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On the Rationality and Fake Degrees of Characters of Cyclotomic Algebras

By Gunter Malle

1. Introduction

Let W be a finite complex reflection group, and $\mathcal{H} = \mathcal{H}(W, \mathbf{u})$ the corresponding generic (cyclotomic) Hecke algebra as introduced in [8] and [9]. In this paper we study the character fields of the irreducible characters of $\mathcal{H}$. In the case that W is a Weyl group, hence a reflection group over the field of rational numbers, it is well known that all other complex irreducible representations of W can also be realized over $\mathbb{Q}$. The corresponding situation for the associated Iwahori-Hecke algebras was investigated by Benson and Curtis [5] and Alvis and Lusztig [1]. They determined the character fields of all absolutely irreducible representations of Iwahori-Hecke algebras. It turned out that sometimes certain square roots of monomials in the parameters have to be adjoined to the ground field. Furthermore they showed that all absolutely irreducible representations can actually be realized over such an extension of the ground field [5,13].

For a complex reflection group W let k denote the character field of the reflection representation of W. It is a result of Benard [4] and Bessis [6] that again all absolutely irreducible complex representations of W can be realized over k and in particular have their character field contained in k. Here we determine the character fields of all generic cyclotomic Hecke algebras associated to complex reflection groups (under a certain assumption known to hold for all but finitely many irreducible types and conjectured to be always true). Partial results in this direction were already obtained in [8] for those complex reflection groups occurring as cyclotomic Weyl groups. Our results show that again the character fields and a splitting field for $\mathcal{H}$ can be obtained by adjoining roots of certain monomials in the parameters to the ground field (Corollary 4.8 and Theorem 5.2).

Our methods are similar to those employed in [5, 6, 8]. As in all of the above references, we use a case-by-case analysis, handling each isomorphism

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type and each irreducible character in turn. As a consequence of our classification we deduce that an irreducible complex reflection group W on $\mathbb{C}^n$ can be generated by n reflections if and only if the reflection character of the associated 1-parameter cyclotomic algebra $\mathcal{H}(W,x)$ has values in $k(x)$ (Corollary 4.9).

The fake degree of an irreducible character $\chi \in \mathrm{Irr}(W)$ is the graded multiplicity of χ in the graded regular representation. It can be defined entirely in terms of W. In Section 6 we prove a remarkable connection between the rationality of characters of cyclotomic algebras and semi-palindromicity of fake degrees, which generalizes a result of Opdam [16] for the real case (Theorem 6.5).

In the last section we take a closer look at the subclass of n-dimensional irreducible reflection groups which can be generated by n of their reflections. These seem to behave nicer than the general reflection groups in several aspects. As an example we give a closed formula for the field of definition k of such a group (Theorem 7.1), and then sharpen some of the results obtained in the previous sections.

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2. Some Prerequisites

We recall the definition of cyclotomic Hecke algebras and some facts about character values.

2A. Cyclotomic Hecke algebras

Let W be a finite irreducible complex reflection group and let k denote the character field of this reflection representation. It is known (see [4, 6]) that then k is a splitting field for W (see also Section 7A for a description of k in an important case). Let $\mathcal{D}$ be the diagram associated to W in [9]. This defines a presentation of W on a set of generators S subject to the order relations $s^{d_s} = 1$ for $s \in S$, together with certain homogeneous relations, the so-called braid relations. The braid group $B = B(W)$ associated to W is by definition the group generated by a set $\{\mathbf{s} \mid s \in S\}$ in bijection with S, subject to the braid relations of $\mathcal{D}$. Let $\mathbf{u} = (u_{s,j} \mid s \in S,\ 0 \le j \le d_s - 1)$ be transcendentals over $\mathbb{Z}$, such that $u_{s,j} = u_{t,j}$ whenever s and t are conjugate in W. The *generic cyclotomic Hecke algebra* $\mathcal{H}(W, \mathbf{u})$ *of* W *with parameter set* $\mathbf{u}$ is defined to be the quotient

$$\mathcal{H}(W, \mathbf{u}) := \mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]B/I, \qquad \text{with } I = (\prod_{j=0}^{d_s-1} (\mathbf{s} - u_{s,j}) \mid s \in S),$$

of the group algebra of B over $A := \mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$ by the ideal I generated by certain deformed order relations. Thus $\mathcal{H} := \mathcal{H}(W, \mathbf{u})$ is a finitely generated algebra over A. We will write $T_{\mathbf{w}}$ for the image in $\mathcal{H}$ of an element $\mathbf{w} = \mathbf{s}_1 \ldots \mathbf{s}_k \in B$. Any ring homomorphism $f : \mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}] \to R$ endows R with an A-module structure, and we write

$$\mathcal{H}_R(W, \mathbf{u}) = \mathcal{H}(W, \mathbf{u}) \otimes_A R$$

for the corresponding specialization of $\mathcal{H}$. Note that such a homomorphism is uniquely determined by the images $f(u_{s,j})$, $s \in S$, $0 \le j \le d_s - 1$. Also note that under the specialization defined by

$$u_{s,j} \mapsto \exp(2\pi i j/d_s) \qquad \text{for } s \in S,\ 0 \le j \le d_s - 1, \tag{2.1}$$

$\mathcal{H}$ maps to the group algebra of the complex reflection group W. Any specialization of $\mathcal{H}$ through which (2.1) factors will be called *admissible*. One particularly important example is the *1-parameter specialization* $\mathcal{H}(W, x)$ of $\mathcal{H}(W, \mathbf{u})$ induced by the map

$$u_{s,j} \mapsto \begin{cases} x & j = 0, \\ \exp(2\pi i j/d_s) & j > 0, \end{cases} \tag{2.2}$$

where x is an indeterminate. This is the analogue of the 1-parameter Iwahori-Hecke algebra for real W.

Henceforth we will make the following assumption:

Assumption 2.3. *The generic cyclotomic algebra* $\mathcal{H}(W, \mathbf{u})$ *defined above is free over* $A = \mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$ *of rank* $|W|$.

This assumption is known to hold for all infinite families of irreducible reflection groups by [3,8,2], and for about half of the remaining 34 exceptional groups (see [8]). We conjecture it to be true in all cases (see [9, Sect. 4]).

Let $L \geq A$ be a splitting field of $\mathcal{H}$. It follows from Assumption 2.3, the fact that k is a splitting field for W, and Tits' deformation theorem that $\mathcal{H}_L$ is isomorphic to the group algebra LW. Thus any extension to L of the specialization (2.1) defines a bijection between $\mathrm{Irr}(\mathcal{H}_L)$ and $\mathrm{Irr}(W)$.

2B. Character values

Let W, k, $\mathcal{H}$ be as above. For $s \in S$, $0 \leq j \leq d_s - 1$, let $v_{s,j}$ be such that $v_{s,j}^{|W|} = u_{s,j}$. Let $L \geq k(\mathbf{v})$ be a splitting field for $\mathcal{H}$. We will see later on (Theorem 5.2) that we may take $L = k(\mathbf{v})$. We extend the specialization (2.1) to the integral closure $\mathcal{O}$ of A in L, such that on $\mathbb{Z}[\mathbf{v}, \mathbf{v}^{-1}]$ we have

$$v_{s,j} \mapsto \exp(2\pi ij/(d_s|W|)) \qquad \text{for } s \in S,\ 0 \leq j \leq d_s - 1\,. \tag{2.4}$$

By Tits' deformation theorem this defines a bijection

$$\mathrm{Irr}(W) \xrightarrow{\sim} \mathrm{Irr}(\mathcal{H}_L), \qquad \chi \mapsto \chi_{\mathbf{v}}\,, \tag{2.5}$$

which furthermore carries over to any admissible specialization of $\mathcal{H}_{\mathcal{O}}$.

In [9] we defined a certain central element $\boldsymbol{\beta} \in Z(B)$ of the braid group whose image β under the canonical epimorphism $B \to W$ generates $Z(W)$. Since we assumed W to be irreducible, $Z(W)$ is cyclic by Schur's lemma. So $\boldsymbol{\pi} := \boldsymbol{\beta}^{|Z(W)|} \in Z(B)$ is the smallest power of $\boldsymbol{\beta}$ which maps to 1 in W. Clearly, the images $T_{\boldsymbol{\beta}}, T_{\boldsymbol{\pi}}$ in $\mathcal{H}$ are also central. Thus, they act as scalars in any (absolutely) irreducible representation of $\mathcal{H}_L$.

Let S' be a system of representatives of the generators in S up to conjugation in W. Write $\boldsymbol{\pi} = \mathbf{s}_1 \ldots \mathbf{s}_l$ in B. For $s \in S'$ let $N_s = |\{j \mid s_j \sim s\}|$ denote the number of factors in the decomposition of $\boldsymbol{\pi}$ conjugate to $\mathbf{s}$. By evaluating linear characters of $\mathcal{H}$ it is easily seen that N_s does not depend on the chosen expression for $\boldsymbol{\pi}$. For an irreducible character χ of W let $m_{s,j}(\chi)$ denote the multiplicity of the eigenvalue $\exp(2\pi ij/d_s)$ of s in a representation affording χ. Springer [5] observed that it is possible to compute character values on central elements (of Iwahori-Hecke algebras) without actually constructing the representations; this was already used in [8] to deal with some of the complex reflection groups. Let $z = |Z(W)|$, so that $\boldsymbol{\beta}^z = \boldsymbol{\pi}$. Then the value of the irreducible character $\chi_{\mathbf{v}} \in \mathrm{Irr}(\mathcal{H}_L)$ on $T_{\boldsymbol{\beta}}$ is given by

$$\chi_{\mathbf{v}}(T_{\boldsymbol{\beta}}) = \chi(\beta) \prod_{s \in S'} \prod_{j=0}^{d_s - 1} (\exp(-2\pi i j/d_s|W|)\, v_{s,j})^{m_{s,j} N_s |W|/z\chi(1)}. \tag{2.6}$$

Indeed, let $\det_\chi$ denote the determinant on a representation affording $\chi_{\mathbf{v}}$. Then $\det_\chi(T_{\boldsymbol{\beta}})^z = \det_\chi(T_{\boldsymbol{\pi}}) = \prod_{s \in S'} \det_\chi(T_{\mathbf{s}}) = \prod_{s\in S'} \prod_{j=0}^{d_s-1} u_{s,j}^{m_{s,j}}$. But $T_{\boldsymbol{\beta}}$ is central, so it acts by a scalar c. This determines c up to a root of unity, which can be computed by specializing to W via (2.4).

2C. Rationality and Schur index

Let $\mathcal{H}_L$ be a cyclotomic algebra as above and $\chi \in \mathrm{Irr}(\mathcal{H}_L)$. The *character field of* χ is the field generated over $k(\mathbf{u})$ by the values of χ on an A-basis of $\mathcal{H}_L$ as provided by Assumption 2.3. It is clear that this does not depend on the choice of basis.

Assume that L is Galois over $K := \mathrm{Quot}(A)$. Then $\mathrm{Gal}(L/K)$ acts on the set $\mathrm{Irr}(\mathcal{H}_L)$ via its action on the character values. We will make use of the following obvious fact. Assume that $\chi, \psi \in \mathrm{Irr}(\mathcal{H}_L)$ have different multiplicity in a character of $\mathcal{H}_L$ whose values are invariant under a subgroup $H \leq \mathrm{Gal}(L/K)$. Then χ, ψ do not lie in the same H-orbit. In particular, if there exists a set of K-rational characters of $\mathcal{H}_L$ such that χ is distinguished from all other irreducible characters by its multiplicities in these characters, then the character field of χ is equal to K.

The field of rationality of a representation of $\mathcal{H}_L$ will be determined by the following:

Lemma 2.7. *Let $\mathcal{H}$ be a finite dimensional algebra over a field K_0 and let B be a basis of $\mathcal{H}$. Let K/K_0 be an extension field such that $\mathcal{H}_K := \mathcal{H} \otimes_{K_0} K$ is split semisimple. Let V be a simple $\mathcal{H}_K$-module with character χ such that $\chi(b) \in K_0$ for all $b \in B$ and assume that there exists an $\mathcal{H}$-module V' such that V has multiplicity 1 in $V' \otimes_{K_0} K$. Then V is realizable over K_0.*

This follows from standard properties of the Schur index (see for example [10, §74]).

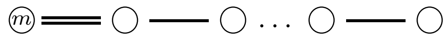
3. Irreducible Characters of Imprimitive Groups

In this section we recall the definition of the imprimitive complex reflection groups and the construction of their complex irreducible characters (see for example [14, Sect. 2A and 5A] or [6, Sect. 1]). From this we deduce some elementary results on multiplicities in induced characters. They are well-known in the case of the real imprimitive reflection groups $W(B_n)$ and $W(D_n)$.

3A. The groups W_n

Throughout this section we fix an integer $m \geq 2$ and let $\zeta_m := \exp(2\pi i/m)$. For any $n \geq 1$ let $W_n := G(m,1,n)$ be the complex linear group on $\oplus_{j=1}^{n} \mathbb{C}e_j$ consisting of all monomial matrices whose non-zero entries lie in $\{\zeta_m^j \mid 0 \leq j \leq m-1\}$. Then W_n is the semidirect product of its subgroup of diagonal matrices with the subgroup of permutation matrices, that is, $W_n = Z_m^n . \mathfrak{S}_n = Z_m \wr \mathfrak{S}_n$. In its above representation W_n is generated by the complex reflection t which sends e_1 to $\zeta_m e_1$ and fixes $e_2, \ldots, e_n$, and the permutation matrices s_l, $1 \leq l \leq n-1$, corresponding to the transpositions $(l, l+1)$. In particular, W_n is an irreducible imprimitive complex reflection group. The generators $\{t, s_1, \ldots, s_{n-1}\}$ together with the relations implied by the diagram

(3.1) $B_n^{(m)}$:



yield a presentation for W_n. The elements $\{t, s_1, \ldots, s_{n-2}\}$ generate a natural subgroup W_{n-1}, the elements $\{s_1, \ldots, s_{n-1}\}$ a natural subgroup $\mathfrak{S}_n$.

We define the following linear character of W_n:

$$\gamma : W_n \to \mathbb{C}, \qquad t \mapsto \zeta_m,\ s_l \mapsto 1 \text{ for } 1 \leq l \leq n-1,$$

For any m-tuple of partitions $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{m-1}) \vdash_m n$ of n we denote by W_α the natural subgroup $W_{n_0} \times \ldots \times W_{n_{m-1}}$ of W_n, where $\alpha_j \vdash n_j$, corresponding to the Young subgroup $\mathfrak{S}_{n_0} \times \ldots \times \mathfrak{S}_{n_{m-1}}$ of $\mathfrak{S}_n$. Via the natural projection $W_{n_j} \to \mathfrak{S}_{n_j}$ the characters of $\mathfrak{S}_{n_j}$ may be regarded as characters of W_{n_j}. The irreducible characters of $\mathfrak{S}_{n_j}$ are indexed by partitions $\alpha_j \vdash n_j$. For any multi-partition $\alpha \vdash_m n$ as above we can thus define a character χ_α of W_n as the induction of the exterior product

$$\chi_\alpha := \operatorname{Ind}_{W_\alpha}^{W_n} \left(\chi_{\alpha_0} \# (\chi_{\alpha_1} \otimes \gamma) \# \ldots \# (\chi_{\alpha_{m-1}} \otimes \gamma^{m-1}) \right) . \tag{3.2}$$

It turns out that χ_α is irreducible, $\chi_\alpha \neq \chi_\beta$ if $\alpha \neq \beta$, and that all irreducible characters of W_n arise in this way, so $\mathrm{Irr}(W_n) = \{\chi_\alpha \mid \alpha = (\alpha_0, \dots, \alpha_{m-1}) \vdash_m n\}$.

LEMMA 3.3. *(a) Let $\alpha = (\alpha_0, \dots, \alpha_{m-1}) \vdash_m n-1$. Then*

$$\mathrm{Ind}_{W_{n-1}}^{W_n}(\chi_\alpha) = \sum \chi_\beta$$

where the sum is over all partitions $\beta = (\beta_0, \dots, \beta_{m-1}) \vdash_m n$ which can be obtained by adding a single 1-hook to α.

(b) For $n \geq 3$ the characters of W_n are distinguished by their restrictions to W_{n-1}. The characters of W_2 are distinguished by their restrictions to W_1 and $\mathfrak{S}_2$.

PROOF. By construction we have $\chi_\alpha = \mathrm{Ind}_{W_\alpha}^{W_{n-1}}(\psi)$ where $\psi = (\psi_0 \# \psi_1 \# \dots \# \psi_{m-1})$ with $\psi_j = \chi_{\alpha_j} \otimes \gamma^j$, thus $\mathrm{Ind}_{W_{n-1}}^{W_n}(\chi_\alpha) = \mathrm{Ind}_{W_\alpha}^{W_n}(\psi)$. Now

$$\mathrm{Ind}_{W_\alpha}^{W_\alpha \times W_1}(\psi) = \sum_{j=0}^{m-1} \psi \# \gamma^j.$$

It is well-known that

$$\mathrm{Ind}_{\mathfrak{S}_{n_j}}^{\mathfrak{S}_{n_j+1}}(\chi_{\alpha_j}) = \sum \chi_{\beta_j}$$

where the sum runs over the partitions $\beta_j \vdash n_j + 1$ which can be obtained from α_j by adding a single 1-hook. We thus obtain that the induced of

$$\psi \# \gamma^j = \psi_0 \# \dots \# (\psi_j \# \gamma^j) \# \dots \# \psi_{m-1}$$

from $W_\alpha \times W_1 = W_{n_0} \times \dots \times (W_{n_j} \times W_1) \times \dots \times W_{n_{m-1}}$ to $W_{n_0} \times \dots \times W_{n_j+1} \times \dots \times W_{n_{m-1}}$ decomposes into the sum of all $\psi_0 \# \dots \# (\chi_{\beta_j} \otimes \gamma^j) \# \dots \# \psi_{m-1}$ with β_j as before. Finally, the induced of

$$\psi_0 \# \dots \# (\chi_{\beta_j} \otimes \gamma^j) \# \dots \# \psi_{m-1}$$

to W_n is by definition the irreducible character χ_β with $\beta = (\alpha_0, \dots, \beta_j, \dots, \alpha_{m-1})$. This completes the proof of (a).

In (b), the assertion for $n = 2$ is easily verified. So now let $n \geq 3$ and $\chi_\alpha, \chi_\beta \in \mathrm{Irr}(W_n)$ with $\alpha \neq \beta$. If the distance $d := \sum_{j=0}^{m-1} ||\alpha_j| - |\beta_j||$ of α and β is larger than 2, then there exists a multi-partition $\nu \vdash_m n-1$ such that α but not β can be obtained by adding a single 1-hook to ν, so by (a) the two characters χ_α, χ_β do not have equal multiplicities in the induced of all irreducible characters of W_{n-1}. Now assume that $|\alpha_j|$ and $|\beta_j|$ only differ for $j = j_1, j_2$, say $|\alpha_{j_1}| = |\beta_{j_1}| - 1$ and $|\alpha_{j_2}| = |\beta_{j_2}| + 1$. If $|\alpha_{j_1}| \neq 0$ there exists a multi-partition $\nu \vdash_m n-1$ such that α, but not β, is obtained from ν by adding a 1-hook (and similarly if $|\beta_{j_2}| \neq 0$). If $|\alpha_{j_1}| = |\beta_{j_2}| = 0$ then since $n > 1$ there exists $j_3 \neq j_1, j_2$ with $|\alpha_{j_3}| = |\beta_{j_3}| > 0$, and by diminishing this part we again find a partition $\nu \vdash_m n-1$ as above. Thus, if the multiplicities of χ_α, χ_β are the same in the induced of each irreducible character of W_{n-1}, then $|\alpha_j| = |\beta_j|$ for all j. Now choose j such that $\alpha_j \neq \beta_j$. Since $n \geq 3$ it follows by elementary combinatorics that there exists a partition $\nu_j \vdash |\alpha_j| - 1$ such that α_j but not β_j can be obtained from ν_j by adding a 1-hook (or vice versa). Hence the characters of W_n are distinguished by their multiplicities in characters induced from W_{n-1}, and the claim follows by Frobenius reciprocity. $\square$

3B. The infinite series $G(m, p, n)$

Note that any element of W_n can be written as the product of a diagonal element $\mathrm{diag}(\zeta_m^{a_1}, \ldots, \zeta_m^{a_n})$ with a permutation matrix corresponding to some $w \in \mathfrak{S}_n$. We denote this element by $(a_1, \ldots, a_n; w)$ for short. Now fix a divisor $p|m$ and let $G(m, p, n)$ be the kernel of the linear character $\gamma^{m/p}$, of index p in W_n:

$$G(m, p, n) = \{(a_1, \ldots, a_n; w) \mid \sum a_j \equiv 0 \pmod{p},\ w \in \mathfrak{S}_n\}.$$

Then $G(m, p, n)$ is again an imprimitive complex reflection group, generated by the reflections $s_1, \tilde{s}_1 := t^{-1} s_1 t, s_2, \ldots, s_{n-1}$, and t^p (if $p \neq m$). Clearly, the reflections $s_1, s_2, \ldots, s_{n-1}$ again generate a natural reflection subgroup $\mathfrak{S}_n$ consisting of all permutation matrices.

We describe the irreducible characters of $G(m, p, n)$ in terms of those of W_n. Denote by τ the cyclic shift on m-tuples of partitions of n, i.e.,

$$\tau(\alpha_0, \ldots, \alpha_{m-1}) = (\alpha_1, \ldots, \alpha_{m-1}, \alpha_0).$$

By definition we then have $\chi_{\tau(\alpha)} \otimes \gamma = \chi_\alpha$. Let $s_p(\alpha)$ denote the order of the stabilizer of α in the cyclic group $\langle \tau^{m/p} \rangle$. Then upon restriction to $G(m,p,n)$ the irreducible character χ_α of W_n splits into $s_p(\alpha)$ different irreducible constituents, and this exhausts the set of irreducible characters of $G(m,p,n)$. More precisely, let $\alpha \vdash_m n$ with $\tilde{p} := s_p(\alpha)$, $W_{\alpha,p} := W_\alpha \cap G(m,p,n)$, and ψ_α the restriction of $\chi_{\alpha_0} \# (\chi_{\alpha_1} \otimes \gamma) \# \ldots \# (\chi_{\alpha_{m-1}} \otimes \gamma^{m-1})$ to $W_{\alpha,p}$. Then ψ_α is invariant under the element $\sigma := (s_1 \cdots s_{n-1})^{n/\tilde{p}}$ (note that $\tilde{p} = s_p(\alpha)$ divides n), and it extends to the semidirect product $W_{\alpha,p}.\langle \sigma \rangle$. The induced of the different extensions of ψ_α then exhaust the irreducible constituents of the restriction of χ_α to $G(m,p,n)$. Thus, we may parametrize $\mathrm{Irr}(G(m,p,n))$ by m-tuples of partitions of n up to cyclic shift by $\tau^{m/p}$ in such a way that any α stands for $s_p(\alpha)$ different characters.

We will need the following consequence of the Littlewood-Richardson rule for the decomposition of characters induced from Young subgroups. Let $\alpha \vdash_m n$ with $\alpha_j = (\alpha_{j1} \geq \alpha_{j2} \geq \ldots)$, and let $\beta = (\beta_1 \geq \beta_2 \geq \ldots)$ be defined by $\beta_l = \sum_j \alpha_{jl}$. Let $\mathfrak{S}_\alpha$ denote the Young subgroup of $\mathfrak{S}_n$ corresponding to $(|\alpha_0|, \ldots, |\alpha_{m-1}|)$. Then we have

$$\langle \chi_\beta, \mathrm{Ind}_{\mathfrak{S}_\alpha}^{\mathfrak{S}_n}(\chi_{\alpha_0} \# \ldots \# \chi_{\alpha_{m-1}}) \rangle = 1 \tag{3.4}$$

for the character $\chi_\beta \in \mathrm{Irr}(\mathfrak{S}_n)$, and all other constituents of the induction are parametrized by partitions strictly smaller than β in the dominance order. This fact can be proved from [12, Cor. 2.8.14] by induction on m.

LEMMA 3.5. *(a) Let $\alpha \vdash_m n$ and $\chi_\alpha \in \mathrm{Irr}(G(m,p,n))$. Then there exists $\beta \vdash n$ such that χ_α has multiplicity 1 in the induced of $\chi_\beta \in \mathrm{Irr}(\mathfrak{S}_n)$ from $t^{-j}\mathfrak{S}_n t^j$ to $G(m,p,n)$, for some $0 \leq j \leq p-1$.*

(b) The irreducible constituents of the restriction of $\chi_\alpha \in \mathrm{Irr}(W_n)$ to $G(m,p,n)$ are distinguished by their restrictions to $t^{-j}\mathfrak{S}_n t^j$, $0 \leq j \leq p-1$.

PROOF. By our above considerations, χ_α is the induced of an irreducible character ψ of $H := W_{\alpha,p}.\langle \sigma \rangle$ with $\sigma := (s_1 \cdots s_{n-1})^{n/\tilde{p}}$, $\tilde{p} = s_p(\alpha)$. By the Mackey theorem, the multiplicity of the induced of χ_β in χ_α is the same as the scalar product of the restrictions of χ_β and of ψ to $H \cap \mathfrak{S}_n$. By Frobenius reciprocity, this is the same as the multiplicity of χ_β in

$(\psi|_{H\cap\mathfrak{S}_n})^{\mathfrak{S}_n}$.

$$\begin{array}{ccccc} W_{\alpha,p} & \text{———} & H & \text{———} & G(m,p,n) \\ | & & | & & | \\ W_{\alpha,p}\cap\mathfrak{S}_n & \text{———} & H\cap\mathfrak{S}_n & \text{———} & \mathfrak{S}_n \end{array}$$

But $W_{\alpha,p}\cap\mathfrak{S}_n$ is a Young subgroup of $\mathfrak{S}_n$, and by the above consequence (3.4) of the Littlewood-Richardson rule, there exists χ_β which has multiplicity 1 in the induced of $\tilde\psi := \psi|_{W_{\alpha,p}\cap\mathfrak{S}_n}$ to $\mathfrak{S}_n$. Thus χ_β occurs with multiplicity 1 in the induced of exactly one of the extensions of $\tilde\psi$ to $H\cap\mathfrak{S}_n$. Hence precisely one of the characters of $G(m,p,n)$ parametrized by α occurs with multiplicity 1 in the induced of χ_β. But all characters parametrized by α are conjugate under t, hence both assertions of the lemma follow. □

4. Determination of the Character Fields

With the preparations in the preceding section we can now determine the character fields of cyclotomic algebras attached to irreducible complex reflection groups. According to the classification by Shephard and Todd [19], the latter fall into an infinite series $G(m,p,n)$ of imprimitive groups, the symmetric groups $\mathfrak{S}_n$, and another 34 primitive groups.

4A. The imprimitive groups

We first consider the imprimitive groups. It is known that the character field k of the reflection representation of $G(m,p,n)$ is equal to $\mathbb{Q}(\zeta_m)$, except if $n=2, m=p$, when k is the maximal totally real subfield of $\mathbb{Q}(\zeta_m)$ (see for example [6]).

Let $\mathcal{H}(W_n;\mathbf{u})$ with $\mathbf{u}=(u_0,\ldots,u_{m-1},x_1,x_2)$ be the generic cyclotomic algebra for W_n. We first determine the splitting field for $\mathcal{H}(W_n;\mathbf{u})$; this result also follows from the explicit construction of all irreducible representations in [3].

Proposition 4.1 (Ariki and Koike). *The field $\mathbb{Q}(\mathbf{u})$ is a splitting field for $\mathcal{H}(W_n;\mathbf{u})$.*

Proof. We proceed by induction on n. If $n=1$ then

$$\mathcal{H}(W_1;\mathbf{u}) = \mathbb{Z}[\mathbf{u},\mathbf{u}^{-1},T]/((T-u_0)\ldots(T-u_{m-1}))$$

is commutative, and the irreducible representations are given by $T \mapsto u_j$, $0 \leq j \leq m-1$. Hence they are all $\mathbb{Q}(\mathbf{u})$-rational. Now assume that $n > 1$. By Lemma 3.3(b) the characters of W_n are distinguished by their restrictions to W_{n-1} (and to $\mathfrak{S}_n$ if $n = 2$). Thus the characters of $\mathcal{H}(W_n;\mathbf{u})$ are distinguished by their restrictions to $\mathcal{H}(W_{n-1};\mathbf{u})$ (and to $\mathcal{H}(\mathfrak{S}_n)$ for $n = 2$). Since the irreducible characters of the latter are $\mathbb{Q}(\mathbf{u})$-rational by induction, the character field is contained in $\mathbb{Q}(\mathbf{u})$. By Lemma 3.3(a) and Lemma 2.7 it now follows again by induction on n that all representations can be realized over $\mathbb{Q}(\mathbf{u})$. □

Let $k := \mathbb{Q}(\zeta_m)$. The symmetric group $\mathfrak{S}_m$ on letters $\{0,\dots,m-1\}$ acts on $N := k(\mathbf{u})$ by permutation of the variables $(u_0,\dots,u_{m-1})$. Let $K := k(\mathbf{u})^{\mathfrak{S}_m}$ be the fixed field, generated over $k(x_1,x_2)$ by the elementary symmetric polynomials $f_j(\mathbf{u})$ in $\mathbf{u}$. Since $\mathcal{H}(W_n;\mathbf{u})$ is already defined over K, each automorphism of N/K permutes the set of representations of $\mathcal{H}(W_n;\mathbf{u})$ over N, hence by Proposition 4.1 in particular it acts on $\mathrm{Irr}(\mathcal{H}(W_n;\mathbf{u}))$. The group $\mathfrak{S}_m$ also acts in a natural way on the set $\{\alpha \vdash_m n\}$ of m-part partitions of n by permuting the m parts. We have the following compatibility between these two actions:

Lemma 4.2. *Let $\sigma \in \mathfrak{S}_m$ and $\alpha \vdash_m n$. Then the character $\chi_\alpha \in \mathrm{Irr}(\mathcal{H}(W_n;\mathbf{u}))$ satisfies $\chi_\alpha = \sigma(\chi_{\sigma(\alpha)})$.*

Proof. For $n = 1$ the (linear) characters of $\mathcal{H}(W_1;\mathbf{u})$ are given by $\chi_\alpha(T) = u_l$ if $|\alpha_j| = \delta_{j,l}$ (see the proof of Proposition 4.1). Let $\sigma \in \mathfrak{S}_m$ and assume that $l = \sigma(l')$. Then $\sigma(\alpha) = (\alpha_{\sigma(j)})$, so $|\sigma(\alpha)_j| = \delta_{j,l'}$, and $\chi_{\sigma(\alpha)}(T) = u_{l'}$. Thus, $\sigma(\chi_{\sigma(\alpha)})(T) = \sigma(u_{l'}) = u_l$ as claimed. Now assume by induction that the assertion is true for $n-1$. Then it also holds for n by Lemma 3.3(b). □

Note that for any admissible specialization $u_j \mapsto u'_j$ $(0 \leq j \leq m-1)$ the Galois group of $k(\mathbf{u}')/k(f_j(\mathbf{u}') \mid 1 \leq j \leq m)$ is in a natural way a subgroup of $\mathfrak{S}_m = \mathrm{Gal}(k(\mathbf{u})/k(f_j(\mathbf{u}) \mid 1 \leq j \leq m))$ and the assertion of Lemma 4.2 remains correct for all elements σ of this Galois group.

It follows from a result of Ariki [2, Prop. 1.6] that the cyclotomic algebra for $G(m,p,n)$ may be obtained as a subalgebra of a specialization of the one for W_n. More precisely, let now $p|m$, $m = pq$, $\mathbf{u} := (u_0,\dots,u_{q-1},x_1,x_2)$

and

$$\tilde{\mathbf{u}} := (\tilde{u}_0, \dots, \tilde{u}_{q-1}, \zeta_p\tilde{u}_0, \dots, \zeta_p\tilde{u}_{q-1}, \dots, \zeta_p^{p-1}\tilde{u}_{q-1}, x_1, x_2), \tag{4.3}$$

where $\zeta_p := \exp(2\pi i/p)$ and $\tilde{u}_j^p = u_j$. Then $\mathcal{H}(G(m,p,n),\mathbf{u})$ is the subalgebra of $\mathcal{H}(W_n,\tilde{\mathbf{u}})$ generated by $T_{\mathbf{t}}^p, T_{\mathbf{t}}^{-1}T_{\mathbf{s}_1}T_{\mathbf{t}}, T_{\mathbf{s}_1}, \dots, T_{\mathbf{s}_{n-1}}$. Note that in the case $n = 2$, p even, this is not the most general generic algebra for W. The latter case will be treated in the next subsection.

In accordance with our previous notation, let $\tau = (0, 1, \dots, m-1) \in \mathfrak{S}_m$, $\eta_p = (0, q, 2q, \dots, (p-1)q)$, and $S_p := \langle \eta_p^{\tau^j} \mid 0 \leq j \leq q-1\rangle \leq \mathfrak{S}_m$ the Galois group of $k(\tilde{\mathbf{u}})/k(\mathbf{u})$. For a multi-partition $\alpha \vdash_m n$ we define its *p-symmetry group*

$$S_p(\alpha) := \langle C_{S_p}(\alpha), \tau^q\rangle \leq S_p.$$

Note that $S_p(\alpha)$ acts naturally on $k(\tilde{\mathbf{u}})$ as a subgroup of $S_p = \mathrm{Gal}(k(\tilde{\mathbf{u}})/k(\mathbf{u}))$. We can now describe the character fields of the irreducible characters of cyclotomic algebras of imprimitive complex reflection groups. Note that Assumption 2.3 is satisfied in the case of imprimitive groups by [3,8,2].

THEOREM 4.4. *Let $n \geq 3$, $W = G(m,p,n)$ and $\alpha \vdash_m n$. Then the character field of $\chi_\alpha \in \mathrm{Irr}(\mathcal{H}(W,\mathbf{u}))$ is the fixed field $k(\tilde{\mathbf{u}})^{S_p(\alpha)}$ of the p-symmetry group of α.*

PROOF. We split the proof into two steps.

In the first step we assume that $s_p(\alpha) = p$. By Proposition 4.1 the character field of the irreducible character χ_α of $\mathcal{H}(W_n,\tilde{\mathbf{u}})$ is contained in $k(\tilde{\mathbf{u}})$. But Lemma 4.2 shows that χ_α is even $k(\mathbf{u})$-rational since $S_p(\alpha) = S_p = \mathrm{Gal}(k(\tilde{\mathbf{u}})/k(\mathbf{u}))$. Thus the irreducible constituents of the restriction of χ_α to $\mathcal{H}(G(m,p,n),\mathbf{u})$ can only be algebraically conjugate among each other. Now Lemma 3.5(b) forces them to be $k(\mathbf{u})$-rational themselves.

In the second step we let $\tilde{p} := s_p(\alpha)$ be arbitrary. By the first step, the constituents of the restriction of χ_α to $\mathcal{H}(G(m,\tilde{p},n),\tilde{\mathbf{u}}^{\tilde{p}})$ are $k(\tilde{\mathbf{u}}^{\tilde{p}})$-rational since $s_{\tilde{p}}(\alpha) = \tilde{p}$. Moreover, by Lemma 3.5(b), they cannot be conjugate among each other. Now by Lemma 4.2 an element $\sigma \in S_p = \mathrm{Gal}(k(\tilde{\mathbf{u}})/k(\mathbf{u}))$ fixes the character χ_α of $\mathcal{H}(G(m,p,n),\mathbf{u})$ precisely if $\sigma \in S_p(\alpha)$ (remember that τ^q fixes χ_α). $\square$

Theorem 4.4 shows that, in contrast to the case of real reflection groups, the character field can become an arbitrarily large extension of the ground field:

Example 4.5. Let $\alpha = (1,2,\dots,q,0,0\dots,0) \vdash_m n$ where $n = q(q+1)/2$ and $m = pq$, and χ_α the corresponding irreducible character of $\mathcal{H}(G(m,p,n),\mathbf{u})$. Then $S_p(\alpha) = \langle \tau^q \rangle$ and hence the character field of χ_α is generated over $k(\mathbf{u})$ by

$$\{ \sqrt[p]{u_0/u_j} \mid 1 \le j \le q-1 \} .$$

In particular it is not a cyclic extension if $q \ge 3$.

4B. Two-dimensional imprimitive groups

The two-dimensional imprimitive groups differ from the other imprimitive reflection groups in that the generic cyclotomic algebra of $G(2m,2p,2)$ has one additional exceptional parameter, thus cannot be obtained as a subalgebra of $\mathcal{H}(W_2,\mathbf{u})$, and secondly the field of rationality for $G(m,m,2)$ is of index 2 in $\mathbb{Q}(\zeta_m)$. Thus the above determination of the character fields of generic cyclotomic algebras does not apply immediately to this case. On the other hand, the irreducible representations have dimension at most 2, so may easily be constructed explicitly. The linear characters are always rational, and the 2-dimensional characters are indexed by multi-partitions $\alpha \vdash_m 2$ with $\alpha_j = 0$ unless $j = j_1$ or $j = j_2 \ne j_1$, and $\alpha_{j_1} = \alpha_{j_2} = 1$.

Theorem 4.6. *Let $W = G(m,p,2)$ and $\alpha \vdash_m 2$ with $\alpha_{j_1} = \alpha_{j_2} = 1$. Then the character field of $\chi_\alpha \in \mathrm{Irr}(\mathcal{H},(u_0,\dots,u_{q-1},x_1,x_2,y_1,y_2))$ is $k(\mathbf{u},\sqrt{x_1x_2y_1y_2}\,\sqrt[p]{u_{j_1}/u_{j_2}})$, where $x_1 = y_1$, $x_2 = y_2$ if p is odd.*

Proof. If $p = m$ then $G(m,m,2) = W(I_2(m))$ is the dihedral Coxeter group, with generators s_1, s_2 say. The 2-dimensional representations of $\mathcal{H}(W,(x_1,x_2,y_1,y_2))$ (with $y_j = x_j$ if m is odd) are given by

$$T_{\mathbf{s}_1} \mapsto \begin{pmatrix} x_1 & 0 \\ 1 & x_2 \end{pmatrix}, \quad T_{\mathbf{s}_2} \mapsto \begin{pmatrix} y_1 & b_l \\ 0 & y_2 \end{pmatrix}, \qquad \text{with } 1 \le l \le \frac{m}{2},$$

where $b_l = -x_1y_1 - x_2y_2 + (\zeta_m^l + \zeta_m^{-l})\sqrt{x_1x_2y_1y_2}$, thus the result follows. If $p < m$ is odd, no additional parameter occurs and the arguments in the proof of Theorem 4.4 apply unchanged. So it remains to consider $G(m,p,2)$ with m,p even and $p<m$. By [15, Prop. 3.8] the corresponding generic cyclotomic algebras can all be realized as subalgebras of suitable specializations of $\mathcal{H}(G(m,2,2),\mathbf{u})$. The irreducible representations of this algebra were determined in loc. cit., 3B, and the claim follows from these explicit matrices. □

4C. The primitive groups

For the primitive Coxeter groups $\mathfrak{S}_n$, $W(E_6)$, $W(E_7)$, $W(E_8)$ and $W(F_4)$ the character fields are known by [5], for $W(H_3)$ and $W(H_4)$ they can be deduced from [1]. In the case of so-called cyclotomic Weyl groups the following result had already been obtained in [8, Satz 6.3].

Theorem 4.7. *Let W be a finite primitive complex reflection group satisfying Assumption 2.3, and χ an irreducible character of $\mathcal{H}_L(W,\mathbf{u})$. Then the character field of χ is contained in $k(\mathbf{u})$ unless $(W,\chi(1))$ are as in Tables 8.1 or 8.2.*

Proof. We discuss the different possibilities case by case. First, it is well known that the assertion holds with $k=\mathbb{Q}$ for the case $W=\mathfrak{S}_n$ of symmetric groups. So we may assume that W is one of the 34 exceptional primitive reflection groups $G_4,\dots,G_{37}$ (in the notation of Shephard and Todd [19]). First we consider the two-dimensional groups, so $W=G_j$ for some $4\le j\le 22$. The proper parabolic subgroups of W are 1-dimensional and the irreducible characters of cyclotomic algebras for 1-dimensional reflection groups are all rational. Thus, by our discussion in Section 2C, only characters with equal restriction to all proper parabolic subgroups can possibly not be $k(\mathbf{u})$-rational. On the other hand, we may compute certain character values on a central element by (2.6) (this was already done for G_7, G_{11}, G_{19} in Tables 3,5,7 of [15], and the values in the other cases can be obtained by specialization according to Tables 4,6,8 of loc. cit.). With these two observations, the result already follows for $j\ne 12,13,14,15,20,21,22$.

For $j=12,13,14,15$ only the character fields of 2-dimensional irreducible characters cannot be deduced by the previous arguments, but the corresponding representations can easily be determined explicitly. In fact, it suffices to do this for $W=G_{15}$ since the other three groups are subgroups of G_{15}, their cyclotomic algebras are subalgebras of suitable specializations of $\mathcal{H}(G_{15}, \mathbf{u})$, and their irreducible representations can be obtained by specialization and restriction of those of $\mathcal{H}(G_{15}, \mathbf{u})$ (see Table 6 in [15]). For $\mathcal{H}(G_{15}, \mathbf{u})$, six 2-dimensional irreducible representations are rational (they factor over the cyclotomic algebra of the imprimitive group $G(2,1,2)$), while the other twelve require the adjunction of a 4th root as in Table 8.1.

For $W = G_{21}$ the 2- and 3-dimensional representations have to be constructed; they involve the irrationalities stated in the table. For $j = 20, 22$ we may again descend from the cyclotomic algebra for G_{21}.

Finally let W be one of $G_{23}, \dots, G_{37}$. By induction and the results of Section 4A we may further assume that the rationality properties of all irreducible characters of proper parabolic subalgebras of $\mathcal{H}(W, \mathbf{u})$ are known. By comparing multiplicities in induced rational characters of parabolic subalgebras we obtain the desired rationality statement for most irreducible characters. For the remaining ones, we at least obtain that certain sums of two or three characters of the same degree must be rational over $k(\mathbf{u})$. For these pairs or triples of characters, we can compute the values on $T_{\boldsymbol{\beta}}$ by the method of Springer (2.6). It turns out that in all cases the values on $T_{\boldsymbol{\beta}}$ are either in $k(\mathbf{u})$ and different for the pair of characters in question, or they are algebraically conjugate over $k(\mathbf{u})$, involving the irrationalities stated in the theorem. We give details of the argument in the case $W = G_{34}$, $\mathcal{H} = \mathcal{H}(W, x)$, the other ones being similar and easier. Here $k = \mathbb{Q}(\zeta_3)$. By using the character tables provided by the computer algebra system Chevie [11] we first check that all irreducible characters of W are uniquely determined by their restrictions to the parabolic subalgebras of types $G(3,3,5)$ and $W(A_5) = \mathfrak{S}_6$, except for certain m_i-tuples of characters of degree d_i with

$$(m_i, d_i) \in \{(2, 896) \text{ (2 pairs)}, (2, 384) \text{ (4 pairs)}, (3, 729) \text{ (2 triples)}\}.$$

Since the characters of these subalgebras are $k(x)$-rational, it follows that at most the above characters can possibly have a character field not contained in $k(x)$. Since the characters of degrees 896 and 384 may be distinguished by their multiplicities in the induced irrational characters of the subalgebra of type G_{33}, their character field over k is equal to $k(\sqrt{x})$. Next we compute the values of the remaining characters on the central element $T_{\boldsymbol{\beta}}$ by (2.6). Here $\boldsymbol{\beta} = (\mathbf{s}_1 \dots \mathbf{s}_6)^7$ by [9, Table 4]. The values of the characters of degree 729 turn out to involve $\sqrt[3]{x}$. This completes the proof for $W = G_{34}$. $\square$

4D. Some consequences

The results in Theorems 4.4, 4.6 and 4.7 show that in all cases the character fields can be generated by roots of suitable monomials in the parameters. Moreover, the order of the root is always bounded by the order of the group of roots of unity in the field of definition k of W; it would be nice to understand this fact by a general argument:

Corollary 4.8. *Let W be a finite complex reflection group with character field k. Let $\mu(k)$ be the group of roots of unity in k. Then the character field of $\mathcal{H}(W, \mathbf{u})$ is contained in*

$$k\Big(\Big(\zeta_{d_s}^{-j} u_{s,j}\Big)^{1/|\mu(k)|} \mid s \in S,\, 0 \le j \le d_s - 1\Big)\,,$$

where $\zeta_{d_s} = \exp(2\pi i/d_s)$.

In the recent preprint [17] Opdam has given a general proof that under Assumption 2.3 the character field of $\mathcal{H}(W, \mathbf{u})$ over $\bar{\mathbb{Q}}(\mathbf{u})$ can be generated by suitable roots of monomials in the parameters.

Another consequence of our results is a characterization of those cases where the reflection representation of the 1-parameter cyclotomic algebra is rational:

Corollary 4.9. *Let W be an n-dimensional irreducible finite complex reflection group with character field k. Then the reflection representation of $\mathcal{H}(W, x)$ is rational over $k(x)$ if and only if W can be generated by n of its reflections.*

Proof. The group $G(m, p, n)$ is generated by n of its reflections if and only if $p = 1$ or $p = m$. The reflection character of the group $G(m, p, n)$ is parametrized by the multi-partition $\alpha = ((n-1), (1), -, \ldots, -)$. Since $S_p(\alpha) = \langle \tau^q \rangle$ is strictly contained in S_p if $p \neq 1, m$ the assertion follows from Theorems 4.4 and 4.6. For the primitive groups, the explicit results show that the reflection character of $\mathcal{H}(W, x)$ is non-rational precisely for

$$W \in \{G_7, G_{11}, G_{12}, G_{13}, G_{15}, G_{19}, G_{22}, G_{31}\}. \tag{4.10}$$

But these are the primitive groups not generated by n of their reflections, where n denotes the dimension of the irreducible reflection representation of W. □

We will come back to the reflection groups generated by n of their reflections in Section 7.

REMARK 4.11. Our results also show that the character field of $\mathcal{H}(W, \mathbf{u})$ is contained in $k(\mathbf{u})$ precisely for the irreducible groups

$$\mathfrak{S}_n,\ G(m,1,n),\ G(m,m,n)\ (n \geq 3),\ G(m,m,2)\ (m \text{ odd}),$$
$$G_4,\ G_{25},\ G_{28} = W(F_4),\ G_{35} = W(E_6).$$

5. Determination of Splitting Fields

The results of the previous section also allow to find the splitting fields of generic cyclotomic algebras.

PROPOSITION 5.1. *Let W be a finite irreducible complex reflection group different from $G_{12}, G_{13}, G_{20}, G_{22}, G_{31}$ and satisfying Assumption 2.3. Let $\chi \in \mathrm{Irr}(\mathcal{H}(W, \mathbf{u}))$ with character field $N \geq k(\mathbf{u})$. Then there exists a representation over N affording χ.*

For W of type $G_{12}, G_{13}, G_{20}, G_{22}, G_{31}$ there exists such a representation over an extension of degree at most 2 of N.

PROOF. The main ingredient will be Lemma 2.7. Let first $G = \mathfrak{S}_n$. Then all character fields are equal to $\mathbb{Q}(\mathbf{u})$ and any non-linear irreducible character occurs with multiplicity 1 in the permutation character of some proper Young subgroup. Thus the result follows by induction from Lemma 2.7. Let now $G = G(m,p,n)$. The assumptions of Lemma 2.7 are satisfied by Lemma 3.5 since by the previous case $k(\mathbf{u})$ is a splitting field for the representations of the Hecke algebra of $\mathfrak{S}_n$. Thus we are left with the primitive exceptional groups. For these it can be checked from the explicit tables that Lemma 2.7 applies except possibly if

$$(W, \chi(1)) \in \{(G_{12}, 4), (G_{13}, 4), (G_{20}, 6), (G_{22}, 4), (G_{22}, 6), (G_{31}, 36)\},$$

when all multiplicities in induced rational representations from proper parabolic subalgebras are divisible by 2,2,2,2,3,2 respectively. The characters of degree 6 of G_{22} can be obtained by specializing $y_j \mapsto \zeta_3^j$ in a 6-dimensional representation of $\mathcal{H}(G_{21}, \mathbf{u})$, hence can be written over an extension of degree 2 of the character field. Thus they can be realized over the character field. Since each of the remaining exceptional characters χ occurs with multiplicity 2 in some representation which can be written over the character field of χ, the last statement of the proposition follows. □

This allows to obtain the following description of splitting fields:

THEOREM 5.2. *Let W be a finite complex reflection group satisfying Assumption 2.3 with character field k. Then a splitting field L of $\mathcal{H}(W,\mathbf{u})$ is contained in*

$$K_W := k(\,(\exp(-2\pi ij/d_s)u_{s,j})^{1/|\mu(k)|} \mid s\in S,\, 0\le j\le d_s-1)\,.$$

PROOF. By Proposition 5.1 and Corollary 4.8 we only have to consider the exceptional characters of the irreducible groups $G_{12}, G_{13}, G_{20}, G_{22}, G_{31}$ listed in the previous proof. For $W = G_{13}$ the representations of $\mathcal{H}(W)$ of degree 4 over K_W can be constructed explicitly. The representation of degree 36 of $W = G_{31}$ has multiplicity 1 in an induced representation of the reflection subgroup $G(4,2,3)$, thus the corresponding representation of $\mathcal{H}(W)$ can be written over K_W.

The representation of $\mathcal{H}(G_{12})$ of degree 4 can be obtained by specializing $y_j \mapsto \zeta_3^j$ in a rational 4-dimensional representation of $\mathcal{H}(G_{14})$ as follows. Let

$$\begin{aligned}\mathcal{H} = \langle T_1, T_2 \mid &(T_1T_2)^4 = (T_2T_1)^4,\\ &(T_1+1)(T_1-x) = (T_2-y_1)(T_2-y_2)(T_2-y_3) = 0\rangle\end{aligned}$$

be the cyclotomic algebra for the complex reflection group G_{14} and let $\tilde{\mathcal{H}}$ denote its image under the specialization $y_j \mapsto \zeta_3^j$. Then $\mathcal{H}(G_{12},(x,-1))$ is the subalgebra of $\tilde{\mathcal{H}}$ generated by the elements T_1, $T_2T_1T_2^{-1}$, $T_2^{-1}T_1T_2$. Now

$$T_1 \mapsto \begin{pmatrix} x & 0 & 0 & 0\\ 0 & x & 0 & 0\\ 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\end{pmatrix},\quad T_2 \mapsto \begin{pmatrix} y_1 & 0 & 0 & y_1^2x^2+y_2y_3\\ 0 & y_1 & -1 & y_2-2xy_1+y_3\\ 0 & 0 & y_2 & (y_2-xy_1)(y_3-y_2)\\ 0 & 0 & 0 & y_3\end{pmatrix}$$

defines a 4-dimensional irreducible representation of $\mathcal{H}$. Under the above specialization this yields the irreducible 4-dimensional representation of

$\mathcal{H}(G_{12},(x,-1))$, over the field $\mathbb{Q}(x,\sqrt{-3})$. This gives rise to an 8-dimensional representation over the field $\mathbb{Q}(x)$. The endomorphism ring of this representation may easily be computed from the explicit matrices given above. The constant coefficient of the characteristic polynomial of an element of this (4-dimensional) endomorphism ring is of the form

$$\left(z_1^2+3z_2^2-(z_3^2+3z_4^2)x\right)^4,$$

with z_j in the ground field. Clearly, with $z_j=\sqrt{x}z_{j+2}$ for $j=1,2$, this quadratic form does represent zero nontrivially. Thus $\mathbb{Q}(\sqrt{-2},\sqrt{x})$ is a splitting field for $\mathcal{H}(G_{12},(x,-1))$.

Similarly, the cyclotomic algebra for G_{22} may be obtained by specializing $y_j\mapsto\zeta_3^j$ in the one for G_{21}. A computation as above shows that the 4-dimensional irreducible representations of $\mathcal{H}(G_{22},(x,-1))$ can be realized over $\mathbb{Q}(\sqrt{x})$.

Finally, the representations of $\mathcal{H}(G_{20})$ of degree 6 can be obtained by specializing $x_j\mapsto(-1)^j$ in representations of $\mathcal{H}(G_{21},\mathbf{u})$ involving the fourth roots of unity. Calculations as above lead to the constant coefficient

$$\left(z_1^2+z_2^2+2z_3^2+2z_4^2\right)^6$$

of the characteristic polynomial of an element of the endomorphism ring, which represents zero nontrivially over $\mathbb{Q}(\sqrt{-3})$. Thus K_W is a splitting field in this case as well. □

6. An Observation on Fake Degrees

6A. Fake degrees

Let W be a finite complex reflection group on the complex vector space V. The ring of invariants of W in the symmetric algebra $S(V)$ of V is a polynomial ring, generated by homogeneous invariants of degrees $d_1,\dots,d_n$, with $n=\dim(V)$. It follows from Molien's formula for $S(V)^W$ that the Poincaré polynomial P_W of W is given by

$$(x-1)^nP_W:=\left(\frac{1}{|W|}\sum_{w\in W}\frac{\det_V(w)}{\det_V(x-w)}\right)^{-1}=\prod_{j=1}^n(x^{d_j}-1),$$

where $\det_V$ denotes the determinant on V. In particular, P_W is a polynomial in x of degree $N^* := \sum_{j=1}^n (d_j - 1)$. For an irreducible character $\chi \in \mathrm{Irr}(W)$ the *fake degree* is defined as

$$R_\chi := (x-1)^n P_W \frac{1}{|W|} \sum_{w\in W} \frac{\det_V(w)\chi(w)}{\det_V(x-w)} \in \mathbb{Z}[x]\,. \tag{6.1}$$

The fake degree has the following symmetry property:

$$R_\chi(x) = x^{N^*} R_{\overline{\det\otimes\chi}}(x^{-1}), \tag{6.2}$$

where $^-$ denotes complex conjugation. Indeed, $P_W(x^{-1}) = x^{-N^*} P_W(x)$, so

$$\begin{aligned} R_\chi(x^{-1}) =& x^{-N^*}(x-1)^n(-x)^{-n} P_W \frac{1}{|W|} \sum_{w\in W} \frac{\det_V(w)\chi(w)}{\det_V(x^{-1}-w)} \\ =& x^{-N^*}(x-1)^n P_W \frac{1}{|W|} \sum_{w\in W} \frac{\det_V(w)\chi(w)}{\det_V(wx-1)} = x^{-N^*} R_{\overline{\det\otimes\chi}}(x). \end{aligned}$$

For example we have $1 = R_1(x^{-1}) = x^{-N^*} R_{\overline{\det}}(x)$, so $R_{\overline{\det}}(x) = x^{N^*}$.

It is possible to express the sum of exponents of the fake degree as follows. By looking at eigenvalues of elements it is easy to see that modulo $(x-1)^2$ the fake degree (6.1) becomes

$$R_\chi(x) = \frac{P_W}{|W|}\left(\chi(1) + \sum_{r\in\mathcal{R}} \frac{\det(r)\chi(r)}{(x-\det(r))}(x-1)\right) \mod (x-1)^2\,.$$

Differentiation yields

$$R'_\chi(x) = \frac{P'_W}{|W|}\chi(1) + \frac{P_W}{|W|}\sum_{r\in\mathcal{R}} \frac{\det(r)\chi(r)}{x-\det(r)} \mod (x-1).$$

Since $P_W = \prod_{j=1}^n (x^{d_j-1} + x^{d_j-2} + \ldots + 1)$ we obtain by evaluating at $x=1$ and using $|W| = \prod_j d_j$ that

$$R'_\chi(1) = \chi(1)N^*/2 + \sum_{r\in\mathcal{R}} \frac{\det(r)\chi(r)}{1-\det(r)}. \tag{6.3}$$

6B. Semi-palindromicity

In the case of real reflection groups W, Beynon and Lusztig [7] observed from their tables that most (but not all) fake degrees have an additional symmetry: they are palindromic. Moreover, the ones which are not palindromic are precisely those for which the corresponding character of the 1-parameter Hecke algebra $\mathcal{H}(W,x)$ is not rational over $\mathbb{Q}(x)$. An a priori explanation of this connection was later given by Opdam [16]. A similar phenomenon can be observed for complex reflection groups. Let us say that $\chi \in \operatorname{Irr}(W)$ has a *semi-palindromic fake degree*, or by abuse of notation that R_χ is semi-palindromic, if there exists a $c = c_\chi \in \mathbb{N}$ such that

$$R_\chi(x) = x^c R_{\overline{\chi}}(x^{-1})\,.$$

Again, it turns out that many, but not all irreducible characters of complex reflection groups have semi-palindromic fake degree. It is possible to extend this to a statement about all irreducible characters. But, although the fake degrees can be defined entirely in terms of the group W it seems necessary to appeal to the cyclotomic Hecke algebra attached to W to phrase the result.

Let us first make the following observation on the exponent c above; here $\mathcal{R}$ denotes the set of all reflections in W:

Proposition 6.4. *Let $\chi, \psi \in \operatorname{Irr}(W)$ be such that $\chi(1) = \psi(1)$, $\chi(r) = \psi(r^{-1})$ for all reflections $r \in \mathcal{R}$, and $R_\chi(x) = x^c R_\psi(x^{-1})$. Then $c = N^* - \sum_{r\in\mathcal{R}} \chi(r)/\chi(1)$. If all reflections in W have order 2 then we also have $2c = R'_\chi(1)/\chi(1)$.*

Proof. By differentiating the equality $R_\chi(x) = x^c R_\psi(x^{-1})$ with respect to x and evaluating at $x = 1$ we get

$$R'_\chi(1) = c\psi(1) - R'_\psi(1).$$

On the other hand, adding (6.3) for χ and ψ and inserting the assumptions gives

$$R'_\chi(1) + R'_\psi(1) = \chi(1)N^* + \sum_{r\in\mathcal{R}} \chi(r)\left(\frac{\det(r)}{1-\det(r)} + \frac{\det(r^{-1})}{1-\det(r^{-1})}\right).$$

The formula for c follows by using that det is a homomorphism and comparing with the first equation.

Now assume that all reflections in W have order 2. Then (6.3) becomes

$$R'_\chi(1) = \frac{1}{2}\left(\chi(1)N^* - \sum_{r\in\mathcal{R}} \chi(r)\right),$$

proving the second statement. □

Since $N^* = |\mathcal{R}|$, the value of c above can also be written $\sum_{r\in\mathcal{R}}(1 - \chi(r)/\chi(1))$.

6C. A symmetry property

Corollary 4.8 implies that the character field of any irreducible character of the 1-parameter cyclotomic algebra $\mathcal{H}(W,x)$ is contained in $k(y)$, where $y^{|\mu(k)|} = x$. Let δ be the automorphism of $k(y)/k(x)$ induced by $y \mapsto \exp(2\pi i/|\mu(k)|)y$. Then δ also acts on $\mathrm{Irr}(\mathcal{H}(W,x))$, so via specialization induces a permutation of the set $\mathrm{Irr}(W)$. The element δ together with complex conjugation generate a dihedral group, so the product of δ with complex conjugation is an involution on $\mathrm{Irr}(W)$. We have the following extension of the previously mentioned observation on semi-palindromicity:

THEOREM 6.5. *Let W be a finite complex reflection group satisfying Assumption 2.3 with set of reflections $\mathcal{R}$ and let $\chi \in \mathrm{Irr}(W)$. Then*

$$R_\chi(x) = x^c R_{\delta(\bar\chi)}(x^{-1}) \qquad \textit{with } c = N^* - \sum_{r\in\mathcal{R}} \chi(r)/\chi(1), \tag{6.6}$$

where δ is the permutation of the set $\mathrm{Irr}(W)$ defined above.

PROOF. If W is of type A_n then all characters of $\mathcal{H}(W,x)$ are $\mathbb{Q}(x)$-rational and all fake degrees are palindromic, so the assertion holds. If W is of exceptional type G_j, $4 \le j \le 37$, the fake degrees can be computed explicitly from (6.1) and the result follows by comparison with the character fields obtained in Theorem 4.7. (Note that for the evaluation of (6.1) it is only necessary to sum over representatives of the conjugacy classes, weighted by the class lengths.)

So we may assume that $W = G(m,p,n)$. In [14, Bem. 2.10] we computed the fake degree of $\chi_\alpha \in \mathrm{Irr}(W_n)$ as

$$R_\alpha(x) = \Gamma(\alpha) \prod_{j=0}^{m-1} x^{j|\alpha_j|}$$

where $\Gamma(\alpha)$ is a palindromic polynomial in x depending only on the set $\{\alpha_j\}$ of components of α. Furthermore, by [14, 5B] the fake degree of $\chi_\alpha \in \mathrm{Irr}(G(m,p,n))$ is given by

$$R_{\chi_\alpha}(x) = \frac{x^{nq}-1}{x^{nm}-1} \frac{1}{s_p(\alpha)} \sum_{j=0}^{p-1} R_{\tau^{jq}(\alpha)}(x)$$

where $m = pq$. To prove the assertion it hence suffices to show that

$$\sum_{j=0}^{m-1} j|\alpha_{j+lq}| + \sum_{j=0}^{m-1} j|(\delta(\overline{\alpha}))_{m-lq+j}| \tag{6.7}$$

is a constant independent of l (here and later on the indices have to be taken modulo m). By our above description of $\mathrm{Irr}(G(m,p,n))$ complex conjugation acts on characters by $\chi_\alpha \mapsto \chi_{\overline{\alpha}}$ where $\overline{\alpha} = (\alpha_0, \alpha_{m-1}, \ldots, \alpha_1)$. Also, δ acts via $\alpha \mapsto \delta(\alpha) = (\alpha'_j)$ with $\alpha'_j = \alpha_{j-q}$ if $q|j$ and $\alpha'_j = \alpha_j$ otherwise. Thus, $\delta(\overline{\alpha}) = \overline{\alpha}'$ with $\overline{\alpha}'_j = \alpha_{m-j-q}$ if $q|j$ and $\overline{\alpha}'_j = \alpha_{m-j}$ otherwise. We write $n_j := |\alpha_j|$. Now let $0 \le l \le p-1$ be fixed. Then (6.7) becomes

$$\begin{aligned}
&\sum_{j=0}^{m-1} j n_{j+lq} + \sum_{j \not\equiv 0 \pmod q} j n_{m-j-(p-l)q} + \sum_{j\equiv 0 \pmod q} j n_{m-q-j-(p-l)q} \\
&= \sum_{j=0}^{m-1} j n_{j+lq} + \sum_{j=1}^{m} j n_{m-j+lq} + \sum_{j\equiv 0 \pmod q} (j-(q+j)) n_{m-q-j+lq} \\
&= m \sum_{j=0}^{m-1} n_j - q \sum_{j \equiv 0 \pmod q} n_j \,,
\end{aligned}$$

so it is indeed independent of l, proving the identity for type $G(m,p,n)$.

The value given for c follows from Proposition 6.4. Indeed, since the character $\delta(\overline{\chi})$ of $\mathcal{H}(W,x)$ is algebraically conjugate to $\bar{\chi}$, we have $\chi(1) = \delta(\overline{\chi})(1)$. Moreover, by the defining relations for $\mathcal{H}(W,x)$ the value of any character on an element $T_{\mathbf{s}}$ with s a generating reflection is $k(x)$-rational, hence $\delta(\overline{\chi})(T_{\mathbf{s}}) = \overline{\chi}(T_{\mathbf{s}})$. Specialization to W shows that $\delta(\overline{\chi})(s) = \overline{\chi}(s)$ for all generating reflections, and hence for all reflections of W. Thus the assumptions of Proposition 6.4 are satisfied. □

REMARK 6.8. It would be nice to have an a priori proof of this theorem, not relying on the classification of the character fields. After reading a preliminary version of this paper, E. Opdam informed the author that he can define (in the spirit of [16]) a map δ^* on the character group of an arbitrary finite complex reflection group W such that the formula in Theorem 6.5 holds with δ replaced by δ^*, without any assumption on the cyclotomic Hecke algebra. However, at the moment there is no a priori proof that δ^* respects irreducibility. Under an additional assumption on the topological braid group of W (which is stronger than our Assumption 2.3) Opdam has a general proof of Theorem 6.5 (see [17, Prop. 7.4]).

Example 6.9. Let $W = G_{32}$ be the primitive 4-dimensional complex reflection group generated by reflections of order 3. Let $\chi \in \mathrm{Irr}(W)$ have degree 64 and $\psi := \delta(\bar{\chi})$. Then $R_\chi(x) = x^c R_\psi(x^{-1})$ by Theorem 6.5. But one finds $\{R'_\chi(1), R'_\psi(1)\} = \{2240, 2560\}$, so $2c = (R'_\chi(1) + R'_\psi(1))/64 = 75 \neq R'_\chi(1)/\chi(1)$. Hence the assumption in the second part of Proposition 6.4 cannot be dropped.

Clearly, if an irreducible character of the cyclotomic algebra $\mathcal{H}$ specialized according to (2.2) has character field not contained in K, the same is true for the corresponding character of the generic cyclotomic algebra. Hence, as an immediate consequence of the preceding results we obtain:

COROLLARY 6.10. *Let W be a finite irreducible complex reflection group with field of definition k. Let χ be an irreducible complex character of W such that the corresponding character of the generic cyclotomic algebra $\mathcal{H}(W,\mathbf{u})$ has character field contained in $k(\mathbf{u})$. Then R_χ is semi-palindromic.*

Example 6.11. By Theorem 6.5 a rational character of the 1-parameter cyclotomic algebra has semi-palindromic fake degree. However, the converse is not always true. Let $\alpha = ((2),(1),(1)^2,-) \vdash_4 5$ and $\chi_\alpha \in \mathrm{Irr}(G(4,2,5))$ a corresponding irreducible character. Then by what we remarked above about the form of the fake degree, R_{χ_α} is semi-palindromic but the corresponding character of $\mathcal{H}(G(4,2,5),x)$ is not rational by Theorem 4.4.

Nevertheless the following can be checked (see Corollary 7.2 for an extension):

Proposition 6.12. *Let W be a finite irreducible complex reflection group with field of definition k and let ρ be the reflection character of W. Then R_ρ is semi-palindromic if and only if the corresponding character of the 1-parameter cyclotomic Hecke algebra $\mathcal{H}(W,x)$ has character field contained in $k(x)$.*

Proof. One direction is clear by Theorem 6.5. Thus we may assume that the character field of ρ is not contained in $k(x)$. If W is of exceptional type, the assertion follows by inspection. So now let $W = G(m,p,n)$ with $p \neq 1, m$ and $\alpha = ((n-1),(1),-,\dots,-)$ (see Corollary 4.9). Then up to a palindromic factor R_{χ_α} is equal to $x + x^{nq+1} + \dots + x^{(p-1)nq+1}$, while the fake degree of the complex conjugate character equals $x^{m-1} + x^{qn-1} + \dots + x^{(p-1)nq-1}$ times the same factor. This proves the assertion. □

Remark 6.13. Let W be a complex reflection group on V with degrees $d_1 \leq \dots \leq d_n$ and ρ the character of the reflection representation. Write

$$R_\rho = \sum_{j=1}^{n} x^{c_j}, \qquad R_{\overline{\rho}} = \sum_{j=1}^{n} x^{d_j - 1}, \tag{6.14}$$

with $1 = c_1 \leq \dots \leq c_n$. The $d_j - 1$ are the so-called exponents of W, the c_j are the coexponents. It was observed by Orlik and Solomon [18] (case by case) that $(d_j - 1) + c_{n-j+1} = d_n$ for $j = 1,\dots,n$ if and only if W can be generated by n reflections. This also follows from our Corollary 4.9 and Proposition 6.12. Indeed, W can be generated by n reflections if and only if the reflection character ρ of $\mathcal{H}(W,x)$ is rational if and only if R_ρ is semi-palindromic, which by (6.14) holds precisely if $(d_j - 1) + c_{n-j+1} = d_n$

for $j = 1, \ldots, n$. Although our proof is also case by case, it seems to shed some new light on the observation of Orlik and Solomon.

Conversely, Corollary 4.9 follows from Proposition 6.12 when we assume the result of Orlik and Solomon, and the latter together with Corollary 4.9 gives Proposition 6.12.

7. Well-Generated Reflection Groups

Let us say that a finite n-dimensional irreducible reflection group $W \leq \mathrm{GL}_n(\mathbb{C})$ is *well-generated* if it can be generated by n of its reflections. Thus in particular real reflection groups are well-generated. In the notation of Shephard and Todd [19] the well generated groups are the symmetric groups, the monomial groups $G(m,1,n), G(m,m,n)$, and those primitive groups G_j, $4 \leq j \leq 37$, which are not listed in (4.10). Well-generated reflection groups seem to behave more nicely in several aspects. For example we saw in Corollary 4.9 that the reflection representation of the 1-parameter cyclotomic algebra $\mathcal{H}(W,x)$ is $k(x)$-rational if and only if W is well-generated.

7A. The field of definition of a well-generated reflection group

First we give a general formula for the splitting field of a well-generated complex reflection group W which seems to have escaped notice until now. As above write k for the character field of its reflection representation, which by [4,6] is a (hence minimal) splitting field for W. It can be checked case by case that in this case k is generated over $\mathbb{Q}$ by the coefficients of the characteristic polynomial of a Coxeter element in the reflection representation. Since the eigenvalues of Coxeter elements in well-generated reflection groups can be expressed in a uniform fashion we obtain the following description of k:

Theorem 7.1. *Let W be a well-generated reflection group with degrees $d_1 \leq \ldots \leq d_n$. For a primitive d_n-th root of unity ζ let G be the setwise stabilizer of $(\zeta^{d_j-1} \mid 1 \leq j \leq n)$ in the Galois group* $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$. *Then k is the fixed field $k = \mathbb{Q}(\zeta)^G$.*

Proof. This can be checked case by case from the known tables for k and the d_j. □

Let us give the connection to the properties of the Coxeter element: By a (case-by-case) result of Shephard and Todd [19, 5.4] there exists an element $c \in W$ which is the product in some suitable order over the elements of a set of n generating reflections of W such that its eigenvalues in the reflection representation are given by $(\zeta^{d_j - 1} \mid 1 \leq j \leq n)$ in our above notation (see also [18] or Remark 6.13 for a description of the eigenvalues in terms of the coexponents). Such an element is usually called a Coxeter element of W. Clearly the fixed field of G is now precisely the field generated by the coefficients of the characteristic polynomial of c.

The theorem is no longer true for n-dimensional groups generated by $n+1$ reflections. As an example take $G = G_{13}$ with degrees $d_1 = 8$, $d_2 = 12$, but where $k = \mathbb{Q}(\zeta_8)$.

7B. Cyclotomic algebras for well-generated groups

The general results in the previous sections have the following stronger versions in the case of well-generated groups:

COROLLARY 7.2. *Let W be a well-generated (irreducible) complex reflection group. Then we have:*

(a) The character field of $\mathcal{H}(W, \mathbf{u})$ is contained in $k((\zeta_{d_s}^{-j} u_{s,j})^{1/z} \mid s \in S,\ 0 \leq j \leq d_s - 1)$, where $z := |Z(W)|$.

(b) The field in (a) is a splitting field for $\mathcal{H}(W, \mathbf{u})$.

(c) Let $\chi \in \mathrm{Irr}(W)$. Then R_χ is semi-palindromic if and only if the corresponding character of $\mathcal{H}(W, x)$ has its character field contained in $k(x)$.

PROOF. Part (a) follows by inspection from the results on character fields and Tables 8.1 and 8.2, where the orders of centers have been included. The second part follows from the first part together with Proposition 5.1. In the third part, let χ_x denote the corresponding character of the 1-parameter cyclotomic algebra. By Theorem 6.5 we may assume that χ_x is not $k(x)$-rational. But then W is a primitive group by Remark 4.11. For these the assertion follows by explicit computation of the fake degrees. □

Since $|Z(W)|$ divides $|\mu(k)|$ for (absolutely) irreducible W by Schur's Lemma, the first part is indeed a strengthening of Corollary 4.8. Note that $z = \gcd(d_1, \ldots, d_n)$, and by Theorem 7.1 the field k can be described in terms of the degrees of W, so the same is true for the splitting field in (b).

Table 8.1. Primitive cases.

W	$\|Z\|$	k	$\chi(1)$	irrationality	$\mathbf{u}$
G_5	6	$\mathbb{Q}(\zeta_3)$	(3×) 3	$\sqrt[3]{y_1y_2y_3z_1z_2z_3}$	$(y_{1,2,3}; z_{1,2,3})$
G_6	4	$\mathbb{Q}(\zeta_{12})$	(6×) 2	$\sqrt{x_1x_2z_1z_2}$	$(x_{1,2}; z_{1,2,3})$
G_7	12	$\mathbb{Q}(\zeta_{12})$	(18×) 2 (6×) 3	$\sqrt{x_1x_2y_1y_2z_1z_2}$ $\sqrt[3]{x_1^2x_2y_1y_2y_3z_1z_2z_3}$	$(x_{1,2}; y_{1,2,3}; z_{1,2,3})$
G_8	4	$\mathbb{Q}(i)$	(2×) 4	$\sqrt{z_1z_2z_3z_4}$	$(z_{1,2,3,4})$
G_9	8	$\mathbb{Q}(\zeta_8)$	(12×) 2 (4×) 4	$\sqrt{x_1x_2z_1z_2}$ $\sqrt[4]{x_1^2x_2^2z_1z_2z_3z_4}$	$(x_{1,2}; z_{1,2,3,4})$
G_{10}	12	$\mathbb{Q}(\zeta_{12})$	(12×) 3 (6×) 4	$\sqrt[3]{y_1y_2y_3z_1z_2z_3}$ $\sqrt{y_2y_3z_1z_2z_3z_4}$	$(y_{1,2,3}; z_{1,2,3,4})$
G_{11}	24	$\mathbb{Q}(\zeta_{24})$	(36×) 2 (24×) 3 (12×) 4	$\sqrt{x_1x_2y_1y_2z_1z_2}$ $\sqrt[3]{x_1^2x_2y_1y_2y_3z_1z_2z_3}$ $\sqrt[4]{x_1^2x_2^2y_1^2y_2y_3z_1z_2z_3z_4}$	$(x_{1,2}; y_{1,2,3}; z_{1,2,3,4})$
G_{12}	2	$\mathbb{Q}(\sqrt{-2})$	(2×) 2	$\sqrt{-x_1x_2}$	$(x_{1,2})$
G_{13}	4	$\mathbb{Q}(\zeta_8)$	(4×) 2 (2×) 4	$\sqrt[4]{x_1^2x_2^2u_1u_2^3}$ $\sqrt{u_1u_2}$	$(x_{1,2}; u_{1,2})$
G_{14}	6	$\mathbb{Q}(\zeta_3, \sqrt{-2})$	(6×) 2 (6×) 3	$\sqrt{-x_1x_2y_1y_2}$ $\sqrt[3]{x_1^2x_2y_1y_2y_3}$	$(x_{1,2}; y_{1,2,3})$
G_{15}	12	$\mathbb{Q}(\zeta_{24})$	(12×) 2 (12×) 3 (6×) 4	$\sqrt[4]{x_1^2x_2^2y_1^2y_2^2u_1u_2^3}$ $\sqrt[3]{x_1^2x_2y_1y_2y_3u_1^2u_2}$ $\sqrt{y_2y_3u_1u_2}$	$(x_{1,2}; y_{1,2,3}; u_{1,2})$
G_{16}	10	$\mathbb{Q}(\zeta_5)$	(10×) 4 (5×) 5	$\sqrt{z_1z_2z_3z_4}$ $\sqrt[5]{z_1z_2z_3z_4z_5}$	$(z_{1,2,3,4,5})$
G_{17}	20	$\mathbb{Q}(\zeta_{20})$	(20×) 2 (20×) 4 (10×) 5 (10×) 6	$\sqrt{x_1x_2z_1z_2}$ $\sqrt[4]{x_1^2x_2^2z_1z_2z_3z_4}$ $\sqrt[5]{x_1^3x_2^2z_1z_2z_3z_4z_5}$ $\sqrt{x_1x_2z_2z_3z_4z_5}$	$(x_{1,2}; z_{1,2,3,4,5})$
G_{18}	30	$\mathbb{Q}(\zeta_{15})$	(30×) 3 (30×) 4 (15×) 5 (15×) 6	$\sqrt[3]{y_1y_2y_3z_1z_2z_3}$ $\sqrt{y_2y_3z_1z_2z_3z_4}$ $\sqrt[5]{y_1^2y_2^2y_3z_1z_2z_3z_4z_5}$ $\sqrt[3]{y_1^2y_2^2y_3^2z_1^2z_2z_3z_4z_5}$	$(y_{1,2,3}; z_{1,2,3,4,5})$
G_{19}	60	$\mathbb{Q}(\zeta_{60})$	(60×) 2 (60×) 3 (60×) 4 (30×) 5 (30×) 6	$\sqrt{x_1x_2y_1y_2z_1z_2}$ $\sqrt[3]{x_1^2x_2y_1y_2y_3z_1z_2z_3}$ $\sqrt[4]{x_1^2x_2^2y_1^2y_2y_3z_1z_2z_3z_4}$ $\sqrt[5]{x_1^3x_2^2y_1^2y_2^2y_3z_1z_2z_3z_4z_5}$ $\sqrt[6]{x_1^3x_2^3y_1^2y_2^2y_3^2z_1^2z_2z_3z_4z_5}$	$(x_{1,2}; y_{1,2,3}; z_{1,2,3,4,5})$

Table 8.2. Primitive cases, continued.

W	$\lvert Z\rvert$	k	$\chi(1)$	irrationality	$\mathbf{u}$
G_{20}	6	$\mathbb{Q}(\zeta_3,\sqrt{5})$	$(6\times)\ 3$ $(6\times)\ 4$ $(3\times)\ 6$	$\sqrt[3]{y_1y_2y_3}$ $\sqrt{y_2y_3}$ $\sqrt[3]{y_1y_2y_3}$	$(y_{1,2,3})$
G_{21}	12	$\mathbb{Q}(\zeta_{12},\sqrt{5})$	$(12\times)\ 2$ $(12\times)\ 3$ $(12\times)\ 4$ $(6\times)\ 6$	$\sqrt{x_1x_2y_1y_2}$ $\sqrt[3]{x_1^2x_2y_1y_2y_3}$ $\sqrt[4]{x_1^2x_2^2y_1^2y_2y_3}$ $\sqrt[6]{x_1^3x_2^3y_1^2y_2^2y_3^2}$	$(x_{1,2};y_{1,2,3})$
G_{22}	4	$\mathbb{Q}(i,\sqrt{5})$	$(4\times)\ 2$ $(4\times)\ 4$ $(2\times)\ 6$	$\sqrt{x_1x_2}$ $\sqrt{x_1x_2}$ $\sqrt{x_1x_2}$	$(x_{1,2})$
$G_{23}=W(H_3)$	2	$\mathbb{Q}(\sqrt{5})$	$(2\times)\ 4$	$\sqrt{x}$	$(x,-1)$
G_{24}	2	$\mathbb{Q}(\sqrt{-7})$	$(2\times)\ 8$	$\sqrt{x}$	$(x,-1)$
G_{26}	6	$\mathbb{Q}(\zeta_3)$	$(6\times)\ 8$	$\sqrt{-x_1x_2y_1y_2}$	$(x_{1,2};y_{1,2,3})$
G_{27}	6	$\mathbb{Q}(\zeta_3,\sqrt{5})$	$(4\times)\ 8$ $(6\times)\ 9$	$\sqrt{x}$ $\sqrt[3]{x}$	$(x,-1)$
G_{29}	4	$\mathbb{Q}(i)$	$(4\times)\ 16$	$\sqrt{x}$	$(x,-1)$
$G_{30}=W(H_4)$	2	$\mathbb{Q}(\sqrt{5})$	$(4\times)\ 16$	$\sqrt{x}$	$(x,-1)$
G_{31}	4	$\mathbb{Q}(i)$	$(4\times)\ 4$ $(2\times)\ 6$ $(4\times)\ 10$ $(8\times)\ 20$ $(2\times)\ 24$ $(2\times)\ 30$ $(4\times)\ 36$ $(6\times)\ 40$	$\sqrt{x}$ $\sqrt{x}$ $\sqrt{x}$ $\sqrt{x}$ $\sqrt{x}$ $\sqrt{x}$ $\sqrt{x}$ $\sqrt{x}$	$(x,-1)$
G_{32}	6	$\mathbb{Q}(\zeta_3)$	$(6\times)\ 64$ $(3\times)\ 81$	$\sqrt{y_1y_2}$ $\sqrt[3]{y_1y_2y_3}$	$(y_{1,2,3})$
G_{33}	2	$\mathbb{Q}(\zeta_3)$	$(2\times)\ 64$	$\sqrt{x}$	$(x,-1)$
G_{34}	6	$\mathbb{Q}(\zeta_3)$	$(8\times)\ 384$ $(6\times)\ 729$ $(4\times)\ 896$	$\sqrt{x}$ $\sqrt[3]{x}$ $\sqrt{x}$	$(x,-1)$
$G_{36}=W(E_7)$	2	$\mathbb{Q}$	$(2\times)\ 512$	$\sqrt{x}$	$(x,-1)$
$G_{37}=W(E_8)$	2	$\mathbb{Q}$	$(4\times)\ 4096$	$\sqrt{x}$	$(x,-1)$

8. Appendix

Here we collect the results on character fields for primitive groups in the form of a table. In the table, we first list the Shephard-Todd name of W, the order of the center $|Z(W)|$ and the splitting field k of W. The next column contains the degrees of irrational characters of the generic cyclotomic algebra $\mathcal{H}(W, \mathbf{u})$, together with the number of such characters. The next column gives the irrationality generating the character field over $k(\mathbf{u})$ (it turns out that for primitive groups the character field is always generated by a single root of a monomial in the parameters). In the last column we give the names of the indeterminates used in describing the irrationalities. For the 2-dimensional groups $G_4, \ldots, G_{22}$, we keep the notation used in [15], for the higher-dimensional groups generated by involutions we have specialized one of the parameters to -1. Note that G_4, G_{25}, G_{28} and G_{35} do not occur in the tables since their cyclotomic algebras have splitting field $k(\mathbf{u})$.

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Fachbereich Mathematik/Informatik
Universität Kassel
Heinrich-Plett-Str. 40
D–34132 Kassel, Germany
E-mail: malle@mathematik.uni-kassel.de