J. Math. Sci. Univ. Tokyo
**6** (1999), 575–598.

# *Residue Formulae for Secondary Characteristic Classes*

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**Abstract.** For a codimension $q$ foliation $\tau$ and a vector field $X$ which preserves $\tau$, we can define the residues of residual secondary characteristic classes as the cohomology classes of the singular set of $X$. We calculate the residues for the examples which are generalizations of those given by Heitsch([He2]).

## §1. Introduction

In this paper we construct examples of residues of secondary characteristic classes, which are generalization of the results of Heitsch([He2]).

Let $\tau$ be a codimension $q$ foliation on a manifold $M$, and $X$ a vector field on $M$. We assume that $X$ preserves $\tau$ and that its singular set (the set of points where $X$ is tangent to $\tau$) is a single leaf $N$ of $\tau$. For a residual element $\varphi \in I_q(WO_q)$, we can define a certain cohomology class in $H^{\deg\varphi - q}(N)$, called the residue of $\tau$, $X$ and $\varphi$ at $N$(see §2).

In [He2], Heitsch constructed examples of many non-trivial residues and showed that they are parametrized by non-zero real numbers $\lambda_1, \lambda_2, \ldots, \lambda_q$, hence they vary continuously. In this paper we generalize the examples of Heitsch. Moreover we can observe that locally they realize the geometrical limits of the examples of Heitsch when some of $\lambda_i$'s go to 0 (see below).

Let $G = \mathrm{SL}_2\mathbf{R} \times \cdots \times \mathrm{SL}_2\mathbf{R}$ ($q$ -times), $K = \mathrm{SO}_2 \times \cdots \times \mathrm{SO}_2$ ($q$ -times) and $\Gamma$ be a discrete subgroup of $G$ such that $\Gamma\backslash G/K$ is a compact manifold. We define a certain action of $G \times \mathbf{R}^n$ on $\mathbf{R}^{2q+n}$ and obtain a codimension $2q+n$ foliation $\tau$ of the foliated $\mathbf{R}^{2q+n}$-bundle $T^nM = (\Gamma \times \mathbf{Z}^n \setminus G \times \mathbf{R}^n) \times_K \mathbf{R}^{2q+n} \longrightarrow N \times T^n = (\Gamma \setminus G/K) \times T^n$. Choose non-zero numbers $\lambda_1, \lambda_2, \ldots, \lambda_q, \mu_1, \mu_2, \ldots, \mu_n \in \mathbf{R}$ and let

$$X_{\lambda,\mu} = \sum_{i=1}^{q} \lambda_i \left( x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i} \right) + \sum_{j=1}^{n} \mu_j z_j \Psi_j(z_j) \frac{\partial}{\partial z_j},$$

1991 *Mathematics Subject Classification*. 57R30.

*Partially supported by JSPS Research Fellowships for Young Scientists.

be a vector field on $\mathbf{R}^{2q+n}$, where $\Psi_1(z_1), \ldots, \Psi_n(z_n)$ are certain functions defined in §3. $X_{\lambda,\mu}$ has an isolated singularity at the origin and commutes with the action of $G \times \mathbf{R}^n$. It induces a vector field $X_{\lambda,\mu}$ on $T^n M$ which preserves $\tau$. The singular set of $X_{\lambda,\mu}$ is now just the zero section $N \times T^n = (\Gamma \setminus G/K) \times T^n$. Let $\varphi \in I_{2q+n}(WO_{2q+n})$ with $\deg\varphi = 4q + 2n$. We can regard $\varphi$ as an element in $I^{2q+n}(gl_{2q+n})$.

The main theorem of this paper is the following.

THEOREM 3.1. *Let $T^n M, N \times T^n, \tau, X_{\lambda,\mu}$ and $\varphi$ be as above. Then*

$$\begin{aligned}
&\mathrm{Res}_\varphi(\tau, X_{\lambda,\mu}, N \times T^n) \\
&\quad = \frac{2^n \pi^q \varphi(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_q, \lambda_q, \overbrace{0, \ldots, 0}^{n-\text{times}})}{(\lambda_1 \cdots \lambda_q)^2 \mu_1 \cdots \mu_n} [W_N \wedge W_{T^n}]
\end{aligned}$$

*where $W_N$ and $W_{T^n} = dt_1 \wedge dt_2 \wedge \cdots \wedge dt_n$ are volume forms on $N$ and $T^n$, respectively.*

The case of $n = 0, 1$ are already given by Heitsch. However we can consider that the example of residues given by this theorem gives the locally geometrical limit of that of Heitsch.

For example, if $n = 0$, we have the following residue in the cohomology class of a product of $p$ surfaces of higher genus $N = \Sigma_1 \times \Sigma_2 \times \ldots \times \Sigma_p$ :

$$\frac{\pi^p \varphi(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_p, \lambda_p)}{(\lambda_1 \cdots \lambda_p)^2} [W_N] \in H^{2p}(N).$$

Note that the coefficient is determined only by the ratio of $\lambda_i$'s since the numerator and the denominator are both homogeneous of degree $2p$ in $\lambda_i$'s. We can assume that the area of the $i$-th surface $\Sigma_i$ is $1/{\lambda_i}^2$. Although when $\lambda_p \to 0$ this coefficient diverges, if we restrict it to some domain $D \subset \Sigma_p$ whose area is constantly equal to 1, the above residue does converge to

$$\begin{aligned}
&\frac{\pi^p \varphi(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_{p-1}, \lambda_{p-1}, 0, 0)}{(\lambda_1 \cdots \lambda_{p-1})^2} [W_{N'} \wedge W_D] \\
&\quad \in H^{2p}(N' \times D),
\end{aligned} \tag{1.1}$$

where $N' = \Sigma_1 \times \Sigma_2 \times \ldots \times \Sigma_{p-1}$ and $W_{N'}$ is its volume form. From the geometrical point of view, $\lambda_p \to 0$ means the curvature of the domain $D$

goes to 0. In the formula of our theorem, by putting $q = p - 1$ and $n = 2$, we have the following residue in the cohomology group of $N' \times T^2$,

$$\begin{aligned}(1.2) \qquad & \frac{4\pi^p \varphi(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_{p-1}, \lambda_{p-1}, 0, 0)}{(\lambda_1 \cdots \lambda_{p-1})^2 \mu_1 \mu_2} [W_{N'} \wedge W_{T^2}] \\ & \in H^{2p}(N' \times T^2).\end{aligned}$$

After taking a suitable domain $D' \subset T^2$ whose curvature is also 0, the restriction of (1.2) to $N' \times D'$ gives the same formula as (1.1). In this sense, we can interpret our formula as a local geometrical limit of that of Heitsch.

In §2, we review the construction of secondary characteristic classes and of residues.

In §3, we construct examples of variation of residues for a certain class of residual secondary characteristic classes.

In §4, we extend the result of §3 for more general residual secondary characteristic classes.

## §2. Preliminaries

We briefly recall the construction of the secondary characteristic classes for foliations and the residue formulae for the residual classes. In this paper we will consider only $C^\infty$- objects.

Let $\mathrm{GL}_q = \mathrm{GL}_q\mathbf{R}$ be the real general linear group and $gl_q = gl_q\mathbf{R}$ its Lie algebra. We define the Chern polynomials $c_1, c_2, \ldots, c_q$ on $gl_q$ by

$$\det\left(tI_q - \frac{1}{2\pi}A\right) = \sum_{i=1}^{q} t^{q-i} c_i(A),$$

where $I_q \in gl_q$ is identity matrix and $A \in gl_q$. Denote by $I^*(\mathrm{GL}_q)$ the graded algebra of adjoint invariant polynomials on $gl_q$. It is well-known that $I^*(\mathrm{GL}_q)$ is a polynomial algebra generated by the Chern polynomials $c_1, c_2, \ldots, c_q$ :

$$I^*(\mathrm{GL}_q) = \mathbf{R}\left[c_1, c_2, \ldots, c_q\right].$$

For any manifold $M$, we denote by $A^*(M)$ the algebra of differential forms on $M$. For any vector bundle $E \longrightarrow M$, we denote by $\Gamma(E)$ the space of smooth sections of $E$

Let $\tau$ be a codimension $q$ foliation on an n-dimensional manifold $M$ and $\nu$ its normal bundle. We call a connection $\theta$ on $\nu$ is basic for $\tau$ if its covariant derivative $\nabla$ satisfies

$$\nabla_Y \Pi(Z) = \Pi([Y, Z])$$

for all $Y \in \Gamma(\tau)$ and $Z \in \Gamma(TM)$. Here $\Pi : TM \longrightarrow \nu = TM/\tau$ is the natural projection.

Choose a basic connection $\theta^b$ on $\nu$ for $\tau$ and a metric connection $\theta^r$ on $\nu$ associated with a fiber metric on $\nu$. For $\varphi \in I^k(\mathrm{GL}_q)$, set

$$\Delta_\varphi(\theta^b, \theta^r) = k \int_0^1 \varphi(\theta^b - \theta^r, \overbrace{\Omega^{\,t}, \ldots, \Omega^{\,t}}^{(k-1)-\text{times}})dt,$$

where $\Omega^{\,t}$ is the curvature of the connection $t\theta^b + (1-t)\theta^r$ on $\nu$. Here $\varphi$ is considered as a homogeneous symmetric tensors of degree $k$. It is well-known that

$$d\Delta_\varphi(\theta^b, \theta^r) = \varphi(\Omega^b) - \varphi(\Omega^r),$$

where $\Omega^b$ and $\Omega^r$ is the curvature of $\theta^b$ and $\theta^r$, respectively. In particular, if $i$ is odd, we have $d\Delta_{c_i}(\theta^b, \theta^r) = c_i(\Omega^b)$ because $c_i(\Omega^r) = 0$.

Let $\wedge(h_1, h_3, \ldots, h_{2[\frac{(q-1)}{2}]+1})$ be the exterior algebra generated by $h_i$'s with $\deg h_i = 2i - 1$, and $\mathbf{R}[c_1, c_2, \ldots, c_q]$ the polynomial algebra generated by $c_j$'s with $\deg c_j = 2j$. Then we define the differential graded algebra $(WO_q, d)$ by

$$WO_q = \wedge(h_1, h_3, \ldots, h_{2[\frac{(q-1)}{2}]+1}) \otimes \mathbf{R}_q[c_1, c_2, \ldots, c_q].$$

where $\mathbf{R}_q[c_1, c_2, \ldots, c_q]$ is the quotient algebra of $\mathbf{R}[c_1, c_2, \ldots, c_q]$ by the ideal generated by elements of degree greater than $2q$. The differential $d : WO_q \longrightarrow WO_q$ is given by

$$\begin{cases} d(h_i \otimes 1) = c_i \\ d(1 \otimes c_i) = 0. \end{cases}$$

Let

$$\alpha_\tau : WO_q \longrightarrow A^*(M),$$

be an algebra homomorphism defined by $\alpha_\tau(h_i) = \Delta_{c_i}(\theta^b, \theta^r)$ and by $\alpha_\tau(c_j) = c_j(\Omega^b)$. It is well-defined since $c_{j_1}(\Omega^b) \cdots c_{j_\ell}(\Omega^b) = 0$ for $j_1 +$

$\cdots + j_\ell > 2q$ by the Bott vanishing theorem. Since $d\alpha_\tau(h_i) = \alpha_\tau(dh_i)$ and $d\alpha_\tau(c_j) = \alpha_\tau(dc_i) = 0$, $\alpha_\tau$ is a cochain map of degree zero and so it induces a homomorphism

$$\alpha_\tau^* : H^*(WO_q) \longrightarrow H^*(M).$$

By the convexity of the set of basic connections and also of metric connections, $\alpha_\tau^*$ is independent of the choices of $\theta^b$ and $\theta^r$.

A vector field $X \in \Gamma(TM)$ is called a $\Gamma$-vector field for $\tau$ if it satisfies $[X, Y] \in \Gamma(\tau)$ for any $Y \in \Gamma(\tau)$. For a $\Gamma$-vector field $X$, the singular set of $X$ is defined to be the set of points of $M$ where $X$ is tangent to $\tau$. Since the normal components of a $\Gamma$-vector field are constant along the leaves of $\tau$, the singular set is a union of leaves of $\tau$. To carry out a theory of residues, the singular set of $X$ has only to be a union of closed and seperated leaves of $\tau$. For simplicity, however, we assume that the singular set of $X$ is a single leaf $N$. Choose an embedded open normal disc bundle $D$ over $N$ in $M$ so that its closure $\bar{D}$ is an embedded normal disc bundle.

For a $\Gamma$-vector field $X$ and a neighborhood $U$ of $M - D$ in$M - N$, a basic connection $\theta$ for $\tau$ is called a basic $X$-connection supported off $U$ if on $U$ its covariant derivative $\nabla$ satisfies

$$\nabla_X \Pi(Y) = \Pi([X, Y]),$$

for any $Y \in \Gamma(TM)$. Here $\Pi : TM \longrightarrow \nu = TM/\tau$ is a natural projection. Such a connection always exists since $\tau$ and $X$ span codimension $q - 1$ foliation on $U$.

Let $I_q(WO_q)$ be the ideal in $WO_q$ generated by the elements of the form $c_J = c_{j_1} \cdots c_{j_k}$ with $|J| = q$. On $M - N$, $\tau$ and $X$ span codimension $(q-1)$ foliation. Thus, by the Bott vanishing theorem, for $\varphi \in I_q(WO_q)$ the restriction $\alpha_\tau(\varphi)|_D$ is a differential form with fiber compact support provided we use a basic $X$-connection $\theta_X$ in the construction of $\alpha_\tau : WO_q \longrightarrow A^*(M)$. Since $d\varphi = 0$, $\alpha_\tau(\varphi)|_D$ is a closed form on $D$ and then $[\alpha_\tau(\varphi)|_D] \in H_c^{\deg\varphi}(D)$. Here $H_c^*$ means an cohomology algebra of closed differential forms with fiber compact support. Let $\gamma_D : H_c^{\deg\varphi}(D) \longrightarrow H^{\deg\varphi - q}(N)$ be the map induced by the integration along the fibers of the $q$ disc bundle $D \longrightarrow N$.

Definition 2.1. Let $M, \tau, N, X, D$ and $\theta_X$ be as above. For $\varphi \in I_q(WO_q)$, the residue of $\varphi, \tau$ and $X$ at $N$ is a cohomology class given by

$$\mathrm{Res}_\varphi(\tau, X, N) = \gamma_D([\alpha_\tau(\varphi)|_D]) \in H^{\deg\varphi - q}(N),$$

where the basic connection used in the construction of $\alpha_\tau : WO_q \longrightarrow A^*(M)$ is the basic $X$-connection $\theta_X$.

In fact $\mathrm{Res}_\varphi(\tau, X, N)$ is independent of the choices of embedded open normal disc bundles and basic $X$-connections. ([He2] Theorem 3.11)

The following lemma will be useful for us later.

LEMMA 2.2 ([He2] Lemma 4.15). *Let $M, \tau, N, X$ and $D$ be as above. Let $\theta$ be a connection on $\nu$ over $M$ with curvature $\Omega$. If $\theta$ restricted to $\nu$ over the boundary $S$ of $D$ agrees with the restriction to $\nu$ over $S$ of some basic $X$-connection supported off a neighborhood of $S$. Then for $c_J \in I_q(WO_q)$ with $|J| = q$,*

$$\mathrm{Res}_{c_J}(\tau, X, N) = \gamma_D\left[c_J(\Omega)|_D\right].$$

## §3. Examples

In this section, we construct our main example. Let $G = \overbrace{\mathrm{SL}_2\mathbf{R} \times \cdots \times \mathrm{SL}_2\mathbf{R}}^{q-\text{times}}$ and $K = \overbrace{\mathrm{SO}_2 \times \cdots \times \mathrm{SO}_2}^{q-\text{times}}$. We define the action of $G \times \mathbf{R}^n$ on $\mathbf{R}^{2q+n}$, as follows. The action of $G \subset G \times \mathbf{R}^n$ is given by the natural inclusion $G \longrightarrow \mathrm{GL}_{2q+n}\mathbf{R}$ with $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1_n \end{pmatrix}$. To define the action of $\mathbf{R}^n \subset G \times \mathbf{R}^n$ on $\mathbf{R}^{2q+n}$, we choose even functions $\Psi_j(z_j)$ for $j = 1, 2, \ldots, n$ such that

i) $0 < \Psi_j(z_j) \leq 1$ for all $z_j \neq 0$,

ii) $\Psi_j(z_j)$ are strictly increasing on the interval $(0, 1/2)$,

iii) $\Psi_j(z_j) = 1$ if $z_j \geq 1/2$,

iv) $\Psi_j(z_j)$ and all its derivatives are zero at $z_j = 0$.

Let $(x_1, y_1, x_2, y_2, \ldots, x_q, y_q, z_1, \ldots, z_n)$ be coordinates on $\mathbf{R}^{2q+n}$ and $(t_1, t_2, \ldots, t_n)$ coordinates on $\mathbf{R}^n \subset G \times \mathbf{R}^n$. On the Lie algebra level, the action of $\mathbf{R}^n \subset G \times \mathbf{R}^n$ on $\mathbf{R}^{2q+n}$ is given by

$$\frac{\partial}{\partial t_j} \longmapsto -|z_j|\,\Psi_j(z_j)\frac{\partial}{\partial z_j}$$

for all $j = 1, 2, \ldots, n$.

Choose non-zero numbers $\lambda_1, \lambda_2, \ldots, \lambda_q, \mu_1, \mu_2, \ldots, \mu_n \in \mathbf{R}$ and set

$$X_{\lambda,\mu} = \sum_{i=1}^{q} \lambda_i \left( x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i} \right) + \sum_{j=1}^{n} \mu_j z_j \Psi_j(z_j) \frac{\partial}{\partial z_j}.$$

$X_{\lambda,\mu}$ has an isolated singularity at the origin and commutes with the action of $G \times \mathbf{R}^n$. Thus we obtain a codimension $2q+n$ foliation $\tau$ of the bundle

$$T^n M = (\Gamma \times \mathbf{Z}^n \setminus G \times \mathbf{R}^n) \times_K \mathbf{R}^{2q+n} \longrightarrow N \times T^n = (\Gamma \setminus G/K) \times T^n$$

transverse to the fibers and a $\Gamma$-vector field $X_{\lambda,\mu}$ for $\tau$ ([He2] p.437-438). Here $\Gamma$ is a uniform discrete subgroup of $G$ such that $\Gamma \setminus G/K$ is a compact manifold. Note that the singular set of $X_{\lambda,\mu}$ is just the zero section $N \times T^n = (\Gamma \setminus G/K) \times T^n$.

Suppose $\varphi \in I_{2q+n}(WO_{2q+n})$ with $\deg\varphi = 4q + 2n$. Then $\varphi$ is a polynomial in the $c_i$'s and thus we can regard $\varphi$ as an element of $I^{2q+n}(\mathrm{GL}_{2q+n})$. We write $\varphi(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_q, \lambda_q, 0, \ldots, 0)$ for $\varphi$ applied to the diagonal matrix diag $(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_q, \lambda_q, 0, \ldots, 0) \in gl_{2q+n}$.

THEOREM 3.1. *Let $T^n M, N \times T^n, \tau, X_{\lambda,\mu}$ and $\varphi$ be as above. Then*

$$\begin{aligned} &\mathrm{Res}_\varphi(\tau, X_{\lambda,\mu}, N \times T^n) \\ &\quad = \frac{2^n \pi^q \varphi(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_q, \lambda_q, \overbrace{0, \ldots, 0}^{n-\text{times}})}{(\lambda_1 \cdots \lambda_q)^2 \mu_1 \cdots \mu_n} [W_N \wedge W_{T^n}], \end{aligned}$$

*where $W_N$ and $W_{T^n} = dt_1 \wedge dt_2 \wedge \cdots \wedge dt_n$ are volume forms on $N$ and $T^n$, respectively.*

PROOF.

Let $\omega_{\lambda,\mu}$ be the 1-form on $\mathbf{R}^{2q+n}$ given by

$$\omega_{\lambda,\mu} = \sum_{i=1}^{q} \lambda_i (x_i dx_i + y_i dy_i) + \sum_{j=1}^{n} \mu_j z_j \Psi_j(z_j) dz_j.$$

The action of $K$ on $\mathbf{R}^{2q+n}$ preserves $\omega_{\lambda,\mu}$ so it induces a 1-form on $T^n M$ also denoted by $\omega_{\lambda,\mu}$. We denote by $T\mathbf{R}^{2q+n} \longrightarrow T^n M$ the tangent bundle along the fiber of $T^nM \longrightarrow N \times T^n$. We can identify $T\mathbf{R}^{2q+n} \longrightarrow T^nM$ with the normal bundle $\nu$ for $\tau$.

Let $\theta$ be the unique basic connection on $T\mathbf{R}^{2q+n}$ whose covariant derivative $\bigtriangledown$ satisfies

$$\bigtriangledown_Y \frac{\partial}{\partial x_i} = \omega_{\lambda,\mu}(Y)\left[X_{\lambda,\mu}, \frac{\partial}{\partial x_i}\right] \text{ for all } Y \in T\mathbf{R}^{2q+n} \text{ and } i = 1, 2, \ldots, q,$$

$$\bigtriangledown_Y \frac{\partial}{\partial y_i} = \omega_{\lambda,\mu}(Y)\left[X_{\lambda,\mu}, \frac{\partial}{\partial y_i}\right] \text{ for all } Y \in T\mathbf{R}^{2q+n} \text{ and } i = 1, 2, \ldots, q,$$

$$\bigtriangledown_Y \frac{\partial}{\partial z_j} = \omega_{\lambda,\mu}(Y)\left[X_{\lambda,\mu}, \frac{\partial}{\partial z_j}\right] \text{ for all } Y \in T\mathbf{R}^{2q+n} \text{ and } j = 1, 2, \ldots, n.$$

$\theta$ is well-defined.

We denote by $S \longrightarrow N \times T^n$ the sphere subbundle in $T^nM \longrightarrow N \times T^n$ which is given by

$$S = \{\rho\left((g, x)\right) \mid \sum_{i=1}^{q} \lambda_i^2 \left(x_i^2 + y_i^2\right) + \sum_{j=1}^{n} \mu_j^2 z_j^2 \Psi_j(z_j)^2 = 1\}.$$

Here $g \in G \times \mathbf{R}^n$, $x = (x_1, y_1, x_2, y_2, \ldots, x_q, y_q, z_1, \ldots, z_n) \in \mathbf{R}^{2q+n}$ and $\rho : G \times \mathbf{R}^n \times \mathbf{R}^{2q+n} \longrightarrow T^nM = (\Gamma \times \mathbf{Z}^n \setminus G \times \mathbf{R}^n) \times_K \mathbf{R}^{2q+n}$ is the natural projection. Restricted to $S$, $\theta$ is the basic $X_{\lambda,\mu}$-connection for $\tau$. By LEMMA 2.2, we can use $\theta$ to compute the residue of $\tau$ and $X_{\lambda,\mu}$ provided we integrate along the fiber of the disc bundle $D \longrightarrow N \times T^n$ given by

$$D = \{\rho\left((g, x)\right) \mid \sum_{i=1}^{q} \lambda_i^2 \left(x_i^2 + y_i^2\right) + \sum_{j=1}^{n} \mu_j^2 z_j^2 \Psi_j(z_j)^2 \leq 1\}.$$

Locally $\tau$ is spanned by vector fields of the forms

$$Y_i + y_i \frac{\partial}{\partial x_i} \text{ for } i = 1, 2, \ldots, q,$$

$$Z_i + \frac{1}{2}\left(x_i \frac{\partial}{\partial x_i} - y_i \frac{\partial}{\partial y_i}\right) \text{ for } i = 1, 2, \ldots, q,$$

and

$$\frac{\partial}{\partial t_j} + |z_j|\, \Psi_j(z_j) \frac{\partial}{\partial z_j} \text{ for } j = 1, 2, \ldots, n.$$

Here $Y_i$ and $Z_i$ are the local vector fields corresponding to the elements $\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$, respectively, which lies in the $i$-th block of the Lie algebra of $G$. Let $\omega_i$, $\gamma_i$ and $dt_j$ be the dual 1-forms of $Y_i$, $Z_i$ and $\partial/\partial t_j$, respectively for $i = 1, 2, \ldots, q$ and $j = 1, 2, \ldots, n$. These 1-forms satisfy

$$d\omega_i = -\omega_i \wedge \gamma_i \text{ for } i = 1, 2, \ldots, q,$$

$$d\gamma_i = 0 \text{ for } i = 1, 2, \ldots, q,$$

and

$$d\,(dt_i) = 0 \text{ for } j = 1, 2, \ldots, n.$$

For the convenience, we use the following notations,

$$\mid a_j \mid = \mid z_j \mid \Psi_j(z_j) \text{ for } j = 1, 2, \ldots, n,$$

and

$$a_j = z_j \Psi_j(z_j) \text{ for } j = 1, 2, \ldots, n.$$

Set

$$\delta_i = x_i y_i \omega_i + \frac{1}{2}\left({x_i}^2 - {y_i}^2\right)\gamma_i \text{ for } i = 1, 2, \ldots, q,$$

$$\varepsilon_i = a_i \mid a_i \mid dt_i \text{ for } i = 1, 2, \ldots, n,$$

$$\lambda\delta = \lambda_1\delta_1 + \lambda_2\delta_2 + \cdots + \lambda_q\delta_q,$$

and

$$\mu\varepsilon = \mu_1\varepsilon_1 + \mu_2\varepsilon_2 + \cdots + \mu_n\varepsilon_n.$$

With these notations, the local connection form $\theta = \left(\theta^i_j\right)_{i,j=1,2,\ldots,2q+n}$, computed with respect to the local basis

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_q}, \frac{\partial}{\partial y_q}, \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}$$

of $T\mathbf{R}^{2q+n}$, is given by

$$\theta^{2i-1}_{2i-1} = \lambda_i\left(-\omega_{\lambda,\mu} + \lambda\delta + \mu\varepsilon\right) - \frac{1}{2}\gamma_i \text{ for } i = 1, 2, \ldots, q,$$

$$\theta^{2i}_{2i} = \lambda_i\left(-\omega_{\lambda,\mu} + \lambda\delta + \mu\varepsilon\right) + \frac{1}{2}\gamma_i \text{ for } i = 1, 2, \ldots, q,$$

$$\theta_{2i}^{2i-1} = -\omega_i \text{ for } i = 1, 2, \ldots, q,$$

$$\theta_{2q+j}^{2q+j} = \mu_j a_j' \left(-\omega_{\lambda,\mu} + \lambda\delta + \mu\varepsilon\right) - |a_j|' dt_j \text{ for } i = 1, 2, \ldots, n,$$

and all other entries are zero. Then the local curvature form $\Omega = \left(\Omega_j^i\right)_{i,j=1,2,\ldots,2q+n}$ is given by

$$\Omega_{2i-1}^{2i-1} = \Omega_{2i}^{2i} = \lambda_i \left(d\lambda\delta + d\mu\varepsilon\right) \text{ for } i = 1, 2, \ldots, q,$$

$$\Omega_{2q+j}^{2q+j} = \mu_j a_j' \left(d\lambda\delta + d\mu\varepsilon\right) + \mu_j a_j'' dz_j \wedge \left(-\omega_{\lambda,\mu} + \lambda\delta + \mu\varepsilon\right) - |a_j|'' dz_j \wedge dt_j \quad \text{for } j = 1, 2, \ldots, n,$$

and all other entries are zero.

We may eliminate the terms $\mu_j a_j'' dz_j \wedge (-\omega_{\lambda,\mu} + \lambda\delta)$ from $\Omega_{2q+j}^{2q+j}$ because these terms can never yield a non-zero element when they are wedged with $\{dt_j\}_{j=1,2,\ldots,n}$ since all $\{dt_j\}_{j=1,2,\ldots,n}$ occur as $\{dz_j \wedge dt_j\}_{j=1,2,\ldots,n}$. $\varphi(\Omega)$ is a volume form on $T^nM$ and it must contain all $dt_j$'s.

Note that all functions of $z_j$'s in $\Omega$ are even functions, we may replace $|a_j|$ by $a$, $|a_j|'$ by $a'$ and $|a_j|''$ by $a''$ provided we integrate $\varphi(\Omega)$ only over the portion of the fiber of the disc bundle where $z_1 \geq 0$, $z_2 \geq 0$, $\ldots$, $z_n \geq 0$.

Now $\varphi(\Omega)$ is hard to compute. So we construct useful polynomial functions on $gl_{2q+n}$ as follows. Define

$$\mathrm{Tr}_k(A) = \mathrm{trace}(A^k) \text{ for } A \in gl_{2q+n} \text{ and } k = 1, 2, \ldots, 2q+n.$$

By the Newton formula, remark that

$$\mathbf{R}[c_1, c_2, \ldots, c_{2q+n}] = \mathbf{R}[\mathrm{Tr}_1, \mathrm{Tr}_2, \ldots, \mathrm{Tr}_{2q+n}].$$

Since $\mathrm{Res}_\varphi(\tau, X_{\lambda,\mu}, N \times T^n)$ is linear in $\varphi$, we can assume that $\varphi$ is of the form

$$\varphi = c_{k_1} c_{k_2} \cdots c_{k_p} \text{ with } k_1 + k_2 + \cdots + k_p = 2q + n.$$

Moreover, using the above remark, we can assume that $\varphi$ is of the form

$$\varphi = \mathrm{Tr}_{m_1} \mathrm{Tr}_{m_2} \cdots \mathrm{Tr}_{m_r} \text{ with } m_1 + m_2 + \cdots + m_r = 2q + n.$$

Since $\Omega = \left(\Omega^i_j\right)_{i,j=1,2,\dots,2q+n}$ is a diagonal matrix, $\Omega^m = \left((\Omega^m)^i_j\right)_{i,j=1,2,\dots,2q+n}$ is computed as follows.

$$\begin{aligned}
(\Omega^m)^{2i-1}_{2i-1} &= (\Omega^m)^{2i}_{2i} = {\lambda_i}^m (d\lambda\delta + d\mu\varepsilon)^m \text{ for } i = 1, 2, \dots, q, \\
(\Omega^m)^{2q+j}_{2q+j} &= \left(\mu_j a'_j\right)^m (d\lambda\delta + d\mu\varepsilon)^m \\
&\quad + m\left(\mu_j a'_j\right)^{m-1} (d\lambda\delta + d\mu\varepsilon)^{m-1} \\
&\quad \wedge \mu_j a''_j dz_j \wedge \left(\mu\varepsilon - \mu_j^{-1} dt_j\right) \text{ for } j = 1, 2, \dots, n.
\end{aligned}$$

If we denote by $P_{\lambda,\mu}$ the diagonal matrix $\mathrm{diag}(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \dots, \lambda_q, \lambda_q, \mu_1 a'_1, \dots, \mu_n a'_n)$, we can write

$$\begin{aligned}
\mathrm{Tr}_m(\Omega) &= \mathrm{Tr}_m(P_{\lambda,\mu}) (d\lambda\delta + d\mu\varepsilon)^m \\
&\quad + \sum_{j=1}^{n} \frac{\partial \mathrm{Tr}_m(P_{\lambda,\mu})}{\partial z_j} (d\lambda\delta + d\mu\varepsilon)^{m-1} \wedge dz_j \wedge \left(\mu\varepsilon - \mu_j^{-1} dt_j\right).
\end{aligned}$$

Let $\mathbf{m} = (m_1, m_2, \dots, m_r)$. By the above assumption of $\varphi$,we have

$$\begin{aligned}
&\varphi(\Omega) \\
&\quad = \mathrm{Tr}_{\mathbf{m}}(\Omega) \\
&\quad = \mathrm{Tr}_{m_1}(\Omega)\,\mathrm{Tr}_{m_2}(\Omega) \cdots \mathrm{Tr}_{m_r}(\Omega) \\
&\quad = \mathrm{Tr}_{\mathbf{m}}(P_{\lambda,\mu}) (d\lambda\delta + d\mu\varepsilon)^{2q+n} \\
&\qquad + \sum_{k_1=1}^{r} \mathrm{Tr}_{\mathbf{m}-\{m_{k_1}\}}(P_{\lambda,\mu}) \sum_{j_1=1}^{n} \frac{\partial \mathrm{Tr}_{m_{k_1}}(P_{\lambda,\mu})}{\partial z_{j_1}} (d\lambda\delta + d\mu\varepsilon)^{2q+n-1} \\
&\qquad\quad \wedge dz_{j_1} \wedge \left(\mu\varepsilon - \mu_{j_1}^{-1} dt_{j_1}\right) \\
&\qquad + \sum_{1 \le k_1 < k_2 \le r} \mathrm{Tr}_{\mathbf{m}-\{m_{k_1}, m_{k_2}\}}(P_{\lambda,\mu}) \\
&\qquad\quad \times \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \frac{\partial \mathrm{Tr}_{m_{k_1}}(P_{\lambda,\mu})}{\partial z_{j_1}} \frac{\partial \mathrm{Tr}_{m_{k_2}}(P_{\lambda,\mu})}{\partial z_{j_2}} (d\lambda\delta + d\mu\varepsilon)^{2q+n-2} \\
&\qquad\quad \wedge dz_{j_1} \wedge \left(\mu\varepsilon - \mu_{j_1}^{-1} dt_{j_1}\right) \wedge dz_{j_2} \wedge \left(\mu\varepsilon - \mu_{j_2}^{-1} dt_{j_2}\right) \\
&\qquad + \cdots \\
&\qquad + \sum_{1 \le k_1 < k_2 < \cdots < k_{r-1} \le r} \mathrm{Tr}_{\mathbf{m}-\{m_{k_1}, m_{k_2}, \dots, m_{k_{r-1}}\}}(P_{\lambda,\mu})
\end{aligned}$$

$$\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\cdots\sum_{j_{r-1}=1}^{n}\frac{\partial \mathrm{Tr}_{m_{k_1}}\left(P_{\lambda,\mu}\right)}{\partial z_{j_1}}\frac{\partial \mathrm{Tr}_{m_{k_2}}\left(P_{\lambda,\mu}\right)}{\partial z_{j_2}}\cdots$$
$$\frac{\partial \mathrm{Tr}_{m_{k_{r-1}}}\left(P_{\lambda,\mu}\right)}{\partial z_{j_{r-1}}}\left(d\lambda\delta+d\mu\varepsilon\right)^{2q+1}$$
$$\wedge dz_{j_1}\wedge\left(\mu\varepsilon-\mu_{j_1}^{-1}dt_{j_1}\right)\wedge dz_{j_2}\wedge\left(\mu\varepsilon-\mu_{j_2}^{-1}dt_{j_2}\right)\wedge\cdots$$
$$\wedge dz_{j_{r-1}}\wedge\left(\mu\varepsilon-\mu_{j_{r-1}}^{-1}dt_{j_{r-1}}\right)$$
$$+\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\cdots\sum_{j_{r}=1}^{n}\frac{\partial \mathrm{Tr}_{m_{k_1}}\left(P_{\lambda,\mu}\right)}{\partial z_{j_1}}\frac{\partial \mathrm{Tr}_{m_{k_2}}\left(P_{\lambda,\mu}\right)}{\partial z_{j_2}}\cdots$$
$$\frac{\partial \mathrm{Tr}_{m_{k_r}}\left(P_{\lambda,\mu}\right)}{\partial z_{j_r}}\left(d\lambda\delta+d\mu\varepsilon\right)^{2q}$$
$$\wedge dz_{j_1}\wedge\left(\mu\varepsilon-\mu_{j_1}^{-1}dt_{j_1}\right)\wedge dz_{j_2}\wedge\left(\mu\varepsilon-\mu_{j_2}^{-1}dt_{j_2}\right)\wedge\cdots$$
$$\wedge dz_{j_r}\wedge\left(\mu\varepsilon-\mu_{j_r}^{-1}dt_{j_r}\right).$$

Since $(d\lambda\delta)^l=0$ for $l>2q$, if we write ${\mu_j}^2{a_j}^2$ by $A_j$ for $j=1,2,\ldots,n$, then we can write $\varphi(\Omega)$ as follows.

$$\begin{aligned}
&\varphi(\Omega)\\
=\;&(\mu_1\mu_2\ldots\mu_n)^{-1}\left\{\frac{(2q+n)!}{2q!}\mathrm{Tr}_{\mathbf{m}}\left(P_{\lambda,\mu}\right)\frac{\partial A_1}{\partial z_1}\cdots\frac{\partial A_n}{\partial z_n}\right.\\
&+(-1)\sum_{j_1=1}^{n}\frac{(2q+n-1)!}{2q!}\frac{\partial \mathrm{Tr}_{\mathbf{m}}\left(P_{\lambda,\mu}\right)}{\partial z_{j_1}}\\
&\quad\times\frac{\partial A_1}{\partial z_1}\cdots\widehat{\frac{\partial A_{j_1}}{\partial z_{j_1}}}\cdots\frac{\partial A_n}{\partial z_n}\left(1-A_{j_1}\right)\\
&+(-1)^2\sum_{1\le j_1<j_2\le n}\frac{(2q+n-2)!}{2q!}\frac{\partial^2 \mathrm{Tr}_{\mathbf{m}}\left(P_{\lambda,\mu}\right)}{\partial z_{j_1}\partial z_{j_2}}\\
&\quad\times\frac{\partial A_1}{\partial z_1}\cdots\widehat{\frac{\partial A_{j_1}}{\partial z_{j_1}}}\cdots\widehat{\frac{\partial A_{j_2}}{\partial z_{j_2}}}\cdots\frac{\partial A_n}{\partial z_n}\times\left(1-A_{j_1}-A_{j_2}\right)\\
&+\cdots\\
&+(-1)^{n-1}\sum_{1\le j_1<\cdots<j_{n-1}\le n}\frac{(2q+1)!}{2q!}\frac{\partial^{n-1} \mathrm{Tr}_{\mathbf{m}}\left(P_{\lambda,\mu}\right)}{\partial z_{j_1}\partial z_{j_2}\ldots\partial z_{j_{n-1}}}
\end{aligned}$$

$$\times \frac{\partial A_1}{\partial z_1} \cdots \frac{\widehat{\partial A_{j_1}}}{\partial z_{j_1}} \cdots \frac{\widehat{\partial A_{j_{n-1}}}}{\partial z_{j_{n-1}}} \cdots \frac{\partial A_n}{\partial z_n} \times \left(1 - A_{j_1} - A_{j_2} - \cdots - A_{j_{n-1}}\right)$$
$$\left. + (-1)^n \frac{\partial^n \mathrm{Tr}_{\mathbf{m}}\left(P_{\lambda,\mu}\right)}{\partial z_1 \partial z_2 \ldots \partial z_n} \left(1 - A_1 - A_2 - \cdots - A_n\right) \right\} (d\lambda\delta)^{2q}$$
$$\wedge dz_1 \wedge dt_1 \wedge \cdots \wedge dz_n \wedge dt_n.$$

Note that this expression of $\varphi(\Omega)$ is valid only on the part of $z_i \geq 0$ for all $i = 1, 2, \ldots, n$.

Since the connection $\theta$ restricted to $S = \partial D$ is basic $X_{\lambda,\mu}$-connection, by LEMMA 2.2 $\mathrm{Res}_{\varphi}\left(\tau, X_{\lambda,\mu}, N \times T^n\right)$ is the cohomology class of $\int_D \varphi(\Omega)$. To integrate $\varphi(\Omega)$ over the disc $D \subset \mathbf{R}^{2q+n}$ we define

$$D' = \{(x_1, y_1, \ldots, x_q, y_q) \in \mathbf{R}^{2q} | \sum_{i=1}^{q} \lambda_i^2 \left(x_i^2 + y_i^2\right) \leq 1 - A_1 - A_2 - \cdots - A_n\},$$

and

$$D_0 = \{(z_1, \ldots, z_n) \in \mathbf{R}^n | A_1 + A_2 + \cdots + A_n \leq 1, z_1 \geq 0, \ldots, z_n \geq 0\}.$$

By a result of Heitsch,

$$\int_{D'} (d\lambda\delta)^{2q} = \frac{\pi^q \left(1 - A_1 - A_2 - \cdots - A_n\right)^{2q}}{\lambda_1^2 \lambda_2^2 \ldots \lambda_q^2} W_N,$$

where $W_N$ is a volume form $\prod_{i=1}^{q} \omega_i \wedge \gamma_i$ of $N$. Thus, using the fact that

$$\frac{\partial A_j}{\partial z_j} = -\frac{\partial\left(1 - A_1 - A_2 - \cdots - A_n\right)}{\partial z_j},$$

we have

$$\int_D \varphi(\Omega)$$
$$= \int_{D_0} \left( \int_{D'} \varphi(\Omega) \right)$$
$$= \frac{2^n \pi^q}{\lambda_1^2 \lambda_2^2 \ldots \lambda_q^2 \mu_1 \mu_2 \ldots \mu_n}$$
$$\times \left[ \int_{D_0} \left\{ \mathrm{Tr}_{\mathbf{m}}\left(P_{\lambda,\mu}\right) \frac{\partial^n \left(1 - A_1 - A_2 - \cdots - A_n\right)^{2q+n}}{\partial z_1 \ldots \partial z_n} \right. \right.$$

$$+\sum_{j_1=1}^{n}\frac{\partial \mathrm{Tr}_{\mathbf{m}}\left(P_{\lambda,\mu}\right)}{\partial z_{j_1}}\frac{\partial^{n-1}\left(1-A_1-A_2-\cdots-A_n\right)^{2q+n-1}}{\partial z_1\ldots\widehat{\partial z_{j_1}}\ldots\partial z_n}\left(1-A_{j_1}\right)$$
$$+\sum_{j_1<j_2}\frac{\partial^2 \mathrm{Tr}_{\mathbf{m}}\left(P_{\lambda,\mu}\right)}{\partial z_{j_1}\partial z_{j_2}}\frac{\partial^{n-2}\left(1-A_1-A_2-\cdots-A_n\right)^{2q+n-2}}{\partial z_1\ldots\widehat{\partial z_{j_1}}\ldots\widehat{\partial z_{j_2}}\ldots\partial z_n}$$
$$\times\left(1-A_{j_1}-A_{j_2}\right)$$
$$+\cdots$$
$$+\sum_{j_1<\cdots<j_{n-1}}\frac{\partial^{n-1} \mathrm{Tr}_{\mathbf{m}}\left(P_{\lambda,\mu}\right)}{\partial z_{j_1}\partial z_{j_2}\ldots\partial z_{j_{n-1}}}\frac{\partial\left(1-A_1-A_2-\cdots-A_n\right)^{2q+1}}{\partial z_1\ldots\widehat{\partial z_{j_1}}\ldots\widehat{\partial z_{j_{n-1}}}\ldots\partial z_n}$$
$$\times\left(1-A_{j_1}-\cdots-A_{j_{n-1}}\right)$$
$$\left.\left.+\frac{\partial^n \mathrm{Tr}_{\mathbf{m}}\left(P_{\lambda,\mu}\right)}{\partial z_1\partial z_2\ldots\partial z_n}\left(1-A_1-A_2-\cdots-A_n\right)^{2q+1}\right\}dz_1\cdots dz_n\right]W_N$$
$$\wedge dt_1\wedge\cdots\wedge dt_n,$$

where the summations $\sum_{j_1<j_2},\ldots,\sum_{j_1<\cdots<j_{n-1}}$ mean $\sum_{1\le j_1<j_2\le n},\ldots,$ $\sum_{1\le j_1<\cdots<j_{n-1}\le n}$, respectively.

Now we need only to compute the integration over $D_0$ in the above large bracket. Let be $F=1-A_1-A_2-\cdots-A_n$. For the convenience, we write $\partial z_{j_1}\ldots\partial z_{j_k}$ by $\partial^k z_{j_1,\ldots,j_k}$.

Lemma 3.2. *For any function $f=f(z_1,\ldots,z_n)$, the following integration*

$$\int_{D_0}\left\{f\frac{\partial^n F^{m+n}}{\partial^n z_{1\ldots n}}+\sum_{j_1=1}^{n}\frac{\partial f}{\partial z_{j_1}}\frac{\partial^{n-1}F^{m+n-1}}{\partial^{n-1}z_{1\ldots\widehat{j_1}\ldots n}}\left(1-A_{j_1}\right)\right.$$
$$+\sum_{j_1<j_2}\frac{\partial^2 f}{\partial^2 z_{j_1j_2}}\frac{\partial^{n-2}F^{m+n-2}}{\partial^{n-2}z_{1\ldots\widehat{j_1}\ldots\widehat{j_2}\ldots n}}\left(1-A_{j_1}-A_{j_2}\right)+\cdots$$
$$\left.+\frac{\partial^n f}{\partial^n z_{12\ldots n}}F^{m+1}\right\}dz_1\cdots dz_n. \tag{3.3}$$

*is equal to $(-1)^n f(0,\ldots,0)$.*

Proof.

We define the $(n-r)-$dimensional faces $D_{k_1\ldots k_r}$ of $D_0$ for $1\le k_1<\ldots<k_r\le n$ by

$$D_{k_1\ldots k_r}=\left\{(z_1,\ldots,z_n)\in D_0\mid z_{k_1}=0,\ldots,z_{k_r}=0\right\}.$$

If $i > p$, using integration by parts repeatedly, for any function $g = g(z_1, \ldots, z_n)$, we have

$$
\begin{aligned}
&\int_{D_0} g \frac{\partial^p F^i}{\partial^p z_{k_1 \ldots k_p}} dz_1 \ldots dz_n \\
&= (-1)^p \Bigg\{ \int_{D_{k_1 \ldots k_p}} g F^i \Big|_{D_{k_1 \ldots k_p}} dz_1 \ldots \widehat{dz_{k_1}} \ldots \widehat{dz_{k_p}} \ldots dz_n \\
&+ \sum_{s_1=1}^{r} \int_{D_{k_1 \ldots \widehat{k_{s_1}} \ldots k_p}} \frac{\partial g}{\partial z_{k_{s_1}}} F^i \Big|_{D_{k_1 \ldots \widehat{k_{s_1}} \ldots k_p}} dz_1 \ldots \widehat{dz_{k_1}} \ldots dz_{k_{s_1}} \ldots \widehat{dz_{k_p}} \ldots dz_n \\
&+ \sum_{s_1 < s_2} \int_{D_{k_1 \ldots \widehat{k_{s_1}} \ldots \widehat{k_{s_2}} \ldots k_p}} \\
&\times \frac{\partial^2 g}{\partial^2 z_{k_{s_1} k_{s_2}}} F^i \Big|_{D_{k_1 \ldots \widehat{k_{s_1}} \ldots \widehat{k_{s_2}} \ldots k_p}} dz_1 \ldots \widehat{dz_{k_1}} \ldots dz_{k_{s_1}} \ldots dz_{k_{s_2}} \ldots \widehat{dz_{k_p}} \ldots dz_n \\
&+ \cdots + \int_{D_0} \frac{\partial^p g}{\partial^p z_{k_1 \ldots k_p}} F^i dz_1 dz_2 \ldots dz_n \Bigg\}.
\end{aligned}
$$

Applying this equation, we can move all the differentials $\partial/\partial z_k$'s of $F^i$ to those of $f$ in (3.3). There integrations over $(n-r)$-dimensional faces appear on terms which range from 1-st to $(n-r+1)$-th, and their sum $S_{n-r}$ is given by

$$
\begin{aligned}
S_{n-r} = &\ (-1)^n \sum_{k_1 < \cdots < k_r} \int_{D_{k_1 \cdots k_r}} \frac{\partial^{n-r} f}{\partial^{n-r} z_{i_1 \ldots i_{n-r}}} F^{m+n} \Big|_{D_{k_1 \cdots k_r}} d^{n-r} z_{\widehat{K}} \\
&+ (-1)^{n-1} \sum_{j_1=1}^{n} \sum_{k_1 < \cdots < k_r} \int_{D_{k_1 \cdots k_r}} \frac{\partial^{n-r} f}{\partial^{n-r} z_{j_1 i_1 \ldots i_{n-r-1}}} F^{m+n-1} \Big|_{D_{k_1 \cdots k_r}} \\
&\qquad \times (1 - A_{j_1})\, d^{n-r} z_{\widehat{K}} \\
&+ (-1)^{n-2} \sum_{j_1 < j_2} \sum_{k_1 < \cdots < k_r} \int_{D_{k_1 \cdots k_r}} \frac{\partial^{n-r} f}{\partial^{n-r} z_{j_1 j_2 i_1 \ldots i_{n-r-2}}} F^{m+n-2} \Big|_{D_{k_1 \cdots k_r}} \\
&\qquad \times (1 - A_{j_1} - A_{j_2})\, d^{n-r} z_{\widehat{K}} \\
&+ \quad \cdots \\
&+ (-1)^{r+1} \sum_{j_1 < \cdots < j_{n-r-1}} \sum_{k_1 < \cdots < k_r} \int_{D_{k_1 \cdots k_r}} \frac{\partial^{n-r} f}{\partial^{n-r} z_{j_1 \ldots j_{n-r-1} i_1}} F^{m+r+1} \Big|_{D_{k_1 \cdots k_r}}
\end{aligned}
$$

$$\times\left(1-A_{j_1}-\cdots-A_{j_{n-r-1}}\right)d^{n-r}z_{\widehat{K}}$$

$$+\quad(-1)^r\sum_{j_1<\cdots<j_{n-r}}\int_{D_{k_1\cdots k_r}}\left.\frac{\partial^{n-r}f}{\partial^{n-r}z_{j_1\ldots j_{n-k}}}F^{m+r}\right|_{D_{k_1\cdots k_r}}$$
$$\times\left(1-A_{j_1}-\cdots-A_{j_{n-r}}\right)d^{n-r}z_{\widehat{K}},$$

where the $l$-th term contains the indices $\{j\}$ $(l-1)$-times, $\{i\}$ $(n-r-l+1)$-times and $\{k\}$ $r$-times. The indices $\{k\}$ are chosen out of the complement of $\{j\}$ in $\{1,2,\ldots,n\}$, and the indices $\{i\}$ are chosen out of the complement of $\{j\}\coprod\{k\}$ in $\{1,2,\ldots,n\}$, and $d^{n-r}z_{\widehat{K}}$ means $dz_1\ldots\widehat{dz_{k_1}}\ldots\widehat{dz_{k_r}}\ldots dz_n$.

Since $F|_{D_{k_1\cdots k_r}}=(1-A_{j_1}-\cdots-A_{j_{l-1}}-\cdots-A_{i_1}-\cdots-A_{i_{n-r-l+1}})$, we can write

$$S_{n-r}=\sum_{j_1<\cdots<j_{n-r}}\int_{D_{k_1\cdots k_r}}\frac{\partial^{n-r}f}{\partial^{n-r}z_{j_1\ldots j_{n-r}}}$$
$$\times\Bigg\{(-1)^n\left(1-A_{j_1}-\cdots-A_{j_{n-r}}\right)^{m+n}$$
$$+(-1)^{n-1}(1-A_{j_1}-\cdots-A_{j_{n-r}})^{m+n-1}$$
$$\times\left(\binom{n-r}{1}-\binom{n-r-1}{0}A_{j_1}-\cdots-\binom{n-r-1}{0}A_{j_{n-r}}\right)$$
$$+(-1)^{n-2}(1-A_{j_1}-\cdots-A_{j_{n-r}})^{m+n-2}$$
$$\times\left(\binom{n-r}{2}-\binom{n-r-1}{1}A_{j_1}-\cdots-\binom{n-r-1}{1}A_{j_{n-r}}\right)$$
$$+\cdots$$
$$+(-1)^{r+1}(1-A_{j_1}-\cdots-A_{j_{n-r}})^{m+r+1}$$
$$\times\left(\binom{n-r}{n-r-1}-\binom{n-r-1}{n-r-2}A_{j_1}-\cdots\right.$$
$$\left.-\binom{n-r-1}{n-r-2}A_{j_{n-r}}\right)$$
$$+(-1)^r(1-A_{j_1}-\cdots-A_{j_{n-r}})^{m+r}$$
$$\times\left(1-A_{j_1}-\cdots-A_{j_{n-r}}\right)\Bigg\}d^{n-r}z_{\widehat{K}}.$$

A simple calculation shows that the polynomial in the above bracket with variables $A_{j_1},\ldots,A_{j_{n-r}}$ is zero if $r<n$. Thus the integration given in (3.3) is equal to $(-1)^n S_0$. However $S_0$ is equal to $f(0,\ldots,0)$. This ends the proof of LEMMA 3.2. □

Now we can conclude that

$$\begin{aligned}
&\mathrm{Res}_{\varphi}\,(\tau, X_{\lambda,\mu}, N\times T^n)\\
&\quad = \left[\int_D \varphi\,(\Omega)\right]\\
&\quad = \frac{2^n\pi^q\varphi\,(\lambda_1,\lambda_1,\lambda_2,\lambda_2,\ldots,\lambda_q,\lambda_q,0,\ldots,0)}{\lambda_1^2\lambda_2^2\ldots\lambda_q^2\mu_1\mu_2\ldots\mu_n}\,[W_N\wedge W_{T^n}]\,.\ \square
\end{aligned}$$

REMARK 3.4. Consider the following elements of degree 8 in $I_4(WO_4)$,

$$c_1^4, c_1^2c_2, c_1c_3 \in I_4(WO_4).$$

Heitsch constructed an example such that the residues of $c_1^4, c_1^2c_2$ and $c_1c_3$ vary as functions $c_1^4(\lambda_1,\lambda_1,\lambda_2,\lambda_2)/(\lambda_1\lambda_2)^2, c_1^2c_2(\lambda_1,\lambda_1,\lambda_2,\lambda_2)/(\lambda_1\lambda_2)^2$ and $c_1c_3(\lambda_1,\lambda_1,\lambda_2,\lambda_2)/(\lambda_1\lambda_2)^2$, respectively([He2] Theorem 5.4). However, since

$$\begin{aligned}
&c_1^4(\lambda_1,\lambda_1,\lambda_2,\lambda_2)/(\lambda_1\lambda_2)^2\\
&\qquad = 4(\lambda_1+\lambda_2)^2\cdot 4(\lambda_1+\lambda_2)^2/\{(2\pi)^4(\lambda_1\lambda_2)^2\}\\
&c_1^2c_2(\lambda_1,\lambda_1,\lambda_2,\lambda_2)/(\lambda_1\lambda_2)^2\\
&\qquad = (\lambda_1^2+4\lambda_1\lambda_2+\lambda_2^2)\cdot 4(\lambda_1+\lambda_2)^2/\{(2\pi)^4(\lambda_1\lambda_2)^2\}\\
&c_1c_3(\lambda_1,\lambda_1,\lambda_2,\lambda_2)/(\lambda_1\lambda_2)^2\\
&\qquad = \lambda_1\lambda_2\cdot 4(\lambda_1+\lambda_2)^2/\{(2\pi)^4(\lambda_1\lambda_2)^2\},
\end{aligned}$$

they vary keeping the linear relation

$$\mathrm{Res}_{c_1^4} - 4\mathrm{Res}_{c_1^2c_2} + 8\mathrm{Res}_{c_1c_3} = 0. \tag{3.5}$$

On the other hand, putting $q=1$ and $n=2$ in THEOREM 3.1, we gain another example such that the residues of $c_1^4, c_1^2c_2$ and $c_1c_3$ vary as functions

$c_1^4(\lambda_1,\lambda_1,0,0)/(\lambda_1^2\mu_1\mu_2)$, $c_1^2c_2(\lambda_1,\lambda_1,0,0)/(\lambda_1^2\mu_1\mu_2)$ and $c_1c_3(\lambda_1,\lambda_1,0,0)/(\lambda_1^2\mu_1\mu_2)$, respectively. However, since

$$\begin{array}{rcll} c_1^4(\lambda_1,\lambda_1,0,0)/(\lambda_1^2\mu_1\mu_2) & = & 4\lambda_1^2 & \cdot 4\lambda_1^2/\{(2\pi)^4(\lambda_1^2\mu_1\mu_2)\} \\ c_1^2c_2(\lambda_1,\lambda_1,0,0)/(\lambda_1^2\mu_1\mu_2) & = & \lambda_1^2 & \cdot 4\lambda_1^2/\{(2\pi)^4(\lambda_1^2\mu_1\mu_2)\} \\ c_1c_3(\lambda_1,\lambda_1,0,0)/(\lambda_1^2\mu_1\mu_2) & = & 0 & \cdot 4\lambda_1^2/\{(2\pi)^4(\lambda_1^2\mu_1\mu_2)\}, \end{array}$$

they vary keeping the linear relation

$$\mathrm{Res}_{c_1^4} - 4\mathrm{Res}_{c_1^2c_2} = 0. \tag{3.6}$$

The relation (3.6) cannot be derived directly from the relation (3.5). Precisely, even if we put $\mathrm{Res}_{c_1c_3} = 0$ in the relation(3.5), we cannot obtain the relation(3.6). Because $\mathrm{Res}_{c_1c_3}$ in (3.5) is zero if and only if $\lambda_1+\lambda_2=0$, since $\lambda_1$ and $\lambda_2$ are non-zero number. However if $\lambda_1+\lambda_2=0$, $\mathrm{Res}_{c_1^4}$ and $\mathrm{Res}_{c_1^2c_2}$ in (3.5) must be also zero. Thus the relation (3.5) vanishes.

Similarly, consider the following elements of degree 10 in $I_5(WO_5)$,

$$c_1^5, c_1^3c_2, c_1^2c_3, c_1c_2^2, c_1c_4, c_2c_3 \in I_5(WO_5).$$

Again Heitsch constructed an example such that the residues of $c_1^5, c_1^3c_2$, $c_1^2c_3, c_1c_2^2, c_1c_4$ and $c_2c_3$ vary keeping the linear relation([He2] Theorem 5.9)

$$\left\{\begin{array}{l} \mathrm{Res}_{c_1^5} - 4\mathrm{Res}_{c_1^3c_2} + 8\mathrm{Res}_{c_1^2c_3} = 0 \\ \mathrm{Res}_{c_1^5} - 16\mathrm{Res}_{c_1c_2^2} + 64\mathrm{Res}_{c_1c_4} - 64\mathrm{Res}_{c_2c_3} = 0 \end{array}\right. . \tag{3.7}$$

On the other hand, putting $q=1$ and $n=3$ in THEOREM 3.1, we gain another example such that the residues of theirs vary keeping the linear relation

$$\left\{\begin{array}{l} \mathrm{Res}_{c_1^5} - 4\mathrm{Res}_{c_1^3c_2} = 0 \\ \mathrm{Res}_{c_1^5} - 16\mathrm{Res}_{c_1c_2^2} = 0 \end{array}\right. . \tag{3.8}$$

Again, the relation (3.8) cannot be derived directly from the relation (3.7).

Of course, similar argument is valid not only for $I_4(WO_4)$ and $I_5(WO_5)$ but also for more general $I_q(WO_q)$ with $q \geq 4$.

## §4. Extended Examples

Let $G = \mathrm{SL}_{2n_1}\mathbf{R} \times \mathrm{SL}_{2n_2}\mathbf{R} \times \cdots \times \mathrm{SL}_{2n_r}\mathbf{R}$ and $K = \mathrm{SO}_{2n_1} \times \mathrm{SO}_{2n_2} \times \cdots \times \mathrm{SO}_{2n_r}$ where $n_1 \leq n_2 \leq \cdots \leq n_r$ are positive integers such that

$2n_1+2n_2+\cdots+2n_r=2q$. Since each component of $G$ is semi-simple, there is a discrete subgroup $\Gamma=\Gamma_1\times\Gamma_2\times\cdots\times\Gamma_r$ such that each $\Gamma_i\backslash\mathrm{SL}_{2n_i}\mathbf{R}/\mathrm{SO}_{2n_i}$ is a compact manifold.

Let $(x_1^i,y_1^i,x_2^i,y_2^i,\ldots,x_{n_i}^i,y_{n_i}^i)$ be coordinates on the $i$-th factor of $\mathbf{R}^{2q}=\mathbf{R}^{2n_1}\times\cdots\times\mathbf{R}^{2n_r}\subset\mathbf{R}^{2q+n}$ and $(z_1,z_2,\ldots,z_n)$ coordinates on the last factor $\mathbf{R}^n\subset\mathbf{R}^{2q+n}$. Choose non-zero numbers $\lambda_1,\lambda_2,\ldots,\lambda_r,\mu_1,\mu_2,\ldots,\mu_n$ and set

$$\begin{aligned} X_{\lambda,\mu} &= \sum_{i=1}^{r}\lambda_i\left(x_1^i\frac{\partial}{\partial x_1^i}+y_1^i\frac{\partial}{\partial y_1^i}+\cdots+x_{n_i}^i\frac{\partial}{\partial x_{n_i}^i}+y_{n_i}^i\frac{\partial}{\partial y_{n_i}^i}\right)\\ &\quad+\sum_{j=1}^{n}\mu_j z_j\Psi_j(z_j)\frac{\partial}{\partial z_j}, \end{aligned}$$

where $\Psi_j$ is a function defined at the beginning of §3. $X_{\lambda,\mu}$ induces a $\Gamma$-vector field $X_{\lambda,\mu}$ for the foliation $\tau$ of the transversely foliated bundle

$$T^nM=(\Gamma\times\mathbf{Z}^n\setminus G\times\mathbf{R}^n)\times_K\mathbf{R}^{2q+n}\longrightarrow N\times T^n=(\Gamma\setminus G/K)\times T^n.$$

The foliation $\tau$ on $T^nM\longrightarrow N\times T^n$ is diffeomorphic to the foliation also denoted by $\tau$ which is obtained from the flat bundle structure

$$((G\times\mathbf{R}^n)/K)\times_{\Gamma\times\mathbf{Z}^n}\mathbf{R}^{2q+n}\longrightarrow N\times T^n=(\Gamma\setminus G/K)\times T^n.$$

Recall that

$$H^*(sl_{2q}\mathbf{R},\mathrm{SO}_{2q})=\wedge(s_3,s_5,\ldots,s_{2q-1},\chi)$$

is an exterior graded algebra with $\deg s_i=2i-1$ and $\deg\chi=2q$. We write $s_I(M)=s_{i_1}s_{i_2}\cdots s_{i_r}(M)\in H^*(N)\subset H^*(N\times T^n)$ for the characteristic class of the flat $\mathrm{SL}_{2q}\mathbf{R}$-bundle $M=(G/K)\times_\Gamma\mathbf{R}^{2q}\longrightarrow N=(\Gamma\setminus G/K)$ corresponding to the element $s_I\in H^*(sl_{2q}\mathbf{R},\mathrm{SO}_{2q})$.

PROPOSITION 4.1. *Let $h_Ic_J=h_{i_1}h_{i_2}\ldots h_{i_r}c_J\in I_{2q+n}(WO_{2q+n})$ where $i_1>1$ and $i_k<2q$. Then*

$$\mathrm{Res}_{h_Ic_J}(\tau,X_{\lambda,\mu},N\times T^n)=s_I(M)\cdot\mathrm{Res}_{c_J}(\tau,X_{\lambda,\mu},N\times T^n).$$

To relate $\mathrm{Res}_{c_J}(\tau,X_{\lambda,\mu},N\times T^n)$ with the previous example, we consider the following scheme. Let

$$i^*:\otimes_{i=1}^{r}H^*(sl_{2n_i}\mathbf{R},\mathrm{SO}_{2n_i})\longrightarrow\otimes^qH^*(sl_2\mathbf{R},\mathrm{SO}_2)$$

be the induced map by the natural inclusion $i : \times^q \mathrm{SL}_2\mathbf{R} \longrightarrow G = \times_{i=1}^r \mathrm{SL}_{2n_i}\mathbf{R}$ and

$$\mu : \otimes_{i=1}^r H^*(sl_{2n_i}\mathbf{R}, \mathrm{SO}_{2n_i}) \longrightarrow H^*(N) = H^*(\Gamma\backslash G/K)$$

be the characteristic map corresponding to the flat $G$-bundle $M \longrightarrow N$.

Proposition 4.2. *For $c_J \in I_{2q+n}(WO_{2q+n})$, modulo $\mu(\mathrm{ker} i^*)$*

$$\begin{aligned}&\mathrm{Res}_{c_J}(\tau, X_{\lambda,\mu}, N \times T^n)\\ &\quad = \beta \cdot \frac{c_J(\lambda_1, \lambda_2, \ldots, \lambda_r, 0, \ldots, 0)}{\lambda_1^{2n_1}\lambda_2^{2n_2}\cdots\lambda_r^{2n_r}\mu_1\mu_2\cdots\mu_n}\chi(M) \cdot [dt_1 \wedge \cdots \wedge dt_n],\end{aligned}$$

*where $\beta$ is a non-zero constant , $\chi(M)$ is the Euler class of $M \longrightarrow N$, and*

$$c_J(\lambda_1, \lambda_2, \ldots, \lambda_r, 0, \ldots, 0)$$

*is the Chern polynomial $c_J$ applied to the diagonal matrix*

$$\mathrm{diag}(\underbrace{\lambda_1, \ldots, \lambda_1}_{2n_1-\text{times}}, \underbrace{\lambda_2, \ldots, \lambda_2}_{2n_2-\text{times}}, \ldots, \underbrace{\lambda_r, \ldots, \lambda_r}_{2n_r-\text{times}}, \underbrace{0, \ldots, 0}_{n-\text{times}}).$$

The proof of PROPOSITION 4.2 is analogous to the proof of Theorem 5.12 and Theorem 5.17 of [He2] and is omitted. Finally ,combining PROPOSITION 4.1 and PROPOSITION 4.2, we have

Theorem 4.3. *Let $h_I c_J = h_{i_1} h_{i_2} \ldots h_{i_r} c_J \in I_{2q+n}(WO_{2q+n})$ where $i_1 > 1$ and $i_k < 2q$. Then modulo $\mu(\mathrm{ker} i^*)$*

$$\begin{aligned}&\mathrm{Res}_{h_I c_J}(\tau, X_{\lambda,\mu}, N \times T^n)\\ &\quad = \beta \cdot \frac{c_J(\lambda_1, \lambda_2, \ldots, \lambda_r, 0, \ldots, 0)}{\lambda_1^{2n_1}\lambda_2^{2n_2}\cdots\lambda_r^{2n_r}\mu_1\mu_2\cdots\mu_n} s_I(M)\chi(M) \cdot [dt_1 \wedge \cdots \wedge dt_n].\end{aligned}$$

Proof of Proposition 4.1.

Let $\theta_T^f$ and $\theta_T^r$ be the flat connection and a metric connection respectively on $T\mathbf{R}^{2q+n} \longrightarrow T^n M$. Here $T\mathbf{R}^{2q+n} \longrightarrow T^n M$ is the tangent bundle along the fiber of $T^n M \longrightarrow N \times T^n$. As in the proof of THEOREM 3.1 we can identify $T\mathbf{R}^{2q+n} \longrightarrow T^n M$ with the normal bundle $\nu$ of $\tau$. Let $\theta_T^b$ be a basic

$X_{\lambda,\mu}$-connection on supported off the complement of $\Pi : D \longrightarrow N \times T^n$ a disc subbundle of $T^nM \longrightarrow N \times T^n$. Denote the curvature of $\theta^b_T$ by $\Omega^b_T$.

Then $\mathrm{Res}_{h_I c_J}(\tau, X_{\lambda,\mu}, N \times T^n)$ is determined by the differential form $\Delta_{c_I}(\theta^b_T, \theta^r_T)c_J(\Omega^b_T)$. $\Delta_{c_I}(\theta^b_T, \theta^r_T)c_J(\Omega^b_T)$ and $\Delta_{c_I}(\theta^f_T, \theta^r_T)c_J(\Omega^b_T)$ determine the same class in $H^*_c(D)$ (see [He2] p.447, [He1], [L]). Now we need to show the following.

Lemma 4.4.

$$\Pi^*(s_I(M))\left[c_J(\Omega^b_T)\right] = \left[\Delta_{c_I}(\theta^f_T, \theta^r_T)c_J(\Omega^b_T)\right] \in H^*_c(D).$$

Proof.
Let

$$\begin{aligned} \Pi_1 : T^nM &= ((G/K) \times_\Gamma \mathbf{R}^{2q}) \times \overbrace{(\mathbf{R} \times_{\mathbf{Z}} \mathbf{R}) \times \cdots \times (\mathbf{R} \times_{\mathbf{Z}} \mathbf{R})}^{n-\text{times}} \longrightarrow M \\ &= ((G/K) \times_\Gamma \mathbf{R}^{2q}) \end{aligned}$$

and

$$\Pi_j : T^nM = ((G/K) \times_\Gamma \mathbf{R}^{2q}) \times \overbrace{(\mathbf{R} \times_{\mathbf{Z}} \mathbf{R}) \times \cdots \times (\mathbf{R} \times_{\mathbf{Z}} \mathbf{R})}^{n-\text{times}} \longrightarrow (\mathbf{R} \times_{\mathbf{Z}} \mathbf{R})$$

be the natural projections onto the $j$-th factor of $T^nM$ for $j = 2, \ldots, n+1$. Then the bundle $T\mathbf{R}^{2q+n} \longrightarrow T^nM$ can be written of the form

$$\Pi_1^*(T\mathbf{R}^{2q}) \oplus \Pi_2^*(T\mathbf{R}) \oplus \cdots \oplus \Pi_{n+1}^*(T\mathbf{R}) \longrightarrow T^nM,$$

where $T\mathbf{R}^{2q} \longrightarrow M = ((G/K) \times_\Gamma \mathbf{R}^{2q})$ is the normal bundle of the foliation of the transversely foliated bundle $M = ((G/K) \times_\Gamma \mathbf{R}^{2q}) \longrightarrow N = \Gamma\backslash G/K$ and $T\mathbf{R} \longrightarrow \mathbf{R} \times_{\mathbf{Z}} \mathbf{R}$ are trivial bundles. Let $\sigma_j$ be global non-zero cross-section of the $j$-th factor $T\mathbf{R}$ for $j = 2, \ldots, n+1$. We give the $j$-th factor $T\mathbf{R}$ the metric $r_j$ so that $\sigma_j$ is length 1 for $j = 2, \ldots, n+1$. Let the metric $r$ on $T\mathbf{R}^{2q+n}$ be induced by the metrics $r_2, \ldots, r_{n+1}$ and some metric $r_1$ on $T\mathbf{R}^{2q}$. If $\theta^{r,1}$ is a metric connection on $T\mathbf{R}^{2q}$ with respect to $r_1$, then $\Pi_1^*\theta^{r,1}$ is also a metric connection on $\Pi_1^*T\mathbf{R}^{2q}$. Let $\theta^j$ be the metric connection on $\Pi_j^*T\mathbf{R}$ so that $\Pi_j^*\sigma_j$ is flat for $j = 2, \ldots, n+1$. Finally we define the metric connection $\theta^r_T$ on $T\mathbf{R}^{2q+n}$ to be $\Pi_1^*\theta^{r,1} \oplus \theta^2 \oplus \cdots \oplus \theta^{n+1}$.

Let $\xi_1, \ldots, \xi_{2q}$ be a local framing of $T\mathbf{R}^{2q}$. With respect to the local framing

$$\zeta = \{\Pi_1^*\xi_1, \ldots, \Pi_1^*\xi_{2q}, \Pi_2^*\sigma_2, \ldots, \Pi_{n+1}^*\sigma_{n+1}\}$$

of $T\mathbf{R}^{2q+n} = \Pi_1^*(T\mathbf{R}^{2q}) \oplus \Pi_2^*(T\mathbf{R}) \oplus \cdots \oplus \Pi_{n+1}^*(T\mathbf{R}) \longrightarrow T^nM$ , the local connection form of $\theta_T^r$ is given by

$$\begin{pmatrix} \Pi_1^*(\theta^{r,1}) & 0 \\ 0 & 0 \end{pmatrix},$$

where $(\theta^{r,1})$ is the local connection form of $\theta^{r,1}$ with respect to the framing $\xi_1, \ldots, \xi_{2q}$.

On the other hand, let $\theta_T^f$ be the flat connection on $T\mathbf{R}^{2q+n}$ induced by the flat connection $\theta^{f,1}$ on $T\mathbf{R}^{2q}$ and the flat structure on $\Pi_2^*(T\mathbf{R}) \oplus \cdots \oplus \Pi_{n+1}^*(T\mathbf{R})$. Then the local connection form of $\theta_T^r$ with respect to the framing $\zeta$ is given by

$$\begin{pmatrix} \Pi_1^*(\theta^{f,1}) & 0 & 0 & 0 \\ 0 & \Pi_2^*(g_2(z_1)dz_1) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \Pi_{n+1}^*(g_{n+1}(z_n)dz_n) \end{pmatrix}.$$

Here $(\theta^{f,1})$ is the local connection form of $\theta^{f,1}$ with respect to the framing $\xi_1, \ldots, \xi_{2q}$ and $g_j(z_{j-1})$ are some functions on $\mathbf{R}$ for $j = 2, \ldots, n+1$.

Thus, with respect to the framing $\zeta$, the local curvature form $\Omega_T^{(t)}$ of the connection $\theta_T^{(t)} = t\theta_T^{f,1} + (1-t)\theta_T^{r,1}$ is given by

$$\begin{pmatrix} \Pi_1^*(\Omega^{(t)}) & 0 \\ 0 & 0 \end{pmatrix},$$

where $(\Omega^{(t)})$ is the local curvature form of the connection $\theta^{(t)} = t\theta^{f,1} + (1-t)\theta^{r,1}$ with respect to the framing $\xi_1, \ldots, \xi_{2q}$.

From the definition of the Chern polynomials, if we denote by $\Theta$ the diagonal matrix $\mathrm{diag}(\Pi_2^*(g_2(z_1)dz_1), \ldots, \Pi_{n+1}^*(g_{n+1}(z_n)dz_n))$, then we have

$$\Delta_{c_i}(\theta_T^{f,1}, \theta_T^{r,1}) = \Pi_1^*\Delta_{c_i}(\theta^{f,1}, \theta^{r,1}) + \beta\Pi_1^*\left\{\int_0^1 c_{i-1}(\Omega^{(t)})dt\right\} \wedge c_1(\Theta)$$

where $\beta$ is some non-zero constant. Since $c_k(\Omega^f) \equiv 0$, $c_k(\Omega^{(t)}) = d\Delta_{c_k}(\theta^{(t)}, \theta^f)$ for $0 \le t \le 1$ and $k = i-n, \ldots, i-1$ ,

$$\Delta_{c_i}(\theta_T^{f,1}, \theta_T^{r,1}) = \Pi_1^* \Delta_{c_i}(\theta^{f,1}, \theta^{r,1}) + d\left[\beta \Pi_1^* \left\{ \int_0^1 \Delta_{c_{i-1}}(\theta^{(t)}, \theta^f) dt \right\} \wedge c_1(\Theta)\right]$$

As $c_J(\Omega_T^b)$ is closed and has compact support in $D$,

$$\Delta_{c_I}(\theta_T^{f,1}, \theta_T^{r,1}) c_J(\Omega_T^b) = \Pi_1^* \Delta_{c_I}(\theta^{f,1}, \theta^{r,1}) c_J(\Omega_T^b) + (\text{exact form}). \tag{4.5}$$

The exact form in (4.5) has also compact support. Now both $\Delta_{c_I}(\theta_T^{f,1}, \theta_T^{r,1}) c_J(\Omega_T^b)$ and $\Pi_1^* \Delta_{c_I}(\theta^{f,1}, \theta^{r,1}) c_J(\Omega_T^b)$ are closed and have compact support in $D$. Thus they determine the same class in $H_c^*(D)$. However $\Pi_1^* \Delta_{c_I}(\theta^{f,1}, \theta^{r,1}) c_J(\Omega_T^b)$ represents $\Pi^*(s_I(M)) \left[ c_J(\Omega_T^b) \right]$. This ends the proof of LEMMA 4.4. $\Box$

By LEMMA 4.4, we can complete the proof of PROPOSITION 4.1, and thus of THEOREM 4.3 as follows.

$$\begin{aligned}
\mathrm{Res}_{h_I c_J}(\tau, X_{\lambda,\mu}, N \times T^n) &= \gamma_D\left(\left[\Delta_{c_I}(\theta_T^b, \theta_T^r) c_J(\Omega_T^b)\Big|_D\right]\right) \\
&= \gamma_D\left(\left[\Delta_{c_I}(\theta_T^f, \theta_T^r) c_J(\Omega_T^b)\Big|_D\right]\right) \\
&= \gamma_D\left(\Pi^*(s_I(M)) \left[c_J(\Omega_T^b)\Big|_D\right]\right) \\
&= s_I(M) \mathrm{Res}_{c_J}(\tau, X_{\lambda,\mu}, N \times T^n). \ \Box
\end{aligned}$$

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(Received September 2, 1998)
(Revised April 2, 1999)

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