J. Math. Sci. Univ. Tokyo
6 (1999), 691–756.

Theory of Fibered 3-Knots in S^5 and its Applications

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Abstract. Let K be a closed connected orientable 3-manifold embedded in S^5 whose complement smoothly fibers over the circle with simply connected fibers. Such an embedded 3-manifold is called a *simple fibered* 3-*knot*. In this paper we study such embedded 3-manifolds and give various new results, which are classified into three types: (1) those which are similar to higher dimensional fibered knots, (2) those which are peculiar to fibered knots in S^5, and (3) applications. Among the results of type (1) are the isotopy criterions via Seifert matrices, determining fibered 3-knots by their exteriors, topological or stable uniqueness of the fibering structures, and the effectiveness of plumbing operations. As results of type (2), we give various explicit examples of fibered 3-knots with the same diffeomorphism type of the abstract 3-manifolds and with congruent Seifert matrices but with different isotopy types. We also give some examples of fibered knots whose exteriors are diffeomorphic but with different isotopy types. We also show that there exist infinitely many embeddings of the punctured K3 surface into S^5 which are fibers of topological fibrations but which can never be a fiber of any smooth fibrations. We construct a fibered 3-knot which is decomposable as a knot such that neither of the factor knots are fibered. As a result of type (3), we study topological isotopies of homeomorphisms of simply connected 4-manifolds with boundary by using the techniques of fibered 3-knots. We also apply our techniques to the embedding problem of simply connected 4-manifolds into S^6. Finally we give some applications to the topological study of isolated hypersurface singularities in $\mathbf{C}^3$.

Contents

1991 *Mathematics Subject Classification*. Primary 57Q45; Secondary 57N35.
Key words: Fibered knot, Seifert matrix, isotopy of 4-manifolds, complex isolated hypersurface singularity.

1. Introduction

Consider a complex isolated hypersurface singularity in $\mathbf{C}^{n+1}$. It is known as Milnor's fibration theorem [59] that the intersection of the hypersurface with a sphere S^{2n+1} centered at the singular point with sufficiently small radius is a nonsingular smooth $(2n-1)$-dimensional manifold and that its complement in S^{2n+1} smoothly fibers over the circle. Furthermore, the isotopy type of the $(2n-1)$-dimensional manifold embedded in S^{2n+1} completely determines the local (embedded) topology of the hypersurface. Thus it is fundamental to study the isotopy types of embedded $(2n-1)$-dimensional manifolds in S^{2n+1} whose complements fiber over the circle. Such embedded manifolds are called *fibered knots* and this class of embeddings has been studied extensively (for example, see [21], [36]). Note that the embedded manifold is not always homeomorphic to the $(2n-1)$-dimensional sphere, although the word "knot" often refers to embeddings of manifolds homeomorphic to the standard spheres.

When $n = 1$, the knot lies in the 3-sphere and the classical knot and link theory plays an important role in this dimension. In fact, the knots which arise around an isolated singular point in $\mathbf{C}^2$ have been completely classified. For example, when the knot consists of one component (i.e., when the hypersurface singularity is irreducible), such knots are completely classified by their Alexander polynomials (see, for example, [51], [90]). On the other hand, when $n \geq 3$, the techniques of higher dimensional differential topology can be applied and many results have been obtained. For example, in [21], [36], it has been shown that the isotopy classes of fibered knots in S^{2n+1} with some connectivity conditions are in one-to-one correspondence with the congruence classes of integral unimodular matrices through Seifert matrix. In particular, the congruence class of a Seifert matrix determines the diffeomorphism type of the embedded manifold. This classification heavily depends on the celebrated h-cobordism theorem of Smale [82], which works only for $n \geq 3$. Another such example can be found in the work of Lê and Ramanujam [52], who have shown that a μ-constant deformation of an isolated hypersurface singularity in $\mathbf{C}^{n+1}$ is topologically constant for $n \neq 2$, by using the h-cobordism theorem. Note that this has not been known to be true or not for $n = 2$.

When $n = 2$, these techniques of higher dimensional differential topology do not work. In fact, it has been known that there exist fibered knots in S^5 with congruent Seifert matrices but with different diffeomorphism types (see [74]). One of the major difficulties here is that the fundamental group of the embedded 3-manifold is not necessarily trivial. Another difficulty is related to the 4-dimensional topology: in this dimension, the fiber of a fibered knot is a simply connected 4-manifold and it is known that the differential topology of 4-dimensional manifolds is totally different from those of the other dimensions as is seen in the works of Freedman [23] and Donaldson [18], [19], [20].

The purpose of the present paper is to study extensively fibered knots in S^5 by using the recently developed theory of 3- and 4-dimensional topology. This is a continuation of the works developed in [74], [75]. Note that, in [74], [75], almost all fibered knots that we could handle were those of homology 3-spheres. In the present paper, we study fibered knots which are not necessarily homology 3-spheres as well. The results can be classified into three types: (1) those which are similar to higher dimensional cases,

(2) those which are peculiar to this dimension, and (3) applications. One of the main results of the first class is as follows.

THEOREM 2.12. *Let K_j $(j = 0, 1)$ be simple fibered knots in S^5 such that K_0 and K_1 are homeomorphic as abstract 3-manifolds. We suppose that $H_1(K_j; \mathbf{Z})$ are a cyclic group of order $r = 1, 2, 4, p^m$ or $2p^m$ with p an odd prime or that K_j are homeomorphic to a connected sum of some copies of $S^2 \times S^1$. Then the two fibered knots K_0 and K_1 are isotopic to each other if and only if their Seifert matrices are congruent.*

One of the main results of the second class is as follows.

THEOREM 4.6. *Let Σ be a nontrivial orientable S^1-bundle over the closed orientable surface of genus $g \geq 2$. Then there exist simple fibered knots $K_0, \dots, K_{g-1}$ in S^5 with the following properties.*

(1) *All K_i are diffeomorphic to Σ as abstract 3-manifolds.*

(2) *The exteriors $E(K_i) = S^5 - \operatorname{Int} N(K_i)$ are all diffeomorphic to each other, where $N(K_i)$ is a tubular neighborhood of K_i in S^5.*

(3) *The Seifert matrices of K_i are all congruent to each other.*

(4) *The knots K_i and K_j are not isotopic to each other if $i \neq j$.*

Using the techniques of fibered knots in S^5, we give some new results on 4-dimensional topology as applications. One of the main results is as follows.

PROPOSITION 9.9. *Let F be a smooth compact 1-connected spin 4-manifold with boundary homeomorphic to a lens space $L(p, q)$ $(p \geq 2)$, the 3-sphere or the Poincaré homology 3-sphere. Let $h_i : F \to F$ $(i = 1, 2)$ be two orientation preserving diffeomorphisms. Then there exists a nonnegative integer k such that $h_i \sharp k(\mathrm{id}) : F \sharp k(S^2 \times S^2) \to F \sharp k(S^2 \times S^2)$ are smoothly isotopic to each other if and only if $(h_1)_* = (h_2)_* : H_2(F; \mathbf{Z}) \to H_2(F; \mathbf{Z})$.*

We also consider some applications to the topology of isolated hypersurface singularities in $\mathbf{C}^3$, which was originally our motivation.

The paper is organized as follows. In §2, we recall the precise definition of simple fibered knots and give some isotopy criterions for fibered knots in S^5. One of the principal results is that two fibered knots in S^5 are isotopic if and only if there exists a homeomorphism between their fibers which gives the congruence of the Seifert matrices (Theorem 2.2). The essential idea used here can already be found in Levine's work [54]; however, we should also use some techniques of 4-dimensional topology, among which is Boyer's result [9] about the existence of homeomorphisms between simply connected 4-manifolds with boundary. As a corollary, we show that for certain cases, the abstract diffeomorphism type of a fibered knot and its Seifert matrix do determine the isotopy type (Theorem 2.12).

In §3, we give some examples of fibered knots in S^5 for which the Seifert matrix and the abstract diffeomorphism type do not determine the isotopy type. Here we give two types of such examples. In the first example, the fibers are diffeomorphic and the Seifert matrices are congruent, but the algebraic isomorphism between the second homology groups of the fibers cannot be realized by any homeomorphism (Theorem 3.1). In this example, we can construct an arbitrary number of such fibered knots. In the second example, the fibers themselves are not homeomorphic to each other (Proposition 3.8, Example 3.10).

In §4, we consider the problem whether a fibered knot in S^5 is determined by its exterior or not. We show that, for fibered knots in S^5 which are certain rational homology 3-spheres, this is true (Theorem 4.1). Note that, for higher dimensional spherical knots, a similar result has been obtained by Levine [54, §22]. However, for fibered knots which are not rational homology 3-spheres, the exterior does not determine the isotopy type in general. We show this by explicitly constructing fibered knots with such properties (Theorem 4.6).

In §5, we study the fibering structures of fibered knots in S^5. For higher dimensions, Durfee [21] and Kato [36] have shown that the fibering structures of fibered knots are unique up to smooth isotopy for higher dimensions. Here we show that the fibering structures of fibered knots in S^5 are unique up to *topological* isotopy (Theorem 5.1) and up to stable equivalence (Proposition 5.7). We also give some examples which show that the smooth fibering structures are not unique in general for fibered knots in S^5.

In §6, we study fibered knots in S^5 whose fibers are punctured K3 surfaces. In fact, in §10, we show that for any smooth closed 1-connected spin 4-manifold N not homotopy equivalent to S^4, there exists a fibered knot whose fiber is diffeomorphic to $N\sharp(S^2 \times S^2) - \mathrm{Int}\, D^4$, by using the techniques developed in [74] (Proposition 10.1). However, a K3 surface cannot be decomposed into a connected sum $N\sharp(S^2 \times S^2)$ (see [19]). In §6, we construct infinitely many fibered knots with fiber a K3 surface by using Matumoto's result about the diffeomorphisms of a K3 surface [58] (Proposition 6.1). As a byproduct, we obtain infinitely many embeddings of a punctured K3 surface in S^5 such that they are fibers of topological fibrations, but which cannot be fibers of any smooth fibrations (Proposition 6.2).

In §7, we study the factorization of fibered knots in S^5. We give an example of a fibered knot in S^5 which is decomposable as a knot, but whose factor knots can never be fibered smoothly (Example 7.1). We also give an example of a fibered knot in S^5 whose Seifert matrix is decomposable, but which is not decomposable as a knot (Proposition 7.4). Note that, in higher dimensions, such a fibered knot is always decomposable. Furthermore, we give some results about the cancellation problem for certain fibered knots (Propositions 7.6 and 7.7).

In §8, we study the plumbing operation of fibered knots in S^5. As is well known, for the other dimensions, the plumbing operation is a powerful way to produce a new fibered knot from given two old ones (see [30], [83], [55], [56]). In this section, we show that the plumbing operation does work also for this dimension (Theorem 8.1). We show this by explicitly constructing the geometric monodromy of the new fibered knot.

In §9, we apply the techniques of fibered knots in S^5 to the study of isotopies of simply connected 4-manifolds with boundary. Quinn [72] has shown that two orientation preserving homeomorphisms of a closed simply connected 4-manifold are homotopic to each other if and only if their induced isomorphisms on the second homology group coincide (see also [17, §5]). Using this, Quinn has shown that the two homeomorphisms with the same isomorphism on the second homology group are actually topologically isotopic to each other. In this section, we use the variation maps of homeomorphisms which are the identity on the boundary and give some topological isotopy criterions (relative to boundary) for homeomorphisms of 1-connected 4-manifolds with boundary (Propositions 9.1 and 9.2). As to topological isotopies which may not fix the boundary points, we also obtain some results when the boundary 3-manifold is homeomorphic to a certain spherical 3-manifold (Proposition 9.7). We also obtain some smooth stable isotopy criterions for diffeomorphisms (Propositions 9.4 and 9.9).

In §10, we apply the techniques of fibered knots in S^5 to the embedding problem of simply connected 4-manifolds into S^6. We first construct fibered knots with fiber constructed from a given 4-manifold and then use this fiber to construct a desired embedding. The main result is Theorem 10.4, which is a special case of Cappell-Shaneson's result [13]. Here, the result itself is not new; however, we have included this section, since the technique for the proof is new and creates interesting embeddings in some cases.

In §11, we apply our results to the study of the topology of isolated hypersurface singularities in $\mathbf{C}^3$. For example, we show the following new results.

PROPOSITION 11.1. *Let $f_t : \mathbf{C}^3, 0 \to \mathbf{C}, 0$ ($t \in [0,1]$) be a μ-constant deformation of an isolated singularity. Suppose that $H_1(K_{f_0}; \mathbf{Z}) \cong \mathbf{Z}/r\mathbf{Z}$, where $r = 1, 2, 4, p^m$ or $2p^m$ with p an odd prime. Furthermore we suppose that $\pi_1(K_{f_0}) \cong \pi_1(K_{f_1})$. Then the fibered knots K_{f_0} and K_{f_1} associated with the hypersurfaces $f_0^{-1}(0)$ and $f_1^{-1}(0)$ respectively are isotopic to each other.*

PROPOSITION 11.4. *Let $f \in \mathbf{C}[x, y, z]$ be a polynomial of three complex variables with $f(0) = 0$ and with an isolated critical point at the origin. We suppose that $H_1(K_f; \mathbf{Z}) \cong \mathbf{Z}/r\mathbf{Z}$, where $r = 1, 2, 4, p^m$ or $2p^m$ with p an odd prime. If f can be connected by a μ-constant deformation to a polynomial with real coefficients, then there exists a homeomorphism germ $\Phi : \mathbf{C}^3, 0 \to \mathbf{C}^3, 0$ such that $\bar{f} = f \circ \Phi$ as germs at the origin, where $\bar{f} : \mathbf{C}^3, 0 \to \mathbf{C}, 0$ is the function defined by $\bar{f}(z) = \overline{f(z)}$ (the complex conjugate of $f(z) \in \mathbf{C}$). (In the terminology of* [42], [77], *f is right equivalent to $\bar{f}$.)*

Note that Szczepanski [84] obtains some results about the topological constancy of μ-constant deformations in three complex variables. However, her arguments contain some essential gaps, mainly caused by ignoring the spin structures of 3-manifolds. We will explain this more in detail in Remark 11.3. Thus Proposition 11.1 is a new result, although it seems weaker compared with Szczepanski's results.

Throughout the paper, all the homology and cohomology groups are with integer coefficients unless otherwise specified. We use the symbol "$\cong$"

to denote a diffeomorphism between smooth manifolds or an appropriate isomorphism between algebraic objects.

The author would like to thank Dr. Eiji Ogasa for bringing Bayer-Fluckiger's paper [4] to his attention.

2. Isotopy Criterion

First we recall the definitions of simple fibered knots and their Seifert forms.

DEFINITION 2.1 ([21]). Let K be a smoothly embedded closed $(2n-1)$-dimensional manifold in S^{2n+1}. Then K is called a *fibered* $(2n-1)$*-knot* if there exists a smooth fibration $\phi : S^{2n+1} - K \to S^1$ such that there exists a trivialization $\alpha : K \times D^2 \to N(K)$ of a tubular neighborhood $N(K)$ of K in S^{2n+1} which makes the following diagram commutative, where p denotes the obvious projection:

$$\begin{array}{ccc} K \times (D^2 - \{0\}) & \xrightarrow{\alpha} & N(K) - K \\ & p \searrow \quad \swarrow \phi & \\ & S^1. & \end{array} \tag{2.1}$$

Furthermore, the fibered knot K is *simple* if the fibers of ϕ are $(n-1)$-connected and K is $(n-2)$-connected.

In this paper, we always assume that fibered knots are simple. Thus in the following we omit the word "simple" for simplicity.

Note that an algebraic knot associated with a complex polynomial function with an isolated critical point is always fibered and is simple in the above sense [59].

Set $F = \overline{\phi^{-1}(1)} = \phi^{-1}(1) \cup K$ $(1 \in S^1)$. We call F a *fiber* of the fibered knot K. Note that F is a smooth $2n$-dimensional compact manifold with $\partial F = K$. In other words, F can be regarded as a Seifert manifold of K in S^{2n+1}.

In the following, we assume that S^1 and S^{2n+1} are oriented. Note that then every fiber and its boundary K have canonical orientations. We define the bilinear form

$$\Gamma_K : H_n(F) \times H_n(F) \to \mathbf{Z} \tag{2.2}$$

by $\Gamma_K(x, y) = \mathrm{lk}(i_*x, y)$ $(x, y \in H_n(F))$, where $i : F \to S^{2n+1} - F$ is the map defined by the translation in the positive normal direction (determined by the orientation of S^1) and "lk" denotes the linking number in S^{2n+1}. We call Γ_K the *Seifert form* of K. Furthermore a matrix representative of Γ_K is called a *Seifert matrix* of K. Here we note that $H_n(F)$ is always a finitely generated free abelian group, since F is $(n-1)$-connected. Note also that a Seifert matrix is always unimodular by virtue of the Alexander duality. Furthermore, it is known that the congruence class of the Seifert matrix is independent of the choice of a fibration $\phi : S^{2n+1} - K \to S^1$ and depends only on the oriented isotopy type of K in S^{2n+1} (see [54], [74]).

Consider the smooth fibration $\phi : S^{2n+1} - \mathrm{Int}\, N(K) \to S^1$, where we identify the fiber with F. Then a *geometric monodromy* $h : F \to F$ is defined up to isotopy relative to boundary; i.e., the total space is obtained from $F \times [0, 1]$ by gluing $F \times \{1\}$ and $F \times \{0\}$ by h. Note that $h|_{\partial F}$ is the identity map by Definition 2.1. Then we define the *variation map* $\Delta_K : H_n(F, \partial F) \to H_n(F)$ as follows. For an element $\gamma \in H_n(F, \partial F)$, take an n-cycle $(D, \partial D)$ in $(F, \partial F)$ representing γ. Then $D \cup (-h(D))$ is an n-cycle in F and we define $\Delta_K(\gamma)$ to be the class represented by $D \cup (-h(D))$. Note that this does not depend on the choice of $(D, \partial D)$ nor on h. In [37], it is shown that Δ_K is always an isomorphism and that giving the Seifert form is equivalent to giving the variation map.

Recall that if $n \geq 3$, two fibered $(2n-1)$-knots are isotopic if and only if their Seifert matrices are congruent [21], [36]. When $n = 2$, we have the following isotopy criterion.

THEOREM 2.2. *Let K_j $(j = 0, 1)$ be fibered 3-knots with fiber F_j. Then the following five are equivalent.*

(1) *The embedded 3-manifolds K_0 and K_1 are (orientation preservingly) isotopic in S^5.*

(2) *There exists a (orientation preserving) homeomorphism $\Psi : F_0 \to F_1$ satisfying $\Gamma_{K_1}(\Psi_*x, \Psi_*y) = \Gamma_{K_0}(x, y)$ for all $x, y \in H_2(F_0)$.*

(3) *There exists a (orientation preserving) homeomorphism $\Psi : F_0 \to F_1$*

which makes the following diagram commutative:

$$\begin{array}{ccc} H_2(F_0, \partial F_0) & \xrightarrow{\Delta_{K_0}} & H_2(F_0) \\ \Psi_* \downarrow & & \Psi_* \downarrow \\ H_2(F_1, \partial F_1) & \xrightarrow{\Delta_{K_1}} & H_2(F_1). \end{array} \tag{2.3}$$

(4) *There exist a* (*orientation preserving*) *homeomorphism* $\psi : K_0 \to K_1$ *and an isomorphism* $\Lambda : H_2(F_0) \to H_2(F_1)$ *which satisfy the following properties* $(a), (b)$ *and* (c)*:*

   (a) *The* 4*-manifold* $F_0 \cup_\psi (-F_1)$ *is a spin manifold, where* $-F_1$ *is the* 4*-manifold* F_1 *with the reversed orientation, and* $F_0 \cup_\psi (-F_1)$ *is the closed* 4*-manifold obtained from* F_0 *and* $-F_1$ *by attaching their boundaries by the homeomorphism* ψ*.*

   (b) *The following diagram is commutative, where the two horizontal sequences are the exact sequences of the pairs* (F_0, K_0) *and* (F_1, K_1) *respectively and* Λ^* *is the adjoint of* Λ *with respect to the identification of* $H_2(F_j, \partial F_j)$ *with* $\mathrm{Hom}(H_2(F_j), \mathbf{Z})$ *arising from the Lefschetz duality:*

$$\begin{array}{ccccccccc} 0 & \to & H_2(K_0) & \to & H_2(F_0) & \to & H_2(F_0, K_0) & \to & H_1(K_1) & \to & 0 \\ & & \psi_* \downarrow & & \Lambda \downarrow & & \Lambda^* \uparrow & & \psi_* \downarrow & & \\ 0 & \to & H_2(K_1) & \to & H_2(F_1) & \to & H_2(F_1, K_1) & \to & H_1(K_1) & \to & 0. \end{array} \tag{2.4}$$

   (c) *We have* $\Gamma_{K_1}(\Lambda x, \Lambda y) = \Gamma_{K_0}(x, y)$ *for all* $x, y \in H_2(F_0)$*.*

(5) *There exist a* (*orientation preserving*) *homeomorphism* $\psi : K_0 \to K_1$ *and an isomorphism* $\Lambda : H_2(F_0) \to H_2(F_1)$ *which satisfy the properties* $(a), (b)$ *above and* (c') *below:*

   (c') *The following diagram is commutative:*

$$\begin{array}{ccc} H_2(F_0, \partial F_0) & \xrightarrow{\Delta_{K_0}} & H_2(F_0) \\ \Lambda^* \uparrow & & \Lambda \downarrow \\ H_2(F_1, \partial F_1) & \xrightarrow{\Delta_{K_1}} & H_2(F_1). \end{array} \tag{2.5}$$

To prove Theorem 2.2, we need some lemmas.

Lemma 2.3. *Let F be a smooth compact 1-connected 4-manifold with connected boundary. Then for some nonnegative integer k, $F\sharp k(S^2 \times S^2)$ admits a handlebody decomposition without 3-handles.*

Proof. Since $(F, \partial F)$ is a 1-connected compact smooth 4-manifold (with connected boundary), there exists a nonnegative integer k such that $(F\natural k(S^2 \times D^2), \partial F\sharp k(S^2 \times S^1))$ admits a handlebody decomposition without 1-handles by [71]. Considering the dual handlebody decomposition, we see that $F' = F\natural k(S^2 \times D^2)$ admits a handlebody decomposition without 3-handles. By attaching k 2-handles to F' appropriately, we get $F\natural k(S^2 \times S^2)^\circ \cong F\sharp k(S^2 \times S^2)$, where $(S^2 \times S^2)^\circ = S^2 \times S^2 - \operatorname{Int} D^4$. Hence $F\sharp k(S^2 \times S^2)$ admits a handlebody decomposition without 3-handles. This completes the proof. □

Lemma 2.4. *Let K_j $(j = 0, 1)$ be fibered $(2n-1)$-knots in S^{2n+1} with fiber F_j. Suppose that F_0 is orientation preservingly homeomorphic to F_1 and that the geometric monodromies of K_j are topologically pseudoisotopic relative to boundary. Then there exists an orientation preserving homeomorphism $\Phi : S^{2n+1} \to S^{2n+1}$ with $\Phi(F_0) = F_1$ and $\Phi \circ i_0 = i_1 \circ \Phi$ on F_0, where $i_j : F_j \to S^{2n+1} - F_j$ are the maps used in the definition of the Seifert form.*

Proof. Let $h_j : F_j \to F_j$ be geometric monodromies of K_j $(j = 0, 1)$. Since h_j are pseudoisotopic, there exists a homeomorphism $\Phi' : F_0 \times I \to F_0 \times I$ $(I = [0, 1])$ such that $\Phi'(x, 0) = (h_0(x), 0)$ and $\Phi'(x, 1) = (\Psi^{-1} \circ h_1 \circ \Psi(x), 1)$ for all $x \in F_0$ and $\Phi'(y, t) = (y, t)$ for all $y \in \partial F_0$ and $t \in I$, where $\Psi : F_0 \to F_1$ is a homeomorphism which preserves the orientations. Then $(\Phi')^{-1}$ induces a homeomorphism

$$\begin{aligned} \Phi'' : F_0 \times I/(x, 1) &\sim (h_0(x), 0) \\ \to F_0 \times I/(x, 1) &\sim (\Psi^{-1} \circ h_1 \circ \Psi(x), 0). \end{aligned} \tag{2.6}$$

It is clear that Φ'' extends to a homeomorphism Φ of S^{2n+1} with the required properties. This completes the proof. □

Lemma 2.5. *Let K be a fibered $(2n-1)$-knot and $\phi_j : S^{2n+1} - K \to S^1$ $(j = 0, 1)$ two smooth fibrations as in Definition 2.1 which are homotopic. Then there exists an orientation preserving homeomorphism $\Phi : S^{2n+1} \to$*

S^{2n+1} with $\Phi(F_0) = F_1$ and $\Phi \circ i_0 = i_1 \circ \Phi$ on F_0, where F_j is a fiber of ϕ_j and $i_j : F_j \to S^{2n+1} - F_j$ is the map used in the definition of the Seifert form.

Remark 2.6. Note that there exist exactly two homotopy classes of fibration maps $S^{2n+1} - K \to S^1$, since $H^1(S^{2n+1} - K)$ is isomorphic to $\mathbf{Z}$ and that there exist exactly two choices for a generator. The two homotopy classes correspond to the two orientations of S^1.

Proof of Lemma 2.5. When $n = 1$, the result is well-known. For $n \geq 2$, by considering the universal cover $\tilde{E}$ of $E = S^{2n+1} - \operatorname{Int} N(K)$, we see that there exists a smooth h-cobordism W between F_0 and F_1 embedded in $\tilde{E} \cong F_0 \times \mathbf{R} \cong F_1 \times \mathbf{R}$. Note that the h-cobordism W has a product structure on the boundary, since the trivialization α as in Definition 2.1 is unique up to homotopy. Then using a result of Freedman [23] when $n = 2$ and the h-cobordism theorem of Smale [82] when $n \geq 3$, we see that F_j are orientation preservingly homeomorphic to each other. Furthermore, using Wall's construction [89, p.140], we can show that the geometric monodromies of ϕ_j are pseudoisotopic relative to boundary (for a precise argument, see [49] or [68, §3].) Now the result follows from Lemma 2.4. This completes the proof. □

Remark 2.7. Using the pseudoisotopy theorem of Cerf [14] ($n \geq 3$) or Perron [69] and Quinn [72] ($n = 2$), we see that the geometric monodromies of ϕ_j are actually topologically isotopic relative to boundary.

Proof of Theorem 2.2. (1) $\Longrightarrow$ (2): This is an immediate consequence of Lemma 2.5.

(2) $\Longrightarrow$ (4): Set $\psi = \Psi|_{\partial F_0}$ and $\Lambda = \Psi_* : H_2(F_0) \to H_2(F_1)$. Then they obviously satisfy the conditions (b) and (c) in (4). The fact that they also satisfy (a) follows from [9].

(4) $\Longrightarrow$ (1): It is well-known that $x \cdot y = \Gamma_{K_j}(x, y) + \Gamma_{K_j}(y, x)$ for all $x, y \in H_2(F_j)$, where $x \cdot y$ denotes the intersection number of x and y in F_j (for example, see [21], [36]). Thus $\Lambda : H_2(F_0) \to H_2(F_1)$ preserves the intersection pairings of F_j. Hence there exists a homeomorphism from F_0 to F_1 which induces Λ and which coincides with ψ on ∂F_0 by [9]. In fact, there exists a smooth h-cobordism W between F_0 and F_1 such that the composite

map

$$H_2(F_0) \xrightarrow{(\iota_0)_*} H_2(W) \xrightarrow{(\iota_1)_*^{-1}} H_2(F_1) \tag{2.7}$$

coincides with Λ, where $\iota_j : F_j \to W$ are the inclusion maps (see [9, p.347]). Then by [50], [71], $W\sharp_c k(S^2 \times S^2 \times I)$ is diffeomorphic to the product $(F_0 \sharp k(S^2 \times S^2)) \times I$ for some nonnegative integer k, where $\sharp_c$ denotes a connected sum along cobordisms. Hence there exists a diffeomorphism $\tilde{\Psi} : F_0 \sharp k(S^2 \times S^2) \to F_1 \sharp k(S^2 \times S^2)$ such that $\tilde{\Psi}_* = \Lambda \oplus \mathrm{id}$ with respect to the decomposition $H_2(F_j \sharp k(S^2 \times S^2)) \cong H_2(F_j) \oplus H_2(\sharp^k(S^2 \times S^2))$. Since we can perform the connected sum operation of $S^2 \times S^2$ in S^5, we obtain Seifert manifolds $\tilde{F}_j$ of K_j and a diffeomorphism $\tilde{\Psi} : \tilde{F}_0 \to \tilde{F}_1$ which preserves the Seifert forms. By Lemma 2.3 we may further assume that $\tilde{F}_j$ have handlebody decompositions without 3-handles. Then by the same argument as in [54, §§18–20], we can smoothly isotope $\tilde{F}_0$ to $\tilde{F}_1$ in S^5. Thus K_0 is isotopic to K_1.

(2) $\Longleftrightarrow$ (3) and (4) $\Longleftrightarrow$ (5): These are obvious in view of the fact that the Seifert form is essentially the same as the variation map [37]. This completes the proof. □

REMARK 2.8. (1) We call a smoothly embedded (closed connected orientable) 3-manifold K in S^5 an *almost fibered knot* if there exists a topological (not necessarily smooth) fibration $\phi : S^5 - K \to S^1$ satisfying the condition in Definition 2.1 and with simply connected fibers such that at least one of the fibers F of ϕ is smooth and smoothly embedded in S^5 (such a fiber F is called an *almost fiber*). Then Theorem 2.2 also holds for almost fibered 3-knots.

(2) If $H_*(K_j; \mathbf{Q}) \cong H_*(S^3; \mathbf{Q})$, then in Theorem 2.2 (4) and (5) the condition (a) can be omitted (see [9, (0.8) Proposition]).

(3) If $H_1(K_j)$ are torsion free, then the diagram (2.4) in Theorem 2.2 (4)(b) and (5)(b) can be replaced by

$$\begin{array}{ccccc} 0 & \longrightarrow & H_2(K_0) & \longrightarrow & H_2(F_0) \\ & & \psi_* \big\downarrow & & \Lambda \big\downarrow \\ 0 & \longrightarrow & H_2(K_1) & \longrightarrow & H_2(F_1). \end{array} \tag{2.8}$$

The remaining commutativity is obtained from the above one by using the Lefschetz duality and the universal coefficient theorem.

As a corollary to Theorem 2.2, we get further isotopy criterions as follows.

Corollary 2.9. *For fibered* 3*-knots* K_j $(j = 0, 1)$, *the following three are equivalent.*

(1) *The embedded* 3*-manifolds* K_0 *and* K_1 *are* (*orientation preservingly*) *isotopic in* S^5.

(2) *The pair* (S^5, K_0) *is* (*orientation preservingly*) *homeomorphic to the pair* (S^5, K_1).

(3) *The fibers of* K_j *are* (*orientation preservingly*) *homeomorphic and the geometric monodromies of* K_j *are homotopic relative to boundary.*

Proof. (1) $\Longrightarrow$ (2): This is obvious.

(2) $\Longrightarrow$ (3): This can be proved by the same argument as in [68]. See also the proof of Lemma 2.5.

(3) $\Longrightarrow$ (1): By [72], the geometric monodromies of K_j are topologically pseudoisotopic relative to boundary. Then by Lemma 2.4 there exists an orientation preserving homeomorphism $\Phi : S^5 \to S^5$ with $\Phi(F_0) = F_1$ and $\Phi \circ i_0 = i_1 \circ \Phi$. The homeomorphism $\Psi = \Phi|_{F_0}$ clearly preserves the Seifert forms. Thus by Theorem 2.2, K_0 and K_1 are isotopic to each other. This completes the proof. $\square$

Remark 2.10. In the other dimensions, an analogue of the above corollary ((1) $\Longleftrightarrow$ (2)) is also true; i.e., two fibered $(2n-1)$-knots K_j $(j = 0, 1)$ are isotopic to each other if and only if (S^{2n+1}, K_0) is orientation preservingly homeomorphic to (S^{2n+1}, K_1). For $n = 1$, this is a well-known fact. For $n \geq 3$ we can prove this using the topological h-cobordism theorem [44] and a result of Durfee [21] and Kato [36] (see the proof of Lemma 2.5. See also [77, Lemma 5]).

As we shall see in §3, a fibered 3-knot is not always determined by the homeomorphism type of the embedded 3-manifold together with the congruence class of a Seifert matrix. However, in the following special cases these invariants do determine the isotopy type.

DEFINITION 2.11. Let Σ be a closed connected orientable 3-manifold. If a fibered 3-knot K is homeomorphic to Σ as an abstract 3-manifold, then we say that K is a *fibered Σ-knot* (see [75]).

THEOREM 2.12. *Let Σ be one of the following 3-manifolds.*

(1) $H_1(\Sigma) \cong \mathbf{Z}/r\mathbf{Z}$, *where $r = 1, 2, 4, p^m$ or $2p^m$ with p an odd prime.*

(2) $\Sigma = \sharp^k(S^2 \times S^1)$ *for some positive integer k.*

Then two fibered Σ-knots are isotopic if and only if their Seifert matrices are congruent.

PROOF. First assume that $H_1(\Sigma) \cong \mathbf{Z}/r\mathbf{Z}$ as in (1). Let K_j $(j = 0, 1)$ be fibered Σ-knots with congruent Seifert matrices. Thus there exists an isomorphism $\Lambda : H_2(F_0) \to H_2(F_1)$ which preserves the Seifert forms, where F_j is the fiber of K_j. Then there exists a unique isomorphism $\alpha : H_1(K_0) \to H_1(K_1)$ which makes the following diagram commutative:

$$\begin{array}{ccccccccc} 0 & \to & H_2(F_0) & \to & H_2(F_0, \partial F_0) & \to & H_1(K_0) & \to & 0 \\ & & \Lambda\downarrow & & \Lambda^*\uparrow & & \alpha\downarrow & & \\ 0 & \to & H_2(F_1) & \to & H_2(F_1, \partial F_1) & \to & H_1(K_1) & \to & 0. \end{array} \tag{2.9}$$

Note that α preserves the linking pairings on $H_1(K_j)$. Using our hypothesis, we can easily deduce that $\alpha = \pm 1$ (see [9, (1.9) Corollary]). Replacing Λ by $-\Lambda$ if necessary, we may assume that $\alpha = 1$. (Note that $-\Lambda$ also preserves the Seifert forms.) Thus α is realized by a homeomorphism (e.g., the identity map). By Theorem 2.2 and Remark 2.8 (2), K_0 is isotopic to K_1.

Next we assume that $\Sigma = S^2 \times S^1$. Let K_j $(j = 0, 1)$ be fibered Σ-knots with congruent Seifert matrices. Thus there exists an isomorphism $\Lambda : H_2(F_0) \to H_2(F_1)$ which preserves the Seifert forms. Then there exists a unique isomorphism α which makes the following diagram commutative:

$$\begin{array}{ccccc} 0 & \longrightarrow & H_2(K_0) & \longrightarrow & H_2(F_0) \\ & & \alpha\downarrow & & \Lambda\downarrow \\ 0 & \longrightarrow & H_2(K_1) & \longrightarrow & H_2(F_1). \end{array} \tag{2.10}$$

Since $H_2(K_j) \cong \mathbf{Z}$, α is realized by a homeomorphism $\psi : K_0 \to K_1$. Let $\tau : S^2 \times S^1 \to S^2 \times S^1$ be the homeomorphism defined by $\tau(u, v) = (uv, v)$, where we identify S^2 with the Riemann sphere $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ and S^1 with the unit circle in $\mathbf{C}$. Note that τ acts on $H_*(S^2 \times S^1)$ trivially. Replacing ψ by $\tau \circ \psi$ if necessary, we may assume that $F_0 \cup_\psi (-F_1)$ is a spin manifold (see [9]). Then by Theorem 2.2 and Remark 2.8 (3), K_0 is isotopic to K_1.

The general case where $\Sigma = \sharp^k(S^2 \times S^1)$ is proved similarly. All we need in addition is that every automorphism of $H_2(\sharp^k(S^2 \times S^1))$ is realized by a homeomorphism, which can be proved by using the fact that $\sharp^k(S^2 \times S^1)$ is diffeomorphic to the boundary of $\natural^k(S^2 \times D^2)$ and by using the techniques of handle-sliding (see [24, proof of Lemma 4] or [48, p.81]). This completes the proof. □

REMARK 2.13. When $H_1(\Sigma) = 0$, the above theorem has been proved in [54], [74].

We have discussed isotopy criterions for fibered 3-knots and have not mentioned their existence. In fact, there exist plenty of fibered 3-knots. For example, every closed connected orientable 3-manifold can be embedded in S^5 as a fibered 3-knot [74]. It seems worthwhile to recall here some methods of constructing fibered 3-knots. We have the algebraic construction [59], the open book construction [37], [74], the cyclic suspension of classical fibered knots [63], and the stabilization of almost fibered 3-knots [75]. Furthermore, almost fibered 3-knots can be constructed by Kervaire's method [41, Chap. II, §6]. We also have plumbing of two fibered 3-knots, which will be discussed later in §8. In the following sections, we often use these methods in order to construct desired fibered 3-knots.

3. Seifert Matrix Does Not Necessarily Determine the Isotopy Type

In §2 we have shown that for certain 3-manifolds Σ, the congruence class of a Seifert matrix is a complete invariant for a fibered Σ-knot. However, this is not the case in general. In this section we give two kinds of such examples.

THEOREM 3.1. *Let Σ be a nontrivial orientable S^1-bundle over the closed orientable surface of genus $g \geq 2$. Then for every positive integer*

n, there exist fibered Σ-knots $K_1, \ldots, K_n$ with diffeomorphic fibers and congruent Seifert matrices such that K_i and K_j $(i \neq j)$ are not isotopic to each other.

Proof. Let $e \in \mathbf{Z}$ be the Euler number of the S^1-bundle Σ. We assume that $e > 0$. (The case where $e < 0$ can be treated similarly.) Then the special handlebody F' whose framed link representation is given in Fig. 1 satisfies $\partial F' \cong \Sigma$, where a *special handlebody* is a handlebody consisting of one 0-handle and some 2-handles attached to the 0-handle simultaneously. This is proved as follows. Let N be the D^2-bundle over the closed orientable surface Σ_g of genus g with Euler number e. Note that $\partial N \cong \Sigma$. Then N is diffeomorphic to the 4-manifold

$$((\Sigma_g - \operatorname{Int} D^2) \times D^2) \cup_h (-D^2 \times D^2), \tag{3.1}$$

where $h : S^1 \times D^2 \to S^1 \times D^2$ is the diffeomorphism defined by $h(t,s) = (t, t^e s)$ and we identify S^1 and D^2 with $\{z \in \mathbf{C} : |z| = 1\}$ and $\{z \in \mathbf{C} : |z| \leq 1\}$ respectively. Hence N admits a handlebody decomposition consisting of one 0-handle, $2g$ 1-handles, and one 2-handle. Thus N is represented by the first picture of Fig. 2, where each pair of small 3-balls with the same index

F'

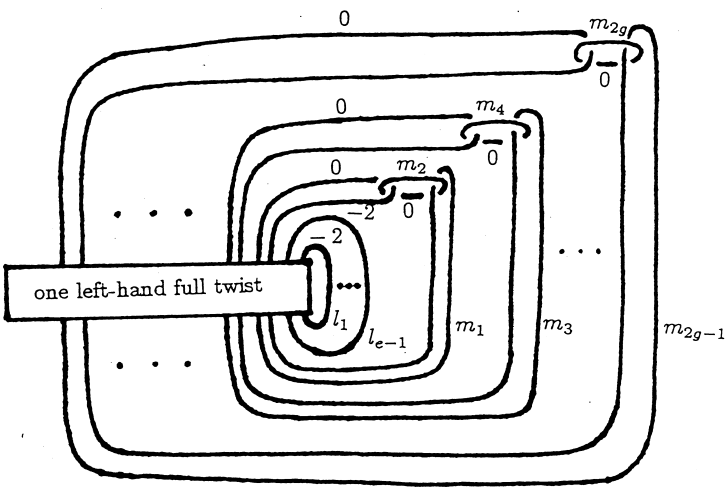

Figure 1.

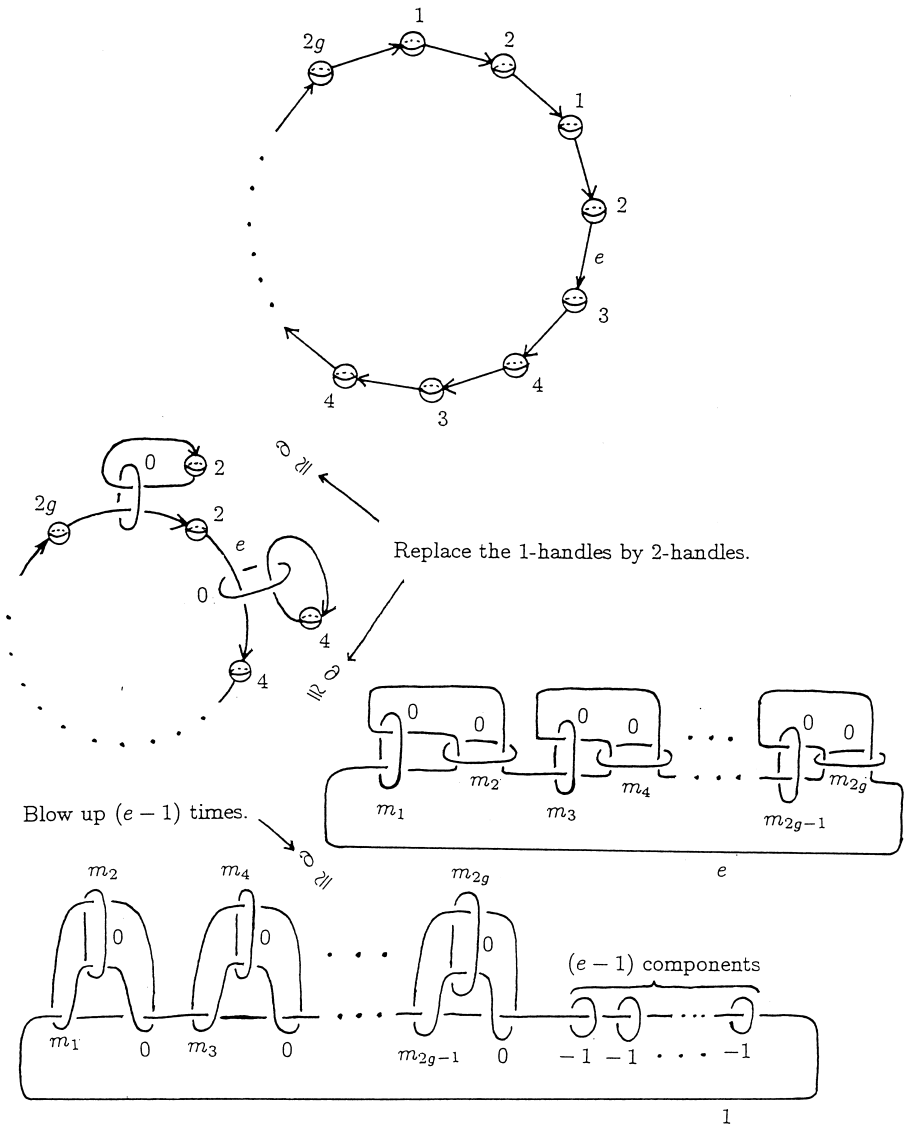
Replace the 1-handles by 2-handles.
Blow up $(e-1)$ times.
$(e-1)$ components

Figure 2.

denotes the attaching disks of a 1-handle. After proceeding as indicated in Fig. 2, consider the last framed link and blow down the $(+1)$-circle. Then the resulting framed link is as in Fig. 1.

Let $\pi : \Sigma \to \Sigma_g$ be the given S^1-bundle projection. Note that $H_1(\Sigma) \cong (\oplus^{2g}\mathbf{Z}) \oplus (\mathbf{Z}/e\mathbf{Z})$ and that $\pi_* : H_1(\Sigma) \to H_1(\Sigma_g)$ is surjective. Let a_i, b_i $(i = 1, 2, \ldots, g)$ be a basis of $H_1(\Sigma_g)$ with $a_i \cdot b_j = \delta_{ij}$ and $a_i \cdot a_j = b_i \cdot b_j = 0$, where "$\cdot$" denotes the intersection number in Σ_g and

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

Then there exists a basis a'_i, b'_i $(i = 1, 2, \ldots, g)$ of the free part of $H_1(\Sigma)$ such that $\pi_*(a'_i) = a_i$ and $\pi_*(b'_i) = b_i$. Let $\bar{a}_i, \bar{b}_i$ $(i = 1, 2, \ldots, g)$ be the basis of $H_2(\Sigma)$ Poincaré dual to a'_i, b'_i. Since the inclusion induces an injection $H_2(\Sigma) \to H_2(F')$ $(\partial F' = \Sigma)$, we identify $\bar{a}_i, \bar{b}_i$ with the corresponding elements of $H_2(F')$.

Let J_j $(j = 1, 2, \ldots, n)$ be unimodular skew-symmetric $2g \times 2g$ integral matrices, which will be given later. (Note that any two such matrices are congruent to each other over the integers.) Let R be the unimodular $(e - 1) \times (e - 1)$ matrix given by

$$R = \begin{pmatrix} -1 & -1 & \cdots & -1 \\ & -1 & \cdots & -1 \\ & & \ddots & \vdots \\ \mathbf{0} & & & -1 \end{pmatrix}. \tag{3.3}$$

Set $L'_j = J_j \oplus R$, then L'_j is unimodular and $L'_j + {}^tL'_j$ is the intersection matrix of F' with respect to a suitable basis of $H_2(F')$ which extends $\{\bar{a}_1, \ldots, \bar{a}_g, \bar{b}_1, \ldots, \bar{b}_g\}$. Using the argument of Kervaire [41], we can construct an embedding $\xi_j : F' \to S^5$ whose Seifert matrix coincides with L'_j. Set $K'_j = \xi_j(\partial F')$, then K'_j is an almost fibered 3-knot. Let K_S be a stabilizer defined in [75]; i.e., K_S is a fibered 3-knot whose fiber is diffeomorphic to $(S^2 \times S^2 \sharp S^2 \times S^2)^\circ (= (S^2 \times S^2 \sharp S^2 \times S^2) - \operatorname{Int} D^4)$. Then $K_j = K'_j \sharp k K_S$ are smoothly fibered 3-knots for some nonnegative integer k [75]. (Note that we can take k uniformly for all $K'_1, \ldots, K'_n$.) Note that K_j is diffeomorphic to Σ and that the fiber is diffeomorphic to $F = F' \sharp 2k(S^2 \times S^2)$. Furthermore, we see easily that the Seifert matrices of $K_1, \ldots, K_n$ are all congruent.

Let P be a unimodular matrix which gives the congruence between L_i and L_j; i.e., ${}^tPL_iP = L_j$, where L_i and L_j are the Seifert matrices of K_i and K_j respectively. Note that $L_i = J_i \oplus R \oplus L_S$ and $L_j = J_j \oplus R \oplus L_S$, where L_S is a Seifert matrix of $\sharp^k K_S$. We need the following lemma.

LEMMA 3.2. *Let Q_i be $m\times m$ skew-symmetric nonsingular integral matrices and R_i $n \times n$ integral matrices such that $R_i + {}^tR_i$ are nonsingular $(i = 1, 2)$. Suppose that there exists an $(m + n) \times (m + n)$ nonsingular matrix P such that ${}^tP(Q_1 \oplus R_1)P = Q_2 \oplus R_2$. Then $P = P_1 \oplus P_2$, where P_1 (resp. P_2) is an $m \times m$ (resp. $n \times n$) nonsingular integral matrix such that ${}^tP_1Q_1P_1 = Q_2$ (resp. ${}^tP_2R_1P_2 = R_2$).*

PROOF. Set

$$P = \begin{pmatrix} P_1 & P_3 \\ P_4 & P_2 \end{pmatrix}, \tag{3.4}$$

where the sizes of P_1, P_2, P_3 and P_4 are $m \times m, n \times n, m \times n$ and $n \times m$ respectively. Then we have

$$\begin{pmatrix} {}^tP_1 & {}^tP_4 \\ {}^tP_3 & {}^tP_2 \end{pmatrix} \begin{pmatrix} Q_1 & 0 \\ 0 & R_1 \end{pmatrix} \begin{pmatrix} P_1 & P_3 \\ P_4 & P_2 \end{pmatrix} = \begin{pmatrix} Q_2 & 0 \\ 0 & R_2 \end{pmatrix}. \tag{3.5}$$

Adding the transpose of the above equation to itself, we obtain

$$\begin{aligned} &\begin{pmatrix} {}^tP_1 & {}^tP_4 \\ {}^tP_3 & {}^tP_2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & R_1 + {}^tR_1 \end{pmatrix} \begin{pmatrix} P_1 & P_3 \\ P_4 & P_2 \end{pmatrix} \\ &= \begin{pmatrix} {}^tP_4(R_1 + {}^tR_1)P_4 & {}^tP_4(R_1 + {}^tR_1)P_2 \\ {}^tP_2(R_1 + {}^tR_1)P_4 & {}^tP_2(R_1 + {}^tR_1)P_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & R_2 + {}^tR_2 \end{pmatrix}. \end{aligned} \tag{3.6}$$

Since $R_i + {}^tR_i$ are nonsingular, we see that P_2 is nonsingular and we have $P_4 = 0$. Therefore we see that P_1 is nonsingular by (3.4). Then by (3.5), we have

$$\begin{pmatrix} {}^tP_1Q_1P_1 & {}^tP_1Q_1P_3 \\ {}^tP_3Q_1P_1 & {}^tP_3Q_1P_3 + {}^tP_2R_1P_2 \end{pmatrix} = \begin{pmatrix} Q_2 & 0 \\ 0 & R_2 \end{pmatrix}. \tag{3.7}$$

Then we see that $P_3 = 0$. This completes the proof. □

We note that the above lemma has an interesting consequence about the cancellation of certain fibered 3-knots. See §7.

Let us go back to the proof of Theorem 3.1. By the above lemma, we see that P is of the form $P_0 \oplus P_1$ with ${}^tP_0 J_i P_0 = J_j$.

Now suppose that K_i and K_j $(i \neq j)$ are isotopic to each other. Then by Theorem 2.2, there exists a homeomorphism $\psi : K_j \to K_i$ which makes the following diagram commutative:

$$\begin{array}{ccccc} 0 & \longrightarrow & H_2(K_j) & \longrightarrow & H_2(F) \\ & & {\scriptstyle \psi_*}\big\downarrow & & {\scriptstyle P}\big\downarrow \\ 0 & \longrightarrow & H_2(K_i) & \longrightarrow & H_2(F). \end{array} \tag{3.8}$$

Note that ψ_* is identified with P_0 with respect to the basis $\{\bar{a}_1, \ldots, \bar{a}_g, \bar{b}_1, \ldots, \bar{b}_g\}$. Since $g \geq 2$, by Waldhausen [86], we may assume that there exists a homeomorphism $\bar{\psi} : \Sigma_g \to \Sigma_g$ which makes the following diagram commutative:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\ \psi\ } & \Sigma \\ {\scriptstyle \pi}\big\downarrow & & {\scriptstyle \pi}\big\downarrow \\ \Sigma_g & \xrightarrow{\ \bar{\psi}\ } & \Sigma_g. \end{array} \tag{3.9}$$

Since $(\bar{\psi})_*$ on $H_1(\Sigma_g)$ preserves the intersection paring on Σ_g, we have ${}^tQJQ = J$, where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{3.10}$$

I is the $g \times g$ unit matrix, and Q is the matrix representative of $(\bar{\psi})_*$ on $H_1(\Sigma_g)$ with respect to the basis $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$. Using Poincaré duality and the universal coefficient theorem, we see easily that $\psi_* = P_0$ on $H_2(\Sigma)$ is identified with ${}^tQ^{-1}$. Hence we have $P_0^{-1} J\, {}^tP_0^{-1} = J$. This implies that ${}^tP_0 J P_0 = J$.

Definition 3.3. Let A and B be unimodular skew-symmetric $2g \times 2g$ integral matrices. We say that A and B are *equivalent* (denoted by $A \sim B$), if there exists a unimodular matrix C such that ${}^tCAC = B$ and ${}^tCJC = J$, where J is the matrix as in (3.10). Note that this is an equivalence relation.

Thus, in order to prove Theorem 3.1, we have only to show that there exist infinitely may equivalence classes in the above sense.

For a unimodular skew-symmetric $2g \times 2g$ integral matrix A, define $\varepsilon_A(t) = \det(tJ - A)$, which is an element of $\mathbf{Z}[t]$.

LEMMA 3.4. *If* $A \sim B$, *then* $\varepsilon_A(t) = \varepsilon_B(t)$.

PROOF. Let C be a matrix as in Definition 3.3. Then we have

$$\begin{aligned} \varepsilon_B(t) &= \det(tJ - B) \\ &= \det(tJ - {}^t\!CAC) \\ &= \det({}^t\!C(tJ - A)C) \\ &= \det(tJ - A) \\ &= \varepsilon_A(t). \end{aligned} \tag{3.11}$$

This completes the proof. □

For a given sequence of integers $\alpha_0, \alpha_1, \ldots, \alpha_{g-1}$, set

$$X = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 & -\alpha_0 \\ 1 & 0 & \cdots & \cdots & 0 & -\alpha_1 \\ & 1 & 0 & \cdots & 0 & -\alpha_2 \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & 1 & 0 & -\alpha_{g-2} \\ \mathbf{0} & & & & 1 & -\alpha_{g-1} \end{pmatrix}. \tag{3.12}$$

Then we can show the following by an induction argument.

LEMMA 3.5.

$$\det(tI - X) = t^g + \alpha_{g-1}t^{g-1} + \cdots + \alpha_1 t + \alpha_0. \tag{3.13}$$

Setting

$$J_j = \begin{pmatrix} 0 & X \\ -{}^tX & 0 \end{pmatrix} \tag{3.14}$$

with $\alpha_0 = 1$, we see that J_j is a unimodular skew-symmetric $2g \times 2g$ integral matrix and that

$$\varepsilon_{J_j}(t) = (t^g + \alpha_{g-1}t^{g-1} + \cdots + \alpha_1 t + 1)^2. \tag{3.15}$$

Thus, varying $\alpha_1, \ldots, \alpha_{g-1}$, we obtain infinitely many unimodular skew-symmetric $2g \times 2g$ integral matrices which are not equivalent to each other in the sense of Definition 3.3 by Lemmas 3.4 and 3.5, since $g \geq 2$. This completes the proof of Theorem 3.1. □

REMARK 3.6. We do not know if there exist *infinitely many* smoothly fibered Σ-knots with congruent Seifert matrices such that they are not isotopic to each other for some Σ. This is true if we can find a uniform integer k as in the above proof for all K_i ($i = 1, 2, \ldots$). See [50], [71], [26]. Note that we can find infinitely many distinct *almost fibered* Σ-knots with congruent Seifert matrices.

REMARK 3.7. Let K and K' be fibered 3-knots with isomorphic Seifert forms. Then the n-fold cyclic suspensions of K and K', which are fibered 5-knots in S^7, are isotopic to each other for all positive integer n, since they have isomorphic Seifert forms (see [63]). In particular, the n-fold cyclic branched covering spaces of S^5 branched along K and K' are diffeomorphic to each other for all n, although K and K' may not be isotopic to each other as has been seen in Theorem 3.1. Compare this with the case of prime knots in S^3, where two such knots are equivalent if and only if the n-fold cyclic branched covering spaces of S^3 branched along the knots are homeomorphic for infinitely many n (see [46]).

Next we give an example which shows that the Seifert matrix does not determine even the homeomorphism type of the fiber, nor the isotopy type of a fibered 3-knot.

PROPOSITION 3.8. *Let Σ be the trivial S^1-bundle over the closed orientable surface of genus $g \geq 2$. Then there exist two fibered Σ-knots with congruent Seifert matrices but with nonhomeomorphic fibers.*

By Theorem 2.2 these fibered Σ-knots are not isotopic to each other.

PROOF OF PROPOSITION 3.8. Let F_0 be the special handlebody whose framed link representation is given in Fig. 3. It is easy to see that ∂F_0 is diffeomorphic to Σ (see the proof of Theorem 3.1). Let $\mathcal{F}_0$ be the almost framing on $\partial F_0 = \Sigma$ induced by that on F_0. Let $\mathcal{F}_1$ be the almost framing on $\partial F_0 = \Sigma$ which is the same framing on $\mathcal{F}_0$ except on γ_{2g+1}, where γ_{2g+1} is

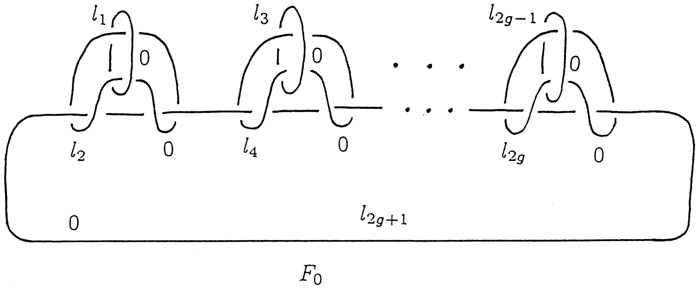

Figure 3.

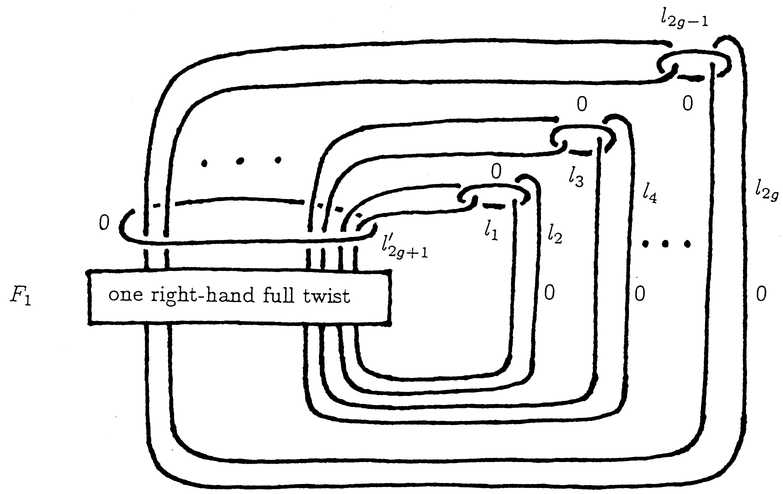

Figure 4.

the homology class in $H_1(\partial F_0; \mathbf{Z}/2\mathbf{Z})$ represented by a meridian loop around the component l_{2g+1} (see Fig. 3). Recall that the almost framings on ∂F_0 are in one-to-one correspondence with the elements of $H^1(\partial F_0; \mathbf{Z}/2\mathbf{Z})$. Then, for $(\partial F_0, \mathcal{F}_1)$, the one component link l_{2g+1} is characteristic in the sense of Kaplan [35]. If we apply Kaplan's algorithm to kill the characteristic sublink without changing the boundary, then we get the framed handlebody F_1 whose framed link representation is given in Fig. 4. Note that the almost framing $\mathcal{F}_1$ is the restriction to ∂F_1 of that on F_1 and that F_j have the same intersection matrix.

We show that F_0 is not homeomorphic to F_1. If F_0 is homeomorphic to F_1, then by [9], there exists a homeomorphism $\psi : \partial F_0 \to \partial F_1$ such that $F_0 \cup_\psi (-F_1)$ is a spin 4-manifold. Thus $\psi_\sharp(\mathcal{F}_0) = \mathcal{F}_1$. (For the definition of $\psi_\sharp(\mathcal{F}_0)$, see [9, (2.3)]. Note that almost framings on Σ are identified with spin structures on Σ in the sense of Boyer [9].) We may assume that ψ preserves the fibers of the trivial fibration $\Sigma \to \Sigma_g$ by [86]. Then it is easy to show that $\psi_\sharp(\mathcal{F}_0)$ is different from $\mathcal{F}_1$ on $\gamma_{2g+1} \in H_1(\Sigma; \mathbf{Z}/2\mathbf{Z})$, since γ_{2g+1} coincides with the class represented by a fiber of $\Sigma \to \Sigma_g$. Thus F_0 is not homeomorphic to F_1.

Since taking connected sum with the spin 4-manifold $S^2 \times S^2$ does not change the almost framing on the boundary, we see that $F_0 \sharp k(S^2 \times S^2)$ is not homeomorphic to $F_1 \sharp k(S^2 \times S^2)$ for any nonnegative integer k.

Since the intersection forms of F_j are isomorphic, we obtain two fibered Σ-knots with the same Seifert matrix and with fibers $F_j \sharp k(S^2 \times S^2)$ for some even integer k, using the technique of the stabilization of almost fibered knots as in [75]. This completes the proof. □

Remark 3.9. We do not know if F_0 and F_1 above are homotopy equivalent relative to boundary.

In all the above examples, the first homology groups of the embedded 3-manifolds are infinite. In the case of rational homology 3-spheres, we have the following example.

Example 3.10. Let V_1 (resp. V_2) be the special handlebody consisting of one 0-handle and one 2-handle attached along the torus knot of type $(3, 19)$ (resp. $(5, 11)$) with the framing 56. By [61], ∂V_i are (orientation preservingly) diffeomorphic to the lens space $L(56, 9) (\cong L(56, 25))$. We fix a diffeomorphism $\partial V_1 \cong \partial V_2 \cong L(56, 9)$. Let $\gamma_i \in H_2(V_i, \partial V_i) \cong \mathbf{Z}$ be generators. Then it is not difficult to show that $\partial\gamma_1 = 19\alpha$ and $\partial\gamma_2 = 5\alpha$ for some generator $\alpha \in H_1(L(56, 9)) \cong \mathbf{Z}/56\mathbf{Z}$. It is known that there exists no homeomorphism of $L(56, 9)$ whose induced isomorphism on $H_1(L(56, 9))$ maps 5α to 19α (for example, see [8], [33]). Thus V_1 is not homeomorphic to V_2. By a similar argument, we can show that $V_1 \sharp k(S^2 \times S^2)$ is not homeomorphic to $V_2 \sharp k(S^2 \times S^2)$ for every nonnegative integer k. Now, using the technique developed in [74, §6], we can construct, for some k, fibered

3-knots K_i $(i = 1, 2)$ with fiber $V_i \sharp k(S^2 \times S^2)$ and with congruent Seifert matrices. By Theorem 2.2 these fibered 3-knots are not isotopic to each other. Note that $r = 56$ does not satisfy the condition in Theorem 2.12 (1).

4. Reflexivity of Fibered 3-Knots

Let K be a 3-manifold embedded in S^5. Set $E(K) = S^5 - \operatorname{Int} N(K)$, where $N(K)$ is a tubular neighborhood of K in S^5. We call $E(K)$ the *exterior* of K. We say that K is *reflexive* if every embedding K' of the 3-manifold in S^5 with exterior diffeomorphic to that of K is isotopic to K. Note that when the embedded 3-manifold is diffeomorphic to S^3, it is known that there exist at most two embeddings with diffeomorphic exteriors (for example, see [54, §22]). Here the terminology "reflexive" comes from this fact, although for general 3-manifolds we may have more than two such embeddings, as will be seen in Theorem 4.6.

In [12], Cappell and Shaneson have shown that there exist infinitely many nonreflexive spherical 3-knots in S^5. (For the case of classical knots in S^3, see [28].) On the other hand, Levine [54, §22] has shown that every simple fibered spherical 3-knot is reflexive. In this section, we show that certain (nonspherical) fibered 3-knots are reflexive (in fact, they are completely determined by the homotopy type of their complements). We also give some examples of nonreflexive fibered 3-knots.

In order to obtain some results on the reflexivity of fibered 3-knots, we consider the following classes of closed orientable 3-manifolds Σ and Σ'.

(I) Σ is of the form $\Sigma = \Sigma_1 \sharp \cdots \sharp \Sigma_r$, where each Σ_i is either $S^1 \times S^2$ or a Haken 3-manifold (i.e., irreducible and sufficiently large [87]). Σ' does not contain any compact contractible 3-dimensional submanifold which is not diffeomorphic to the standard 3-ball (i.e., it contains no fake 3-cell).

(II) Σ is a Seifert fibered space with infinite π_1 and Σ' is irreducible.

(III) Σ and Σ' are hyperbolic 3-manifolds.

(IV) Σ is a hyperbolic 3-manifold which contains an embedded hyperbolic tube of radius $(\log 3)/2 = 0.549306 \cdots$ about a closed geodesic and Σ' is irreducible (see [25]).

Note that for each pair of closed orientable 3-manifolds Σ and Σ' as above the following holds: every homotopy equivalence $\psi : \Sigma' \to \Sigma$ is homotopic to a homeomorphism. See [87], [48], [80], [66], [67], [62], [25] and [68, Lemme 9].

THEOREM 4.1. *Suppose that Σ and Σ' are a closed orientable* 3*-manifold pair as in* (I), (II), (III)*, or* (IV) *described above and that $H_1(\Sigma; \mathbf{Q}) = 0 = H_1(\Sigma'; \mathbf{Q})$. Let K_0 be a fibered Σ-knot and K_1 a fibered Σ'-knot. Then K_0 is isotopic to K_1 if and only if $(E_0, \partial E_0)$ is orientation preservingly homotopy equivalent to $(E_1, \partial E_1)$, where $E_j = S^5 - \operatorname{Int} N(K_j)$ $(j = 0, 1)$.*

PROOF. Suppose that $(E_0, \partial E_0)$ is orientation preservingly homotopy equivalent to $(E_1, \partial E_1)$. Let $\pi_j : (\tilde{E}_j, \partial \tilde{E}_j) \to (E_j, \partial E_j)$ be the infinite cyclic coverings induced by the fibrations $(j = 0, 1)$. Since $(E_0, \partial E_0)$ is homotopy equivalent to $(E_1, \partial E_1)$, there exists a homotopy equivalence $\lambda : (\tilde{E}_0, \partial \tilde{E}_0) \to (\tilde{E}_1, \partial \tilde{E}_1)$. Note that $(\tilde{E}_j, \partial \tilde{E}_j)$ is diffeomorphic to $(F_j, \partial F_j) \times \mathbf{R}$. Set $\bar{\lambda} = r_1 \circ \lambda \circ \iota_0$, where $\iota_0 : (F_0, \partial F_0) \to (\tilde{E}_0, \partial \tilde{E}_0)$ is an inclusion map and $r_1 : (\tilde{E}_1, \partial \tilde{E}_1) \to (F_1, \partial F_1)$ is a natural retraction map. Furthermore let $h_j : (F_j, \partial F_j) \to (F_j, \partial F_j)$ be the geometric monodromy of K_j. Then the following diagram is commutative up to homotopy:

$$\begin{array}{ccc} (F_0, \partial F_0) & \xrightarrow{h_0} & (F_0, \partial F_0) \\ \bar{\lambda}\downarrow & & \bar{\lambda}\downarrow \\ (F_1, \partial F_1) & \xrightarrow{h_1} & (F_1, \partial F_1). \end{array} \tag{4.1}$$

Thus the homotopy equivalence $\bar{\lambda}$ preserves the intersection forms and the algebraic monodromies. Let S_j, H_j and L_j be the intersection matrix for F_j, the monodromy matrix for K_j and the Seifert matrix of K_j respectively with respect to a fixed basis of $H_2(F_j)$. Then by [21], [36], we have $L_j(I - H_j) = S_j$, where I is the unit matrix. On the other hand, we have $\det(I - H_j) \neq 0$, since $H_1(K_j; \mathbf{Q}) = 0$. Hence we have $L_j = S_j(I - H_j)^{-1}$. Thus $\bar{\lambda}$ also induces an isomorphism on the second homology group which preserves the Seifert forms.

By our hypothesis on Σ and Σ', we see that K_0 is homeomorphic to K_1 and that the homotopy equivalence $\bar{\lambda}|_{K_0} : K_0 \to K_1$ is homotopic to a homeomorphism. Thus, changing $\bar{\lambda}$ homotopically if necessary, we may assume that $\bar{\lambda}|_{K_0}$ is a homeomorphism. Then by Theorem 2.2, K_0 is isotopic to K_1 (see also Remark 2.8 (2)). The converse is trivial. This completes the proof. □

COROLLARY 4.2. *Let* Σ *be a closed orientable* 3*-manifold as in* (I), (II) *or* (III) *described just before Theorem* 4.1 *and suppose that* $H_1(\Sigma;\mathbf{Q}) = 0$. *Then every fibered* Σ*-knot is reflexive.*

As a corollary to the above result, we obtain the following.

COROLLARY 4.3. *Let* Σ *and* Σ' *be a closed orientable* 3*-manifold pair as in* (I), (II), (III), *or* (IV) *and suppose that* $H_1(\Sigma;\mathbf{Q}) = 0 = H_1(\Sigma';\mathbf{Q})$. *Let* K_0 *be a fibered* Σ*-knot,* K_1 *a fibered* Σ'*-knot and* $\phi_i : S^5 - K_i \to S^1$ *the fibrations as in Definition* 2.1 ($i = 0, 1$). *Suppose that there exists an* (*orientation preserving*) *embedding* $\theta : E(K_0) \to E(K_1)$ *which makes the following diagram commutative:*

$$\begin{array}{ccc} E(K_0) & \xrightarrow{\theta} & E(K_1) \\ \phi_0 \searrow & & \swarrow \phi_1 \\ & S^1. & \end{array} \tag{4.2}$$

Furthermore, we assume that $F_1 - \operatorname{Int}\theta(F_0)$ *is an h-cobordism between* ∂F_1 *and* $\theta(\partial F_0)$, *where* F_i *is the fiber of* $\phi_i : E(K_i) \to S^1$ *over* $1 \in S^1$. *Then* K_0 *and* K_1 *are isotopic to each other.*

PROOF. Under the hypothesis, it is not difficult to show that $(E(K_0), \partial E(K_0))$ and $(E(K_1), \partial E(K_1))$ are orientation preservingly homotopy equivalent. Then the result follows directly from Theorem 4.1. □

Compare Corollary 4.3 with the corresponding higher dimensional result (for example, see the proof of [52, Lemma 2.2]).

REMARK 4.4. In Theorem 4.1, the hypothesis on the first homology group is essential. See the example below (Theorem 4.6).

REMARK 4.5. For $n \geq 3$, every simple fibered $(2n-1)$-knot K satisfying $H_*(K;\mathbf{Q}) \cong H_*(S^{2n-1};\mathbf{Q})$ is reflexive. This can be proved by using the same argument as in the proof of Theorem 4.1 together with the fact that

K is simply connected (see also [54, §22]). We do not know if this is true without the assumption on the homology group.

Next we give examples of nonreflexive fibered 3-knots.

THEOREM 4.6. *Let Σ be a nontrivial orientable S^1-bundle over the closed orientable surface of genus $g \geq 2$. Then there exist simple fibered knots $K_0, \dots, K_{g-1}$ in S^5 with the following properties.*

(1) *All K_i are diffeomorphic to Σ as abstract* 3*-manifolds.*

(2) *The exteriors $E(K_i) = S^5 - \operatorname{Int} N(K_i)$ are all diffeomorphic to each other, where $N(K_i)$ is a tubular neighborhood of K_i in S^5.*

(3) *The Seifert matrices of K_i are all congruent to each other.*

(4) *The knots K_i and K_j are not isotopic to each other if $i \neq j$.*

PROOF. In the following, we use the same notation as in the proof of Theorem 3.1. Let K_0 be a fibered Σ-knot constructed in the proof of Theorem 3.1 by using a unimodular skew-symmetric $2g \times 2g$ matrix J_0. (The matrix J_0 will be chosen more explicitly later.) Note that the Seifert matrix L_0 of K_0 is of the form $L_0 = J_0 \oplus R \oplus L_S$. Let F be the fiber of the fibered 3-knot K_0 and $h : F \to F$ the geometric monodromy.

It is not difficult to see that there exist $\tilde{a}_i$ and $\tilde{b}_i$ in $H_2(F, \partial F)$ such that $\partial \tilde{a}_i = a'_i$, $\partial \tilde{b}_i = b'_i$, and $\bar{a}_i \cdot \tilde{a}_j = \bar{b}_i \cdot \tilde{b}_j = \delta_{ij}, \bar{a}_i \cdot \tilde{b}_j = \bar{b}_i \cdot \tilde{a}_j = 0$, where "$\cdot$" denotes the intersection number in F. Note that $H_2(F, \partial F)$ naturally decomposes as $H_2(F', \partial F') \oplus H_2((\sharp^{2k} S^2 \times S^2)^\circ, \partial(\sharp^{2k} S^2 \times S^2)^\circ)$. We may assume that $\tilde{a}_i, \tilde{b}_i \in H_2(F', \partial F')$. Furthermore, $H_2(F')$ and $H_2(F', \partial F')$ decomposes as $H_2(F') = A \oplus B$ and $H_2(F', \partial F') = A' \oplus B'$ respectively in accordance with the decomposition $J_0 \oplus R$. Then we may further assume that $\tilde{a}_i$ and $\tilde{b}_i$ are in A', since in the exact sequence of the pair $(F', \partial F')$

$$(4.3) \qquad 0 \longrightarrow H_2(\partial F') \xrightarrow{\alpha} A \oplus B \xrightarrow{\beta} A' \oplus B' \xrightarrow{\partial} H_1(\partial F') \longrightarrow 0,$$

$\operatorname{Im} \alpha = A$, $\beta(B) \subset B'$ and $B'/\beta(B) (\subset H_1(\partial F'))$ is finite cyclic of order e (generated by the class represented by a fiber of the fibration $\pi : \Sigma \to \Sigma_g$).

Since Σ is a principal S^1-bundle, we have a 1-parameter family of diffeomorphisms $\varphi_t : \Sigma \to \Sigma$ parametrized by $t \in S^1 = [0,1]/\{0,1\}$ which is induced by the S^1-action on Σ. Let $c : \Sigma \times [0,1] \to F$ be a collar neighborhood of ∂F in F, where $c(x,1) = x$ for all $x \in \Sigma = \partial F$. Then define the diffeomorphism $h'' : F \to F$ by

$$h''(x) = \begin{cases} x, & \text{if } x \notin c(\Sigma \times [0,1]), \\ c(\varphi_t(y), t), & \text{if } x = c(y,t). \end{cases} \tag{4.4}$$

Furthermore, set $h' = h'' \circ h$. Note that h' is isotopic to h (not relative to boundary). Let Δ and $\Delta' : H_2(F, \partial F) \to H_2(F)$ be the variation maps of h and h' respectively. Then we have

$$\Delta'(\tilde{a}_i) = \Delta(\tilde{a}_i) - \bar{b}_i, \tag{4.5}$$

$$\Delta'(\tilde{b}_i) = \Delta(\tilde{b}_i) + \bar{a}_i, \tag{4.6}$$

$$\Delta'(x) = \Delta(x), \tag{4.7}$$

where x is an arbitrary element in $B' \oplus H_2((\sharp^{2k} S^2 \times S^2)^\circ, \partial(\sharp^{2k} S^2 \times S^2)^\circ)$. Thus the variation map $\Delta' : H_2(F', \partial F') \oplus H_2((\sharp^{2k} S^2 \times S^2)^\circ, \partial(\sharp^{2k} S^2 \times S^2)^\circ) \to H_2(F') \oplus H_2((\sharp^{2k} S^2 \times S^2)^\circ)$ decomposes as the direct sum of two maps $H_2(F', \partial F') \to H_2(F')$ and $H_2((\sharp^{2k} S^2 \times S^2)^\circ, \partial(\sharp^{2k} S^2 \times S^2)^\circ) \to H_2((\sharp^{2k} S^2 \times S^2)^\circ)$ as Δ does. Furthermore, we see that Δ and Δ' preserve the decompositions $H_2(F') = A \oplus B$ and $H_2(F', \partial F') = A' \oplus B'$ and that the restrictions to $B' \oplus H_2((\sharp^{2k} S^2 \times S^2)^\circ, \partial(\sharp^{2k} S^2 \times S^2)^\circ)$ of Δ and Δ' coincide. Thus in the following, we concentrate ourselves to the part corresponding to A and A'. Note that $\{\bar{a}_i, \bar{b}_i\}$ constitutes a basis of A and $\{\tilde{a}_i, \tilde{b}_i\}$ a basis of A'. With respect to these bases, we have

$$\Delta' = \Delta + J, \tag{4.8}$$

where J is as in (3.10). On the other hand, by [37], we have $\Delta = J_0^{-1}$. Thus we have

$$\Delta' = J_0^{-1} + J = J_0^{-1}(J_0 - J)J. \tag{4.9}$$

Now let J_0 be the matrix of the form

$$J_0 = \begin{pmatrix} 0 & X \\ -{}^tX & 0 \end{pmatrix}, \tag{4.10}$$

where X is a unimodular $g \times g$ integral matrix of the form as in (3.12) constructed from a sequence of integers $\alpha_0, \ldots, \alpha_{g-1}$. We choose the sequence as follows. Set

$$C = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2^2 & \cdots & 2^{g-1} \\ 1 & 3 & 3^2 & \cdots & 3^{g-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & g & g^2 & \cdots & g^{g-1} \end{pmatrix}, \tag{4.11}$$

and let C^* be the transpose of the cofactor matrix of C; in other words, $C^*C = CC^* = (\det C)I$, where I is the $g \times g$ unit matrix. Note that $\det C = \prod_{1 \le i < j \le g}(j-i)$. We set

$$\begin{pmatrix} \alpha_{g-1} \\ \alpha_{g-2} \\ \vdots \\ \alpha_1 \\ \alpha_0 \end{pmatrix} = \frac{1}{\det C} C^* \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ (g-1)! \end{pmatrix}. \tag{4.12}$$

Setting $C^* = (c^*_{ij})_{1 \le i,j \le g}$, we have

$$\begin{aligned} \begin{pmatrix} \alpha_{g-1} \\ \vdots \\ \alpha_1 \\ \alpha_0 \end{pmatrix} &= \frac{(g-1)!}{\det C} \begin{pmatrix} c^*_{1g} \\ \vdots \\ c^*_{g-1g} \\ c^*_{gg} \end{pmatrix} \\ &= \frac{1}{\prod_{1 \le i < j \le g-1}(j-i)} \begin{pmatrix} c^*_{1g} \\ \vdots \\ c^*_{g-1g} \\ c^*_{gg} \end{pmatrix}. \end{aligned} \tag{4.13}$$

Furthermore, we have

$$\begin{aligned} c^*_{ig} = (-1)^{i+g} \\ \cdot \det \begin{pmatrix} 1 & 1 & 1 & \cdots & \check{1} & \cdots & 1 \\ 1 & 2 & 2^2 & \cdots & 2^{i-1} & \cdots & 2^{g-1} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 1 & (g-1) & (g-1)^2 & \cdots & (g-1)^{i-1} & \cdots & (g-1)^{g-1} \end{pmatrix}, \end{aligned} \tag{4.14}$$

where the column with "ˇ" is deleted.

Lemma 4.7. *The product* $\prod_{1\le j<k\le g-1}(k-j)$ *divides* c^*_{ig} *in* $\mathbf{Z}$.

Proof. For each i with $1 \le i \le g$, set

$$f_i(x_1,\dots,x_{g-1}) = \det\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & \check{x_1^{i-1}} & \cdots & x_1^{g-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{i-1} & \cdots & x_2^{g-1} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 1 & x_{g-1} & x_{g-1}^2 & \cdots & x_{g-1}^{i-1} & \cdots & x_{g-1}^{g-1} \end{pmatrix}, \tag{4.15}$$

which is an element of $\mathbf{Z}[x_1,\dots,x_{g-1}]$. Then we have only to show that for all j and k with $1 \le j < k \le g-1$, $x_k - x_j$ divides $f_i(x_1,\dots,x_{g-1})$. Let us denote by $\mathcal{S}_{g-1}$ the symmetric group on the $g-1$ letters $\{1,2,\dots,g-1\}$. For an element $\sigma \in \mathcal{S}_{g-1}$, let us denote by ε_σ the signature of σ and by $\tilde{\sigma} : \{1,2,\dots,g-1\} \to \{0,1,\dots,i-2,i,\dots,g-1\}$ the composition $\tau\circ\sigma$, where $\tau : \{1,2,\dots,g-1\} \to \{0,1,\dots,i-2,i,\dots,g-1\}$ is the order preserving bijection. Then we have

$$\begin{aligned} (4.16)\quad & f_i(x_1,\dots,x_{g-1}) \\ &= \sum_{\sigma\in\mathcal{S}_{g-1}} \varepsilon_\sigma x_1^{\tilde\sigma(1)}\cdots x_{g-1}^{\tilde\sigma(g-1)} \\ &= \sum_{\sigma(j)<\sigma(k)} \varepsilon_\sigma x_1^{\tilde\sigma(1)}\cdots x_{g-1}^{\tilde\sigma(g-1)} + \sum_{\sigma(j)>\sigma(k)} \varepsilon_\sigma x_1^{\tilde\sigma(1)}\cdots x_{g-1}^{\tilde\sigma(g-1)} \\ &= \sum_{\sigma(j)<\sigma(k)} \varepsilon_\sigma \big(x_1^{\tilde\sigma(1)}\cdots x_j^{\tilde\sigma(j)}\cdots x_k^{\tilde\sigma(k)}\cdots x_{g-1}^{\tilde\sigma(g-1)} \\ &\qquad\qquad - x_1^{\tilde\sigma(1)}\cdots x_j^{\tilde\sigma(k)}\cdots x_k^{\tilde\sigma(j)}\cdots x_{g-1}^{\tilde\sigma(g-1)}\big) \\ &= \sum_{\sigma(j)<\sigma(k)} \varepsilon_\sigma x_1^{\tilde\sigma(1)}\cdots x_j^{\tilde\sigma(j)}\cdots x_k^{\tilde\sigma(j)}\cdots x_{g-1}^{\tilde\sigma(g-1)}\big(x_k^{\tilde\sigma(k)-\tilde\sigma(j)} - x_j^{\tilde\sigma(k)-\tilde\sigma(j)}\big), \end{aligned}$$

which implies that $x_k - x_j$ divides $f_i(x_1,\dots,x_{g-1})$. This completes the proof. □

By the above lemma, $\alpha_0,\dots,\alpha_{g-1}$ are integers. Furthermore, since

$$c^*_{gg} = \prod_{1\le j<k\le g-1}(k-j), \tag{4.17}$$

we have $\alpha_0 = 1$.

LEMMA 4.8. *For the above constructed sequence of integers $\alpha_0, \ldots, \alpha_{g-1}$, we have*

$$1 + n\alpha_{g-1} + n^2\alpha_{g-2} + \cdots + n^{g-1}\alpha_1 + n^g\alpha_0 = 1 \tag{4.18}$$

for $n = 0, 1, \ldots, g-1$.

PROOF. Since we have

$$C \begin{pmatrix} \alpha_{g-1} \\ \vdots \\ \alpha_1 \\ \alpha_0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (g-1)! \end{pmatrix} \tag{4.19}$$

by (4.12), the result easily follows. □

For $n = 0, 1, \ldots, g-1$, set $h'_n = (h'')^n \circ h$, where $(h'')^n$ denotes the composite of n copies of h''. Note that $h'_1 = h'$. Let Δ'_n denote the variation map of h'_n restricted to the direct summands A' and A of $H_2(F, \partial F)$ and $H_2(F)$ respectively. Then we have

$$\Delta'_n = \Delta + nJ = J_0^{-1} + nJ = J_0^{-1}(nJ_0 - J)J, \tag{4.20}$$

where

$$nJ_0 - J = \begin{pmatrix} 0 & nX - I \\ -n^tX + I & 0 \end{pmatrix}. \tag{4.21}$$

It is easy to show that

$$\begin{aligned} &\det(nX - I) \\ = \; &(-1)^g(1 + n\alpha_{g-1} + n^2\alpha_{g-2} + \cdots + n^{g-1}\alpha_1 + n^g\alpha_0) \end{aligned} \tag{4.22}$$

and that this is equal to $(-1)^g$ by the above lemma. Thus Δ'_n is an isomorphism. Then by the open book construction [37], [74], we can construct a fibered knot K_n using the diffeomorphism h'_n. If we put $L'_n = (\Delta'_n)^{-1}$, then the Seifert matrix of K_n is of the form $L'_n \oplus R \oplus L_S$.

By the above construction, the properties (1), (2) and (3) of Theorem 4.6 obviously hold. Now we show that K_i and K_j are not isotopic to each other for $i \neq j$. Consider $\varepsilon_{L'_n}(t) = \det(tJ - L'_n)$. Then it is not difficult to show that $\varepsilon_{L'_n} = f_n(t)^2$, where

$$
\begin{aligned}
f_n(t) &= t^g + (nt+1)t^{g-1}\alpha_{g-1} + \cdots \\
&\quad + (nt+1)^{g-1}t\alpha_1 + (nt+1)^g.
\end{aligned}
\tag{4.23}
$$

Thus the coefficient of t in $f_n(t)$ is equal to $\alpha_1 + gn$. This implies that $\varepsilon_{L'_i}(t) \neq \varepsilon_{L'_j}(t)$ if $i \neq j$. Then by the same argument as in the proof of Theorem 3.1, we see that K_i and K_j are not isotopic to each other. This completes the proof of Theorem 4.6. □

Remark 4.9. In the above construction, the exteriors $E = E(K_i)$ of K_i are diffeomorphic to each other and if we attach $\Sigma \times D^2$ to E appropriately, then we obtain K_i as the core $\Sigma \times \{0\}$ of $\Sigma \times D^2$. In this construction, the attaching maps are essentially different from each other.

Remark 4.10. By the above calculation, we see that when $e = 1$, the variation map of $h'' : F' \to F'$ is equal to J, which is unimodular. In other words, we can construct a fibered Σ-knot by using $h'' : F' \to F'$ as the geometric monodromy; in other words, we do not need stabilizations. This is a fibered Σ-knot which has the minimal second betti number of the fiber.

Remark 4.11. We do not know if there exist infinitely many fibered Σ-knots which satisfy the conditions (1)—(4) (or (1) and (4)) of Theorem 4.6 for some 3-manifold Σ. See also Remark 3.6.

5. Fibering Structures

In higher dimensions, fibering structures of fibered knots are unique up to isotopy [21], [36]. In this section we consider the problem whether or not fibering structures for fibered 3-knots are unique (see [21, p.53]). First we show that fibering structures are unique up to topological isotopy.

Theorem 5.1. *Let K be a fibered* 3*-knot and $\phi_j : S^5 - K \to S^1$ $(j = 0, 1)$ two smooth fibrations of its complement as in Definition* 2.1*. Then there exists a homeomorphism $\Phi : S^5 \to S^5$ with $\Phi(K) = K$ which is* *topologically isotopic to the identity and which makes the following diagram commutative:*

$$
\begin{array}{ccc}
S^5 - K & \xrightarrow{\Phi} & S^5 - K \\
\phi_0 \searrow & & \swarrow \phi_1 \\
& S^1. &
\end{array}
\tag{5.1}
$$

Proof. By the proof of Lemma 2.5 together with Remark 2.7, we see that there exists an orientation preserving homeomorphism $\Phi : (S^5, K) \to (S^5, K)$ which makes the diagram (5.1) commutative. Then the result follows from the following lemma.

Lemma 5.2. *Every orientation preserving homeomorphism $h : S^m \to S^m$ is topologically isotopic to the identity for all m.*

Proof. We prove the lemma by the induction on m. For $m = 0$, the result is obvious. Assume that $m \geq 1$ and that the conclusion of the lemma is true for $m - 1$. Let D_+ and D_- be the hemispheres of S^m separated by the equatorial $(m-1)$-dimensional sphere S_0^{m-1}. Note that $D_\pm$ are homeomorphic to the m-dimensional disk. By a topological isotopy, we may assume that $h(D_+) \subset \operatorname{Int} D_+$. Then by the solution of the annulus conjecture (see [43], [70]), we see that $D_+ - h(\operatorname{Int} D_+)$ is homeomorphic to $S^{m-1} \times [0,1]$. Hence, by the induction hypothesis, we may assume that $h(S_0^{m-1}) = h(S_0^{m-1}), h(D_+) = D_+$ and $h(D_-) = D_-$. Then for $h|_{D_\pm}$ use the Alexander trick to obtain a topological isotopy to the identity map. This completes the proof of Lemma 5.2 and hence that of Theorem 5.1 as well. □

Remark 5.3. In [21], it has been shown that the fibering structures are unique up to smooth isotopy for a fibered $(2n-1)$-knot K with $n \geq 3$. More precisely, if ϕ_j $(j = 0, 1)$ are two fibrations, then there exists a diffeomorphism $\Phi : S^{2n+1} \to S^{2n+1}$ with $\Phi(K) = K$ smoothly isotopic to the identity such that $\phi_0 = \phi_1 \circ \Phi$. This is also true for $n = 1$.

If we work in the differentiable category, fibering structures of fibered 3-knots are not unique. To see this, we use the following proposition, which is essentially due to Donaldson.

PROPOSITION 5.4. *Let f be a polynomial in $\mathbf{C}^3$ with $f(0) = 0$ and with an isolated critical point at the origin. Let F_f be a fiber of the Milnor fibration associated with f (see §11). If $\partial F_f = K_f$ is a homology 3-sphere, then F_f is not diffeomorphic to a connected sum $X_1 \sharp X_2$, where X_j are smooth 4-manifolds with $b_2^+ > 0$ (b_2^+ is the rank of the positive part of the intersection form on the second homology group).*

PROOF. By [59], F_f is diffeomorphic to the manifold given by $D_\varepsilon^6 \cap f^{-1}(\delta)$ ($0 < \delta << \varepsilon << 1$). Thus F_f can be smoothly embedded (as a smooth manifold) in a nonsingular algebraic surface in $\mathbf{C}P^3$ with sufficiently high degree. Then the result follows directly from [20, Theorem A]. This completes the proof. □

PROPOSITION 5.5. *Set $f(x, y, z) = x^2 + y^3 + z^{12k-1}$ ($k \geq 4$). Then the algebraic knot K_f associated with f has at least two smooth fibrations of its complement with nondiffeomorphic fibers.*

PROOF. Let F_f be a fiber of the Milnor fibration associated with f. Then by [10], ∂F_f is a homology 3-sphere and the intersection form of F_f is isomorphic to $k(-2E_8 \oplus 3U) \oplus (k-2)U$, where E_8 is the unique positive definite symmetric unimodular bilinear form of even type and of rank 8 and U is the bilinear form represented by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5.2}$$

(See, for example, [60].) Furthermore by [57], the 3-manifold K_f bounds a smooth compact 1-connected 4-manifold N with intersection form U. Set $F = kV_4 \sharp (k-3)(S^2 \times S^2) \sharp N$, where V_4 is diffeomorphic to a K3 surface (for example, a nonsingular complex surface of degree 4 in $\mathbf{C}P^3$). Note that the intersection forms of F_f and F are isomorphic and $\partial F_f \cong \partial F$. Let L be a Seifert matrix of the algebraic knot K_f. Then by [74], there exists a fibered 3-knot K with fiber F and Seifert matrix L. (This is obtained by the open book construction.) By Theorem 2.12 (or by [74]), K_f is isotopic to K. However, the fibers of K_f and K are not diffeomorphic to each other by Proposition 5.4. This completes the proof. □

REMARK 5.6. Other examples as in Proposition 5.5 can be found in [74]. We do not know if there exists a fibered 3-knot which admits an infinitely many smoothly distinct fibering structures. It is probable that we can construct such an example by using the method of the next section together with a result of [27] about exotic K3 surfaces.

Let K_S be a stabilizer defined in [75]. Furthermore let $\phi_S : S^5 - K_S \to S^1$ be a fixed fibration of its complement whose fiber F_S is diffeomorphic to $((S^2 \times S^2)\sharp(S^2 \times S^2))^\circ$. If we work in the differentiable category, all we can say is that fibering structures of fibered 3-knots are stably unique as follows.

PROPOSITION 5.7. *Let K be a fibered 3-knot and $\phi_j : S^5 - K \to S^1$ $(j = 0, 1)$ two smooth fibrations of its complement as in Definition 2.1. Then there exist a nonnegative integer k and a diffeomorphism $\Phi : S^5 \to S^5$ with $\Phi(K\sharp kK_S) = \Phi(K\sharp kK_S)$ smoothly isotopic to the identity such that the two fibrations $\phi_j\sharp k\phi_S : S^5 - (K\sharp kK_S) \to S^1$ satisfy $(\phi_1\sharp k\phi_S) \circ \Phi = \phi_0\sharp k\phi_S$, where $\phi_j\sharp k\phi_S$ is the fibration of the complement of the fibered 3-knot $K\sharp kK_S$ associated with ϕ_j and ϕ_S.*

PROOF. Let F_j be a fiber of $\phi_j|_{S^5-\mathrm{Int}\,N(K)}$, where $N(K)$ is a tubular neighborhood of K in S^5. Furthermore let $E = E(K) = S^5 - \mathrm{Int}\,N(K)$ be the exterior of K in S^5 and $\pi : \tilde{E} \to E$ the universal cover of E. Then by the same argument as in the proof of Lemma 2.5, we see that there exists a smooth h-cobordism W between F_0 and F_1 embedded in $\tilde{E}$. Then by [50], [71], $W\sharp_c 2k'(S^2 \times S^2 \times I)$ is diffeomorphic to $(F_0\sharp 2k'(S^2 \times S^2)) \times I$ for a sufficiently large nonnegative integer k'. In particular, the fibers F'_j of $\phi_j\sharp k'\phi_S$ are diffeomorphic to each other and we also see that their geometric monodromies are smoothly pseudoisotopic relative to boundary by using Wall's construction [89, p.140]. Hence, by the same argument as in the proof of Lemma 2.4, we see that there exists an orientation preserving diffeomorphism $\Psi : S^5 \to S^5$ with $\Psi(F'_0) = F'_1$ such that $\Psi|_{F'_0}$ preserves the Seifert forms. By Lemma 2.3 we may also assume that F'_j have handlebody decompositions without 3-handles. Then by using Levine's argument [54, §§18–20], we see that F'_0 and F'_1 are smoothly isotopic to each other in S^5. Then by an argument similar to that in [21, Proof of 3.2], we see that the geometric monodromies of $\phi_j\sharp k'\phi_S$ are smoothly pseudoisotopic to each other. By [69], [72], this pseudoisotopy is smoothly isotopic to an isotopy after a connected sum with k'' additional copies of K_S to $K\sharp k'K_S$ for some

k''. Then using the same argument as in [21], we see that there exists a desired diffeomorphism $\Phi : S^5 \to S^5$ for $k = k' + k''$. This completes the proof. $\square$

6. Fibered 3-Knots with Fiber a Punctured K3 Surface

In §10 we will show that for any smooth closed 1-connected spin 4-manifold N not homotopy equivalent to S^4, there exists a fibered 3-knot whose fiber is diffeomorphic to $(N\sharp(S^2 \times S^2))^\circ$. Let V_4 be a K3 surface; i.e., V_4 is a 1-connected compact complex surface with trivial canonical bundle. Then it is known that V_4 is not diffeomorphic to a connected sum $N\sharp(S^2 \times S^2)$ [19]. Thus it arises the question whether or not a punctured K3 surface V_4° can be a fiber of a fibered 3-knot. In this section we answer to this question affirmatively, using a result of Matumoto [58].

Suppose $A \in GL(3, \mathbf{Z})$ satisfies $\det(I - A) = \pm 1$, where $I \in GL(3, \mathbf{Z})$ is the unit matrix. Set

$$B = \begin{pmatrix} I - A & 0 \\ 0 & I - {}^tA^{-1} \end{pmatrix}, \tag{6.1}$$

then it is easy to see that B is unimodular. Set

$$C = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} B^{-1}. \tag{6.2}$$

Then C is unimodular and we have

$$C + {}^tC = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{6.3}$$

and

$$-C^{-1}\,{}^tC = \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}. \tag{6.4}$$

Then $D = -C^{-1}\,{}^tC$ is an isometry of

$$H = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{6.5}$$

i.e., ${}^tDHD = H$. Thus D preserves the positive part of H (here we identify H with the symmetric bilinear form $\mathbf{R}^6 \times \mathbf{R}^6 \to \mathbf{R}$ with matrix representative H). Then we can show that a matrix representative of D restricted to the positive part of H is equal to $A + {}^tA^{-1} = A(I + A^{-1}{}^tA^{-1})$. Since the symmetric matrix $I + A^{-1}{}^tA^{-1}$ is positive definite, we see that D preserves the orientation of the positive part of H if and only if $\det A = 1$. Assume $\det A = 1$ in the following.

Let L_1 be a unimodular 8×8 matrix satisfying $L_1 + {}^tL_1 = -E_8$; for example, a Seifert matrix for the algebraic knot associated with the polynomial $x^2 + y^3 + z^5$. Set $L = L_1 \oplus L_1 \oplus C$. Then $L + {}^tL$ is congruent to $-2E_8 \oplus 3U$ and the isometry $-L^{-1}{}^tL$ of $L + {}^tL$ preserves the orientation of the positive part of $L + {}^tL$. By [58], $-L^{-1}{}^tL$ can be realized by a self-diffeomorphism of a punctured K3 surface V_4° which is the identity on the boundary. Then using the open book construction [37], [74], we get a fibered 3-knot whose fiber is a punctured K3 surface. Thus we obtain the following.

Proposition 6.1. *There exist infinitely many fibered 3-knots with fiber diffeomorphic to a punctured $K3$ surface.*

Proof. For integers α_1 and α_2 with $\alpha_1 + \alpha_2 = \pm 1$, set

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -\alpha_1 \\ 0 & 1 & -\alpha_2 \end{pmatrix}. \tag{6.6}$$

Note that there exist infinitely many such pairs (α_1, α_2). By Lemma 3.5, we see that $\det A = 1$ and $\det(I - A) = \alpha_1 + \alpha_2 = \pm 1$. Using these matrices in the above construction, we obtain infinitely many fibered 3-knots with fiber diffeomorphic to a punctured K3 surface. We can show that all these knots are not isotopic to each other, since their Alexander polynomials (or the characteristic polynomials of the algebraic monodromy) are distinct. This completes the proof. □

If we use a matrix with $\det A = -1$ in the above construction, we get the following.

Proposition 6.2. *There exist infinitely many smooth embeddings $\xi_i : V_4^\circ \to S^5$ $(i = 1, 2, 3, \ldots)$ with the following properties.*

(1) *$\xi_i(\partial V_4^\circ)$ is an almost fibered 3-knot with $\xi_i(V_4^\circ)$ a fiber.*

(2) *There exists no smooth fibering of the complement of $\xi_i(\partial V_4^\circ)$ in S^5 with $\xi_i(V_4^\circ)$ a fiber.*

Proof. For integers α_1 and α_2 with $\alpha_1+\alpha_2+2=\pm 1$, set

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -\alpha_1 \\ 0 & 1 & -\alpha_2 \end{pmatrix}. \tag{6.7}$$

Note that there exist infinitely many such pairs (α_1, α_2). We see by Lemma 3.5 that $\det A = -1$ and $\det(I - A) = \pm 1$. Then we can find an embedding $\xi : V_4^\circ \to S^5$ whose Seifert matrix is $L = L_1 \oplus L_1 \oplus C$, using Kervaire's method [41]. (Note that V_4° has a special handlebody decomposition [29].) Using the unimodularness of L and Freedman's theorem [23], we see that the exterior of $\xi(\partial V_4^\circ)$ in S^5 fibers over S^1 topologically with $\xi(V_4^\circ)$ a fiber. Since the monodromy does not preserve the orientation of the positive part of $H_2(V_4^\circ; \mathbf{R})$, it cannot be realized by any diffeomorphism [20]. Thus the complement of $\xi(\partial V_4^\circ)$ in S^5 never fibers over S^1 smoothly with $\xi(V_4^\circ)$ a fiber.

We see that the almost fibered knots constructed above are all distinct by the same argument as in the proof of Proposition 6.1. This completes the proof. □

Remark 6.3. We do not know whether $\xi(\partial V_4^\circ)$ in Proposition 6.2 is a smooth fibered 3-knot or not. If there exists a smooth homotopy K3 surface with a self-diffeomorphism realizing the above algebraic monodromy, then $\xi(\partial V_4^\circ)$ does fiber smoothly, and vice versa. See [27].

Remark 6.4. We do not know if there exists an almost fibered 3-knot which is not smoothly fibered. Note that there does exist an *algebraically fibered* 3-knot which is not smoothly fibered [40]. In fact, Kearton's example is not an almost fibered 3-knot. Note also that an almost fibered 3-knot is always algebraically fibered and that every algebraically fibered 3-knot is topologically fibered [32].

7. Decomposition of Fibered 3-Knots

We say that a knot is *decomposable* if it is the connected sum of two nontrivial knots. Here a *trivial knot* refers to a standardly embedded sphere embedded in a sphere in codimension two. In [74] we have given an example of a decomposable fibered 3-knot such that one of its factor knots is not smoothly fibered. In this section we give an example of a decomposable fibered 3-knot such that neither of the factor knots is smoothly fibered.

Example 7.1. Let Σ be the Brieskorn homology 3-sphere $\Sigma(2,7,13)$ (for example, see [57]). Then Σ bounds a smooth compact 1-connected 4-manifold M with intersection form isomorphic to $-E_{16}$ [57]. Set $F = M\sharp 2(S^2 \times S^2)$ and let L be a Seifert matrix of the algebraic 3-knot associated with the polynomial $x^2 + y^3 + z^{11}$. Then by [10], we see that $L + {}^tL$ is congruent to $-2E_8 \oplus 2U \cong -E_{16} \oplus 2U$. Thus there exists a fibered Σ-knot K with fiber F and Seifert matrix L [74]. Since L is nontrivial and the Rohlin invariant of Σ is zero, the knot K is decomposable by [75].

Suppose $K = K_1 \sharp K_2$, where both K_j are nontrivial. We show that neither of K_j is smoothly fibered. It is easy to see that $\pi_1(S^5 - K_j)$ are isomorphic to $\mathbf{Z}$, since so is $\pi_1(S^5 - K)$. Since Σ is an irreducible 3-manifold, we may assume that K_1 is an S^3-knot and K_2 is a Σ-knot. Since $\Delta_L(t) = \det(tL + {}^tL) = \phi_{66}(t)$ (the 66-th cyclotomic polynomial) is irreducible, L cannot be decomposed into a direct sum. Thus we see that a Seifert matrix of K_1 must coincide with L and a Seifert matrix of K_2 with zero.

If K_1 is smoothly fibered, then its fiber F_1 is a smooth compact 1-connected 4-manifold with $\partial F_1 \cong S^3$ and intersection form isomorphic to $-2E_8 \oplus 2U$. This contradicts a result of Donaldson [19]. Thus K_1 is not smoothly fibered.

If K_2 is smoothly fibered, then its fiber F_2 is a smooth contractible 4-manifold with $\partial F_2 \cong \Sigma$. Then $M \cup_\Sigma (-F_2)$ is a smooth closed 1-connected 4-manifold with intersection form isomorphic to $-E_{16}$. This contradicts Donaldson's theorem [18]. Thus K_2 is not smoothly fibered.

Thus the fibered 3-knot K is the connected sum of two nontrivial knots and neither of the factor knots are smoothly fibered.

Remark 7.2. Using Freedman's results [23], we see that both of K_j do fiber topologically (see also [32]).

Next we consider the decomposition of fibered 3-knots constructed by the cyclic suspension ([63]) of iterated torus knots (for example, see [91]). By using the results in [75], it is easy to prove the following.

Proposition 7.3. *Let k be an iterated torus knot which is not a torus* *knot, and let K be the r-fold cyclic suspension of k. Then if K is a homology 3-sphere, then the knot K is decomposable.*

In the above proposition, the condition that K should be a homology 3-sphere is essential. If K is not a homology 3-sphere, the conclusion of the above proposition does not hold in general as follows.

PROPOSITION 7.4. *Let K be the 2-fold cyclic suspension of the $\{(3,2),(7,3)\}$-iterated torus knot. Then $H_1(K) \cong \mathbf{Z}/3\mathbf{Z}$ and the knot K is not decomposable as a knot.*

PROOF. The Alexander polynomial $\Delta_K(t)$ of K is easily computed and we get $\Delta_K(t) = \phi_{42}(t)\phi_9(t)$, where $\phi_m(t)$ is the m-th cyclotomic polynomial. Since $\Delta_K(1) = 3$, we see that $H_1(K) \cong \mathbf{Z}/3\mathbf{Z}$.

Suppose $K = K_1 \sharp K_2$, where both of K_j are nontrivial. It is easy to see that $\pi_1(S^5 - K_j)$ are isomorphic to $\mathbf{Z}$, since so is $\pi_1(S^5 - K)$. Since the 3-manifold K is the 2-fold branched covering space of S^3 branched along a prime knot, it is irreducible ([47], [45, p.493]). Thus we may assume that K_1 is homeomorphic to S^3 and K_2 is homeomorphic to K as abstract 3-manifolds. Let L_j be a Seifert matrix of K_j and L a Seifert matrix of K. Then $L_1 \oplus L_2$ is S-equivalent to L (for details, see [54]). Let $\Delta_j(t)$ be the Alexander polynomial of K_j. Since L_j is not S-equivalent to the zero matrix, $\Delta_j(t) = \det(tL_j + {}^tL_j)$ is nontrivial. Since $\Delta_K(t) = \Delta_1(t)\Delta_2(t)$ and $\Delta_1(1) = 1$, we have $\Delta_1(t) = \phi_{42}(t)$ and $\Delta_2(t) = \phi_9(t)$.

On the other hand, L is of the form $A \oplus B$, where A is a Seifert matrix of the algebraic knot associated with the polynomial $x^2 + y^3 + z^7$,

$$B = \begin{pmatrix} B_1 & {}^tB_1 & {}^tB_1 \\ B_1 & B_1 & {}^tB_1 \\ B_1 & B_1 & B_1 \end{pmatrix}, \tag{7.1}$$

and B_1 is a Seifert matrix of the $(2,3)$-torus knot (see [81]). We have $\mathrm{sign}(A + {}^tA) = -8$ by [10] and $\mathrm{sign}(B + {}^tB) = -2$ by a direct computation, where "sign" denotes the signature of a symmetric matrix. Thus $\mathrm{sign}(L + {}^tL) = -10$.

Since $\Delta_K(-1) = 1$, we have $\det(L_j - {}^tL_j) = \pm 1$. Then using Trotter's argument [85], we see that L_j are S-equivalent to nonsingular matrices L'_j. Since the degrees of $\Delta_j(t) = \det(tL'_j + {}^tL'_j)$ for $j = 1, 2$ are 12 and 6 respectively, we have $|\mathrm{sign}(L_1 + {}^tL_1)| = |\mathrm{sign}(L'_1 + {}^tL'_1)| \leq 12$ and $|\mathrm{sign}(L_2 + {}^tL_2)| = |\mathrm{sign}(L'_2 + {}^tL'_2)| \leq 6$. Since the Rohlin invariant of S^3 is equal to zero, we have $\mathrm{sign}(L_1 + {}^tL_1) \equiv 0 \bmod 16$, and hence $\mathrm{sign}(L_1 + {}^tL_1) = 0$. Since $\mathrm{sign}(L + {}^tL) = \mathrm{sign}(L_1 + {}^tL_1) + \mathrm{sign}(L_2 + {}^tL_2)$, we have $|\mathrm{sign}(L + {}^tL)| \leq 6$. This is a contradiction. Thus K cannot be decomposed into a nontrivial connected sum. This completes the proof. □

Remark 7.5. The $\{(3,2),(7,3)\}$-iterated torus knot is not algebraic (for a precise definition of an algebraic knot, see §11). Thus we do not know whether or not there exists a counter-example to Corollary 3.8 of [75] when the 3-knot is not a homology 3-sphere. Furthermore, we do not know if the knot in the above proposition is *topologically* decomposable or not.

Using Lemma 3.2 of §3, we obtain the following interesting result about the cancellation of fibered 3-knots.

Proposition 7.6. *Let K and K' be fibered 3-knots in S^5 such that the intersection forms of the fibers are the zero forms. If there exists a fibered S^3-knot K_0 such that $K \sharp K_0$ is isotopic to $K' \sharp K_0$, then K is isotopic to K'.*

Proof. We denote by F, F' and F_0 the fibers of K, K' and K_0 respectively. By Theorem 2.2, we see that there exist a homeomorphism $\psi : K \to K'$ and an isomorphism $\Lambda : H_2(F \natural F_0) \to H_2(F' \natural F_0)$ which satisfy certain conditions. By Lemma 3.2, this isomorphism decomposes into two isomorphisms $\Lambda_1 : H_2(F) \to H_2(F')$ and $\Lambda_2 : H_2(F_0) \to H_2(F_0)$. Now it is easy to see that the homeomorphism ψ and the isomorphism Λ_1 satisfy the required conditions of Theorem 2.2 (4) for K and K'. Hence K and K' are isotopic to each other. This completes the proof. □

We say that a fibered 3-knot K is *definite* if the intersection form of its fiber is (positive or negative) definite. For the cancellation problem for such fibered 3-knots, we have the following.

Proposition 7.7. *Let K and K' be definite fibered 3-knots in S^5 such that K and K' are homology 3-spheres as abstract 3-manifolds. If there exists a definite fibered S^3-knot K_0 such that $K \sharp K_0$ is isotopic to $K' \sharp K_0$, then K is isotopic to K'.*

The above proposition can be proved by using an argument similar to that in the proof of Proposition 7.6 together with the result in [5, §2].

About the cancellation problem for simple spherical $(2n-1)$-knots in S^{2n+1} $(n \geq 2)$, Bayer-Fluckiger has obtained numerous results. In some cases, the cancellation works, while in other cases, it does not. For details, see [1, Proposition 6.6], [2], [3, Corollary 1], and [4].

REMARK 7.8. About the factorization of simple (not necessarily fibered) S^3-knots, the following is already known.

(1) Every simple S^3-knot decomposes into finitely may irreducible knots [5].

(2) Every simple S^3-knot has only finitely many factors. In particular, such a 3-knot can factorize in only finitely many ways [6].

(3) Irreducible factorization of a simple S^3-knot is not unique in general [38], [39], [5].

8. Plumbing Fibered 3-Knots

It is known that for fibered $(2n-1)$-knots with $n \neq 2$, the plumbing operation is a powerful way to construct a new fibered knot from given two fibered knots (see [30], [83], [55], [56]). In this section we show that the plumbing operation does work also for fibered 3-knots.

First we recall the plumbing operation. Let K_i $(i = 1, 2)$ be two fibered 3-knots with fiber F_i. Divide S^5 into two hemispheres D_i^5 $(i = 1, 2)$, so that S^5 is the union of the two 5-dimensional disks D_i^5 attached along their common boundary $S(\cong S^4)$. We suppose that $F_i \subset D_i^5$ and that there exists an embedding $\psi : D^2 \times D^2 \to S$ with the following properties.

(1) $F_1 \cap S = F_2 \cap S = F_1 \cap F_2 = \psi(D^2 \times D^2)$.

(2) $\psi(\partial D^2 \times D^2) = \partial F_1 \cap \psi(D^2 \times D^2)$ and $\psi(D^2 \times \partial D^2) = \partial F_2 \cap \psi(D^2 \times D^2)$.

(3) The orientations of F_1 and F_2 coincide on $\psi(D^2 \times D^2)$.

Define $F_1 \Box F_2$ to be the union of F_1 and F_2 with the corner smoothed and define $K = K_1 \Box K_2 = \partial(F_1 \Box F_2)$. We say that K is obtained from K_1 and K_2 by a *plumbing* construction (see [55, §2]).

By the same argument as in [55, Proposition 2.1], we see easily that K is algebraically fibered; i.e., K has a Seifert matrix S-equivalent to a unimodular matrix (see [75]).

Our main result of this section is the following.

THEOREM 8.1. *If K_i $(i = 1, 2)$ are fibered 3-knots with fiber F_i, then the knot $K = K_1 \Box K_2$ constructed from K_1 and K_2 by a plumbing construction is also fibered with fiber $F_1 \Box F_2$.*

PROOF. Set $F = F_1 \Box F_2$ and let $h_i : F_i \to F_i$ be the geometric monodromy of K_i. Note that h_i is the identity on the boundary ∂F_i. We define the diffeomorphisms $\tilde{h}_i : F \to F$ $(i = 1, 2)$ by

$$\tilde{h}_i(x) = \begin{cases} h_i(x), & \text{if } x \in F_i, \\ x, & \text{if } x \notin F_i. \end{cases} \tag{8.1}$$

It is easy to see that $\tilde{h}_i$ are diffeomorphisms and that they are the identity map on ∂F. Set $h = \tilde{h}_2 \circ \tilde{h}_1 : F \to F$, which is a diffeomorphism with $h|_{\partial F}$ the identity.

We need the following lemma.

LEMMA 8.2. *Set*

$$F_1' = F_1 - \psi(D^2 \times \operatorname{Int} D^2), \tag{8.2}$$
$$F_2' = F_2 - \psi(\operatorname{Int} D^2 \times D^2), \tag{8.3}$$
$$Y_1 = \partial F_1 - \psi(\partial D^2 \times \operatorname{Int} D^2), \tag{8.4}$$
$$Y_2 = \partial F_2 - \psi(\operatorname{Int} D^2 \times \partial D^2). \tag{8.5}$$

Then $(j_i)_* : H_2(F_i', Y_i) \to H_2(F_i, \partial F_i)$ is an isomorphism, where $j_i : (F_i', Y_i) \to (F_i, \partial F_i)$ is the inclusion map $(i = 1, 2)$.

PROOF. By Lefschetz duality we have an isomorphism $H_2(F_1', Y_1) \cong H^2(F_1', \psi(D^2 \times \partial D^2))$. Furthermore, the latter group is isomorphic to $H^2(F_1, \psi(D^2 \times D^2))$ by excision. Then since $\psi(D^2 \times D^2)$ is contractible, we have $H^2(F_1, \psi(D^2 \times D^2)) \cong \tilde{H}^2(F_1) \cong H_2(F_1, \partial F_1)$. It is not difficult to show that the composition of the above isomorphisms $H_2(F_1', Y_1) \to H_2(F_1, \partial F_1)$ coincides with $(j_1)_*$. Similarly we can show that $(j_2)_*$ is an isomorphism. This completes the proof. □

Now let $\{x_1,\dots,x_k\}$ and $\{y_1,\dots,y_l\}$ be fixed bases of $H_2(F_1)$ and $H_2(F_2)$ respectively. Let $\{x_1^*,\dots,x_k^*\}$ and $\{y_1^*,\dots,y_l^*\}$ be their dual bases of $H_2(F_1,\partial F_1)\cong H^2(F_1)$ and $H_2(F_2,\partial F_2)\cong H^2(F_2)$ respectively. Note that $x_i\cdot x_j^*=\delta_{ij}$ and $y_i\cdot y_j^*=\delta_{ij}$. By the above lemma, we may assume that the supports of the representatives of $x_1^*,\dots,x_k^*,y_1^*,\dots,y_l^*$ do not intersect with $\psi(D^2\times D^2)$. Furthermore, it is easy to see that $\{x_1,\dots,x_k,y_1,\dots,y_l\}$ and $\{x_1^*,\dots,x_k^*,y_1^*,\dots,y_l^*\}$ constitute bases of $H_2(F)$ and $H_2(F,\partial F)$ respectively. Set

$$\Delta_1(x_j^*)=\sum_{i=1}^{k}D_{ij}^1x_i \quad\text{and}\quad \Delta_2(y_j^*)=\sum_{i=1}^{l}D_{ij}^2y_i, \tag{8.6}$$

where Δ_i is the variation map of h_i. Furthermore, set Q_{ij} to be the intersection number $x_i\cdot y_j$ in $F=F_1\Box F_2$ and set $L_1(x_i,x_j)=L_{ij}^1$ and $L_2(y_i,y_j)=L_{ij}^2$, where L_i are the Seifert forms of K_i. Here we assume that the map $i:F\to S^5-F$ used in the definition of the Seifert form satisfies that $i(\psi(D^2\times D^2))\subset D_1^5$. Then the Seifert matrix L of F with respect to the basis $\{x_1,\dots,x_k,y_1,\dots,y_l\}$ is equal to

$$\begin{pmatrix} L^1 & 0\\ {}^tQ & L^2\end{pmatrix}, \tag{8.7}$$

where $L^s=(L_{ij}^s)$ $(s=1,2)$ and $Q=(Q_{ij})$. Thus L is unimodular, which implies that K is an almost fibered knot (see Remark 2.8 (1)).

Now let us calculate the variation map Δ_h of h. We have

$$\begin{aligned}
\Delta_h(x_i^*) &= x_i^*-(\tilde h_2\circ\tilde h_1)_*(x_i^*) && (8.8)\\
&= x_i^*-(\tilde h_2)_*(x_i^*)+(\tilde h_2)_*(x_i^*-(\tilde h_1)_*(x_i^*)) && (8.9)\\
&= (\tilde h_2)_*(x_i^*-(\tilde h_1)_*(x_i^*)) && (8.10)\\
&= (\tilde h_2)_*(\Delta_1(x_i^*)) && (8.11)\\
&= \Delta_1(x_i^*)-\Delta_2\left(\sum_{j=1}^{l}(\Delta_1(x_i^*)\cdot y_j)y_j^*\right) && (8.12)\\
&= \sum_{j=1}^{k}D_{ji}^1x_j-\sum_{j=1}^{l}\sum_{m=1}^{k}D_{mi}^1(x_m\cdot y_j)\sum_{n=1}^{l}D_{nj}^2y_n && (8.13)\\
&= \sum_{j=1}^{k}D_{ji}^1x_j-\sum_{n=1}^{l}\left(\sum_{j=1}^{l}\sum_{m=1}^{k}D_{mi}^1Q_{mj}D_{nj}^2\right)y_n, && (8.14)
\end{aligned}$$

where (8.10) is valid since the support of a representative of x_i^* does not intersect with F_2. Furthermore we have

$$\begin{aligned}
&(8.15) & \Delta_h(y_j^*) &= y_j^* - (\tilde{h}_2)_* \circ (\tilde{h}_1)_*(y_j^*) \\
&(8.16) & &= y_j^* - (\tilde{h}_2)_*(y_j^*) \\
&(8.17) & &= \Delta_2(y_j^*) \\
&(8.18) & &= \sum_{n=1}^{l} D_{nj}^2 y_n.
\end{aligned}$$

Thus the matrix representative of Δ_h is equal to

$$(8.19) \qquad \begin{pmatrix} D^1 & 0 \\ -D^2\,{}^t\!QD^1 & D^2 \end{pmatrix},$$

where $D^s = (D_{ij}^s)$ $(s = 1, 2)$. Since K_1 and K_2 are fibered knots, D^1 and D^2 are unimodular. Thus Δ_h is an isomorphism. Then using the open book construction [37], we can construct a fibered knot K' with fiber F and geometric monodromy h. Furthermore, the Seifert matrix L' of K' satisfies

$$\begin{aligned}
&(8.20) & L' &= D^{-1} \\
&(8.21) & &= \begin{pmatrix} (D^1)^{-1} & 0 \\ {}^t\!Q & (D^2)^{-1} \end{pmatrix} \\
&(8.22) & &= \begin{pmatrix} L^1 & 0 \\ {}^t\!Q & L^2 \end{pmatrix} \\
&(8.23) & &= L.
\end{aligned}$$

Thus the fibered 3-knot K' is isotopic to the almost fibered knot K by Theorem 2.2 and Remark 2.8 (1), and hence K is a fibered 3-knot. To show that the fiber of K is isotopic to $F_1 \Box F_2$, we need the following.

LEMMA 8.3. *Let $\xi_i : F \to S^5$ $(i = 1, 2)$ be two smooth embeddings, where F is a compact 1-connected 4-manifold with connected boundary. Suppose that the Seifert forms of $\xi_i(F)$ coincide with each other. Then ξ_1 and ξ_2 are isotopic as embedding maps.*

PROOF. By Lemma 2.3, $F \natural kN$ has a handlebody decomposition without 3-handles for some nonnegative integer k, where $N = (S^2 \times S^2)^\circ$. Then we can extend ξ_i to the embedding $\tilde{\xi}_i : F\natural kN \to S^5$ by using the trivial embedding $N \to S^5$. Note that the Seifert forms of $\tilde{\xi}_i$ coincide. Then by using the same argument as in [54, §§18–20], we see that $\tilde{\xi}_i$ are isotopic to each other. Then their restrictions ξ_i to F are also isotopic. This completes the proof. □

By the above lemma, we see that a fiber of K' is isotopic to $F_1 \square F_2$ embedded in S^5. Thus K is a fibered 3-knot with fiber $F_1 \square F_2$. This completes the proof of Theorem 8.1. □

Remark 8.4. By [65, 2.4. Theorem], the fibered 3-knot $K_1 \square K_2$ *unfolds* into K_1 and K_2.

Remark 8.5. Suppose that F_1 and F_2 are compact 1-connected 4-manifolds with connected boundary embedded in S^5 such that they are separated by a standardly embedded 4-sphere. Then we can construct $F = F_1 \square F_2$ by the plumbing construction. Furthermore, suppose that ∂F is a fibered 3-knot with fiber F. Then is it true that ∂F_i are fibered 3-knots with fiber F_i? The answer is "no" in general as follows. Let F_1 be the punctured K3 surface embedded in S^5 as in Proposition 6.2. Furthermore let F_2 be the fiber of a stabilizer as defined in [75, §4]. Note that F_2 is diffeomorphic to $(S^2 \times S^2 \sharp S^2 \times S^2)^\circ$. Then the connected sum of ∂F_1 and ∂F_2 is a fibered 3-knot with fiber $F_1 \natural F_2$ by [74] and Lemma 8.3. Note that the connected sum operation is realized as a special case of a plumbing operation. Thus the plumbing of F_1 and F_2 creates a fiber of a fibered knot, although F_1 is not a fiber of a fibered knot. This example shows that the de-plumbing operation does not work well for fibered 3-knots. Note that the corresponding de-plumbing operation works for higher dimensions (see [55, Proposition 2.1]).

Let K_H be a fibered 3-knot whose Seifert matrix is equal to (± 1). The fiber of such a fibered knot is called a *Hopf band*. For example, consider the algebraic 3-knot associated with the polynomial $x^2 + y^2 + z^2 \in \mathbf{C}[x, y, z]$ and let F be the fiber. Then it is known that F is a Hopf band. In this case, ∂F is diffeomorphic to the lens space $L(2, 1)(\cong \mathbf{R}P^3)$.

Definition 8.6. We say that a fibered 3-knot K is *obtained by plumbing* if there exists a sequence of compact 1-connected 4-manifolds with connected boundary $F_0, F_1, \dots, F_s$ embedded in S^5 such that $F_0 = D^4, \partial F_s = K$ and F_{i+1} is obtained by plumbing together F_i and a Hopf band. For the classical case (i.e., for 1-knots) see [83], [30]. For higher dimensions, see [55].

Then we can show the following by the same argument as in [55, Proposition 2.4].

PROPOSITION 8.7. *If a fibered* 3*-knot* K *is obtained by plumbing, then it has a unimodular triangular Seifert matrix.*

We do not know if the converse is also true for fibered 3-knots.

By the same argument as in [55, §3], we obtain the following.

PROPOSITION 8.8. *Let* K *be a fibered* 3*-knot obtained by plumbing such that* $H_1(K) = 0$*, that the fiber* F *has a positive definite intersection form* Q*. Then* Q *is an orthogonal direct sum of copies of the form* E_8*.*

We end this section by posing some problems.

Problem 8.9. (1) Does there exist a fibered S^3-knot which cannot be obtained by plumbing?

(2) Let K be a fibered 3-knot. Then does there exist a sequence F_0, $F_1, \dots, F_s$ of compact 1-connected 4-manifolds with connected boundary embedded in S^5 such that $F_0 = D^4, \partial F_s = K$ and either F_{i+1} is obtained by plumbing together F_i and a Hopf band or F_i is obtained by plumbing together F_{i+1} and a Hopf band (see [56])?

9. Application to Isotopy of 1-Connected 4-Manifolds with Boundary

In [72], it has been shown that two (orientation preserving) homeomorphisms of a *closed* simply connected 4-manifold are homotopic if and only if the induced isomorphisms on the second homology group coincide (see also [17, §5]). Using this, Quinn [72] has shown that two homeomorphisms with the same isomorphism on the second homology group are actually topologically isotopic. In this section we apply our results in the previous sections to the case where the simply connected 4-manifold has nonempty connected boundary.

First we show the following.

Proposition 9.1. *Let F be a smooth compact 1-connected 4-manifold with connected boundary. Suppose that $h_i : F \to F$ $(i = 1, 2)$ are orientation preserving diffeomorphisms which are the identity on the boundary and whose associated variation maps $\Delta_{h_i} : H_2(F, \partial F) \to H_2(F)$ are isomorphisms. Then $\Delta_{h_1} = \Delta_{h_2}$ if and only if h_1 and h_2 are topologically isotopic relative to boundary.*

Proof. Since the variation maps are isomorphisms, we can construct fibered 3-knots K_i with fiber F and geometric monodromy h_i by the open book construction [37]. If $\Delta_{h_1} = \Delta_{h_2}$, then by Theorem 2.2, K_1 and K_2 are isotopic to each other. In fact, by Lemma 8.3, we may assume that they have a common fiber F embedded in S^5. Let $N(F)$ be a regular neighborhood of F in S^5 and set $V = S^5 - \operatorname{Int} N(F)$. Then we have diffeomorphisms $\psi_i : F \times I \to V$ which arise from the fibrations $S^5 - K_i \to S^1$. Then the composition $\psi_2^{-1} \circ \psi_1 : F \times I \to F \times I$ gives a smooth pseudoisotopy relative to boundary between the identity and $h_2^{-1} \circ h_1$. Thus h_1 and h_2 are smoothly pseudoisotopic relative to boundary. Finally by [72], [69], h_1 and h_2 are topologically isotopic relative to boundary.

Conversely, if h_1 and h_2 are topologically isotopic relative to boundary, then we see easily that $\Delta_{h_1} = \Delta_{h_2}$ by the definition of the variation map. This completes the proof. □

Proposition 9.2. *Let F be a smooth compact 1-connected 4-manifold with connected boundary such that there exists a unimodular matrix L with $L + {}^tL$ an intersection matrix of F. Suppose that $h_i : F \to F$ $(i = 1, 2)$ are homeomorphisms such that $h_1 \circ h_2^{-1}$ is topologically isotopic relative to boundary to a diffeomorphism. Then h_1 and h_2 are topologically isotopic relative to boundary if and only if $\Delta_{h_1} = \Delta_{h_2}$.*

Proof. By using Kervaire's method [41] we can construct an embedding $\xi : F \to S^5$ such that the Seifert matrix coincides with L. Then, since L is unimodular, $K = \xi(\partial F)$ is almost fibered; i.e., $V = S^5 - \operatorname{Int} N(\xi(F))$ is an h-cobordism relative to boundary between two copies $F_\pm$ of F, where

$N(\xi(F))$ is a regular neighborhood of $\xi(F)$ in S^5. In other words, if we identify the two ends $F_\pm$ of V by the identity map and then attach $\partial F \times D^2$ along the boundary, then we get S^5. Note that V is homeomorphic to $F \times I$ by [23].

Now suppose that $\Delta_{h_1} = \Delta_{h_2}$. By our hypothesis, we may assume that $h_1 \circ h_2^{-1}$ is a diffeomorphism. By using a homological argument, we see that if we use $h_1 \circ h_2^{-1}$ when identifying the two ends of V instead of the identity map, we get S^5 again. Let K' be the almost fibered 3-knot in S^5 thus constructed. Furthermore let h and $h' : F \to F$ be the geometric monodromies of K and K' respectively arising from the homeomorphism $V \to F \times I$. Note that h and h' are homeomorphisms. Then we see that $h' = (h_1 \circ h_2^{-1}) \circ h$.

LEMMA 9.3. *Let F be a smooth compact 1-connected 4-manifold with connected boundary and let $g_i : F \to F$ $(i = 0, 1, 2)$ be homeomorphisms which are the identity on the boundary. If $\Delta_{g_1} = \Delta_{g_2}$, then we have $\Delta_{g_0 \circ g_1} = \Delta_{g_0 \circ g_2}$ and $\Delta_{g_1 \circ g_0} = \Delta_{g_2 \circ g_0}$.*

PROOF. Let $(D, \partial D)$ be an arbitrary 2-cycle in $(F, \partial F)$. Then the 2-cycles $D - g_1(D)$ and $D - g_2(D)$ are homologous in F by our hypothesis. Hence $g_0(D) - g_0 \circ g_1(D)$ and $g_0(D) - g_0 \circ g_2(D)$ are also homologous in F. Thus $D - g_0 \circ g_1(D) = (D - g_0(D)) + (g_0(D) - g_0 \circ g_1(D))$ and $D - g_0 \circ g_2(D) = (D - g_0(D)) + (g_0(D) - g_0 \circ g_2(D))$ are homologous in F. This implies that $\Delta_{g_0 \circ g_1} = \Delta_{g_0 \circ g_2}$. The other equality can be proved similarly. This completes the proof. □

By the above lemma together with our assumption that $\Delta_{h_1} = \Delta_{h_2}$, we see that $\Delta_h = \Delta_{h'}$. This implies that the Seifert forms of K and K' coincide with each other and hence by Theorem 2.2, K and K' are isotopic to each other. In fact, by Lemma 8.3, they have a common fiber F embedded in S^5. Then by the same argument as in the proof of Proposition 9.1, we see that h and h' are topologically pseudoisotopic relative to boundary. Then by [72], [69], they are topologically isotopic relative to boundary. Since $h' = (h_1 \circ h_2^{-1}) \circ h$, we see that h_1 and h_2 are topologically isotopic relative to boundary. This completes the proof. □

PROPOSITION 9.4. *Let F be a smooth compact 1-connected spin 4-manifold with connected boundary and $h_i : F \to F$ $(i = 1, 2)$ two diffeo-*

morphisms which are the identity on the boundary. Then there exists a nonnegative integer k such that $h_i\sharp k(\mathrm{id}) : F\sharp k(S^2 \times S^2) \to F\sharp k(S^2 \times S^2)$ are smoothly isotopic relative to boundary if and only if $\Delta_{h_1} = \Delta_{h_2}$.

PROOF. First suppose that $\Delta_{h_1} = \Delta_{h_2}$. In [74], it has been shown that there exists a nonnegative integer k_1 and a fibered 3-knot K such that $F\sharp k_1(S^2 \times S^2)$ is a fiber of K. Let h be the geometric monodromy of K. Then by an argument similar to the proofs of Propositions 9.1 and 9.2, we see that h and $((h_1\sharp k_1(\mathrm{id})) \circ (h_2\sharp k_1(\mathrm{id}))^{-1}) \circ h$ are smoothly pseudoisotopic relative to boundary. This implies that $h_1\sharp k_1(\mathrm{id})$ and $h_2\sharp k_1(\mathrm{id})$ are smoothly pseudoisotopic relative to boundary. Then by [72] there exists a nonnegative integer k_2 such that $h_1\sharp(k_1 + k_2)\mathrm{id}$ and $h_2\sharp(k_1 + k_2)\mathrm{id}$ are smoothly isotopic relative to boundary.

Conversely suppose that there exists a nonnegative integer k such that $h_1\sharp k(\mathrm{id})$ and $h_2\sharp k(\mathrm{id})$ are smoothly isotopic relative to boundary. Then we have $\Delta_{h_1\sharp k(\mathrm{id})} = \Delta_{h_2\sharp k(\mathrm{id})}$. It is easy to see that $\Delta_{h_i\sharp k(\mathrm{id})} : H_2(F\natural k(S^2 \times S^2)^\circ, \partial(F\natural k(S^2 \times S^2)^\circ)) \to H_2(F\natural k(S^2 \times S^2)^\circ)$ decomposes into the direct sum of the two maps $\Delta_{h_i} : H_2(F, \partial F) \to H_2(F)$ and $\Delta_{k(\mathrm{id})} : H_2(\natural^k(S^2 \times S^2)^\circ, \partial(\natural^k(S^2 \times S^2)^\circ)) \to H_2(\natural^k(S^2 \times S^2)^\circ)$. Thus we have $\Delta_{h_1} = \Delta_{h_2}$. This completes the proof. $\square$

As we have used the techniques of fibered 3-knots in S^5 in the above propositions, we have needed some technical assumptions on F and on the homeomorphisms. It is probable that the following conjectures hold.

CONJECTURE 9.5. *Let F be a compact* 1*-connected topological* 4*-manifold* (*possibly with boundary*) *and $h_i : F \to F$ $(i = 1, 2)$ two homeomorphisms which are the identity on the boundary. Then h_1 and h_2 are topologically isotopic relative to boundary if and only if $\Delta_{h_1} = \Delta_{h_2}$.*

CONJECTURE 9.6. *Let F be a smooth compact* 1*-connected* 4*-manifold* (*possibly with boundary*) *and $h_i : F \to F$ $(i = 1, 2)$ two diffeomorphisms which are the identity on the boundary. Then there exists a nonnegative integer k such that $h_i\sharp k(\mathrm{id}) : F\sharp k(S^2 \times S^2) \to F\sharp k(S^2 \times S^2)$ $(i = 1, 2)$ are smoothly isotopic relative to boundary if and only if $\Delta_{h_1} = \Delta_{h_2}$.*

As to isotopies which may not fix the boundary points, we have the following result.

PROPOSITION 9.7. *Let F be a smooth compact 1-connected 4-manifold with connected boundary such that there exists a unimodular matrix L with $L + {}^tL$ an intersection matrix of F. We suppose that ∂F is homeomorphic to a lens space $L(p,q)$ $(p \geq 2)$, the 3-sphere S^3, or the Poincaré homology 3-sphere $\Sigma(2,3,5)$. Suppose that $h_i : F \to F$ $(i = 1,2)$ are homeomorphisms such that $h_1 \circ h_2^{-1}$ is topologically isotopic to a diffeomorphism. Then h_1 and h_2 are topologically isotopic (not necessarily relative to boundary) if and only if $(h_1)_* = (h_2)_* : H_2(F) \to H_2(F)$.*

For the proof, we need the following.

LEMMA 9.8. *Let F be a compact 1-connected 4-manifold with $H_1(\partial F; \mathbf{Q}) = 0$ and $g_i : F \to F$ $(i = 1,2)$ homeomorphisms which are the identity on the boundary. Then $\Delta_{g_1} = \Delta_{g_2}$ if and only if $(g_1)_* = (g_2)_* : H_2(F) \to H_2(F)$.*

PROOF. If $\Delta_{g_1} = \Delta_{g_2}$, then for every $\gamma \in H_2(F)$, we have $\gamma - (g_1)_*(\gamma) = \gamma - (g_2)_*(\gamma)$, and hence $(g_1)_* = (g_2)_* : H_2(F) \to H_2(F)$. Conversely, suppose that $(g_1)_* = (g_2)_*$ on $H_2(F)$. By Poincaré-Lefschetz duality and the universal coefficient theorem, $H_2(F, \partial F)$ is isomorphic to $\mathrm{Hom}(H_2(F), \mathbf{Z})$ and hence $(g_1)_* = (g_2)_* : H_2(F, \partial F) \to H_2(F, \partial F)$. Let $(D, \partial D)$ be a 2-cycle in $(F, \partial F)$. Then there exists a 3-chain $\tilde{D}$ in F such that $D' = \partial\tilde{D} - (g_1(D) - g_2(D))$ is a 2-chain in ∂F. Furthermore, we have $\partial D' = -g_1(\partial D) + g_2(\partial D) = 0$, since g_i are the identity on the boundary. Thus D' is a 2-cycle in ∂F. Since by our hypothesis we have $H_2(\partial F) = 0$, there exists a 3-chain $\tilde{D}'$ in ∂F such that $\partial\tilde{D}' = D'$. Thus $\tilde{D} - \tilde{D}'$ is a 3-chain in F and we have $\partial(\tilde{D} - \tilde{D}') = g_1(D) - g_2(D)$. This implies that $D - g_1(D)$ and $D - g_2(D)$ are homologous in F. Thus $\Delta_{g_1} = \Delta_{g_2} : H_2(F, \partial F) \to H_2(F)$. This completes the proof. □

PROOF OF PROPOSITION 9.7. First suppose that $(h_1)_* = (h_2)_*$ on $H_2(F)$. We may assume that $g = h_1 \circ h_2^{-1}$ is a diffeomorphism. Then g induces the identity on $H_2(F)$ and on $H_2(F, \partial F)$. Thus $g|_{\partial F}$ also induces the identity on $H_1(\partial F)$. Then by [33], [8], [31], [7], we see that $g|_{\partial F}$ is isotopic to the identity on ∂F. Thus we may assume that $g|_{\partial F}$ is already the identity. Then by the above lemma, we see that the variation maps of g and the identity coincide. Then by Proposition 9.2, g and the identity are topologically isotopic relative to boundary. Hence h_1 and h_2 are topologically isotopic.

Conversely, if h_1 and h_2 are topologically isotopic, then we have $(h_1)_* = (h_2)_*$ on $H_2(F)$. This completes the proof. □

Using a similar argument together with Proposition 9.4, we obtain the following.

PROPOSITION 9.9. *Let F be a smooth compact 1-connected spin 4-manifold with boundary homeomorphic to a lens space $L(p,q)$ $(p \geq 2)$, the 3-sphere S^3, or the Poincaré homology 3-sphere $\Sigma(2,3,5)$. Suppose that $h_i : F \to F$ $(i = 1,2)$ are orientation preserving diffeomorphisms. Then there exists a nonnegative integer k such that $h_i \sharp k(\mathrm{id}) : F \sharp k(S^2 \times S^2) \to F \sharp k(S^2 \times S^2)$ are smoothly isotopic (not necessarily relative to boundary) if and only if $(h_1)_* = (h_2)_* : H_2(F) \to H_2(F)$.*

REMARK 9.10. Let $K_1, \ldots, K_n$ be the fibered 3-knots constructed in Theorem 3.1 and let h_j be the geometric monodromy of K_j. Then $h_j|_{\partial F} = \mathrm{id}$ and $(h_i)_* = (h_j)_*$ on $H_*(F)$ and on $H_*(F, \partial F)$ for all i and j. This is seen as follows. It is known that if L_j is a Seifert matrix of K_j, then the matrix representative of $(h_j)_*$ on $H_2(F)$ is given by $-L_j^{-1}{}^tL_j$ (see, for example, [21], [36]). Furthermore, it is easy to see that $-J_j^{-1}{}^tJ_j$ is equal to the unit matrix, since J_j is skew-symmetric. Thus the matrix $-L_j^{-1}{}^tL_j$ does not depend on j. However h_i is not homotopic relative to boundary to h_j by Corollary 2.9 if $i \neq j$. In fact the variation maps Δ_{h_i} and $\Delta_{h_j} : H_2(F, \partial F) \to H_2(F)$ are different $(i \neq j)$. We do not know if they are topologically isotopic (not necessarily relative to boundary).

10. Application to Embeddings of 4-Manifolds into S^6

First we prove the following.

PROPOSITION 10.1. *Let M be a smooth closed 1-connected spin 4-manifold. Then there exists a fibered 3-knot whose fiber is diffeomorphic to*

$$(10.1) \qquad \begin{cases} (M \sharp (S^2 \times S^2))^\circ, & \textit{if } M \textit{ is not a homotopy 4-sphere, and} \\ M^\circ, & \textit{if } M \textit{ is a homotopy 4-sphere,} \end{cases}$$

where X° denotes $X - \operatorname{Int} D^4$ for a 4-manifold X.

REMARK 10.2. If M is a homotopy 4-sphere, then $(M\sharp(S^2 \times S^2))^\circ$ cannot be a fiber of any simple fibered 3-knot. This is because there exists no unimodular matrix L such that

$$L + {}^tL = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{10.2}$$

PROOF OF PROPOSITION 10.1. By Donaldson's theorem [18], the intersection matrix of M is indefinite. First suppose that M is not a homotopy 4-sphere and set $F = (M\sharp(S^2 \times S^2))^\circ$. Let Q be the intersection form of F. Then by [88] every isometry of Q is induced by some self-diffeomorphism of F which is the identity on the boundary. Therefore, if we find a unimodular matrix L such that $L + {}^tL$ is a matrix representative of Q, then the conclusion follows by the same argument as in [74, §3].

By Serre's theorem (see, for example, [60]), Q is isomorphic to $\alpha E_8 \oplus \beta U$ with $\beta \geq 2$. Set

$$L_0 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}, L_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{pmatrix} \quad \text{and}$$

$$L_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{10.3}$$

then L_i $(i = 0, 1, 2)$ are unimodular, $L_0 + {}^tL_0$ is a matrix representative of $U \oplus U$, $L_1 + {}^tL_1$ is a matrix representative of $U \oplus U \oplus U$, and $L_2 + {}^tL_2$ is a matrix representative of E_8. Combining these matrices, we obtain a unimodular matrix L such that $L + {}^tL$ is a matrix representative of Q. This completes the proof for the case where M is not a homotopy 4-sphere.

When M is a homotopy 4-sphere, we can perform the open book construction [37] by using the identity map $M^\circ \to M^\circ$ and get a desired fibered 3-knot. This completes the proof. □

REMARK 10.3. Let K be a fibered 3-knot with fiber F. Suppose F' is a compact 1-connected smooth 4-manifold with boundary diffeomorphic to ∂F. Then F' can be embedded into S^5 as an *almost fiber* of K (see Remark 2.8 (1)) if and only if F' is smoothly h-cobordant to F relative to boundary.

Using the above theorem, we can give an alternative proof of Cappell-Shaneson's embedding theorem [13] in the case where the 4-manifold is simply connected.

THEOREM 10.4. *Let M be a smooth closed 1-connected spin 4-manifold. Then the following two hold.*

(1) *The 4-manifold M admits a PL embedding into S^6 which is locally flat except possibly at one point of M.*

(2) *If the signature of M vanishes, then M embeds smoothly into S^6.*

PROOF. (1) We can embed $(M \sharp 2(S^2 \times S^2))^\circ$ smoothly into S^5 by Proposition 10.1. This implies that M° embeds smoothly in S^5. Putting the cone on the punctured hole in $S^6 = D^6 \cup_{S^5} D^6$, we see that M embeds in S^6 in the required manner.

(2) By Proposition 10.1 and its proof, M° embeds smoothly in S^5 so that its Seifert matrix L is the direct sum of some copies of

$$L_0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix}. \tag{10.4}$$

(If M is a homotopy 4-sphere, then L is the zero matrix.) Clearly this matrix L is null-cobordant in the sense of Levine [53]. Thus ∂M° bounds a 4-disk which is smoothly embedded in D^6 ($\partial D^6 = S^5$). Putting together the embedded M° in S^5 and the 4-disk in D^6, we get a required embedding of M into S^6. This completes the proof. □

Remark 10.5. (1) We can prove the above theorem by a similar argument without using Donaldson's theorem.

(2) If M° itself is a fiber of a fibered 3-knot K in S^5, then the embedding of M into D^6 constructed in the proof of Theorem 10.4 (1) is closely related to the trivial unfolding of K as discussed in [65, §1].

Remark 10.6. Cappell and Shaneson [13] state the above theorem without the assumption that M should be simply connected (see also [73]).

Remark 10.7. In fact, the 4-manifolds as in Theorem 10.4 (2) embed into S^5 (see [15], [16]).

11. Application to the Topology of Hypersurface Singularities in $\mathbf{C}^3$

Let $f \in \mathbf{C}[z_1, \dots, z_{n+1}]$ be a complex polynomial of $n+1$ complex variables. We suppose that $f(0) = 0$ and that f has an isolated critical point at the origin. For a sufficiently small positive real number ε, we set $K_f = S^{2n+1}_\varepsilon \cap f^{-1}(0)$, where S^{2n+1}_ε is the $(2n+1)$-sphere in $\mathbf{C}^{n+1}$ centered at the origin with radius ε. It is known that K_f is a smooth closed $(2n-1)$-manifold and that the isotopy class of K_f in S^{2n+1} does not depend on the choice of ε, provided that it is sufficiently small. We call the isotopy class of K_f the *algebraic knot associated with* f. In [59], it has been shown that

$$\phi_f = f/|f| : S^{2n+1}_\varepsilon - K_f \to S^1 \tag{11.1}$$

is a locally trivial smooth fibration and that, with this fibration, K_f is a simple fibered $(2n-1)$-knot in the sense of Definition 2.1. The fiber F_f of the fibration ϕ_f is called the *Milnor fiber* of f and the n-th betti number $\mu(f)$ of F_f is called the *Milnor number* of f.

By [59], it has been shown that the topological type of $f^{-1}(0)(\subset \mathbf{C}^{n+1})$ at the origin is completely determined by the algebraic knot $K_f(\subset S^{2n+1})$. Thus, the study of algebraic knots plays an important role in the topological study of isolated hypersurface singularities in $\mathbf{C}^{n+1}$. (For various definitions of the topological type of such an isolated hypersurface singularity, see [77].)

In this section, we apply our results in the previous sections to the study of algebraic knots in S^5 and hence to the study of isolated hypersurface singularities in $\mathbf{C}^3$.

Let $f_t : \mathbf{C}^{n+1}, 0 \to \mathbf{C}, 0$ ($t \in [0,1]$) be a *μ-constant deformation*; i.e., each f_t is a polynomial of $n+1$ complex variables with $f_t(0) = 0$ and with an isolated critical point at the origin such that the Milnor numbers $\mu(f_t)$ are constant. It has been known that if $n \neq 2$, then the algebraic knots K_{f_t} are isotopic to each other [52]. When $n = 2$, we have a partial result as follows.

PROPOSITION 11.1. *Let $f_t : \mathbf{C}^3, 0 \to \mathbf{C}, 0$ ($t \in [0,1]$) be a μ-constant deformation of an isolated singularity. Suppose that $H_1(K_{f_0}) \cong \mathbf{Z}/r\mathbf{Z}$, where $r = 1, 2, 4, p^m$ or $2p^m$ with p an odd prime. Furthermore we suppose that $\pi_1(K_{f_0}) \cong \pi_1(K_{f_1})$. Then the algebraic knots K_{f_0} and K_{f_1} are isotopic to each other.*

PROOF. It is not difficult to see that the isomorphism class of the Seifert form is constant under a μ-constant deformation (for details, see [78]). This implies that the homology groups of K_{f_t} are also constant. If $\pi_1(K_{f_i})$ are finite groups, then the result follows from [22], [76]. If $\pi_1(K_{f_i})$ are infinite, then it has been shown that if K_{f_0} and K_{f_1} have isomorphic fundamental groups, then they are diffeomorphic [64] (not necessarily preserving the orientations). More precisely, in [64], Neumann considers diffeomorphisms which preserve the orientations, and the case where the link 3-manifold is a torus bundle over the circle is excluded. In this case, it is a Haken 3-manifold and the above result follows from [87]. Now by our hypothesis, $H_1(K_{f_i})$ are finite, and hence K_{f_i} are not torus bundles over the circle. Thus K_{f_0} and K_{f_1} are orientation preservingly diffeomorphic. Then by Theorem 2.12, we have the conclusion. This completes the proof. □

REMARK 11.2. In fact, in Proposition 11.1 the polynomial functions f_0 and f_1 are *right equivalent* as germs at the origin; i.e., there exists a homeomorphism germ $\Phi : \mathbf{C}^3, 0 \to \mathbf{C}^3, 0$ such that $f_0 = f_1 \circ \Phi$ as germs at the origin. This is true, since the fibrations associated with K_{f_0} and K_{f_1} are homotopic. See [42], [68].

Remark 11.3. Szczepanski [84] claims a more general result in which she drops the assumption on $H_1(K_{f_0})$. Unfortunately her proof contains some gaps as follows.

(1) In the fourth paragraph of the proof of Theorem 2 of [84], she claims that there exists a $\mathbf{Z}$-homology cobordism between V and $K \times I$. In order to prove this, she assumes that $V \cup (K \times I)$ is homologically indistinguishable from $K \times S^1$ (see [84, Appendix]). However, this is not always the case, since she assumes only the existence of a diffeomorphism ψ between the two boundaries K_0 and K_1 of V. Let $\varphi : H_*(K_0) \to H_*(K_1)$ be the isomorphism induced by the inclusions into the homology cobordism V. Then $V \cup (K \times I)$ has the same homology as $K \times S^1$ if and only if $\psi_* = \varphi$. Therefore ψ should satisfy this homological condition. See also [84, Remark 1 (p.521)].

(2) The 4-manifold $V \cup (K \times I)$ must be a spin manifold. In the proof of Proposition 3.1 of [84], she claims that $\alpha \cup \beta : V \cup (K \times I) \to S^3 \times S^1$ gives a degree one normal map (see also [11]). If this is the case, the stable normal bundle of $V \cup (K \times I)$ should be trivial, since that of $S^3 \times S^1$ is trivial. This is equivalent to that $V \cup (K \times I)$ is spin; i.e., $w_2(V \cup (K \times I)) = 0$. Note that such a 4-manifold is not always spin; for example, consider the $S^1 \times S^2$-bundle over S^1 whose monodromy is the diffeomorphism defined by $(z, w) \mapsto (z, zw)$, where we identify S^1 with the unit circle in $\mathbf{C}$ and S^2 with $\bar{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. Thus one needs to assume that the diffeomorphism ψ preserves the spin structures of K_i associated with the corresponding Milnor fibers.

Consequently Theorems A, B, 1, 2, and 3 of [84] need some corrections. Here we note that the part (1) above corresponds to the condition (4)(b) in our Theorem 2.2 and the part (2) above to the condition (4)(a). We also note that the two conditions are independent as is seen from the example in (2) above.

Note that the condition on $H_1(K_{f_0})$ in Proposition 11.1 eliminates the two problems above.

We also note that Szczepanski [84] misunderstands something about the orientations of link 3-manifolds. For example, in the proof of [84, Theorem 2], one should note that the embedding of a Milnor fiber into the other which homologically identifies the strongly distinguished bases is automatically orientation preserving. Thus $\partial V = K_0 \cup (-K_1)$. Therefore, the diffeomorphism $\psi : K_0 \to K_1$ should be orientation preserving so that $V \cup (K \times I)$, which we construct by using id : $K \times \{0\} \to K_0$ and $\psi : K_0 \times \{1\} \to K_1$, becomes orientable.

Furthermore, when the link 3-manifolds are lens spaces, Szczepanski [84] uses an argument a little bit complicated (see [84, Theorem 3]). (For example, she uses a result of [34] which is valid only in the case where $H_1(K_i)$ are of prime power order.) We can find a shorter proof as in our proof of Proposition 11.1.

PROPOSITION 11.4. *Let $f \in \mathbf{C}[x, y, z]$ be a polynomial of three complex variables with $f(0) = 0$ and with an isolated critical point at the origin. We suppose that $H_1(K_f) \cong \mathbf{Z}/r\mathbf{Z}$, where $r = 1, 2, 4, p^m$ or $2p^m$ with p an odd prime. If f can be connected by a μ-constant deformation to a polynomial with real coefficients, then f is right equivalent to $\bar{f}$, where $\bar{f} : \mathbf{C}^3, 0 \to \mathbf{C}, 0$ is the function defined by $\bar{f}(z) = \overline{f(z)}$ (the complex conjugate of $f(z) \in \mathbf{C}$).*

PROOF. Define the diffeomorphisms $\Phi_0 : \mathbf{C}^3 \to \mathbf{C}^3$ and $\tau : \mathbf{C} \to \mathbf{C}$ by $\Phi_0(z_1, z_2, z_3) = (\bar{z}_1, \bar{z}_2, \bar{z}_3)$ and $\tau(z) = \bar{z}$ respectively. Let g be a polynomial with real coefficients which is connected to f by a μ-constant deformation. Then we have $g = \tau \circ g \circ \Phi_0$. Thus K_g is orientation preservingly isotopic to its mirror image. (Note that the fibration associated with the map $\tau \circ g \circ \Phi_0$ coincides with the fibration associated with g but with reversed orientations of S^5 and S^1. In particular, the orientations of the fibers and the knots coincide.) In other words, K_g is $(+1)$-amphicheiral. This implies that the Seifert matrix L_g of K_g is congruent to ${}^tL_g$. Since the congruence class of the Seifert matrix is invariant under μ-constant deformations, we see that L_f is also congruent to ${}^tL_f$, where L_f is the Seifert matrix of K_f. Then by Theorem 2.12, K_f and its mirror image are isotopic to each other. Thus the fibered 3-knots associated with the maps f and $\tau \circ f \circ \Phi_0$ are isotopic. Then by Perron [68], we see that there exits a homeomorphism germ $\Phi : \mathbf{C}^3, 0 \to \mathbf{C}^3, 0$ such that $\bar{f} = f \circ \Phi$ as germs at the origin. This completes the proof. □

Compare the above proposition with [77, Proposition 6].

PROPOSITION 11.5. *Let f and $g \in \mathbf{C}[x, y, z]$ be polynomials of three complex variables with $f(0) = g(0) = 0$ and with isolated critical points at* *the origin. We suppose that $H_1(K_f;\mathbf{Q}) = 0 = H_1(K_g;\mathbf{Q})$. Then if there exists a homeomorphism germ $\Phi : \mathbf{C}^3, 0 \to \mathbf{C}^3, 0$ such that $\Phi(f^{-1}(0)) = g^{-1}(0)$ as germs at the origin, then the algebraic knots K_f and K_g are isotopic.*

Proof. It is not difficult to see that, under the above hypothesis, there exists a topological (oriented) invertible cobordism between (S^5, K_f) and (S^5, K_g) (for details, see [77]). Then we see that $(E(K_f), \partial E(K_f))$ and $(E(K_g), \partial E(K_g))$ are orientation preservingly homotopy equivalent. Then the result follows from Theorem 4.1 when the fundamental groups are infinite (see [64], [66]). When the fundamental groups are finite, we can use an argument similar to that in the proof of Proposition 11.1. This completes the proof. □

Remark 11.6. In the above proposition, the assumption on $H_1(K_f;\mathbf{Q})$ and $H_1(K_g;\mathbf{Q})$ are in fact redundant. For details, see [77]. Note that our proof is different from that given in [77].

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(Received March 15, 1999)

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