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Markov Property of Kusuoka-Zhou's Dirichlet Forms on Self-Similar Sets

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Abstract. The main purpose of this note is to fill a gap in Kusuoka-Zhou's construction of self-similar Dirichlet forms on self-similar sets. Unfortunately, it is not quite clear whether or not the self-similar closed form $\mathcal{E}$ obtained in the proof of Theorem 6.9 of [KZ] satisfies the Markov property. We will use a kind of fixed point theorem of order preserving additive maps on a cone to prove existence of a self-similar closed form with the Markov property. The fixed point theorem will be introduced in § 1. It is also applicable to other problems, for example, the existence problem of a harmonic structure on a p.c.f. self-similar set. In § 2, we will apply the fixed point theorem to show existence of self-similar Dirichlet forms on self-similar sets.

1. A Fixed Point Theorem

In this section, we will introduce a fixed point theorem on an ordered topological cone.

Definition 1.1 (Topological cone). A Hausdorff topological space U is called a topological cone if it satisfies the following conditions.

(1) U is a commutative semigroup with a unity. We use $u + v$ to denote the semigroup sum of u and v in U. The unity is denoted by 0.

(2) There exists a map $[0, \infty) \times U \to U$, $(s, u) \to su$, that satisfies the standard properties of a scalar multiplication with respect to the semigroup structure:

(a) $s_1(s_2u) = (s_1s_2)u$ and $(s_1 + s_2)u = s_1u + s_2u$ for any $s_1, s_2 \in [0, \infty)$ and any $u \in U$.

(b) $s(u + v) = su + sv$ for any $s \in [0, \infty)$ and any $u, v \in U$.

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(c) $0u = 0$ and $1u = u$ for any $u \in U$.

(3) The group sum and the scalar multiplication are continuous with respect to the topology of U and $[0, \infty)$.

Immediate examples of topological cones are positive valued functions on a set and positive definite quadratic forms on a vector space. For those examples, there are natural orders associated with cone structures.

Definition 1.2 (Order on a topological cone). A topological cone U is said to be ordered if there is a partial order $\leq$ on U that satisfies the following conditions.

(1) For any $u, v \in U$ with $u \neq 0$, there exists $s \in (0, \infty)$ such that $sv \leq u$.

(2) If $u_i \leq v_i$ for $i = 1, 2$, then $u_1 + u_2 \leq v_1 + v_2$.

(3) If $u \leq v$ for $u, v \in U$ and $s_1 \leq s_2$ for $s_1, s_2 \in [0, \infty)$, then $s_1 u \leq s_2 v$.

(4) If $\lim_{n \to \infty} u_n = u$, $\lim_{n \to \infty} v_n = v$ and $u_n \leq v_n$ for any n, then $u \leq v$.

(5) For any sequence $\{u_n\}_{n \geq 1} \subset U$, if there exists $v, w \in U$ such that $v \leq u_n \leq w$ for any $n \geq 1$, then we can choose a subsequence $\{u_{n_i}\}_{i \geq 1}$ and $u \in U$ so that $u_{n_i} \to u$ as $i \to \infty$.

The assumption (1) of the above definition is rather strong. For example, it is not satisfied for the case of positive valued functions on $\mathbb{R}$. This condition may not be necessary later in the fixed point theorem if the mapping is additive. See the remark after Theorem 1.5.

For an ordered topological cone, we can prove an analogy of a fundamental fact in real analysis : "a bounded monotonic sequence has a limit."

Lemma 1.3. *Let U be an ordered topological cone. If $\{u_n\}_{n \geq 1} \subset U$ is a monotonically increasing (resp. decreasing) sequence with an upper (resp. lower) bound, i.e. there exists $u \in U$ such that $u_n \leq u_{n+1} \leq u$ (resp. $u \leq u_{n+1} \leq u_n$) for all n, then $\{u_n\}_{n \geq 1}$ is convergent as $n \to \infty$.*

Proof. Assume that $u_n \leq u_{n+1} \leq u$ for all n. Then by (5) of Definition 1.2, any subsequence $\{u_{n_i}\}_{i \geq 1}$ contains a convergent subsequence.

Using (4) of Definition 1.2, we see that the limits of such convergent sequences are all the same. Now a standard argument of general topology implies that $\{u_n\}_{n\geq 1}$ is convergent as $n \to \infty$. □

DEFINITION 1.4. Let U be an ordered topological cone. Let $T : U \to U$.

(1) T is said to be order preserving if $Tu \leq Tv$ whenever $u \leq v$.

(2) T is said to be super(resp. sub)-additive if $T(u+v) \leq Tu+Tv$ (resp. $Tu+Tv \leq T(u+v)$) for any $u, v \in U$.

(3) T is said to be homogeneous if $T(su) = sTu$ for any $(s, u) \in [0, \infty) \times U$.

The following fixed point theorem is the main theorem of this section.

THEOREM 1.5. *Let U be an ordered topological cone. Assume that $T : U \to U$ is continuous, order preserving, homogeneous and super(or sub)-additive. If there exists $u \in U$ such that $\{T^n u\}_{n\geq 0}$ is bounded from both above and below, i.e. there exists $u_1, u_2 \in U$ such that $u_1 \leq T^n u \leq u_2$ for any n, then there exist a fixed point $u_* \in U$ of T that satisfies $\alpha u_1 \leq u_* \leq \beta u_2$ for some positive constants α and β. Moreover if $u \in V$ for a closed sub-cone V of U and $T(V) \subseteq V$, then $u_* \in V$.*

PROOF. Assume that T is super-additive. Set $v_N = N^{-1} \sum_{n=0}^{N-1} T^n u$. Then $u_1 \leq v_N \leq u_2$ for any N. Hence by (5) of Definition 1.2, there exists a subsequence $\{v_{N_i}\}_{i\geq 0}$ and $v \in U$ such that $v_{N_i} \to v$ as $i \to \infty$. Now note that

$$Tv_N + \frac{1}{N}u \leq \frac{1}{N}\sum_{n=0}^{N} T^n u = v_N + \frac{1}{N}T^N u.$$

So letting $N = N_i$, as $\lim_{i\to\infty} N_i{}^{-1} u = \lim_{i\to\infty} N_i{}^{-1} T^{N_i} u = 0$, we see that $Tv \leq v$. As T is order preserving, $\{T^n v\}_{n\geq 0}$ is monotonically decreasing. By (1) of Definition 1.2, we can choose $\alpha > 0$ so that $\alpha u \leq v$. Then $\alpha u_1 \leq \alpha T^n u \leq \alpha T^n v$. Hence by Lemma 1.3, $\{T^n\}_{n\geq 0}$ is convergent as $n \to \infty$. Let u_* be the limit, then $Tu_* = u_*$ and $\alpha u_1 \leq u_* \leq v \leq u_2$.

If $u \in V$ where V is a closed subcone of U, then obviously v and u_* belong to V.

If T is sub-additive, a similar argument implies that there exists $u_* \in U$ that satisfies $Tu_* = u_*$ and $u_1 \leq u_* \leq \beta u_2$ for some $\beta > 0$. □

Remark. If T is additive, i.e. $T(u+v) = Tu + Tv$, in addition to the conditions in Theorem 1.5, then the above proof implies that $Tv = v$ and $u_1 \leq v = u_* \leq u_2$. Also note that we don't need (1) of Definition 1.2 in such a case.

In many cases, to find u with $\{T^n u\}_{n\geq 0}$ bounded from both above and below is as difficult as to show existence of a fixed point. So, Theorem 1.5 is not quite useful in such cases. It helps, however, to find another fixed point from a known fixed point as follows.

Corollary 1.6. *Let U be an ordered topological cone and let $T : U \to U$ be continuous, order preserving, homogeneous and super(or sub)-additive. Also let V be a closed subcone of U which is invariant under T, i.e. $T(V) \subseteq V$, and $V\backslash\{0\} \neq \emptyset$. If there exists a non-trivial fixed point $u \in U\backslash\{0\}$ of T, then there exists a non-trivial fixed point $v \in V\backslash\{0\}$ of T.*

In fact, we will use this corollary to show existence of a self-similar Dirichlet form in the next section.

Proof. Choose $w \in V\backslash\{0\}$. Then there exist positive numbers α and β such that $\alpha u \leq w \leq \beta u$. As u is a fixed point of T, we see that $\alpha u = \alpha T^n u \leq T^n w \leq \beta T^n u = u$. Hence Theorem 1.5 implies that there exists $v \in V$ such that $Tv = v$ and $\alpha u \leq v \leq \beta u$. □

2. Existence of Self-Similar Dirichlet Forms

In this section, we will apply Theorem 1.5 to fill a gap in the proof of Theorem 6.9 of Kusuoka-Zhou[KZ]. First we will briefly introduce the setting in [KZ]. We will use the notations and definitions in [KZ] in the following without further notice. ψ_i are α-similitudes in $\mathbb{R}^D$ for $i \in I$, where $I = \{1, \cdots, N\}$, and E is the self-similar set with respect to $\{\psi_i\}_{i\in I}$: E is non-empty compact set that satisfies $E = \cup_{i\in I}\psi_i(E)$. Assume that $(E, \{\psi_i\}_{i\in I})$ satisfies (A.1), (A.2), (A.3) and (A.4). ν is the self-similar measure on E with $\nu(\psi_i(E)) = 1/N$ for any $i \in I$.

Now by the results in Section 4 - 6 of [KZ], we see

THEOREM 2.1. *Assume* (B.1), (B.2), (GB) *and* $\underline{\lim}_{n\to\infty} \frac{\log R_n}{n} + \log M_1 > 0$. *Then*

(1) *There exists a Dirichlet form* $\mathcal{E}_0$ *on* $L^2(K,\nu)$. *Let* $\mathcal{D}_0$ *be the domain of* $\mathcal{E}_0$. *Then* $f \circ \psi_i \in \mathcal{D}_0$ *for any* $f \in \mathcal{D}_0$ *and any* $i \in I$.

(2) *Define a collection of closed forms* $\mathcal{CF}$ *by*

$$\{A : A \text{ is a closed quadratic form on } L^2(E,\nu),\ \mathrm{Dom}(A) = \mathcal{D}_0 \text{ and there exists } c_1, c_2 > 0 \text{ such that } c_1\mathcal{E}_0(f,f) \le A(f,f) \le c_2\mathcal{E}_0(f,f) \text{ for any } f \in \mathcal{D}_0\}.$$

Then, there exists $\mathcal{E} \in \mathcal{CF}$ *such that*

$$\mathcal{E}(f,f) = \frac{\rho}{N}\sum_{i\in I}\mathcal{E}(f\circ\psi_i, f\circ\psi_i)$$

for all $f \in \mathcal{D}_0$.

It seems rather difficult to see that the original $\mathcal{E}$ constructed by Kusuoka-Zhou has the Markov property. Set

$$\mathcal{DF} = \{A : A \in \mathcal{CF}, A \text{ has the Markov property}\}.$$

Now we recall the definition of the Markov property.

DEFINITION 2.2. $A \in \mathcal{CF}$ has the Markov property if and only if $\bar{f} \in \mathcal{D}_0$ and $A(\bar{f},\bar{f}) \le A(f,f)$ for any $f \in \mathcal{D}_0$, where $\bar{f}$ is defined by

$$\bar{f}(x) = \begin{cases} 1 & \text{if } f(x) > 1 \\ f(x) & \text{if } 0 \le f(x) \le 1, \\ 0 & \text{if } f(x) < 0.. \end{cases}$$

There are several ways of describing the Markov property. The above definition is one of the strongest versions. See [FOT] for details.

The following is the main theorem.

THEOREM 2.3. *Under the same assumptions in Theorem 2.1, there exists a Dirichlet form* $\mathcal{E}_* \in \mathcal{DF}$ *on* $L^2(E,\nu)$ *that satisfies*

$$\mathcal{E}_*(f,f) = \frac{\rho}{N}\sum_{i\in I}\mathcal{E}_*(f\circ\psi_i, f\circ\psi_i)$$

for all $f \in \mathcal{D}_0$.

In the rest of this section, we will give a proof of Theorem 2.3. First we will introduce a topology in $\mathcal{CF}$.

Definition 2.4. For $\{A_n\}_{n\geq 1} \subset \mathcal{CF}$, we say that $A_n \to A$ as $n \to \infty$ for $A \in \mathcal{CF}$ if and only if $A_n(f,f) \to A(f,f)$ as $n \to \infty$ for any $f \in \mathcal{D}_0$.

Remark. For any $f \in \mathcal{D}_0$, define a map $G_f : \mathcal{CF} \to \mathbb{R}$ by $G_f(A) = A(f,f)$ for any $A \in \mathcal{CF}$. Then the convergence in the above definition is induced by the weakest topology where G_f is continuous for any $f \in \mathcal{D}_0$.

Now define a partial order $\leq$ in $\mathcal{CF}$ by letting $A \leq B$ if and only if $A(f,f) \leq B(f,f)$ for any $f \in \mathcal{D}_0$.

Lemma 2.5. *Let* $\{A_n\}_{n\geq 1}$ *be a sequence in* $\mathcal{CF}$. *Assume that for some* $E_1, E_2 \in \mathcal{CF}$, $E_1 \leq A_n \leq E_2$ *for any* $n \geq 1$. *Then there exist a subsequence* $\{A_{n_i}\}_{i\geq 1}$ *which is convergent as* $i \to \infty$.

Proof. As $\mathcal{D}_0$ is separable, we can choose a countable dense subset $\{f_j\}_{j\geq 1}$ of $\mathcal{D}_0$, where the norm of $\mathcal{D}_0$ is defined by $||f|| = \sqrt{\int_E f^2 d\nu} + \sqrt{\mathcal{E}_0(f,f)}$. By the diagonal argument, we can choose a subsequence $\{A_{n_i}\}_{i\geq 1}$ so that $\{A_{n_i}(f_j,f_j)\}_{i\geq 1}$ is convergent as $i \to \infty$ for all j. Then it is routine to see that $\{A_{n_i}\}_{i\geq 1}$ is convergent as $i \to \infty$ for all $f \in \mathcal{D}_0$. Now define $A(f,f) = \lim_{i\to\infty} A_{n_i}(f,f)$. Then we see that $A \in \mathcal{CF}$. $\square$

By the above lemma, it follows that $\mathcal{CF}$ becomes an ordered topological cone and $\mathcal{DF}$ is a closed subcone of $\mathcal{CF}$.

Lemma 2.6. *For* $A \in \mathcal{CF}$, *define a symmetric form* TA *on* $\mathcal{D}_0$ *by*

$$(TA)(f,g) = \frac{\rho}{N} \sum_{i\in I} A(f \circ \psi_i, g \circ \psi_i)$$

for any $f, g \in \mathcal{D}_0$. *Then* $TA \in \mathcal{CF}$.

Proof. There exist $c_1, c_2 > 0$ such that $c_1\mathcal{E} \leq A \leq c_2\mathcal{E}$. Since $TA \leq TB$ if $A \leq B$ and $T(cA) = cT(A)$ for any $c \geq 0$, it follows that $c_1 T\mathcal{E} \leq TA \leq c_2 T\mathcal{E}$. Note that $T\mathcal{E} = \mathcal{E}$ by Theorem 2.1. Hence $TA \in \mathcal{CF}$. $\square$

Proof of Theorem 2.3. By the above theorem, we see that $T : \mathcal{CF} \to \mathcal{CF}$ and T is order-preserving, homogeneous and additive. Also by Definition 2.2, it follows that $T(\mathcal{DF}) \subseteq \mathcal{DF}$. Hence applying Corollary 1.6, we conclude that there exits $\mathcal{E}_* \in \mathcal{DF}$ such that $T\mathcal{E}_* = \mathcal{E}_*$. □

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