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Existence and Uniqueness Theorems for a Class of Linear Fuchsian Partial Differential Equations

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Abstract. We will consider linear Fuchsian partial differential operators of the form $\mathcal{P} = (tD_t)^m + \sum_{j=0}^{m-1} \sum_{|\alpha| \le m-j} a_{j,\alpha}(t,z) \cdot (\mu(t)D_z)^\alpha (tD_t)^j$. For this class of operators, we will formulate existence and uniqueness theorems which are slightly more general than the ones obtained by Baouendi and Goulaouic in 1973. We will specify some properties of the function $\mu(t)$ and assert that a property is necessary for our theorems to hold.

1. Introduction

In 1973, Baouendi and Goulaouic defined Fuchsian partial differential operators, considered to be "natural" generalizations of Fuchsian ordinary differential operators, and gave some generalizations in [1] of the classical Cauchy-Kowalewski and Holmgren theorems for this class of operators. Since then, many have applied and extended their arguments to various cases, as can be seen in the works of Elschner [2],[3], Tahara [8] and Oaku [6]. Recent related results include those of Mandai [5], Uokawa [10] and Yamane [11].

In this paper, we shall prove similar existence and uniqueness theorems as in [1], but for a slightly wider class of linear Fuchsian partial differential operators. To this end, we shall make use of the concept of weight functions introduced by Tahara in [9]. Finally, we shall assert that a certain property of the weight function is necessary for the uniqueness theorem to hold.

2. Statement of Results

Denote by N the set of all natural numbers and let $\mathsf{N}^* = \mathsf{N} - \{0\}$. Let (t,z) be an element of $\mathsf{R} \times \mathsf{C}^n$. The respective derivatives will be denoted by

$$D_t = \frac{\partial}{\partial t} \qquad \text{and} \qquad D_z^\alpha = \left(\frac{\partial}{\partial z_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial z_n}\right)^{\alpha_n},$$

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where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathsf{N}^n$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Let $\mathcal{V}$ be a connected neighborhood of $0 \in \mathsf{C}^n$, and let Ω be a connected subset of R^n contained in $\mathcal{V}$. For a sufficiently small $s > 0$, define

$$\Omega_s = \bigcup_{a \in \Omega} D(a, s),$$

where $D(a, s) = \{z \in \mathsf{C}^n; |z - a| < s\}$. Choose $s_1 > 0$ such that $\overline{\Omega}_s \subset \mathcal{V}$ for all $0 < s \leq s_1$. For simplicity, we will assume that $s_1 < 1$.

For an open, bounded set $X \subset \mathsf{C}^n$, denote by $\mathcal{A}(X)$ the space of complex-valued functions which are continuous over $\overline{X}$ and holomorphic on X. Consider in particular $\mathcal{A}(\Omega_s)$. This is a Banach space with norm

$$\|f\|_{\mathcal{A}(\Omega_s)} = \sup_{z \in \overline{\Omega}_s} |f(z)|.$$

For brevity, we will denote this space by $\mathcal{A}_s$, invoking the notation $\mathcal{A}(\Omega_s)$ only when the need arises. Note that for $r > s > 0$, we have $\mathcal{A}_r \hookrightarrow \mathcal{A}_s$.

Let E be a topological vector space, $\varepsilon > 0$ and $p \in \mathsf{N}$. We will follow the notation in [1], denoting by $C_p^0([0, \varepsilon], E)$ the set of all functions $u(t)$ in $C^p((0, \varepsilon), E)$ which satisfy $(tD_t)^j u(t) \in C^0([0, \varepsilon], E)$ for $0 \leq j \leq p$.

By a *weight function* $\mu(t)$ on $(0, T)$, we mean a real-valued function satisfying the following (see [9]):

(A–1) $\mu(t) \in C^0((0, T))$,

(A–2) $\mu(t) > 0$ and is increasing on $(0, T)$,

(A–3) $\displaystyle\int_0^T \frac{\mu(s)}{s}\, ds < \infty$.

Note that conditions (A–2) and (A–3) imply $\lim_{t \to 0} \mu(t) = 0$.

We will only deal with linear *Fuchsian partial differential operators* of order $m \in \mathsf{N}^*$ given by

$$\mathcal{P} = (tD_t)^m + \sum_{j=0}^{m-1} \sum_{|\alpha| \leq m-j} a_{j,\alpha}(t, z)(\mu(t) D_z)^\alpha (tD_t)^j. \tag{1}$$

In what follows, we will assume that $\mu(t)$ is a weight function and that all the coefficients $a_{j,\alpha}(t, z)$ are contained in $C^0([0, T], \mathcal{A}(\mathcal{V}))$. We define the *characteristic polynomial* associated with the above operator $\mathcal{P}$ as follows:

$$\mathcal{C}(\lambda, z) = \lambda^m + a_{m-1,0}(0, z)\lambda^{m-1} + \cdots + a_{0,0}(0, z). \tag{2}$$

Its roots will be referred to as *characteristic exponents* and will be denoted $\lambda_1(z), \ldots, \lambda_m(z)$. All throughout this paper, we will assume that

$$\textbf{(NE)} \qquad \Re\lambda_j(z) < 0 \qquad \text{for all } z \in \mathcal{V},\ 1 \le j \le m.$$

The following are the main results of this paper.

THEOREM 1. *Let $\mathcal{P}$ be the Fuchsian partial differential operator described in* (1). *Then given any $s \in (0, s_1)$, there exists $\varepsilon \in (0, T)$ such that for any $f(t,z) \in C^0([0,T], \mathcal{A}_{s_1})$, the equation $\mathcal{P}u = f$ has a unique solution $u(t,z)$ in $C^0_m([0,\varepsilon], \mathcal{A}_s)$.*

The case when $\mu(t) = t^\eta$ $(\eta > 0)$ was established by Baouendi and Goulaouic in [1], while the case when $\mu(t) = (-\log t)^{-\rho}$ $(\rho > 1)$ was treated by Uokawa in [10]. Aside from these, we can also consider the following family of weight functions: set $L_0(t) = -\log t$, $L_1(t) = \log(L_0(t))$, $L_2(t) = \log(L_1(t))$, and so on. Then for all $n \in \mathsf{N}$ and $d > 1$, the function $\{L_0(t)L_1(t)\cdots L_n(t)^d\}^{-1}$ is a weight function.

Next, we consider the restriction of the operator $\mathcal{P}$ to R^n, i.e.,

$$\mathcal{P}_R = (tD_t)^m + \sum_{j=0}^{m-1} \sum_{|\alpha| \le m-j} a_{j,\alpha}(t,x)(\mu(t)D_x)^\alpha (tD_t)^j, \tag{3}$$

where $x = (x_1, \ldots, x_n) \in \mathsf{R}^n$. Denote by B_R a ball on R^n centered at the origin and of radius R, small enough to be contained in Ω. (It follows that (NE) also holds for the operator $\mathcal{P}_R$.) Furthermore, denote by $\mathcal{D}'(\mathsf{R}^n)$ the space of distributions on R^n equipped with the natural topology (see [7]). We now give the following uniqueness result:

THEOREM 2. *Suppose a function $u(t) \in C^0_m([0,T], \mathcal{D}'(\mathsf{R}^n))$ satisfies the equation $\mathcal{P}_R u = 0$ on $[0,\varepsilon) \times B_\rho$ for some $\varepsilon > 0$ and $\rho > 0$. Then we have $u = 0$ on $[0,\delta) \times B_r$ for some $\delta > 0$ and $r > 0$.*

Again, the case $\mu(t) = t^\eta$ $(\eta > 0)$ was given in [1]. By considering equations in the real domain, Baouendi and Goulaouic claimed that their results do not necessarily hold when η is nonpositive. This assumption on η translates to assumption (A–3) on the weight function, namely, $\int_0^T (\mu(s)/s)\, ds < \infty$. Section 5 is devoted to the justification of this assumption.

3. Preliminaries to the Proof

We first state some inequalities which are particularly useful in establishing some estimates in the proof.

Lemma 1. *For $0 < s < r$, $0 \leq |\alpha| \leq m$ and for $a(t, \cdot) \in C^0([0,T], \mathcal{A}_s)$, there exists a constant $M > 0$ such that for any $u(t) \in C^0([0,T], \mathcal{A}_r)$, the following inequality holds over $[0,T]$:*

$$\|a(t,\cdot)D_z^\alpha u(t,\cdot)\|_{\mathcal{A}_s} \leq \frac{M}{(r-s)^{|\alpha|}}\|u(t)\|_{\mathcal{A}_r}. \tag{4}$$

The proof is an easy application of Cauchy's Inequality associated with discs of radius $r - s$ in C^n.

In what follows, it will be convenient to set

$$\varphi(t) = \int_0^t \frac{\mu(s)}{s}\, ds.$$

It is clear that the nonnegative function $\varphi(t)$ is continuous and increasing on $[0,T]$, and differentiable on $(0,T)$. The following simple inequality involving the weight function plays a very important role in the proof of Theorem 1.

Lemma 2. *Let $p \geq 0$ and $m \geq 1$. Then for a weight function $\mu(t)$ on $[0,T]$ and for any $t \in [0,T]$, the following inequalities hold:*

$$(a) \qquad \int_0^1 \mu(\sigma t)^m \varphi(\sigma t)^p\, d\sigma \leq \mu(t)^m \varphi(t)^p$$

$$(b) \qquad \int_0^1 \mu(\sigma t)^m \varphi(\sigma t)^p\, d\sigma \leq \frac{1}{p+1}\mu(t)^{m-1}\varphi(t)^{p+1}.$$

The proof consists in simple applications of the properties of $\mu(t)$ and $\varphi(t)$, and hence will be omitted.

Next, we will define the operators which will be used in the proof. Let $g \in C^0([0,T], \mathcal{A}_s)$ and consider the ordinary differential equation

$$(tD_t + 1)^j u = (D_t t)^j u = g. \tag{5}$$

This has a unique solution $u(t,z)$ in $C_j^0([0,T],\mathcal{A}_s)$ given by

$$u(t,z) = \mathcal{H}_j(g)(t,z) \stackrel{\text{def}}{=} \int_{[0,1]^j} g(\sigma_1 \cdots \sigma_j t, z)\, d\sigma_1 \cdots d\sigma_j\, . \tag{6}$$

Since the solution is unique, it follows that $\mathcal{H}_j(D_t t)^j$ is the identity operator in $C_j^0([0,T],\mathcal{A}_s)$.

From the operator $\mathcal{P}$, define the operator $\mathcal{Q}$ by

$$\mathcal{Q} = (tD_t)^m + a_{m-1,0}(t,z)(tD_t)^{m-1} + \cdots + a_{0,0}(t,z) \tag{7}$$

and consider the ordinary differential equation $\mathcal{Q}u = g$ with a parameter z. Clearly, (NE) also holds for the operator $\mathcal{Q}$. By suitably modifying the arguments in [1], we have the following proposition (see also [10]).

Proposition 1. *There exists a sufficiently small positive T_0 such that the following hold:*

(*i*) *For any $s \in (0,s_1)$ and $g \in C^0([0,T_0],\mathcal{A}_s)$, the equation $\mathcal{Q}u = g$ has a unique solution $u(t,z)$ in $C_m^0([0,T_0],\mathcal{A}_s)$. Let the operator R be defined by $R[g](t,z) = u(t,z)$.*

(*ii*) *The operator $R\mathcal{Q}$ is the identity operator in $C_m^0([0,T_0],\mathcal{A}_s)$.*

(*iii*) *We have $R \in \mathcal{L}(C^0([0,T_0],\mathcal{A}_s), C_m^0([0,T_0],\mathcal{A}_s))$.*

(*iv*) *The operator $(D_t t)^m R$ is bounded, that is,*

$$B = \sup_{0<\tau\le T_0} \|(D_t t)^m R\|_{\mathcal{L}(C^0([0,\tau],\mathcal{A}_s),C^0([0,\tau],\mathcal{A}_s))} < +\infty.$$

4. Proof of Theorem 1

In the following, it is convenient to rewrite the operator $\mathcal{P}$ as

$$\mathcal{P} = \mathcal{Q} + \sum_{j=0}^{m-1} \sum_{1\le|\alpha|\le m-j} c_{j,\alpha}(t,z)(\mu(t)D_z)^\alpha (D_t t)^j.$$

All the $c_{j,\alpha}(t,z)$'s above are linear combinations of the $a_{j,\alpha}(t,z)$'s and as such, they are also in $C^0([0,T],\mathcal{A}(\mathcal{V}))$. Thus, the original equation $\mathcal{P}u = f$ can now be written as

$$\mathcal{Q}u = f - \sum_{j=0}^{m-1} \sum_{1\le|\alpha|\le m-j} c_{j,\alpha}(t,z)(\mu(t)D_z)^\alpha (D_t t)^j u.$$

This is further equivalent to

$$u = R\left[f - \sum_{j=0}^{m-1} \sum_{1\le|\alpha|\le m-j} c_{j,\alpha}(t,z)(\mu(t)D_z)^{\alpha}(D_t t)^j u\right]. \tag{8}$$

Hence, to solve the original equation, it is sufficient to solve the equation above.

Let $f(t,z) \in C^0([0,T],\mathcal{A}_{s_1})$. We shall assume that $T > 0$ is small enough so that $\mu(t) \le 1$ and $\varphi(t) \le 1$ on $(0,T)$, and that $0 \le T \le T_0$ where T_0 is the one mentioned in Proposition 1.

We will use the method of successive approximations to solve equation (8). Define the approximate solutions as follows: $u_0 = R[f]$, and for $p \ge 1$,

$$u_p = R[f] - R\left[\sum_{j=0}^{m-1} \sum_{1\le|\alpha|\le m-j} c_{j,\alpha}(t,z)(\mu(t)D_z)^{\alpha}(D_t t)^j u_{p-1}\right].$$

By construction, $u_p(t,z) \in C_m^0([0,T],\mathcal{A}_s)$ for all p and all $s \in (0,s_1)$. Define, too, the sequence of functions $\{v_p(t,z),\ p = 0,1,\ldots\}$ by

$$v_p(t,z) = (D_t t)^m (u_p - u_{p-1})(t,z),$$

where $u_{-1}(t,z) \equiv 0$. We see that for all $p \in \mathsf{N}$ and for all $s \in (0,s_1]$, we have $v_p(t,z) \in C^0([0,T],\mathcal{A}_s)$.

We now establish some estimates satisfied by the functions $v_p(t,z)$.

Case I. There exist positive constants K and C such that for any $0 \le p \le m$ and $s \in (0,s_1)$, the following inequality holds:

$$\|v_p(t)\|_{\mathcal{A}_s} \le K \frac{C^p\, e^{mp} \mu(t)^p}{(s_1 - s)^{mp}}. \tag{9}$$

We will show this by induction. When $p = 0$, $v_0(t) = (D_t t)^m R[f]$ and so by Proposition 1, $\|v_0(t)\|_{\mathcal{A}_s} \le B \sup_{0\le t\le T} \|f(t)\|_{\mathcal{A}_{s_1}}$. Hence, take $K = B \sup_{0\le t\le T} \|f(t)\|_{\mathcal{A}_{s_1}}$.

Assume that the above claim is true for $p = k < m$. Using the fact that $\mathcal{H}_{m-j}(D_t t)^{m-j}$ is the identity operator in $C_{m-j}^0([0,T],\mathcal{A}_s)$, we have

$$v_{k+1}(t) = -(D_t t)^m R\left[\sum_{j=0}^{m-1} \sum_{1\le|\alpha|\le m-j} c_{j,\alpha}(t,z)(\mu(t)D_z)^{\alpha}\mathcal{H}_{m-j}(v_k(t))\right].$$

Taking norms and applying Lemma 1, we get

$$\begin{aligned}\|v_{k+1}(t)\|_{\mathcal{A}_s} &\leq B \sup_{0\leq\tau\leq t} \sum_{j=0}^{m-1} \sum_{1\leq|\alpha|\leq m-j} \|c_{j,\alpha}(\tau,z)(\mu(\tau)D_z)^\alpha \mathcal{H}_{m-j}(v_k(\tau))\|_{\mathcal{A}_s} \\ &\leq B\,\mu(t) \sum_{j=0}^{m-1} \sum_{1\leq|\alpha|\leq m-j} \frac{M}{\eta^{|\alpha|}} \sup_{0\leq\tau\leq t} \|\mathcal{H}_{m-j}(v_k(\tau))\|_{\mathcal{A}_{s+\eta}}\end{aligned}$$

for some M and any $\eta \in (0, s_1 - s)$. Here, we used the assumptions that $\mu(t)$ is increasing but less than one on $[0,T]$.

Applying the induction hypothesis and (a) of Lemma 2, and setting $\eta = (s_1 - s)/(k+1)$,

$$\|v_{k+1}(t)\|_{\mathcal{A}_s} \leq BK \frac{C^k e^{mk} \mu(t)^{k+1}}{(s_1-s)^{mk}} \left(\frac{k+1}{k}\right)^{mk} \sum_{j=0}^{m-1} \sum_{1\leq|\alpha|\leq m-j} \frac{M(k+1)^{|\alpha|}}{(s_1-s)^{|\alpha|}}.$$

Using the fact that $(1+1/k)^k$ approaches e monotonically from below, and choosing $C \geq \sum_{j=0}^{m-1} \sum_{1\leq|\alpha|\leq m-j} BM(m+1)^m$, we get the desired result.

Case II. The following inequality holds for all $p \geq m$ and $s \in (0, s_1)$:

$$\|v_p(t)\|_{\mathcal{A}_s} \leq K \frac{C^p e^{mp}}{(s_1-s)^{mp}} \mu(t)^m [\varphi(t)]^{p-m}. \tag{10}$$

Note that from the previous case, the above estimate holds for $p = m$. Assume now that the above is true for $p = k \geq m$. Proceeding as before, we can more precisely estimate as follows:

$$\begin{aligned}\|v_{k+1}(t)\|_{\mathcal{A}_s} &\leq B \sum_{j=0}^{m-1} \sum_{1\leq|\alpha|\leq m-j} \frac{M\mu(t)^{|\alpha|}}{\eta^{|\alpha|}} \sup_{0\leq\tau\leq t} \|\mathcal{H}_{m-j}(v_k(\tau))\|_{\mathcal{A}_{s+\eta}} \\ &\leq BK \frac{C^k e^{mk}}{(s_1-s-\eta)^{mk}} \sum_{j=0}^{m-1} \sum_{1\leq|\alpha|\leq m-j} \frac{M\mu(t)^{|\alpha|}}{\eta^{|\alpha|}} \times \\ &\qquad \times \sup_{0\leq\tau\leq t} \left\{ \int_{[0,1]^{m-j}} \mu(\sigma_1\cdots\sigma_{m-j}\tau)^m \times \right. \\ &\qquad \left. \times \varphi(\sigma_1\cdots\sigma_{m-j}\tau)^{k-m}\, d\sigma_1\cdots d\sigma_{m-j} \right\}.\end{aligned}$$

Applying (b) of Lemma 2 exactly $|\alpha|$ times and (a) in the remaining $m-j-|\alpha|$ times to the integral on the right, and again setting $\eta = (s_1 - s)/(k+1)$, we get

$$\begin{aligned} \|v_{k+1}(t)\|_{\mathcal{A}_s} \leq\ & K \frac{C^k e^{mk} \mu(t)^m}{(s_1 - s)^{mk}} \left(\frac{k+1}{k}\right)^{mk} \sum_{j=0}^{m-1} \sum_{1 \leq |\alpha| \leq m-j} BM \times \\ & \times \left(\frac{k+1}{s_1 - s}\right)^{|\alpha|} \frac{\varphi(t)^{k-m+|\alpha|}}{(k-m+1)\cdots(k-m+|\alpha|)} . \end{aligned}$$

Having assumed that $\varphi(t) \leq 1$ on $(0, T)$ and noting that for all natural numbers $k \geq m$,

$$\frac{(k+1)^{|\alpha|}}{(k-m+1)\cdots(k-m+|\alpha|)} \leq P$$

for some constant P, we can arrive at the desired inequality by further specifying that

$$C \geq \sum_{j=0}^{m-1} \sum_{1 \leq |\alpha| \leq m-j} BMP.$$

Using the estimates (9) and (10), we can now show the convergence of the approximate solutions $u_p(t, z)$. Fix any $s \in (0, s_1)$. Since $\mu(t)$ and $\varphi(t)$ monotonically decrease to zero, we can choose $\varepsilon \in (0, T)$ such that

$$\frac{Ce^m \mu(t)}{(s_1 - s)^m} < 1 \qquad \text{and} \qquad \frac{Ce^m \varphi(t)}{(s_1 - s)^m} < 1 \tag{11}$$

for all $0 \leq t \leq \varepsilon$.

Thus, by the estimates in the two cases above, we see that $\sum_{p=0}^{\infty} v_p(t, z)$ is majorized by a convergent geometric series and is therefore convergent in $C^0([0, \varepsilon], \mathcal{A}_s)$. From here, it is easy to see that $\{u_p(t)\}$ converges to some $u(t) \in C^0_m([0, \varepsilon], \mathcal{A}_s)$ and that this limit is indeed a solution of the original partial differential equation.

The uniqueness of the solution can be proved in a similar manner.

5. Proof of Theorem 2

As in [1], we will make use of the fact that any distribution $u \in \mathcal{D}'(\mathsf{R}^n)$ with compact support can be regarded as a continuous linear functional $\tilde{u}$

over the space $\mathcal{H}(\mathsf{C}^n)$ of holomorphic functions over C^n by defining

$$\langle \tilde{u}, \theta \rangle = \left\langle u, \theta|_{\mathsf{R}^n} \right\rangle_{\mathcal{E}' \times C^\infty} \qquad \text{for any } \theta \in \mathcal{H}(\mathsf{C}^n).$$

Let $F_s(\Omega)$ be the closure in $\mathcal{A}(\Omega_s)$ of $\mathcal{H}(\mathsf{C}^n)$. Then $F_s(\Omega)$ is also a Banach space. Let $F'_s(\Omega)$ be the dual space of $F_s(\Omega)$. Note that $\{F'_s(\Omega)\}$ is an increasing scale of Banach spaces. By simply modifying the arguments used in proving Theorem 1, we can establish the following existence and uniqueness result in the dual spaces.

Proposition 2. *Let $\mathcal{P}$ be the Fuchsian partial differential operator described in* (1) *and let $s_0 \in (0, s_1)$. Then given any $s \in (s_0, s_1)$, there exists $\varepsilon \in (0, T)$ such that for any $f(t, z) \in C^0([0, T], F'_{s_0}(\Omega))$, the equation $\mathcal{P}u = f$ has a unique solution $u(t, z)$ in $C^0_m([0, \varepsilon], F'_s(\Omega))$.*

With this result, Theorem 2 can be proved using the arguments used in the proof of Theorem 4 in [1]. We may omit the details.

6. Necessity of Assumption (A–3)

We now present a situation in which Theorem 2 does not hold.

Let us consider the equation

$$(tD_t + \sigma + \mu(t)D_x)u = 0$$

with two independent variables $(t, x) \in \mathsf{R}^2$. Here $\sigma > 0$ and $\mu(t)$ is a function on $(0, T)$ satisfying (A–1) and (A–2) but not (A–3), i.e., $\int_0^T (\mu(s)/s)\, ds = \infty$. Note that the characteristic exponent of the above operator is $-\sigma$ (< 0). This equation has solutions of the form

$$u(t, x) = t^{-\sigma} F\left(e^{-x - \int_t^c \frac{\mu(s)}{s}\, ds} \right)$$

where $0 < c < T$ and $F(y)$ is any differentiable function. In particular, if $F(y)$ is nonnegative and increasing in y, there exists an $A > 0$ such that for all $x \in (-A, \infty)$, we have

$$|u(t, x)| \leq t^{-\sigma} F\left(e^{A - \int_t^c \frac{\mu(s)}{s}\, ds} \right).$$

One such suitable choice is $F(y) = G(y)^{\sigma+1}$, where $G(y)$ is the inverse function of $\exp(A - \int_t^c (\mu(s)/s)\, ds)$. With this choice of $F(y)$, it easily follows that $u(t,x) \in C_1^0([0,T], \mathcal{D}'(-A,\infty))$, thereby negating the claim of Theorem 2.

In fact, more could be said about the necessity of assumption (A–3). Let us modify the assumptions in the original formulation to be able to apply the result of Mandai in [4].

Consider the operator

$$\mathcal{P}_* = (tD_t)^m + \sum_{j=0}^{m-1} \sum_{|\alpha| \le m-j} a_{j,\alpha}(t,x)(\mu(t)D_x)^\alpha (tD_t)^j, \tag{12}$$

with coefficients $a_{j,\alpha}(t,x)$ in $C^0([0,T], \mathcal{A}(\mathcal{V})) \cap C^\infty((0,T), \mathcal{A}(\mathcal{V}))$. As for the real-valued function $\mu(t)$, we will require it to satisfy

(A$_*$–1) $\mu(t) \in C^0([0,T]) \cap C^\infty((0,T))$,

(A$_*$–2) $\mu(t) > 0$ and is increasing on $(0,T)$.

We shall further assume that the roots $\tau_1(t,x,\xi)$, …, $\tau_m(t,x,\xi)$ of the *reduced* principal symbol

$$\tau^m + \sum_{j=0}^{m-1} \left(\sum_{|\alpha|=m-j} a_{j,\alpha}(t,x)\xi^\alpha \right) \tau^j$$

satisfy the following conditions:

(B–1) $\tau_1(t,x,\xi)$, …, $\tau_m(t,x,\xi)$ are real and distinct for all $t > 0$, $x \in \mathcal{V} \cap \mathsf{R}^n$ and $\xi \in \mathsf{R}^n - \{0\}$,

(B–2) There exists a j and a $\xi^0 \ne 0$ such that $\tau_j(0,0,\xi^0) \ne 0$.

Note that if we consider the principal symbol of the operator $\mathcal{P}_*$, its roots actually are

$$\tilde{\tau}_i(t,x,\xi) = \tau_i\left(t, x, \tfrac{\mu(t)}{t}\xi\right) = \frac{\mu(t)}{t}\, \tau_i(t,x,\xi),$$

since $\tau_i(t,x,\xi)$ is homogeneous of degree 1 in ξ.

Proposition 3. *Suppose that* (NE) *and all the above assumptions on the operator* $\mathcal{P}_*$ *hold. Then, local uniqueness in* $C_m^0([0,T],\mathcal{D}'(\mathsf{R}^n))$ *is valid for the equation* $\mathcal{P}_*u = 0$ *if and only if* $\displaystyle\int_0^T \frac{\mu(s)}{s}\,ds < \infty$.

The sufficiency part follows from Theorem 2. On the other hand, suppose $\int_0^T (\mu(s)/s)\,ds = \infty$. Then we can use the arguments in Proposition 5 of [4] to show that $\mathcal{P}_*$ has the escape property. Theorem B of the same paper then states that a null-solution exists, and thus establishes the necessity part.

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