_{\mathbf{R}} |\sigma|^{3/2+\alpha}|\tilde{g}|^2 d\zeta$.

Furthermore, we get the following corollary replacing ξ by $1-\xi$ in Proposition 2.2.

Corollary 2.3. *Even if the domain* $(0,\infty)^2$ *and the condition* $\boldsymbol{u}(0,t) = \boldsymbol{b}(t)$ *are replaced by* $(-\infty, 1) \times (0, \infty)$ *and* $\boldsymbol{u}(1,t) = \boldsymbol{b}(t)$, *respectively, Proposition* 2.2 *remains valid.*

Next, we consider the problem

$$\boldsymbol{v}_t = L\boldsymbol{v} + \boldsymbol{F}(\xi, t) \equiv \epsilon \boldsymbol{v}_{\xi\xi} + \boldsymbol{a}(\xi, t) \times \boldsymbol{v}_{\xi\xi} + \boldsymbol{F}(\xi, t), \tag{2.2}$$

$$\boldsymbol{v}(\xi, 0) = \boldsymbol{v}_0(\xi), \tag{2.3}$$

$$\text{(2.4)} \qquad \boldsymbol{v}(0,t)=\boldsymbol{b}_0(t), \quad \boldsymbol{v}(1,t)=\boldsymbol{b}_1(t).$$

For this problem, we have the following proposition.

PROPOSITION 2.4. *For given $\epsilon > 0$, $r \in (0,1/2)$, $T \in (0,\infty)$, $\boldsymbol{a} \in W_2^{2+r,1+r/2}(I_T)$, $\boldsymbol{F} \in W_2^{1+r,(1+r)/2}(I_T)$, $\boldsymbol{v}_0 \in W_2^{2+r}(I)$ and $\boldsymbol{b}_0, \boldsymbol{b}_1 \in W_2^{5/4+r/2}(0,T)$, there exists a unique solution $\boldsymbol{v} \in W_2^{3+r,(3+r)/2}(I_T)$ of* (2.2)–(2.4). *At this time, the estimate*

$$\text{(2.5)} \ \|\boldsymbol{v}\|_{W_2^{3+r,(3+r)/2}(I_T)} \le c_1 \Big(\|\boldsymbol{v}_0\|_{W_2^{2+r}(I)} + \|\boldsymbol{F}\|_{W_2^{1+r,(1+r)/2}(I_T)} + \|\boldsymbol{b}_0\|_{W_2^{5/4+r/2}(0,T)} + \|\boldsymbol{b}_1\|_{W_2^{5/4+r/2}(0,T)} \Big)$$

holds, where c_1 increases monotonically with T.

PROOF. First, we define the set of N intervals $\{I^k\}_{k=1}^N$ in $[0,1]$ by

$$I^1=[0,3\lambda), \quad I^j=((j-2)\lambda,\,(j+2)\lambda) \quad (j=2,3,\dots,N-1),$$
$$I^N=((N-2)\lambda,\,1],$$

where N is a large natural number and $\lambda = 1/(N+1)$. We also denote, by $\{p^k(\xi)\}_{k=1}^N$, the set of N scalar functions belonging to $C^\infty[0,1]$ such that $p^k(\xi)=1$ for $\xi \in [(k-1)\lambda,\,(k+1)\lambda]$, $0<p^k(\xi)<1$ for $\xi \in I^k-[(k-1)\lambda,\,(k+1)\lambda]$, $p^k(\xi)=0$ for $\xi \notin I^k$ and $|(d/d\xi)^n p^k| \le c\lambda^{-n}$ for $n=1,2,3,\dots$. Let $q^k(\xi)=p^k(\xi)/\sum_{l=1}^N (p^l(\xi))^2$. Then $\sum_{k=1}^N p^k q^k = 1$ follows for any $\xi \in [0,1]$ (see [7, § 13]).

Let $\widetilde{\boldsymbol{v}} \in W_2^{3+r,(3+r)/2}(I_T)$ be an extension of $\boldsymbol{v}_0$ satisfying $\widetilde{\boldsymbol{v}}|_{t=0}=\boldsymbol{v}_0$ and $\widetilde{\boldsymbol{v}}_t|_{t=0}=L\boldsymbol{v}_0+\boldsymbol{F}(\cdot,0)$. Then

$$\|\widetilde{\boldsymbol{v}}\|_{W_2^{3+r,(3+r)/2}(I_T)} \le c\left(\|\boldsymbol{v}_0\|_{W_2^{2+r}(I)} + \|\boldsymbol{F}\|_{W_2^{1+r,(1+r)/2}(I_T)} \right)$$

holds. According to [8, Lemma 6.3], the inequality

$$\int_0^T \|\boldsymbol{G}(\cdot,t)\|_{L^2(I)}^2 t^{-1-r}\,dt \le c\int_0^1 \langle\!\langle \boldsymbol{G}(\xi,\cdot)\rangle\!\rangle^2_{W_2^{(1+r)/2}(0,T)}\,d\xi$$

is valid for $\boldsymbol{G} = \boldsymbol{F} + L\widetilde{\boldsymbol{v}} - \widetilde{\boldsymbol{v}}_t$, which satisfies $\boldsymbol{G}(\xi, 0) = \mathbf{0}$. Therefore, we have

$$\begin{aligned} \|\boldsymbol{G}\|_{H_\gamma^{1+r,(1+r)/2}(I_T)} &\leq c\|\boldsymbol{G}\|_{W_2^{1+r,(1+r)/2}(I_T)} \\ &\leq c'\left(\|\boldsymbol{v}_0\|_{W_2^{2+r}(I)} + \|\boldsymbol{F}\|_{W_2^{1+r,(1+r)/2}(I_T)}\right) \end{aligned}$$

for any $\gamma > 0$. In the same way,

$$\begin{aligned} &\|\boldsymbol{b}_i - \widetilde{\boldsymbol{v}}(i,\cdot)\|_{H_\gamma^{5/4+r/2}(0,T)} \\ &\quad \leq c\|\boldsymbol{b}_i - \widetilde{\boldsymbol{v}}(i,\cdot)\|_{W_2^{5/4+r/2}(0,T)} \\ &\quad \leq c'\left(\|\boldsymbol{v}_0\|_{W_2^{2+r}(I)} + \|\boldsymbol{F}\|_{W_2^{1+r,(1+r)/2}(I_T)} + \|\boldsymbol{b}_i\|_{W_2^{5/4+r/2}(0,T)}\right) \end{aligned}$$

$(i = 0, 1)$ is obtained.

Extending the domain of definition for $\boldsymbol{b}_0 - \widetilde{\boldsymbol{v}}(0,t)$ and $\boldsymbol{b}_1 - \widetilde{\boldsymbol{v}}(1,t)$ to $(0,\infty)$ and setting $\boldsymbol{w}^{k,(0)}(\xi,t) = \mathbf{0}$, we define $\boldsymbol{w}^{k,(m)}(\xi,t)$ for $k = 1, 2, \dots, N$ and $m \in \mathbf{N}$ as follows.

- $\boldsymbol{w}^{1,(m)}$ is the solution in Proposition 2.2 with $\boldsymbol{\alpha} = \boldsymbol{a}(\lambda, 0)$, $\boldsymbol{f} = \boldsymbol{f}^{1,(m-1)}$ and $\boldsymbol{b} = \boldsymbol{b}_0 - \widetilde{\boldsymbol{v}}(0,t)$.
- $\boldsymbol{w}^{j,(m)}$ $(j = 2, 3, \dots, N-1)$ are the solutions in Proposition 2.1 with $\boldsymbol{\alpha} = \boldsymbol{a}(j\lambda, 0)$ and $\boldsymbol{f} = \boldsymbol{f}^{j,(m-1)}$.
- $\boldsymbol{w}^{N,(m)}$ is the solution in Corollary 2.3 with $\boldsymbol{\alpha} = \boldsymbol{a}(N\lambda, 0) = \boldsymbol{a}(1-\lambda, 0)$, $\boldsymbol{f} = \boldsymbol{f}^{N,(m-1)}$ and $\boldsymbol{b} = \boldsymbol{b}_1 - \widetilde{\boldsymbol{v}}(1,t)$.

Here $\boldsymbol{f}^{k,(m-1)} = p^k \sum_{l=1}^N q^l (L - L_0^l)\boldsymbol{w}^{l,(m-1)} + p^k \sum_{l=1}^N (L(q^l \boldsymbol{w}^{l,(m-1)}) - q^l L\boldsymbol{w}^{l,(m-1)}) + p^k \boldsymbol{G}$ on I_T with L_0^l defined by $L_0^l \boldsymbol{w} = \epsilon \boldsymbol{w}_{\xi\xi} + \boldsymbol{a}(l\lambda, 0)\boldsymbol{w}_{\xi\xi}$ and it is extended to $\mathbf{R}_\infty$ with preservation of class.

Using the embedding theorem and the interpolation inequality, we get

$$\begin{aligned} &\|p^k q^l (L - L_0^l)\boldsymbol{w}^{l,(m-1)}\|^2_{H_\gamma^{1+r,(1+r)/2}(I_T)} \\ &\quad \leq \sup_{(\xi,t)\in I_T^k \cap I_T^l} |\boldsymbol{a}(\xi,t) - \boldsymbol{a}(l\lambda, 0)|^2 \|\boldsymbol{w}_{\xi\xi}^{l,(m-1)}\|^2_{H_\gamma^{1+r,(1+r)/2}(I_T^k \cap I_T^l)} \end{aligned}$$

$$
\begin{aligned}
&+\delta(1+\lambda^{-4})\int_0^\tau e^{-2\gamma t}\langle\!\langle \boldsymbol{w}_{\xi\xi}^{l,(m-1)}(\cdot,t)\rangle\!\rangle^2_{W_2^{1+r}(I^k\cap I^l)}dt\\
&+\delta\int_{I^k\cap I^l}\langle\!\langle e^{-\gamma t}\boldsymbol{w}_{\xi\xi}^{l,(m-1)}(\xi,\cdot)\rangle\!\rangle^2_{W_2^{(1+r)/2}(0,\tau)}d\xi\\
&+c\delta^{-c'}(1+\lambda^{-4})\|e^{-\gamma t}\boldsymbol{w}^{l,(m-1)}\|^2_{L^2(I^k_\tau\cap I^l_\tau)}\\
\le\, & c''(\lambda^{1+2r}+\tau^{1/2+r}+\delta\lambda^{-4}+\delta^{-c'}\lambda^{-4}\gamma^{-3-r})\\
&\times\|\boldsymbol{w}^{l,(m-1)}\|^2_{H_\gamma^{3+r,(3+r)/2}(I^k_\tau\cap I^l_\tau)}
\end{aligned}
$$

if $I^k\cap I^l\neq\emptyset$, where δ is an arbitrary positive number. In the same way, we have

$$
\begin{aligned}
&\|p^k(L(q^l\boldsymbol{w}^{l,(m-1)})-q^lL\boldsymbol{w}^{l,(m-1)})\|^2_{H_\gamma^{1+r,(1+r)/2}(I_\tau)}\\
&\quad\le(\delta\lambda^{-8}+c\delta^{-c'}\lambda^{-8}\gamma^{-3-r})\|\boldsymbol{w}^{l,(m-1)}\|^2_{H_\gamma^{3+r,(3+r)/2}(I^k_\tau\cap I^l_\tau)}\quad(I^k\cap I^l\neq\emptyset).
\end{aligned}
$$

It follows from the inequalities in Propositions 2.1, 2.2 and Corollary 2.3 that

$$
\begin{aligned}
&\sum_{k=1}^N\|\boldsymbol{w}^{k,(m)}\|_{H_\gamma^{3+r,(3+r)/2}(I_\tau)}\\
&\quad\le c\Bigg(\sum_{k=1}^N\|\boldsymbol{f}^{k,(m-1)}\|_{H_\gamma^{1+r,(1+r)/2}(I_\tau)}\\
&\qquad\qquad+\|\boldsymbol{b}_0-\widetilde{\boldsymbol{v}}(0,\cdot)\|_{H_\gamma^{5/4+r/2}(0,\tau)}+\|\boldsymbol{b}_1-\widetilde{\boldsymbol{v}}(1,\cdot)\|_{H_\gamma^{5/4+r/2}(0,\tau)}\Bigg)\\
&\quad\le c'(\lambda^{1/2+r}+\tau^{1/4+r/2}+\delta\lambda^{-4}+\delta^{-c''}\lambda^{-4}\gamma^{-(3+r)/2})\\
&\qquad\times\sum_{k=1}^N\sum_{l=1}^N\|\boldsymbol{w}^{l,(m-1)}\|_{H_\gamma^{3+r,(3+r)/2}(I^k_\tau\cap I^l_\tau)}\\
&\qquad+c'\Bigg(\sum_{k=1}^N\|p^k\boldsymbol{G}\|_{H_\gamma^{1+r,(1+r)/2}(I_\tau)}\\
&\qquad\qquad+\|\boldsymbol{b}_0-\widetilde{\boldsymbol{v}}(0,\cdot)\|_{H_\gamma^{5/4+r/2}(0,\tau)}+\|\boldsymbol{b}_1-\widetilde{\boldsymbol{v}}(1,\cdot)\|_{H_\gamma^{5/4+r/2}(0,\tau)}\Bigg).
\end{aligned}
$$

Note that the number of I^k such that $I^k\cap I^l\neq\emptyset$ with fixed l ($\neq k$) is less than or equal to six. Then, choosing sufficiently small λ, τ, δ and large γ, we have the convergence of $\sum_{k=1}^N \|\boldsymbol{w}^{k,(m)}\|_{H_\gamma^{3+r,(3+r)/2}(I_\mathcal{T})}$ as $m \nearrow \infty$. The limit satisfies

$$\begin{aligned}
&\sum_{k=1}^N \|\boldsymbol{w}^{k,(\infty)}\|_{H_\gamma^{3+r,(3+r)/2}(I_\mathcal{T})} \\
&\qquad \le c\left(\|\boldsymbol{G}\|_{H_\gamma^{1+r,(1+r)/2}(I_\mathcal{T})} + \|\boldsymbol{b}_0 - \widetilde{\boldsymbol{v}}(0,\cdot)\|_{H_\gamma^{5/4+r/2}(0,\mathcal{T})}\right. \\
&\qquad\qquad \left. + \|\boldsymbol{b}_1 - \widetilde{\boldsymbol{v}}(1,\cdot)\|_{H_\gamma^{5/4+r/2}(0,\mathcal{T})}\right) \\
&\qquad \le c'\left(\|\boldsymbol{v}_0\|_{W_2^{2+r}(I)} + \|\boldsymbol{F}\|_{W_2^{1+r,(1+r)/2}(I_\mathcal{T})}\right. \\
&\qquad\qquad \left. + \|\boldsymbol{b}_0\|_{W_2^{5/4+r/2}(0,\mathcal{T})} + \|\boldsymbol{b}_1\|_{W_2^{5/4+r/2}(0,\mathcal{T})}\right).
\end{aligned}$$

Since

$$\sum_{k=1}^N q^k \boldsymbol{w}_t^{k,(\infty)} = \sum_{k=1}^N q^k (L_0^k \boldsymbol{w}^{k,(\infty)} + \boldsymbol{f}^{k,(\infty)}) = \sum_{l=1}^N L(q^l \boldsymbol{w}^{l,(\infty)}) + \boldsymbol{F} + L\widetilde{\boldsymbol{v}} - \widetilde{\boldsymbol{v}}_t,$$

the function $\sum_{k=1}^N q^k \boldsymbol{w}^{k,(\infty)} + \widetilde{\boldsymbol{v}}$ is a unique solution to (2.2)–(2.4).

By repeating the same argument $[T/\mathcal{T}] + 1$-times for $t > \mathcal{T}$ and noting that

$$\|\boldsymbol{v}\|_{W_2^{3+r,(3+r)/2}(I_T)} \le ce^{\gamma T}\|\boldsymbol{v}\|_{H_\gamma^{3+r,(3+r)/2}(I_T)},$$

the proof is completed. □

3. Parabolic Regularization

In this section, we prove the existence of a unique solution to the vector equation

$$\boldsymbol{X}_t = \epsilon \boldsymbol{X}_{\xi\xi} + \frac{\boldsymbol{X}_\xi \times \boldsymbol{X}_{\xi\xi}}{|\boldsymbol{X}_\xi|^3 + \epsilon^\alpha} + \boldsymbol{V}(\boldsymbol{X}, t) \tag{3.1}$$

($\epsilon > 0$, $\alpha > 0$) with (1.2) and (1.3). First, we obtain the following proposition.

Proposition 3.1. *Let $\epsilon > 0$, $\alpha > 0$, $T \in (0,\infty)$, $r \in (0, 1/2)$, $\boldsymbol{V} \in C([0,T]; C_b^3(\mathbf{R}^3))$, $\boldsymbol{V}_t \in C([0,T]; C_b^1(\mathbf{R}^3))$, $\boldsymbol{X}_0 \in W_2^{2+r}(I)$ and $\boldsymbol{B}_0, \boldsymbol{B}_1 \in$*

$W_2^{5/4+r/2}(0,T)$. *Then there exists a positive constant* T_0 *such that the equation* (3.1) *with* (1.2) *and* (1.3) *has a unique solution in* $W_2^{3+r,(3+r)/2}(I_{T_0})$.

PROOF. Let $\boldsymbol{X}^{(m)}$ ($m \in \mathbf{N}$) be a solution in Proposition 2.4 with $\boldsymbol{a} = \boldsymbol{a}^{(m-1)}(\xi,t)$, $\boldsymbol{F} = \boldsymbol{V}^{(m-1)}(\xi,t)$, $\boldsymbol{v}_0 = \boldsymbol{X}_0$, $\boldsymbol{b}_0 = \boldsymbol{B}_0$ and $\boldsymbol{b}_1 = \boldsymbol{B}_1$, where $\boldsymbol{a}^{(0)} = \mathbf{0}$, $\boldsymbol{a}^{(m)} = (|\boldsymbol{X}_\xi^{(m)}|^3 + \epsilon^\alpha)^{-1}\boldsymbol{X}_\xi^{(m)}$, $\boldsymbol{V}^{(0)} = \mathbf{0}$ and $\boldsymbol{V}^{(m)} = \boldsymbol{V}(\boldsymbol{X}^{(m)}(\xi,t),t)$. Then $\boldsymbol{X}^{(m)}$ is shown to be well-defined for all m as follows.

Let c_2 be a constant such that

$$c_2 > c_1\left(\|\boldsymbol{X}_0\|_{W_2^{2+r}(I)} + \|\boldsymbol{B}_0\|_{W_2^{5/4+r/2}(0,T)} + \|\boldsymbol{B}_1\|_{W_2^{5/4+r/2}(0,T)}\right).$$

Then $c_2 > \|\boldsymbol{X}^{(1)}\|_{W_2^{3+r,(3+r)/2}(I_T)}$ holds. Moreover, assume that $\|\boldsymbol{X}^{(m)}\|_{W_2^{3+r,(3+r)/2}(I_{T_1})} \le c_2$ holds for an m and a constant $T_1 \in (0,T]$. Noting that $\boldsymbol{X}_\xi^{(m)} \in C([0,T_1];C^{1/2+r}(I))$ and $\boldsymbol{X}_{\xi\xi}^{(m)}$, $\boldsymbol{X}_t^{(m)} \in C^{r/2}([0,T_1];L^2(I))$ are deduced from Sobolev's embedding theorem, we obtain

$$\|\boldsymbol{V}^{(m)}\|_{W_2^{1+r,(1+r)/2}(I_\tau)} \le c\|\boldsymbol{V}^{(m)}\|_{W_2^{2,1}(I_\tau)} \le c'\tau^{1/2}$$

for $\tau \in (0,T_1]$. It follows from this and (2.5) that $\|\boldsymbol{X}^{(m+1)}\|_{W_2^{3+r,(3+r)/2}(I_{T_2})} \le c_2$ holds if sufficiently small T_2 ($\in (0,T_1]$) is chosen. Therefore, by induction, $\|\boldsymbol{X}^{(m)}\|_{W_2^{3+r,(3+r)/2}(I_{T_2})} \le c_2$ is valid for all m.

Next, set $\boldsymbol{Z}^{(m+1)} = \boldsymbol{X}^{(m+1)} - \boldsymbol{X}^{(m)}$. Then

$$\begin{aligned}\boldsymbol{Z}_t^{(m+2)} &= \epsilon\boldsymbol{Z}_{\xi\xi}^{(m+2)} + \frac{\boldsymbol{X}_\xi^{(m+1)} \times \boldsymbol{Z}_{\xi\xi}^{(m+2)}}{|\boldsymbol{X}_\xi^{(m+1)}|^3 + \epsilon^\alpha} + \frac{\boldsymbol{Z}_\xi^{(m+1)} \times \boldsymbol{X}_{\xi\xi}^{(m+1)}}{|\boldsymbol{X}_\xi^{(m+1)}|^3 + \epsilon^\alpha}\\ &\quad + \left(\frac{1}{|\boldsymbol{X}_\xi^{(m+1)}|^3 + \epsilon^\alpha} - \frac{1}{|\boldsymbol{X}_\xi^{(m)}|^3 + \epsilon^\alpha}\right)\boldsymbol{X}_\xi^{(m)} \times \boldsymbol{X}_{\xi\xi}^{(m+1)}\\ &\quad + \boldsymbol{V}^{(m+1)} - \boldsymbol{V}^{(m)}\end{aligned}$$

is obtained with $\boldsymbol{Z}^{(m+2)}(\xi,0) = \boldsymbol{Z}^{(m+2)}(0,t) = \boldsymbol{Z}^{(m+2)}(1,t) = \mathbf{0}$. By the above boundedness of $\|\boldsymbol{X}^{(m)}\|_{W_2^{3+r,(3+r)/2}(I_{T_2})}$ and the embedding theorem, we can prove that

$$\Bigg\| \frac{\boldsymbol{Z}_\xi^{(m+1)} \times \boldsymbol{X}_{\xi\xi}^{(m+1)}}{|\boldsymbol{X}_\xi^{(m+1)}|^3 + \epsilon^\alpha} + \left(\frac{1}{|\boldsymbol{X}_\xi^{(m+1)}|^3 + \epsilon^\alpha} - \frac{1}{|\boldsymbol{X}_\xi^{(m)}|^3 + \epsilon^\alpha} \right) \boldsymbol{X}_\xi^{(m)} \times \boldsymbol{X}_{\xi\xi}^{(m+1)} \Bigg\|_{W_2^{1+r,(1+r)/2}(I_\tau)}$$

is bounded from above by $c\|\boldsymbol{Z}_\xi^{(m+1)}\|_{W_2^{2,1}(I_\tau)}$ for $\tau \in (0, T_2]$. To verify this, we should note that

$$\begin{aligned}
&\int_0^\tau dt \int_0^1\!\!\int_0^1 |\boldsymbol{X}_{\xi\xi\xi}^{(m+1)}(\xi_1,t)|^2 \Big(|\boldsymbol{X}_\xi^{(m+1)}(\xi_1,t)|^3 - |\boldsymbol{X}_\xi^{(m)}(\xi_1,t)|^3 \\
&\qquad\qquad - |\boldsymbol{X}_\xi^{(m+1)}(\xi_2,t)|^3 + |\boldsymbol{X}_\xi^{(m)}(\xi_2,t)|^3 \Big)^2 \\
&\qquad \times \frac{d\xi_1 d\xi_2}{|\xi_1 - \xi_2|^{1+2r}} \\
&\qquad \le c \Big\| |\boldsymbol{X}_\xi^{(m+1)}|^3 - |\boldsymbol{X}_\xi^{(m)}|^3 \Big\|_{W_2^{2,1}(I_\tau)}^2 \\
&\qquad\quad \times \int_0^\tau dt \int_0^1\!\!\int_0^1 |\boldsymbol{X}_{\xi\xi\xi}^{(m+1)}(\xi_1,t)|^2 \frac{d\xi_1 d\xi_2}{|\xi_1 - \xi_2|^{2r}} \\
&\qquad \le c' \Big\| |\boldsymbol{X}_\xi^{(m+1)}|^3 - |\boldsymbol{X}_\xi^{(m)}|^3 \Big\|_{W_2^{2,1}(I_\tau)}^2,
\end{aligned}$$

where $W_2^{2,1}(I_\tau) \subset C([0,\tau]; C^{1/2}(I))$ is taken into account,

$$\begin{aligned}
&\int_0^\tau\!\!\int_0^\tau \sup_{\xi \in I} \Big(|\boldsymbol{X}_\xi^{(m+1)}(\xi,t_1)|^3 - |\boldsymbol{X}_\xi^{(m)}(\xi,t_1)|^3 \\
&\qquad\qquad - |\boldsymbol{X}_\xi^{(m+1)}(\xi,t_2)|^3 + |\boldsymbol{X}_\xi^{(m)}(\xi,t_2)|^3 \Big)^2 \frac{dt_1 dt_2}{|t_1 - t_2|^{2+r}} \\
&\qquad \le c \Big\| |\boldsymbol{X}_\xi^{(m+1)}|^3 - |\boldsymbol{X}_\xi^{(m)}|^3 \Big\|_{W_2^{2,1}(I_\tau)}^2,
\end{aligned}$$

$$\begin{aligned}
&\Big\| |\boldsymbol{X}_\xi^{(m+1)}|^3 - |\boldsymbol{X}_\xi^{(m)}|^3 \Big\|_{W_2^{2,1}(I_\tau)} \\
&\qquad \le c \|\boldsymbol{Z}_\xi^{(m+1)}\|_{W_2^{2,1}(I_\tau)}
\end{aligned}$$

$$
\begin{aligned}
&+c\left\|\frac{(\boldsymbol{X}_\xi^{(m+1)}\cdot\boldsymbol{X}_{\xi\xi}^{(m+1)})^2}{|\boldsymbol{X}_\xi^{(m+1)}|}-\frac{(\boldsymbol{X}_\xi^{(m)}\cdot\boldsymbol{X}_{\xi\xi}^{(m)})^2}{|\boldsymbol{X}_\xi^{(m)}|}\right\|_{L^2(I_\tau)}\\
\le\ & c\|\boldsymbol{Z}_\xi^{(m+1)}\|_{W_2^{2,1}(I_\tau)}\\
&+c\Bigg\|\frac{1}{|\boldsymbol{X}_\xi^{(m+1)}||\boldsymbol{X}_\xi^{(m)}|}\\
&\quad\times\Big((\boldsymbol{Z}_\xi^{(m+1)}\cdot\boldsymbol{X}_{\xi\xi}^{(m+1)})|\boldsymbol{X}_\xi^{(m)}|(\boldsymbol{X}_\xi^{(m+1)}\cdot\boldsymbol{X}_{\xi\xi}^{(m+1)})\\
&\qquad+(\boldsymbol{X}_\xi^{(m)}\cdot\boldsymbol{Z}_{\xi\xi}^{(m+1)})|\boldsymbol{X}_\xi^{(m)}|(\boldsymbol{X}_\xi^{(m+1)}\cdot\boldsymbol{X}_{\xi\xi}^{(m+1)})\\
&\qquad+(\boldsymbol{X}_\xi^{(m)}\cdot\boldsymbol{X}_{\xi\xi}^{(m)})(|\boldsymbol{X}_\xi^{(m)}|-|\boldsymbol{X}_\xi^{(m+1)}|)(\boldsymbol{X}_\xi^{(m+1)}\cdot\boldsymbol{X}_{\xi\xi}^{(m+1)})\\
&\qquad+(\boldsymbol{X}_\xi^{(m)}\cdot\boldsymbol{X}_{\xi\xi}^{(m)})|\boldsymbol{X}_\xi^{(m+1)}|(\boldsymbol{Z}_\xi^{(m+1)}\cdot\boldsymbol{X}_{\xi\xi}^{(m+1)})\\
&\qquad+(\boldsymbol{X}_\xi^{(m)}\cdot\boldsymbol{X}_{\xi\xi}^{(m)})|\boldsymbol{X}_\xi^{(m+1)}|(\boldsymbol{X}_\xi^{(m)}\cdot\boldsymbol{Z}_{\xi\xi}^{(m+1)})\Big)\Bigg\|_{L^2(I_\tau)}\\
\le\ & c'\|\boldsymbol{Z}_\xi^{(m+1)}\|_{W_2^{2,1}(I_\tau)}.
\end{aligned}
$$

The reason to remark these inequalities is that the author does not know whether or not the norm of $|\boldsymbol{X}_\xi^{(m+1)}|^3-|\boldsymbol{X}_\xi^{(m)}|^3$ in $W_2^{2s,s}(I_\tau)$ with s non-integer can be estimated from above by $c\|\boldsymbol{Z}_\xi^{(m+1)}\|_{W_2^{2s,s}(I_\tau)}$.

Furthermore, we estimate $\boldsymbol{V}^{(m+1)}-\boldsymbol{V}^{(m)}$ as

$$
\begin{aligned}
&\|\boldsymbol{V}^{(m+1)}-\boldsymbol{V}^{(m)}\|_{W_2^{1+r,(1+r)/2}(I_\tau)}\\
&\quad\le\|\boldsymbol{V}^{(m+1)}-\boldsymbol{V}^{(m)}\|_{W_2^{2,1}(I_\tau)}\\
&\quad\le c\|\boldsymbol{Z}^{(m+1)}\|_{W_2^{2,1}(I_\tau)}\left(\sup_{0<t<\tau}|\boldsymbol{V}(\cdot,t)|_3+\sup_{0<t<\tau}|\boldsymbol{V}_t(\cdot,t)|_1\right)\\
&\quad\le c'\|\boldsymbol{Z}^{(m+1)}\|_{W_2^{2,1}(I_\tau)}.
\end{aligned}
$$

Therefore, using (2.5) and the interpolation inequality, we derive

$$
\begin{aligned}
\|\boldsymbol{Z}^{(m+2)}\|_{W_2^{3+r,(3+r)/2}(I_\tau)}&\le c\|\boldsymbol{Z}^{(m+1)}\|_{W_2^{3,3/2}(I_\tau)}\\
&\le(\delta+c'\delta^{-c''}\tau)\|\boldsymbol{Z}^{(m+1)}\|_{W_2^{3+r,(3+r)/2}(I_\tau)},
\end{aligned}
$$

where $0 < \tau \leq T_2$ and δ is an arbitrary positive number. Since $\|\boldsymbol{Z}^{(2)}\|_{W_2^{3+r,(3+r)/2}(I_{T_2})} \leq 2c_2$, the norm $\|\boldsymbol{Z}^{(m)}\|_{W_2^{3+r,(3+r)/2}(I_\tau)}$ converges to zero as $m \nearrow \infty$ if sufficiently small δ and $\tau = T_0$ are chosen. Thus $\boldsymbol{X}^{(m)}$ converges to a solution of (3.1) with (1.2) and (1.3) in $W_2^{3+r,(3+r)/2}(I_{T_0})$.

Estimating the difference between two solutions of (3.1) with identical initial-boundary conditions in the same way as $\boldsymbol{Z}^{(m+2)}$, we can prove that it is equal to zero, or the solution is unique. □

In the above proof, we did not use (1.4). But it is important in the following lemma.

Lemma 3.2. *Let all the assumptions in Proposition* 3.1 *be satisfied. Furthermore, assume $\alpha < 3/4$. Let $\boldsymbol{X}$ be a solution of* (3.1) *with* (1.2) *and* (1.3) *in $W_2^{3+r,(3+r)/2}(I_T)$. Then there exists a positive constant E_0 depending only on α, T, $\|(d/d\xi)^2\boldsymbol{X}_0\|_{L^2(I)}$ and $\int_0^T |\boldsymbol{V}(\cdot,t)|_2 dt$ such that, for any $\epsilon \in (0, E_0]$, the solution $\boldsymbol{X}$ satisfies the estimates*

$$\sup_{0<t<T} \|\boldsymbol{X}(\cdot,t)\|_{W_2^2(I)} \leq c_*, \tag{3.2}$$

$$\|\boldsymbol{X}\|_{W_2^{3+r,(3+r)/2}(I_T)} \leq c. \tag{3.3}$$

In (3.2), *$\boldsymbol{X}_0$ and $\boldsymbol{V}$ control the constant c_* through the values $\|\boldsymbol{X}_0\|_{W_2^2(I)}$ and $\int_0^T |\boldsymbol{V}(\cdot,t)|_2 dt$, respectively.*

Proof. It follows from (3.1) and the boundary conditions that

$$\epsilon \boldsymbol{X}_{\xi\xi} + \frac{\boldsymbol{X}_\xi \times \boldsymbol{X}_{\xi\xi}}{|\boldsymbol{X}_\xi|^3 + \epsilon^\alpha}\bigg|_{\xi=0,1} = \boldsymbol{0},$$

which implies $\boldsymbol{X}_{\xi\xi}|_{\xi=0,1} = \boldsymbol{0}$. Since

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\boldsymbol{X}_\xi(\cdot,t)\|^2_{L^2(I)} &= \int_0^1 \boldsymbol{X}_\xi \cdot \boldsymbol{X}_{\xi t} d\xi \\
&= -\epsilon\|\boldsymbol{X}_{\xi\xi}\|^2_{L^2(I)} + \int_0^1 \boldsymbol{X}_\xi \cdot (\boldsymbol{X}_\xi \cdot \nabla)\boldsymbol{V}(\boldsymbol{X},t) d\xi \\
&\leq |\boldsymbol{V}(\cdot,t)|_1 \, \|\boldsymbol{X}_\xi\|^2_{L^2(I)}
\end{aligned}$$

is obtained, we deduce

$$\sup_{0<t<T} \|\boldsymbol{X}_\xi(\cdot,t)\|_{L^2(I)} \le c_* \tag{3.4}$$

by Gronwall's inequality and (1.5). Here $\nabla \boldsymbol{V}(\cdot,t)$ stands for the Jacobian matrix of $\boldsymbol{V}(\cdot,t)$ with t fixed.

Let us define the norm $\|\!|\cdot|\!\|$ by

$$|\!|\!|\boldsymbol{f}|\!|\!| = \left\|(|\boldsymbol{X}_\xi(\cdot,t)|^3 + \epsilon^\alpha)^{-1/2}\boldsymbol{f}\right\|_{L^2(I)}$$

and assume that $0<\epsilon<1$. Notice that the inequality

$$\sup_{\xi\in I}|g(\xi)| \le \int_0^1 |g(\xi)|d\xi + \int_0^1 |g_\xi(\xi)|d\xi \tag{3.5}$$

for any smooth scalar function g is derived from

$$g(\xi) = \int_{\xi_0}^{\xi} g_\xi(\eta)d\eta + \int_0^1 g(\xi)d\xi$$

with $g(\xi_0) = \int_0^1 g(\xi)d\xi$. Then, using (3.4), (3.5) and Young's inequality, we have

$$\begin{aligned}
\sup_{\xi\in I}|\boldsymbol{X}_\xi(\xi,t)|^2 &\le \|\boldsymbol{X}_\xi\|^2_{L^2(I)} + 2\|\boldsymbol{X}_\xi\|_{L^2(I)}\|\boldsymbol{X}_{\xi\xi}\|_{L^2(I)} \\
&\le c_* + c_*\left(\sup_{\xi\in I}|\boldsymbol{X}_\xi(\xi,t)|^3 + \epsilon^\alpha\right)^{1/2} |\!|\!|\boldsymbol{X}_{\xi\xi}|\!|\!| \\
&\le c_* + \frac{1}{4}\left(\sup_{\xi\in I}|\boldsymbol{X}_\xi(\xi,t)|^3 + 1\right)^{2/3} + c_*'|\!|\!|\boldsymbol{X}_{\xi\xi}|\!|\!|^4,
\end{aligned}$$

which leads to

$$\sup_{\xi\in I}|\boldsymbol{X}_\xi(\xi,t)| \le c_*(|\!|\!|\boldsymbol{X}_{\xi\xi}(\cdot,t)|\!|\!|^2 + 1). \tag{3.6}$$

Moreover, (3.6) yields

$$\|\boldsymbol{X}_{\xi\xi}(\cdot,t)\|_{L^2(I)} \le c_*(|\!|\!|\boldsymbol{X}_{\xi\xi}(\cdot,t)|\!|\!|^3 + 1)|\!|\!|\boldsymbol{X}_{\xi\xi}(\cdot,t)|\!|\!|, \tag{3.7}$$

$$\|\boldsymbol{X}_{\xi\xi\xi}(\cdot,t)\|_{L^2(I)} \le c_*(|\!|\!|\boldsymbol{X}_{\xi\xi}(\cdot,t)|\!|\!|^3 + 1)|\!|\!|\boldsymbol{X}_{\xi\xi\xi}(\cdot,t)|\!|\!|. \tag{3.8}$$

Since $\boldsymbol{X}_{\xi\xi}$ vanishes on the boundary and $\sup_{\xi\in I}|\boldsymbol{X}_{\xi\xi}(\xi,t)|^2 \le 2\|\boldsymbol{X}_{\xi\xi}\|_{L^2(I)}\|\boldsymbol{X}_{\xi\xi\xi}\|_{L^2(I)}$ is derived in the same way as (3.5), the inequalities (3.7) and (3.8) yield

$$\sup_{\xi\in I}|\boldsymbol{X}_{\xi\xi}(\xi,t)| \le c_*(\|\!|\boldsymbol{X}_{\xi\xi}(\cdot,t)\|\!|^3+1)\|\!|\boldsymbol{X}_{\xi\xi}(\cdot,t)\|\!|^{1/2}\|\!|\boldsymbol{X}_{\xi\xi\xi}(\cdot,t)\|\!|^{1/2}. \tag{3.9}$$

The constant c_* in (3.6)–(3.9) depends only on $\int_0^T|\boldsymbol{V}(\cdot,t)|_1dt$ as in (3.4).

The reason to use the norm $\|\!|\cdot\|\!|$ is that we can estimate $(d/dt)\|\!|\boldsymbol{X}_{\xi\xi}\|\!|^2$ successfully by

$$\begin{aligned}
&\int_0^1 \frac{\boldsymbol{X}_{\xi\xi}}{|\boldsymbol{X}_\xi|^3+\epsilon^\alpha}\cdot\left(\frac{\boldsymbol{X}_\xi\times\boldsymbol{X}_{\xi\xi}}{|\boldsymbol{X}_\xi|^3+\epsilon^\alpha}\right)_{\xi\xi}d\xi\\
&\quad= -\int_0^1\left(\frac{\boldsymbol{X}_{\xi\xi}}{|\boldsymbol{X}_\xi|^3+\epsilon^\alpha}\right)_\xi\cdot\left(\frac{\boldsymbol{X}_\xi\times\boldsymbol{X}_{\xi\xi}}{|\boldsymbol{X}_\xi|^3+\epsilon^\alpha}\right)_\xi d\xi\\
&\quad= \int_0^1\left(\frac{(|\boldsymbol{X}_\xi|^3+\epsilon^\alpha)_\xi\boldsymbol{X}_{\xi\xi}}{(|\boldsymbol{X}_\xi|^3+\epsilon^\alpha)^2}\cdot\frac{\boldsymbol{X}_\xi\times\boldsymbol{X}_{\xi\xi\xi}}{|\boldsymbol{X}_\xi|^3+\epsilon^\alpha}\right.\\
&\qquad\qquad\left.+\frac{\boldsymbol{X}_{\xi\xi\xi}}{|\boldsymbol{X}_\xi|^3+\epsilon^\alpha}\cdot\frac{(|\boldsymbol{X}_\xi|^3+\epsilon^\alpha)_\xi\boldsymbol{X}_\xi\times\boldsymbol{X}_{\xi\xi}}{(|\boldsymbol{X}_\xi|^3+\epsilon^\alpha)^2}\right)d\xi\\
&\quad= 0.
\end{aligned}$$

In fact, using (3.1), (3.6), (3.9) and Young's inequality, we have

$$\begin{aligned}
&\frac{d}{dt}\|\!|\boldsymbol{X}_{\xi\xi}(\cdot,t)\|\!|^2\\
&\quad= \int_0^1\left(\frac{2\boldsymbol{X}_{\xi\xi}\cdot\boldsymbol{X}_{\xi\xi t}}{|\boldsymbol{X}_\xi|^3+\epsilon^\alpha}-\frac{3|\boldsymbol{X}_\xi||\boldsymbol{X}_{\xi\xi}|^2\boldsymbol{X}_\xi\cdot\boldsymbol{X}_{\xi t}}{(|\boldsymbol{X}_\xi|^3+\epsilon^\alpha)^2}\right)d\xi\\
&\quad\le -2\epsilon\|\!|\boldsymbol{X}_{\xi\xi\xi}\|\!|^2+c_*\epsilon\int_0^1\frac{|\boldsymbol{X}_\xi|^2|\boldsymbol{X}_{\xi\xi}|^2|\boldsymbol{X}_{\xi\xi\xi}|}{(|\boldsymbol{X}_\xi|^3+\epsilon^\alpha)^2}d\xi\\
&\qquad+c_*|\boldsymbol{V}(\cdot,t)|_1\int_0^1\frac{|\boldsymbol{X}_{\xi\xi}|^2}{|\boldsymbol{X}_\xi|^3+\epsilon^\alpha}d\xi+c_*|\boldsymbol{V}(\cdot,t)|_2\int_0^1\frac{|\boldsymbol{X}_\xi|^2|\boldsymbol{X}_{\xi\xi}|}{|\boldsymbol{X}_\xi|^3+\epsilon^\alpha}d\xi\\
&\quad\le -2\epsilon\|\!|\boldsymbol{X}_{\xi\xi\xi}\|\!|^2+c_*\epsilon\int_0^1\frac{|\boldsymbol{X}_{\xi\xi}|^2|\boldsymbol{X}_{\xi\xi\xi}|}{(|\boldsymbol{X}_\xi|^3+\epsilon^\alpha)^{4/3}}d\xi\\
&\qquad+c_*|\boldsymbol{V}(\cdot,t)|_1\int_0^1\frac{|\boldsymbol{X}_{\xi\xi}|^2}{|\boldsymbol{X}_\xi|^3+\epsilon^\alpha}d\xi+c_*|\boldsymbol{V}(\cdot,t)|_2\int_0^1\frac{|\boldsymbol{X}_\xi|^{1/2}|\boldsymbol{X}_{\xi\xi}|}{(|\boldsymbol{X}_\xi|^3+\epsilon^\alpha)^{1/2}}d\xi
\end{aligned}$$

$$
\begin{aligned}
&\le -2\epsilon|||\boldsymbol{X}_{\xi\xi\xi}|||^2 + c'_*\epsilon^{1-\alpha/3}(|||\boldsymbol{X}_{\xi\xi}|||^3+1)|||\boldsymbol{X}_{\xi\xi}|||^{3/2}|||\boldsymbol{X}_{\xi\xi\xi}|||^{3/2}\\
&\quad +c'_*|\boldsymbol{V}(\cdot,t)|_2(|||\boldsymbol{X}_{\xi\xi}|||+1)|||\boldsymbol{X}_{\xi\xi}|||\\
&\le -\epsilon|||\boldsymbol{X}_{\xi\xi\xi}|||^2 + c_3\epsilon^{1-4\alpha/3}(|||\boldsymbol{X}_{\xi\xi}|||^{12}+1)|||\boldsymbol{X}_{\xi\xi}|||^6\\
&\quad +c_4|\boldsymbol{V}(\cdot,t)|_2(|||\boldsymbol{X}_{\xi\xi}|||^2+1).
\end{aligned}
$$

Here the positive constants c_3 and c_4 depend only on $\int_0^T|\boldsymbol{V}(\cdot,t)|_1dt$. Therefore, we obtain

$$
(3.10)\qquad \frac{d}{dt}(|||\boldsymbol{X}_{\xi\xi}(\cdot,t)|||^2+1) \le -\epsilon|||\boldsymbol{X}_{\xi\xi\xi}|||^2 + c_3\epsilon^{1-4\alpha/3}(|||\boldsymbol{X}_{\xi\xi}|||^2+1)^9 +c_4|\boldsymbol{V}(\cdot,t)|_2(|||\boldsymbol{X}_{\xi\xi}|||^2+1).
$$

Setting $q(t)=(|||\boldsymbol{X}_{\xi\xi}|||^2+1)^{-8}$ in (3.10), we get

$$
\frac{dq}{dt}\ge -8c_4|\boldsymbol{V}(\cdot,t)|_2q-8c_3\epsilon^{1-4\alpha/3}.
$$

This leads to

$$
\begin{aligned}
q(t) &\ge \exp\left(-8c_4\textstyle\int_0^t|\boldsymbol{V}(\cdot,t_1)|_2dt_1\right)\\
&\quad\times\left(q(0)-8c_3\epsilon^{1-4\alpha/3}\int_0^t\exp\left(8c_4\textstyle\int_0^{t_1}|\boldsymbol{V}(\cdot,t_2)|_2dt_2\right)dt_1\right)\\
&\ge q(0)\exp\left(-8c_4\textstyle\int_0^T|\boldsymbol{V}(\cdot,t)|_2dt\right)-8c_3\epsilon^{1-4\alpha/3}T\\
&\ge (\|(d/d\xi)^2\boldsymbol{X}_0\|^2_{L^2(I)}+1)^{-8}\exp\left(-8c_4\textstyle\int_0^T|\boldsymbol{V}(\cdot,t)|_2dt\right)\\
&\quad -8c_3\epsilon^{1-4\alpha/3}T.
\end{aligned}
$$

If E_0 is chosen so small that

$$
(3.11)\qquad 0<E_0^{1-4\alpha/3}<\min\left(1,\quad\frac{\exp\left(-8c_4\int_0^T|\boldsymbol{V}(\cdot,t)|_2dt\right)}{8c_3T(\|(d/d\xi)^2\boldsymbol{X}_0\|^2_{L^2(I)}+1)^8}\right),
$$

then $q(t)\ge(\|(d/d\xi)^2\boldsymbol{X}_0\|^2_{L^2(I)}+1)^{-8}\exp\left(-8c_4\int_0^T|\boldsymbol{V}(\cdot,t)|_2dt\right)-8c_3E_0^{1-4\alpha/3}T>0$, or

$$
\begin{aligned}
(3.12)\qquad &\sup_{0<t<T}|||\boldsymbol{X}_{\xi\xi}(\cdot,t)|||\\
&\le\left(\left(\frac{\exp\left(-8c_4\int_0^T|\boldsymbol{V}(\cdot,t)|_2dt\right)}{(\|(d/d\xi)^2\boldsymbol{X}_0\|^2_{L^2(I)}+1)^8}-8c_3E_0^{1-4\alpha/3}T\right)^{-1/8}-1\right)^{1/2}\\
&\le c_*
\end{aligned}
$$

is obtained for all $\epsilon \in (0, E_0]$.

Thus we get (3.2) from (3.4), (3.7) and (3.12). The estimate $\sup_{0<t<T} \|\boldsymbol{X}(\cdot,t)\|_{L^2(I)} \le c_*$ is derived from

$$\begin{aligned}\sup_{0<t<T} \|\boldsymbol{X}(\cdot,t)\|_{L^2(I)} &\le \sup_{0<t<T} \|\boldsymbol{X}_\xi(\cdot,t)\|_{L^2(I)} + \sup_{0<t<T} |\boldsymbol{B}_0(t)| \\ &\le \sup_{0<t<T} \|\boldsymbol{X}_\xi(\cdot,t)\|_{L^2(I)} + |\boldsymbol{B}_0(0)| + \int_0^T |\boldsymbol{V}(\cdot,t)|_0 dt.\end{aligned}$$

It is easy to obtain $\sup_{0<t<T} \|\boldsymbol{X}_t(\cdot,t)\|_{L^2(I)} \le c$ from (3.1) and (3.2). Since it yields $\|\boldsymbol{V}(\boldsymbol{X},t)\|_{W_2^{2,1}(I_T)} \le c$, we have (3.3) using (2.5) with $\boldsymbol{F} = \boldsymbol{V}(\boldsymbol{X},t)$. $\square$

Since the above a priori estimates are valid with T arbitrarily given by $\boldsymbol{V}$, we can prove the unique solvability of (3.1) on $[0,T]$ applying the standard continuation argument.

PROPOSITION 3.3. *Let all the assumptions in Proposition* 3.1 *be satisfied. Furthermore, assume $\alpha < 3/4$. Then, for each $\epsilon \in (0, E_0]$ with E_0 satisfying* (3.11), *there exists a unique solution $\boldsymbol{X}$ of* (3.1) *with* (1.2) *and* (1.3) *in $W_2^{3+r,(3+r)/2}(I_T)$ and* (3.2) *is valid.*

The above solution has the following property.

LEMMA 3.4. *For the solution $\boldsymbol{X}$ in Proposition* 3.3,

$$\sup_{(\xi,t)\in I_T} \frac{1}{|\boldsymbol{X}_\xi(\xi,t)|^3 + \epsilon^\alpha} \le c_* \tag{3.13}$$

and

$$\int_{t_1}^{t_2} \|\boldsymbol{X}_t(\cdot,t)\|_{L^2(I)} dt \le c_*|t_2 - t_1| + \int_{t_1}^{t_2} |\boldsymbol{V}(\cdot,t)|_0 dt \tag{3.14}$$

hold, where the dependency of c_* on $\boldsymbol{X}_0$ and $\boldsymbol{V}$ is the same as in (3.2).

PROOF. We obtain

$$\epsilon \int_0^T \|\boldsymbol{X}_{\xi\xi\xi}(\cdot,t)\|^2 dt \le c_* \tag{3.15}$$

from (3.10) and (3.12). Since

$$\begin{aligned}\frac{d}{dt}\int_0^1 \frac{d\xi}{(|\boldsymbol{X}_\xi(\xi,t)|^3+\epsilon^\alpha)^{1/3}} &= -\int_0^1 \frac{|\boldsymbol{X}_\xi|\boldsymbol{X}_\xi\cdot\boldsymbol{X}_{\xi t}}{(|\boldsymbol{X}_\xi|^3+\epsilon^\alpha)^{4/3}}d\xi \\ &\le \int_0^1 \frac{\epsilon|\boldsymbol{X}_{\xi\xi\xi}|+|(\boldsymbol{X}_\xi\cdot\nabla)\boldsymbol{V}(\boldsymbol{X},t)|}{(|\boldsymbol{X}_\xi|^3+\epsilon^\alpha)^{2/3}}d\xi \\ &\le \epsilon^{1-\alpha/6}\|\!|\boldsymbol{X}_{\xi\xi\xi}|\!\| \\ &\quad +|\boldsymbol{V}(\cdot,t)|_1\int_0^1 \frac{d\xi}{(|\boldsymbol{X}_\xi|^3+\epsilon^\alpha)^{1/3}},\end{aligned}$$

we have

$$\sup_{0<t<T}\int_0^1 \frac{d\xi}{(|\boldsymbol{X}_\xi(\xi,t)|^3+\epsilon^\alpha)^{1/3}} \le c_*$$

by (1.5) and (3.15). Therefore, (3.5) yields

$$\begin{aligned}&\sup_{\xi\in I}\frac{1}{(|\boldsymbol{X}_\xi(\xi,t)|^3+\epsilon^\alpha)^{1/3}} \\ &\quad\le \int_0^1 \frac{d\xi}{(|\boldsymbol{X}_\xi|^3+\epsilon^\alpha)^{1/3}} + \left\|\left(\frac{1}{(|\boldsymbol{X}_\xi|^3+\epsilon^\alpha)^{1/3}}\right)_\xi\right\|_{L^2(I)} \\ &\quad\le c_* + \left(\sup_{\xi\in I}\frac{1}{(|\boldsymbol{X}_\xi(\xi,t)|^3+\epsilon^\alpha)^{1/3}}\right)^{1/2}\|\!|\boldsymbol{X}_{\xi\xi}|\!\| \\ &\quad\le c_*' + \frac{1}{2}\sup_{\xi\in I}\frac{1}{(|\boldsymbol{X}_\xi(\xi,t)|^3+\epsilon^\alpha)^{1/3}}\end{aligned}$$

for $t\in[0,T]$. Thus (3.13) follows.

The inequality (3.14) is derived from (3.1) with (3.2) and (3.13). $\square$

4. Proof of Theorem

Finally, letting $\epsilon \searrow 0$, we establish the existence of a solution to (1.1)–(1.3).

Proof of Theorem 1.1. Let $\boldsymbol{V}^\epsilon$ and $\boldsymbol{X}_0^\epsilon$ be mollified functions which converge to $\boldsymbol{V}$ in $L^1(0,T;C_b^2(\mathbf{R}^3))$ and to $\boldsymbol{X}_0$ in $W_2^2(I)$, respectively, as $\epsilon\searrow 0$. Suppose that $\boldsymbol{B}_j^\epsilon$ $(j=0,1)$ are solutions to $(d/dt)\boldsymbol{B}_j^\epsilon = \boldsymbol{V}^\epsilon(\boldsymbol{B}_j^\epsilon,t)$ with $\boldsymbol{B}_j^\epsilon(0)=\boldsymbol{X}_0^\epsilon(j)$. Furthermore, let $\boldsymbol{X}^\epsilon$ be the solution to (3.1), (1.2) and

(1.3) with $\boldsymbol{V}$, $\boldsymbol{X}_0$, $\boldsymbol{B}_0$ and $\boldsymbol{B}_1$ replaced by $\boldsymbol{V}^\epsilon$, $\boldsymbol{X}_0^\epsilon$, $\boldsymbol{B}_0^\epsilon$ and $\boldsymbol{B}_1^\epsilon$, respectively. Then (3.2) and (3.13) are still valid for $\epsilon \ll E_0$ and there exists a sequence $\{\epsilon_m\}_{m=1}^\infty$ $(E_0 \gg \epsilon_1 > \epsilon_2 > \epsilon_3 > \ldots \searrow 0)$ such that $\boldsymbol{X}^{\epsilon_m}$ converges to some function $\boldsymbol{Y}$ in $L^\infty(0,T;W_2^2(I))$ weak-* as $m \nearrow \infty$. Since (3.14) with $\boldsymbol{V}$ replaced by $\boldsymbol{V}^{\epsilon_m}$ implies that $\boldsymbol{X}^{\epsilon_m}$ is equicontinuous in $L^2(I)$ on $[0,T]$, $\boldsymbol{X}^{\epsilon_m}$ converges to $\boldsymbol{Y}$ strongly in $L^2(I)$, and $W_2^s(I)$ $(0<s<2)$, uniformly on $[0,T]$. From this, the weak convergence of $\boldsymbol{X}^{\epsilon_m}$ follows in $W_2^2(I)$ uniformly on $[0,T]$. Noting that (3.2) and (3.14) also give the equicontinuity of $\boldsymbol{X}_\xi^{\epsilon_m}$ on I_T as

$$\begin{aligned}
&|\boldsymbol{X}_\xi^{\epsilon_m}(\xi_2,t_2) - \boldsymbol{X}_\xi^{\epsilon_m}(\xi_1,t_1)| \\
&\quad \le \left| \int_{\xi_1}^{\xi_2} \boldsymbol{X}_{\xi\xi}^{\epsilon_m}(\xi,t_2)d\xi \right| \\
&\qquad + c_* \|\boldsymbol{X}^{\epsilon_m}(\cdot,t_2) - \boldsymbol{X}^{\epsilon_m}(\cdot,t_1)\|_{L^2(I)}^{1/4} \|\boldsymbol{X}^{\epsilon_m}(\cdot,t_2) - \boldsymbol{X}^{\epsilon_m}(\cdot,t_1)\|_{W_2^2(I)}^{3/4} \\
&\quad \le c_*'(|\xi_2-\xi_1|^{1/2} + |t_2-t_1|^{1/4}) + c_*' \left(\int_{t_1}^{t_2} |\boldsymbol{V}^{\epsilon_m}(\cdot,t)|_0 dt \right)^{1/4} \\
&\quad \le c_*'(|\xi_2-\xi_1|^{1/2} + |t_2-t_1|^{1/4}) \\
&\qquad + c_*' \left(\int_{t_1}^{t_2} (|\boldsymbol{V}(\cdot,t)|_0 + |\boldsymbol{V}^{E_0}(\cdot,t) - \boldsymbol{V}(\cdot,t)|_0) dt \right)^{1/4},
\end{aligned}$$

we get not only $\sup_{(\xi,t)\in I_T} |\boldsymbol{Y}_\xi(\xi,t)| \le c_*$ but also $\sup_{(\xi,t)\in I_T} |\boldsymbol{Y}_\xi(\xi,t)|^{-1} \le c_*$ from (3.13). Therefore, we obtain

$$\frac{\boldsymbol{X}_\xi^{\epsilon_m} \times \boldsymbol{X}_{\xi\xi}^{\epsilon_m}}{|\boldsymbol{X}_\xi^{\epsilon_m}|^3 + \epsilon_m^\alpha} \to \frac{\boldsymbol{Y}_\xi \times \boldsymbol{Y}_{\xi\xi}}{|\boldsymbol{Y}_\xi|^3} \qquad \text{weakly in } L^2(I) \text{ uniformly on } [0,T].$$

Next, integrating (3.1) with $\epsilon = \epsilon_m$ over $[t_1,t_2] \subset [0,T]$, we have

$$\boldsymbol{X}^{\epsilon_m}(\xi,t_2) - \boldsymbol{X}^{\epsilon_m}(\xi,t_1) = \int_{t_1}^{t_2} \left(\epsilon_m \boldsymbol{X}_{\xi\xi}^{\epsilon_m} + \frac{\boldsymbol{X}_\xi^{\epsilon_m} \times \boldsymbol{X}_{\xi\xi}^{\epsilon_m}}{|\boldsymbol{X}_\xi^{\epsilon_m}|^3 + \epsilon_m^\alpha} + \boldsymbol{V}^{\epsilon_m}(\boldsymbol{X}^{\epsilon_m},t) \right) dt.$$

Letting $m \nearrow \infty$, we deduce

$$\boldsymbol{Y}(\xi,t_2) - \boldsymbol{Y}(\xi,t_1) = \int_{t_1}^{t_2} \left(\frac{\boldsymbol{Y}_\xi \times \boldsymbol{Y}_{\xi\xi}}{|\boldsymbol{Y}_\xi|^3} + \boldsymbol{V}(\boldsymbol{Y},t) \right) dt$$

in $L^2(I)$ from the above convergence and the fact that

$$\begin{aligned}\int_0^T &\|\boldsymbol{V}^{\epsilon_m}(\boldsymbol{X}^{\epsilon_m},t)-\boldsymbol{V}(\boldsymbol{Y},t)\|_{L^2(I)}dt\\ &\leq \int_0^T |\boldsymbol{V}^{\epsilon_m}(\cdot,t)-\boldsymbol{V}(\cdot,t)|_0 dt\\ &\quad+\int_0^T |\boldsymbol{V}(\cdot,t)|_1 dt \sup_{0<t<T}\|\boldsymbol{X}^{\epsilon_m}(\cdot,t)-\boldsymbol{Y}(\cdot,t)\|_{L^2(I)}\\ &\searrow 0 \text{ as } m\nearrow\infty.\end{aligned}$$

Hence $\boldsymbol{Y}$ is a solution of (1.1) with (1.2) and (1.3) in $\mathrm{Lip}([0,T];L^2(I))$. □

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