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The Generalized Whittaker Functions for $SU(2,1)$ *and the Fourier Expansion of Automorphic Forms*

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Abstract. Explicit form of Fourier expansion of automorphic forms plays an important role in the theory. Here we investigate the case of $SU(2,1)$ and give an explicit formula of generalized Whittaker functions for the standard representations of the group. Together with a result of [K-O], we obtain a form of fully developed Fourier expansion of automorphic forms belonging to arbitrary standard representations.

Introduction

In the theory of automorphic forms, Fourier expansion of modular forms is a fundamental tool for investigation. For example, coefficients of the expansion can be used for construction of L-functions. In spite of this importance, the theory of Fourier expansion of automorphic forms seems still in very primitive state.

Our concern in this paper is to have a theory of fully developed Fourier expansion of modular forms on $SU(2,1)$, the real special unitary group of signature $(2+,1-)$. To have such a theory we need Whittaker functions and generalized Whittaker functions of the standard representations of $SU(2,1)$. A quite explicit result is obtained by Koseki-Oda [K-O] for Whittaker functions. The remaining problem for our purpose is to consider the generalized Whittaker functions. This is the theme of the present paper.

The peculiarity of the case of $SU(2,1)$, different from the case of $SL_2(\mathbb{R})$, is that the maximal unipotent subgroup N is not abelian. It is isomorphic to the Heisenberg group of dimension three, and has infinite-dimensional irreducible unitary representations σ, which are called Stone von Neumann representations. Together with unitary characters they constitute the unitary dual of N. The Fourier expansion of automorphic forms on $SU(2,1)$ is

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to consider irreducible decomposition of the restriction $\pi|_N$ of automorphic representations π with respect to N. Therefore we have to handle those terms which corresponds to the Stone von Neumann representations.

Naive formulation of the problem is to investigate intertwiners in $\mathrm{Hom}_N(\pi|_N, \sigma)$ which is isomorphic to $\mathrm{Hom}_G(\pi, \mathrm{Ind}_N^G \sigma)$ by Frobenius reciprocity. But this fails in general, since the intertwining space in question is infinite-dimensional. The right formulation of the problem is given by introducing a larger group R containing N.

Here is the formulation of our main result. Let P be the minimal parabolic subgroup of $SU(2,1)$ with a Levi decomposition $L \ltimes N$. Let S be the maximal closed subgroup of L which acts trivially on the center $Z(N)$ of N. The group R is the semidirect product S and N. We want to investigate the intertwining space $\mathrm{Hom}_G(\pi, \mathrm{Ind}_R^G \eta)$ for certain unitary irreducible representations η of R, and the images of intertwiners: these are the space of generalized Whittaker functionals and the space of generalized Whittaker functions, respectively.

Our main result is to obtain an explicit formula for the radial part of such generalized Whittaker functions with special K-type. Simultaneously we have the archimedean local multiplicity one theorem for the intertwining space, which generalizes that of Shalika [Sha] to the setting above. As a bonus, we obtain a sufficient condition for one-dimensionality of the intertwining space in terms of the parameters of representations (Theorem 3.3.5, Theorem 4.2.4). Consequently we have an explicit form of the Fourier expansion of automorphic forms, which separates the finite part (*i.e.* the coefficients) and the archimedean part (*i.e.* the generalized Whittaker functions) in each term of the expansion (Theorem 5.3.1).

We should remark that Piatetski-Shapiro announced the multiplicity one theorem of "Heisenberg model" for the irreducible representations of $U(3)$ over local fields more than two decades ago ([PS] p.589). Gelbart and Rogawski used this result as a crucial step to investigate automorphic L-functions obtained from Fourier-Jacobi expansion ([Ge-Ro] p.452). Over the real field, this result is a part of our Thorems (3.3.5) and (4.2.4). However, up to the present, any of these authors did not publish their proof.

The difficult part of our investigation is the case where π is of the large discrete series representation of $SU(2,1)$. For such representations our method of proof uses fundamental results of Yamashita [Ya2].

Though the author needs the result of this paper for investigation of automorphic forms, he also believes that it is interesting for the problem of realization in generalized Gelfand-Graev representations.

Let us explain the contents of this paper more in detail. Firstly in §1, after recalling some basic results about generalized Gelfand-Graev representations, we define generalized Whittaker functions with given K-type for irreducible admissible representations π of a real semisimple Lie group G. In §§2.1 we fix some notation for the structure of the group $G = SU(2,1)$ and its Lie algebra. In §§2.2 we shortly summarize necessary facts on parameterization of irreducible K-module τ_λ in $\widehat{K}$ and Clebsch-Gordan decomposition of $\tau_\lambda \otimes \mathrm{Ad}_{\mathfrak{p}_\mathbb{C}}$. In §§2.3 we construct irreducible unitary representation $\eta_{\mu,\psi}$ of R concretely on $L^2(\mathbb{R})$ by using the theory of Weil representations. We calculate explicitly the η-action of $\mathfrak{n} = \mathrm{Lie}N$ on a basis of $L^2(\mathbb{R})$ consisting of Hermite functions for the computation of the A-radial part of shift operators in later sections. In §§2.4 we briefly recall Harish-Chandra 's parameterization of the discrete series representations and the principal series representations of $SU(2,1)$. The core of this paper is the section §3, which treats the case of a discrete series representation. Here we use a fundamental result of Yamashita which characterizes the A-radial part of generalized Whittaker functions of discrete series representations by means of Schmid operators [Ya2]. We recall in §§3.1 the definition of the Schmid operators and a result of Yamashita as Proposition 3.1.1. The first half of §§3.2 is devoted to explicit calculation of the A-radial part of Schmid operators in terms of the coefficient functions c_k's of a generalized Whittaker function for discrete series representation with minimal K-type. We obtain a system of difference-differential equations satisfied by c_k's (Proposition 3.2.4, Proposition 3.2.5, Proposition 3.2.6). Finally in §§3.3 we give an explicit formula for c_k's (Theorem 3.3.2, Theorem 3.3.4) by solving the differential equations in Proposition 3.3.1, Proposition 3.3.3. As an immediate corollary we have the multiplicity one theorem for the generalized Whittaker model for the discrete series representations (Theorem 3.3.5). The case of a principal series representation is treated in §4. We also obtain an explicit formula of the generalized Whittaker functions and the multiplicity one theorem in this case (Theorem 4.2.4). The rest of this paper §5 is an application of the explicit formula obtained in previous sections to the theory of Fourier expansion of automorphic forms on $SU(2,1)$. Among others we can define normalized Fourier coefficients of automorphic forms in Theorem 5.3.1.

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1. Generalized Whittaker Models

1.1. The space of the generalized Whittaker functionals

Firstly in this subsection, we define the space of the generalized Whittaker functionals for an irreducible admissible representation $(\pi, \mathcal{H}_\pi)$ of a real semisimple group.

Let G be a connected real semisimple Lie group with finite center and K its maximal compact subgroup. We denote by θ the Cartan involution associated to K. Take a minimal parabolic subgroup P of G with a Levi decomposition: $P = L \ltimes N$, where N is the unipotent radical of P and L is the θ-invariant reductive part of P (*i.e.* the Levi subgroup). The action of L on N by conjugation induces its action on the unitary dual $\widehat{N}$ of N. Hence putting S_ξ the stabilizer of the class of a unitary representation $(\xi, \mathcal{S})$ of N in L, we can extend $(\xi, \mathcal{S})$ to a unique projective representation $(\overline{\xi}, \mathcal{S})$ of $S_\xi \ltimes N$.

Under some condition, we can get $(\overline{\xi}, \mathcal{S})$ as a representation of $S_\xi \ltimes N$, not a projective representation. Let $\mathfrak{n}$ be the Lie algebra of N. Assume ξ corresponds to the coadjoint orbit of $X^* \in \mathfrak{n}^*$ in the Kirillov theory (*cf* [Co-Gr]). Let X be the element of $\mathfrak{g} = \mathrm{Lie}G$ determined by $\langle X^*, Z\rangle = B(\theta X, Z)$. Here $Z \in \mathfrak{n}$ and B is the Killing form of $\mathfrak{g}$. Denote by H the semisimple element of the $\mathfrak{sl}_2$-triple containing X.

Proposition 1.1.1 ([Ya] Prop 2.2). *When the subspace $\mathfrak{g}(1) := \{X \in \mathfrak{g} \mid [H, X] = X\}$ of Lie algebra $\mathfrak{g}$ admits an $\mathrm{Ad}(S_\xi)$-invariant complex structure, $(\overline{\xi}, \mathcal{S})$ is extendable to a unitary representation $\widetilde{\xi}$ of $S_\xi \ltimes N$ acting on the same representation space with $\overline{\xi}$.*

We remark that the assertion in the proposition above is valid in the case of $G = SU(2, 1)$.

We assume the condition of Proposition 1.1.1 is satisfied throughout this section. Denote the group $S_\xi \ltimes N$ by R_ξ. Let η be a unitary representation of R_ξ defined by $c' \otimes \widetilde{\xi}$. Here, $c' := c \otimes 1_N$ is the extension of an irreducible representation c of S_ξ trivially on N.

Consider a space

$$C_\eta^\infty(R_\xi \backslash G) := \{f : G \to \mathcal{S}^\infty \mid f \text{ is a } C^\infty\text{-function satisfying,} \\ f(rg) = \eta(r).f(g),\ \forall r \in R_\xi\ , \forall g \in G\}$$

on which G acts via right translation. We call this C^∞-induced representation $\mathrm{Ind}_{R_\xi}^G \eta$ of G *the reduced generalized Gelfand-Graev representation*. Here, we used the standard notation that $\mathcal{S}^\infty$ means the subspace consisting of all smooth vectors in $\mathcal{S}$.

We can now define the space of the generalized Whittaker functionals as the space of intertwining operators.

Definition. For an irreducible admissible representation $(\pi, \mathcal{H}_\pi)$ of G, we denote the underlining $(\mathfrak{g}_\mathbb{C}, K)$-module of π by the same symbol π. We call the space of intertwiners

$$I_{\pi,\eta} \ := \ \mathrm{Hom}_{(\mathfrak{g}_\mathbb{C},K)}(\pi^*, \mathrm{Ind}_{R_\xi}^G \eta)$$

of $(\mathfrak{g}_\mathbb{C}, K)$-modules *the space of the algebraic generalized Whittaker functionals*. Here $\mathfrak{g}_\mathbb{C}$ is the complexification of the Lie algebra $\mathfrak{g}$ of G and π^* denotes the contragredient $(\mathfrak{g}_\mathbb{C}, K)$-module of π.

1.2. Generalized Whittaker functions with fixed K-type

In order to investigate algebraic generalized Whittaker functionals $l \in I_{\pi,\eta}$, we study the functions $l(v^*) \in \mathrm{Ind}_{R_\xi}^G \eta$: the image of vectors v^* belonging to $(\pi^*, \mathcal{H}_{\pi^*})$ by l. To describe these functions explicitly, we specify a K-type of π and consider vectors v^* belonging to this K-type.

Let (τ, V_τ) be a K-type of π, that is, τ occurs in the decomposition of π as K-module: $\pi|_K = \oplus_{\tau \in \widehat{K}} [\pi : \tau]\tau$. Choose a K-equivariant injection $\iota_\tau : \ \tau \hookrightarrow \pi$, and pullback a generalized Whittaker functional l by this injection ι_τ:

$$\mathrm{Hom}_{(\mathfrak{g}_\mathbb{C},K)}(\pi^*, \mathrm{Ind}_{R_\xi}^G \eta) \ni l \mapsto \iota_\tau^*(l) \in \mathrm{Hom}_K(\tau^*, \mathrm{Ind}_{R_\xi}^G \eta|_K).$$

Here we note the isomorphism

$$\mathrm{Hom}_K(\tau^*, \mathrm{Ind}_{R_\xi}^G \eta|_K) \cong (\mathrm{Ind}_{R_\xi}^G \eta|_K \otimes \tau)^K.$$

The latter space is defined by

$$\begin{aligned}\Big(C_\eta^\infty(R_\xi\backslash G) \otimes_{\mathbb{C}} V_\tau\Big)^K \ &\cong \ C_{\eta,\tau}^\infty(R_\xi\backslash G/K) \\ := &\left\{ \varphi : G \to \mathcal{S}^\infty \otimes_{\mathbb{C}} V_\lambda \ \middle| \ \begin{array}{l} \varphi \text{ is a } C^\infty\text{-function satisfying} \\ \varphi(rgk) = \eta(r)\tau_\lambda(k)^{-1}.\varphi(g), \\ \forall r \in R\ , \forall g \in G\ , \forall k \in K \end{array} \right\}.\end{aligned}$$

We study functions $F \in C_{\eta,\tau}^\infty(R_\xi\backslash G/K)$ representing $\iota_\tau^*(l)$. By definition,

$$l(v^*)(g) \ = \ \langle v^*, F(g)\rangle_K,$$

$v^* \in V_\tau^*$. Here $\langle\ ,\ \rangle_K$ means the canonical pairing of K-modules V_τ^* and V_τ.

DEFINITION. We call the above function F corresponding to $\iota_\tau^*(l)$, $l \in I_{\pi,\eta}$ *the algebraic generalized Whittaker function associated to representation π with K-type τ*. Moreover if we impose the slowly increasing condition for the A-radial part $F|_A$ of F, such a function is called *the generalized Whittaker function*(see subsection 3.3). Here A is the vector subgroup of G of which Lie algebra is the maximal abelian subalgebra of $\mathfrak{g}$.

We investigate these functions in the following setting

$G = SU(2,1)$
- ξ : an infinite-dimensional irreducible unitary representation of N
- π : a discrete series representation of G (resp. a principal series representation of G)
- τ : the minimal K-type of π (resp. the corner K-type of π)

and give an explicit formula for F and the multiplicity one property of $I_{\pi,\eta}$ simultaneously by constructing F.

2. The Structure of Lie Groups and Parameterization of Representations

In this section we give a glossary on the group structure of $SU(2,1)$ and representations for later use. We first fix realizations and give explicit coordinates of various subgroups and their Lie algebras. It is crucial for our explicit calculation of generalized Whittaker functions. We also recall parameterization of representations of these groups.

2.1. Subgroups, subalgebras and root space decomposition

Subgroups and their realizations. We denote by $\mathrm{diag}(X_1, X_2, X_3)$ a diagonal matrix of degree 3 with (i,i)-entry X_i for each i $(1 \leqslant i \leqslant 3)$. Put $I_{2,1} := \mathrm{diag}(1,1,-1)$. Then we realize the identity component of the stabilizer group $SU(2,1)$ of the Hermitian form of three variables with signature $(2+,1-)$ as

$$\{g \in SL(3,\mathbb{C}) | {}^t\bar{g} I_{2,1} g = I_{2,1}\}.$$

Here ${}^tg$ is the transpose of g, and $\bar{g}$ the complex conjugate of g. We denote the group by G. Let

$$G = NAK$$

be the Iwasawa decomposition of G. Then in this realization, a maximal compact subgroup K of G can be written as

$$K = \{\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \in G \mid k_1 \in U(2), k_2 \in U(1),\ k_2 \det k_1 = 1\}.$$

Here $U(n)$ is the unitary group of degree n. The Euclidean subgroup A is

$$A = \{a_r := \begin{pmatrix} \frac{r+r^{-1}}{2} & & \frac{r-r^{-1}}{2} \\ & 1 & \\ \frac{r-r^{-1}}{2} & & \frac{r+r^{-1}}{2} \end{pmatrix} \mid r \in \mathbb{R}_{>0}\}.$$

The maximal unipotent subgroup N is isomorphic to the Heisenberg group $H(\mathbb{R}^2)$ of dimension 3. Here $H(\mathbb{R}^2)$ is the set $\{(x,y;t) \in \mathbb{R}^3\}$ with a group law $(x,y;t)(x',y';t') = (x+x', y+y'; t+t'+xy'-yx')$. Note that the group law above is different from usual one. See (2.1.1).

Lie algebras and root space decompositions. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G corresponding to a Cartan involution $\theta : X \mapsto I_{2,1} X I_{2,1}^{-1},\ X \in \mathfrak{g}$. Then

$$\mathfrak{k} = \{\begin{pmatrix} X_1 & 0 \\ 0 & X_3 \end{pmatrix} \Bigg| \begin{array}{l} X_1 \in \mathfrak{u}(2),\ X_3 \in \mathfrak{u}(1), \\ \mathrm{tr} X_1 + X_3 = 0 \end{array} \},$$

$$\mathfrak{p} = \{\begin{pmatrix} 0 & X_2 \\ {}^t\bar{X}_2 & 0 \end{pmatrix} \mid X_2 \in M_{2,1}(\mathbb{C})\}.$$

Since G/K is Hermitian, we have a decomposition $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ such that $\mathfrak{p}_+$ is identified with the holomorphic tangent space at the origin $1 \cdot K \in G/K$, corresponding to the complex structure of G/K. In our realization we have

$$\mathfrak{p}_+ = \{\begin{pmatrix} 0 & X_2 \\ 0 & 0 \end{pmatrix} \mid X_2 \in M_{2,1}(\mathbb{C})\}, \qquad \mathfrak{p}_- = \{\begin{pmatrix} 0 & 0 \\ {}^tY_2 & 0 \end{pmatrix} \mid Y_2 \in M_{2,1}(\mathbb{C})\}.$$

We fix a compact Cartan subalgebra $\mathfrak{t}$ in $\mathfrak{k}$ by

$$\mathfrak{t} = \{\mathrm{diag}(\sqrt{-1}h_1, \sqrt{-1}h_2, \sqrt{-1}h_3) \mid h_i \in \mathbb{R}, \sum_{i=1}^{3} h_i = 0\}$$

and take a basis $\{H'_{12}, H'_{13}\}$ of a compact Cartan subalgebra $\mathfrak{t}$ as

$$H'_{12} = \mathrm{diag}(1, -1, 0), \qquad H'_{13} = \mathrm{diag}(1, 0, -1).$$

Define linear forms β_{ij} on $\mathfrak{t}_{\mathbb{C}} (i \neq j, 1 \leq i, j \leq 3)$ by

$$\beta_{ij} : \mathfrak{t}_{\mathbb{C}} \ni \mathrm{diag}(t_1, t_2, t_3) \mapsto t_i - t_j \in \mathbb{C}.$$

Then the root system Σ associated to $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ is given by $\{\beta_{ij} \mid i \neq j,\ 1 \leq i, j \leq 3\}$. We fix a positive system Σ_+ as $\{\beta_{ij} \mid i < j\}$.

Let $\mathfrak{g}_\beta$ be the root space associated to $\beta \in \Sigma$: $\mathfrak{g}_\beta := \{X \in \mathfrak{g} \mid [H, X] = \beta(H)X,\ \forall H \in \mathfrak{t}_{\mathbb{C}}\}$. We denote Σ_c and Σ_n the sets of compact and noncompact roots, respectively. In our choice of coordinates,

$$\Sigma_c = \{\beta_{12}, \beta_{21}\}, \quad \Sigma_n = \{\beta_{13}, \beta_{23}, \beta_{31}, \beta_{32}\},$$

and matrix element E_{ij} $(1 \leq i, j \leq 3)$ generates the root space $\mathfrak{g}_{\beta_{ij}}$. We put

$$X_{\beta_{ij}} = \begin{cases} E_{ij} & \text{when}(i,j) \neq (2,1); \\ -E_{ij} & \text{when}(i,j) = (2,1), \end{cases}$$

and take it as a root vector in $\mathfrak{g}_{\beta_{ij}}$. This choice is natural, since the complex conjugation with respect to our real form of $\mathfrak{sl}(3,\mathbb{C})$ converts two root vectors $X_{\beta_{ij}}$ and $X_{\beta_{ji}}$ mutually. Put $\Sigma_{c,+} := \Sigma_c \cap \Sigma_+$ and $\Sigma_{n,+} := \Sigma_n \cap \Sigma_+$.

A maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ is given by

$$\mathfrak{a} = \mathbb{R}H \qquad \text{with} \quad H := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The restricted root system Ψ associated to $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}})$ is given by $\Psi := \{\pm e, \pm 2e\}$, where e is an $\mathbb{R}$-linear form on $\mathfrak{a}$ defined by $e : aH \mapsto a$. We fix a positive root system Ψ_+ as $\{e, 2e\}$. Then $\sum_{\alpha \in \Psi_+} \mathfrak{g}_\alpha$ is a maximal nilpotent subalgebra $\mathfrak{n} = \text{Lie } N$ of $\mathfrak{g}$. We get the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$.

We choose root vectors for $\mathfrak{g}_\alpha$, $\alpha \in \Psi_+$ as follows,

$$\mathfrak{g}_{2e} = \mathbb{R}E_1, \qquad \mathfrak{g}_e = \mathbb{R}E_{2,+} \oplus \mathbb{R}E_{2,-},$$

$$E_1 := i \begin{pmatrix} 1 & & -1 \\ & 0 & \\ 1 & & -1 \end{pmatrix}, \qquad E_{2,+} := \begin{pmatrix} & -1 & \\ 1 & & -1 \\ & -1 & \end{pmatrix},$$

$$E_{2,-} := \begin{pmatrix} & -i & \\ -i & & i \\ & -i & \end{pmatrix},$$

where the symbol i means the imaginary unit $\sqrt{-1}$. Then according to the Iwasawa decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{a}_{\mathbb{C}} \oplus \mathfrak{n}_{\mathbb{C}}$, the root vectors X_β associated to noncompact roots $\beta \in \Sigma_n$ decompose as

$$X_{\beta_{13}} = \frac{1}{2}H'_{13} + \frac{1}{2}H + \frac{i}{2}E_1; \quad X_{\beta_{31}} = \frac{-1}{2}H'_{13} + \frac{1}{2}H - \frac{i}{2}E_1;$$

$$X_{\beta_{23}} = -X_{\beta_{21}} - \frac{1}{2}E_{2,+} - \frac{i}{2}E_{2,-}; \quad X_{\beta_{32}} = -X_{\beta_{12}} - \frac{1}{2}E_{2,+} + \frac{i}{2}E_{2,-}.$$

This will be used to calculate the action of Schmid operators, defined in subsection 3.1, on the A-radial part of generalized Whittaker functions.

The exponential coordinate of N. We prepare *the exponential coordinate* of $N = \exp\mathfrak{n}$ which will be used for description of generalized theta functions in subsection 5.2. As is well-known, for a connected simply connected nilpotent Lie group the exponential map gives a diffeomorphism between the group and its Lie algebra (*cf.* [Co-Gr] p.13). Therefore we can take the basis $\{E_{2,\pm}, E_1\}$ of $\mathfrak{n}$ as a coordinate of N. The group law of N is translated into

$$Y * X \ := \ \log\Big((\exp Y)(\exp X)\Big).$$

The Campbell-Baker-Hausdorff formula says that

$$\begin{aligned} Y * X &= Y + X + \tfrac{1}{2}[Y,X] + \tfrac{1}{12}[Y,[Y,X]] - \tfrac{1}{12}[X,[Y,X]] + \cdots \\ &= Y + X + \tfrac{1}{2}[Y,X]. \end{aligned}$$

The second equality follows from the 2-step nilpotency of $\mathfrak{n}$. If $Y = mE_{2,+} + nE_{2,-} + kE_1$, $X = xE_{2,+} + yE_{2,-} + zE_1$, then

$$\begin{aligned} (2.1.1) \quad Y * X &= Y + X + \tfrac{1}{2}2(my - nx)E_1 \\ &= (m+x)E_{2,+} + (n+y)E_{2,-} + (k + z + (my - nx))E_1, \end{aligned}$$

because of the relation $[E_{2,+}, E_{2,-}] = 2E_1$. We abbreviate the element $\exp(xE_{2,+} + yE_{2,-} + zE_1)$ of N as $(x, y; z)$. This is utilized for solving the differential equations for generalized theta functions (Theorem 5.2.1).

2.2. Representations of maximal compact subgroup K

Here we summarize necessary facts on representations of K from [K-O].

Parameterization of irreducible K-modules. The set L_T^+ of $\Sigma_{c,+}$-dominant T-integral weights is given by $L_T^+ = \{(m,n) \in \mathbb{Z}^{\oplus 2} | m \geq n\}$. For each $\mu = (\mu_1, \mu_2) \in L_T^+$, the vector space V_μ spanned by $\{v_k^\mu \mid 0 \leq k \leq d_\mu\}$ with $\mathfrak{k}_{\mathbb{C}}$-action as

$$\begin{gathered} \tau_\mu(Z)v_k^\mu = (\mu_1 + \mu_2)v_k^\mu, \\ \tau_\mu(H'_{12})v_k^\mu = (2k - d_\mu)v_k^\mu, \quad \tau_\mu({H'}_{13})v_k^\mu = (k + \mu_2)v_k^\mu, \\ \tau_\mu(X_{\beta_{12}})v_k^\mu = (k+1)v_{k+1}^\mu, \quad \tau_\mu(X_{\beta_{21}})v_k^\mu = (k - d_\mu - 1)v_{k-1}^\mu \end{gathered}$$

gives an irreducible K-module (τ_μ, V_μ) via the highest weight theory. Here Z denotes a generator $2H'_{13} - H'_{12}$ of the center.

Tensor products with $\mathfrak{p}_\mathbb{C}$. We regard the 4-dimensional vector space $\mathfrak{p}_\mathbb{C}$ as a $\mathfrak{k}_\mathbb{C}$-module via the adjoint representation ad. Then $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are invariant subspaces, and

$$\mathfrak{p}_+ = \mathbb{C}X_{\beta_{13}} \oplus \mathbb{C}X_{\beta_{23}} \cong V_{\beta_{13}}, \qquad \mathfrak{p}_- = \mathbb{C}X_{\beta_{32}} \oplus \mathbb{C}X_{\beta_{31}} \cong V_{\beta_{32}}.$$

Given an irreducible K-module V_μ we have $V_\mu \otimes \mathfrak{p}_\mathbb{C} = (V_\mu \otimes \mathfrak{p}_+) \oplus (V_\mu \otimes \mathfrak{p}_-)$, and Clebsch-Gordan's theorem tells us the following decomposition of $V_\mu \otimes \mathfrak{p}_\pm$:

$$V_\mu \otimes \mathfrak{p}_+ \cong V_{\mu+\beta_{13}} \oplus V_{\mu+\beta_{23}}, \quad V_\mu \otimes \mathfrak{p}_- \cong V_{\mu+\beta_{32}} \oplus V_{\mu+\beta_{31}}$$

Here we understand $V_\nu = (0)$ if $\nu \in L_T$ is not dominant. We hence have

$$V_\mu \otimes \mathfrak{p}_\mathbb{C} \cong V_\mu^+ \oplus V_\mu^-;$$

$$V_\mu^+ := V_{\mu+\beta_{13}} \oplus V_{\mu+\beta_{32}}, \qquad V_\mu^- := V_{\mu-\beta_{13}} \oplus V_{\mu-\beta_{32}},$$

under the above convention.

The decompositions of $V_\mu \otimes \mathfrak{p}_\mathbb{C}$ induce the following projectors:

$$p^+_{\beta_{13}}(\mu) : V_\mu \otimes \mathfrak{p}_\mathbb{C} \to V_{\mu+\beta_{13}}, \quad p^+_{\beta_{23}}(\mu) : V_\mu \otimes \mathfrak{p}_\mathbb{C} \to V_{\mu-\beta_{32}},$$

$$p^-_{\beta_{23}}(\mu) : V_\mu \otimes \mathfrak{p}_\mathbb{C} \to V_{\mu+\beta_{32}}, \quad p^-_{\beta_{13}}(\mu) : V_\mu \otimes \mathfrak{p}_\mathbb{C} \to V_{\mu-\beta_{13}}.$$

In terms of $\{v_k^\mu\}$, they are expressed as follows:

Proposition 2.2.1 ([K-O] Prop 2.3).

$$\begin{aligned}
p^+_{\beta_{13}}(\mu)(v_k^\mu \otimes X_{\beta_{13}}) &= (k+1)v_{k+1}^{\mu+\beta 13}, & p^+_{\beta_{23}}(\mu)(v_k^\mu \otimes X_{\beta_{13}}) &= -v_k^{\mu-\beta 32},\\
p^+_{\beta_{13}}(\mu)(v_k^\mu \otimes X_{\beta_{23}}) &= (d_\mu - k + 1)v_k^{\mu+\beta 13}, & p^+_{\beta_{23}}(\mu)(v_k^\mu \otimes X_{\beta_{23}}) &= v_{k-1}^{\mu-\beta 32},\\
p^-_{\beta_{23}}(\mu)(v_k^\mu \otimes X_{\beta_{32}}) &= -(k+1)v_{k+1}^{\mu+\beta 32}, & p^-_{\beta_{13}}(\mu)(v_k^\mu \otimes X_{\beta_{32}}) &= v_k^{\mu-\beta 13},\\
p^-_{\beta_{23}}(\mu)(v_k^\mu \otimes X_{\beta_{31}}) &= (d_\mu - k + 1)v_k^{\mu+\beta 32}, & p^-_{\beta_{13}}(\mu)(v_k^\mu \otimes X_{\beta_{31}}) &= v_{k-1}^{\mu-\beta 13},
\end{aligned}$$

for $k = 1, \ldots, d_\lambda$. Here one should note that $d_{\mu\pm\beta_{13}} = d_{\mu\pm\beta_{32}} = d_\mu \pm 1$.

2.3. Representation theory of the group R

We construct the unitary representations of R with nontrivial central characters, which are necessary for our purpose.

The stabilizer S_ξ of representation with nontrivial central character. Here we give an explicit form of the stabilizing group S_ξ of the class of ξ when ξ is an infinite-dimensional irreducible unitary representation of N in order to construct concretely the unitary representation η explained in subsection 1.1.

Note that the maximal unipotent subgroup N of G is the Heisenberg group $H(\mathbb{R}^2)$ of dimension 3. Recall that the unitary dual $\widehat{N}$ of N consists of unitary characters and infinite-dimensional irreducible unitary representations by the Stone-von Neumann theorem:

Proposition 2.3.1. *Every irreducible unitary representation σ of $H(\mathbb{R}^2)$ is either of two cases*:
a) *If the central character ψ of σ is trivial, σ is a one-dimensional representation i.e. a unitary character.*
b) *If the central character ψ of σ is non-trivial, σ is a unique infinite-dimensional irreducible unitary representation, up to unitary equivalence.*

We call σ in the case(b) *a Stone von Neumann representation*. Let σ be a Stone von Neumann representation of N. The equivalence class of σ in $\widehat{N}$ is completely determined by its central character ψ. Let L be the Levi subgroup of P, and $Z(N)$ the center of N. Then L acts both on $\widehat{N}$ and $Z(N)$ by conjugation. Hence the stabilizer S of σ in L is the centralizer of $Z(N)$. In particular S is independent of σ and of the following form

$$S \ = \ \{\mathrm{diag}(\alpha,\beta,\bar{\alpha}^{-1}) \in G' \mid \alpha,\beta \in U(1), \alpha\beta\bar{\alpha}^{-1} = 1\}.$$

Since the action of S on N by conjugation is faithful, S can be regarded as a subgroup of the automorphism group of N. Passing to the abelianized subgroup $N^{ab} = N/[N:N]$ of N, we have

$$S \ \hookrightarrow \ \mathrm{Aut}\ N \to \mathrm{Aut}\ N^{ab}.$$

Since N^{ab} is identified with $\mathbb{R}^{\oplus 2}$, Aut $N^{ab} \cong GL_2(\mathbb{R})$. Composing all these identifications, we get an isomorphism between S and $SO(2)$

$$\begin{array}{ccccccc} S & \hookrightarrow & \mathrm{Aut}N & \to & GL_2(\mathbb{R}) & \supset & SO(2), \\ \mathrm{diag}(\alpha,\beta,\bar{\alpha}^{-1}) & & & \mapsto & & & R(3\theta) \end{array}$$

by putting $\alpha = e^{i\theta}, \beta = e^{-2i\theta}, \theta \in [0, 2\pi)$, where $R(3\theta)$ means the rotation of angle -3θ.

Representations of R with nontrivial central characters. Here we fix a model of a Stone von Neumann representation σ. By $L^2(\mathbb{R})$ we denote the space of square integrable functions on $\mathbb{R}$. For each element $(x, y; t) \in H(\mathbb{R}^2)$ and a function $\Phi \in L^2(\mathbb{R})$, define

$$\rho_\psi\Big((x, y; t)\Big).\Phi(\xi) \;=\; \psi(t + 2\xi y + xy)\Phi(\xi + x). \tag{2.3.1}$$

Then $\rho_\psi : H(\mathbb{R}^2) \to \mathrm{Aut}(L^2(\mathbb{R}))$ is an infinite-dimensional irreducible unitary representation with central character ψ, called *the Shrödinger model* of a Stone von Neumann representation. Our model differs from usual one. Since the commutation relation $[E_{2,+}, E_{2,-}] = 2E_1$ between the basis of $\mathfrak{n}$ is different from the Heisenberg commutation relation by 2.

We extend the representation ρ_ψ of N to the representation of R by using the theory of Weil representation. As is well-known, the two-fold covering $\widetilde{SL}_2(\mathbb{R})$ of $SL_2(\mathbb{R})$ has a unitary representation $(\omega_\psi, L^2(\mathbb{R}))$. This is obtained as the intertwiner of representations ρ_ψ and ρ_ψ^g of the Heisenberg group $H(\mathbb{R}^2)$ whose central characters are the same, by virtue of Proposition 2.3.1(b). Here the representation ρ_ψ^g is given by

$$\rho_\psi^g : H(\mathbb{R}^2) \ni (x, y; t) \mapsto \rho_\psi((x, y)g; t) \in \mathrm{Aut}(L^2(\mathbb{R})).$$

From the construction above, we have the canonical extension

$$\omega_\psi \times \rho_\psi :\; \widetilde{SL}_2(\mathbb{R}) \ltimes H(\mathbb{R}^2) \to \mathrm{Aut}(L^2(\mathbb{R})).$$

Identifying S, N with $SO(2)$, $H(\mathbb{R}^2)$ respectively, the semidirect product $R = S \ltimes N$ is regarded as a subgroup of $\widetilde{SL}_2(\mathbb{R}) \ltimes H(\mathbb{R}^2)$. Let $\widetilde{R}$ denote the pullback $\widetilde{S} \ltimes N \cong \widetilde{SO}(2) \ltimes H(\mathbb{R}^2)$ of R by the covering

$$pr \times id \;: \widetilde{SL}_2(\mathbb{R}) \ltimes H(\mathbb{R}^2) \twoheadrightarrow SL_2(\mathbb{R}) \ltimes H(\mathbb{R}^2).$$

Then tensoring an odd character $\widetilde{\chi}$ of $\widetilde{SO}(2)$ to $(\omega_\psi \times \rho_\psi)|_{\widetilde{R}}$, we have a representation of R

$$\widetilde{\chi} \otimes (\omega_\psi \times \rho_\psi)|_{\widetilde{R}} \;:\; R = S \ltimes N \;\to\; \mathrm{Aut}(L^2(\mathbb{R})).$$

We denote this representation by $(\eta, L^2(\mathbb{R}))$. A character of $\widetilde{SO}(2)$ is called odd, if it does not factor through the covering $\widetilde{SO}(2) \twoheadrightarrow SO(2)$.

Now we fix a description of odd characters. The covering $\widetilde{SO}(2) \twoheadrightarrow SO(2)$ is identified with the twice homomorphism $SO(2) \to SO(2)$; $R(\theta) \mapsto R(2\theta)$. Hence the characters of $\widetilde{SO}(2)$ is identified with $\frac{1}{2}\mathbb{Z}$, if we identify these of $SO(2)$ with $\mathbb{Z}$. Therefore odd characters $\widetilde{\chi}_\mu$ of $\widetilde{SO}(2)$ is parameterized by some elements $\mu = m + \frac{1}{2}$ in $\frac{1}{2}\mathbb{Z} \backslash \mathbb{Z}$ $(m \in \mathbb{Z})$.

Here is a diagram explaining the above construction

$$\begin{array}{ccccc} \widetilde{R} = \widetilde{S} \ltimes N & & \widetilde{SL}_2(\mathbb{R}) \ltimes H(\mathbb{R}^2) & \xrightarrow{\omega_\psi \times \rho_\psi} & \mathrm{Aut}(L^2(\mathbb{R})) \\ \downarrow & & {\scriptstyle pr\times id}\downarrow & & \\ R = S \ltimes N & \longrightarrow & SL_2(\mathbb{R}) \ltimes H(\mathbb{R}^2). & & \end{array}$$

The representations of R with non-trivial central characters are exhausted by these representations constructed above.

Lemma 2.3.2. *The unitary induced representation* $\mathrm{Ind}_N^R \rho_\psi$ *of* R *has a direct sum decomposition*:

$$\mathrm{Ind}_N^R \rho_\psi \ \cong \ \bigoplus_{\mu \in \frac{1}{2}\mathbb{Z} \backslash \mathbb{Z}} \widetilde{\chi}_\mu \otimes (\omega_\psi \times \rho_\psi).$$

Proof. The induced representation $\mathrm{Ind}_N^{\tilde{R}} \rho_\psi$ of $\widetilde{R} = \widetilde{S} \ltimes N$ is isomorphic to $\mathrm{Reg}_{\tilde{S}} \otimes (\omega_\psi \times \rho_\psi)$, where $\mathrm{Reg}_{\tilde{S}}$ is the regular representation of $\widetilde{S}$ (*cf.* [Ya] p.399, Lemma 2.5). Since $\widetilde{S} \cong \widetilde{SO}(2)$, $\mathrm{Reg}_{\tilde{S}}$ decomposes as a direct sum of characters:

$$\mathrm{Reg}_{\tilde{S}} \ \cong \ \bigoplus_{\mu \in \frac{1}{2}\mathbb{Z}} \widetilde{\chi}_\mu.$$

Therefore $\mathrm{Ind}_N^R \rho_\psi$, which is identified with $\mathrm{Ker}(\widetilde{R} \to R)$-invariant part of $\mathrm{Ind}_N^{\tilde{R}} \rho_\psi$, has the decomposition stated above. □

A basis of η and the action of $\mathrm{Lie}N$. It is well known that Hermite functions

$$h_j(\xi) \ := \ (-1)^j e^{\xi^2/2} \frac{d^j}{d\xi^j} e^{-\xi^2},$$

$j = 0, 1, 2, \ldots$ form an orthogonal Hilbert basis of $L^2(\mathbb{R})$.

Lemma 2.3.3. *The subspace of smooth vectors in $L^2(\mathbb{R})$ is the Schwartz space $\mathcal{S}(\mathbb{R})$, and the action of root vectors E_1, $E_{2,+}$, $E_{2,-}$ on $\mathcal{S}(\mathbb{R})$ through the underlining Harish-Chandra module of $\eta_{\mu,\psi_s} = \widetilde{\chi}_\mu \otimes (\omega_\psi \times \rho_\psi)|_{\widetilde{R}}$ are as follows*:

$$\eta(E_1).h_j = \sqrt{-1}sh_j,$$

$$\eta(E_{2,+}).h_j = \frac{-1}{2}h_{j+1} + jh_{j-1}, \quad \eta(E_{2,-}).h_j = \sqrt{-1}sh_{j+1} + 2\sqrt{-1}sjh_{j-1},$$

where ψ_s is an additive character $\mathbb{R} \ni t \mapsto e^{ist} \in U(1)$ of $\mathbb{R}$ with parameter $s \in \mathbb{R}\backslash\{0\}$.

Proof. Differentiate Schödinger model (2.3.1) in the exponential coordinate. Then we have

$$\rho_{\psi_s}(E_1).\Phi(\xi) = \sqrt{-1}s\Phi(\xi),$$

$$\rho_{\psi_s}(E_{2,+}).\Phi(\xi) = \frac{d}{d\xi}\Phi(\xi), \quad \rho_{\psi_s}(E_{2,-}).\Phi(\xi) = 2\sqrt{-1}s\xi \cdot \Phi(\xi),$$

where $\Phi \in \mathcal{S}(\mathbb{R})$. The well-known recurrence relations on Hermite polynomials:

$$\begin{aligned} H_j'(\xi) &= 2j \cdot H_{j-1}(\xi), \\ H_{j+1}(\xi) - 2\xi \cdot H_j(\xi) + 2j \cdot H_{j-1}(\xi) &= 0 \end{aligned}$$

(*cf.* [Er] p.193) tell us the assertion of Lemma. □

These formulae will be used in calculation of the radial parts of the Schmid operators and the Casimir operator (Proposition 3.2.3, Proposition 4.2.1).

2.4. The standard representations of G

Since G is of real rank one, the standard representations of G consist of discrete series representations and principal series representations. We briefly recall Harish-Chandra's parameterization of the discrete series, and the K-type decomposition of the standard representations.

The discrete series representations. Let Ξ denote the set of all Σ-regular $\Sigma_{c,+}$-dominant T-integral weights $\Lambda \in L_T$ of $\mathfrak{t}_{\mathbb{C}}$ and let $\widehat{G}_d$ denote the set of all equivalence classes of discrete series representations of G. By a result of Harish-Chandra, there is a bijection between Ξ and $\widehat{G}_d$. A member belonging to the class corresponding to $\Lambda \in \Xi$ is said to have the Harish-Chandra parameter Λ and denoted by π_Λ.

Under identification of the weight lattice L_T with $\mathbb{Z}^{\oplus 2}$, we have

$$\Xi \;=\; \{\Lambda = (\Lambda_1, \Lambda_2) \in \mathbb{Z}^{\oplus 2} \mid \Lambda_1 > \Lambda_2,\ \Lambda_1\Lambda_2 \neq 0\}.$$

This set decomposes into three disjoint subsets $\Xi_J = \{\Lambda \in \Xi \mid \langle \Lambda, \beta \rangle > 0,\ \forall \beta \in \Sigma_J^+\}$ correspond to positive root systems $\Sigma_J^+ (J = I, II, III)$ compatible with the positive compact root system $\Sigma_{c,+}$ fixed before as $\{\beta_{12}\}$. We fix Σ_J^+'s as follows

$$\begin{aligned} \Sigma_I^+ &:= \{\beta_{12}, \beta_{13}, \beta_{23}\}, \\ \Sigma_{II}^+ &:= \{\beta_{12}, \beta_{32}, \beta_{13}\}, \\ \Sigma_{III}^+ &:= \{\beta_{12}, \beta_{32}, \beta_{31}\}. \end{aligned}$$

Note these three are translate into each other by the action of the Weyl group $W(\mathfrak{g}, \mathfrak{t})/W(\mathfrak{k}, \mathfrak{t})$. Then by the inner product on L_T induced from the Killing form we can see easily

$$\begin{aligned} \Xi_I^+ &= \{(\Lambda_1, \Lambda_2) \in \mathbb{Z}^{\oplus 2} |\ \Lambda_1 > \Lambda_2 > 0\ \}, \\ \Xi_{II}^+ &= \{(\Lambda_1, \Lambda_2) \in \mathbb{Z}^{\oplus 2} |\ \Lambda_1 > 0 > \Lambda_2\ \}, \\ \Xi_{III}^+ &= \{(\Lambda_1, \Lambda_2) \in \mathbb{Z}^{\oplus 2} |\ 0 > \Lambda_1 > \Lambda_2\ \}. \end{aligned}$$

Representations parameterized by Ξ_I^+ (resp.Ξ_{III}^+) are called the holomorphic discrete series representations (resp. the antiholomorphic discrete series representations). In the remaining case, discrete series representations whose Harish-Chandra parameters Λ's belong to Ξ_{II}^+ are the large discrete series representations in the sense of Vogan [Vo].

The Blattner formula tells us the K-type decomposition of the discrete series representation π_Λ as follows.

$$\pi_\Lambda|_K \;=\; \oplus_{\mu \in L_T^+(\Lambda)} [\pi_\Lambda : \tau_\mu] \tau_\mu$$

where the set $L_T^+(\Lambda)$ of parameters of the K-types of π_Λ is given by

$$L_T^+(\Lambda) = \begin{cases} \{\lambda + m_1\beta_{13} + m_2\beta_{23} \mid m_1, m_2 \geq 0 \in \mathbb{Z}\} & \text{when } \Lambda \in \Xi_I^+, \\ \{\lambda + m_1\beta_{13} + m_2\beta_{32} \mid m_1, m_2 \geq 0 \in \mathbb{Z}\} & \text{when } \Lambda \in \Xi_{II}^+, \\ \{\lambda + m_1\beta_{31} + m_2\beta_{32} \mid m_1, m_2 \geq 0 \in \mathbb{Z}\} & \text{when } \Lambda \in \Xi_{III}^+. \end{cases}$$

When $\Lambda \in \Xi_J$, λ in the formula above is described as

$$\lambda = \Lambda - \rho_c + \rho_n^J \begin{cases} (\Lambda_1 + 1, \Lambda_2 + 2) & \text{when } J = I, \\ (\Lambda_1, \Lambda_2) & \text{when } J = II, \\ (\Lambda_1 - 2, \Lambda_2 - 1) & \text{when } J = III, \end{cases}$$

where ρ_c is the half-sum of the compact positive roots and ρ_n^J of the non-compact ones in Σ_J^+. This λ is the highest weight of the minimal K-type of π_Λ which is called the Blattner parameter. About the multiplicity we remark that all $[\pi_\Lambda : \tau_\mu]$ is one for $G = SU(2,1)$. That is the multiplicity free property for K-types is valid.

The principal series representations. Let

$$P \ = \ NAM$$

be the Langlands decomposition of P. We note M is isomorphic to $U(1)$. For characters $e^\nu : a_r \mapsto r^{\nu+2}$ ($\nu \in \mathbb{C}$) of A and $\chi_{\lambda_0} : \mathrm{diag}(e^{i\theta}, e^{-2i\theta}, e^{i\theta}) \mapsto e^{i\lambda_0\theta}$ ($\lambda_0 \in \mathbb{Z}$) of M, the induced representation $\pi_{\lambda_0,\nu} = \mathrm{Ind}_P^G(1_N \otimes e^\nu \otimes \chi_{\lambda_0})$ is called the principal series representation of G. The representation space of $\pi_{\lambda_0,\nu}$ is given by

$$\{f : G \to \mathbb{C} \ \mid \ f \text{ is a } C^\infty\text{-function satisfying,} \\ f(na_rmg) = r^{\nu+2}\chi_{\lambda_0}(m)f(g), \ \forall na_rm \in P \ , \forall g \in G\}.$$

By the Frobenius reciprocity, we have the K-type decomposition:

$$\pi_{\lambda_0,\nu}|_K \ = \ \oplus_{\mu \in L_T^+(\lambda_0)} [\pi_{\lambda_0,\nu} : \tau_\mu]\tau_\mu$$

with multiplicities $[\pi_{\lambda_0,\nu} : \tau_\mu] = 1$, where the set $L_T^+(\lambda_0)$ of parameters of the K-types of $\pi_{\lambda_0,\nu}$ is given by

$$L_T^+(\lambda_0) = \{(-\lambda_0, -\lambda_0) + m_1\beta_{13} + m_2\beta_{32} \mid m_1, m_2 \geq 0 \in \mathbb{Z}\}.$$

The principal series representation $\pi_{\lambda_0,\nu}$ has only one K-type $\tau_{(-\lambda_0,-\lambda_0)}$ whose dimension is one. We call it *the corner K-type of* $\pi_{\lambda_0,\nu}$.

3. The Case of Discrete Series Representations

Here we treat discrete series representations and have the multiplicity one theorem for such representations. Our method is to solve the system of partial differential equations which characterize generalized Whittaker functions and to give an explicit form of these functions.

3.1. Yamashita's characterization

In this subsection we give a variant of a result of Yamashita, in a suitable form for our use, which characterizes the space of the algebraic minimal K-type generalized Whittaker functionals for discrete series representations. This is fundamental for our purpose.

We denote the space of V_λ-valued functions on G with the τ_λ-equivariance by

$$C^\infty_{\tau_\lambda}(G/K) := \{\varphi : G \to V_\lambda \mid \varphi \text{ is a } C^\infty\text{-function satisfying} \\ \varphi(gk) = \tau_\lambda(k)^{-1}.\varphi(g) \ , \forall g \in G \ , \forall k \in K\}.$$

We can regard $\mathfrak{p}_\mathbb{C}$ as a K-module through the adjoint representation $\mathrm{Ad}_{\mathfrak{p}_\mathbb{C}}$. The differential operator

$$\nabla_{\tau_\lambda} \ : \ C^\infty_{\tau_\lambda}(G/K) \to C^\infty_{\tau_\lambda \otimes \mathrm{Ad}_{\mathfrak{p}_\mathbb{C}}}(G/K)$$

defined by

$$\nabla_{\tau_\lambda}\varphi \ := \ \sum_{i=1}^{4} R_{X_i}\varphi \otimes X_i,$$

is a K-homomorphism. Here $\{X_i \ (i = 1, \dots, 4)\}$ is an orthonormal basis of $\mathfrak{p}$ with respect to the Killing form on $\mathfrak{g}$ and $R_X\varphi$ means the right differential of function φ by $X \in \mathfrak{g}$: $R_X\varphi(g) = \frac{d}{dt}\varphi(g \exp tX)|_{t=0}$. The operator ∇_{τ_λ} is called *the Schmid operator*. We take $C(X_\beta + X_{-\beta})$, $C\sqrt{-1}(X_\beta - X_{-\beta})$ as orthonormal basis of $\mathfrak{p}_\mathbb{C}$, where β is β_{13} or β_{23} and C is a positive constant depending on the normalization of the fixed Killing form. Then, using this basis, the Schmid operator ∇_{τ_λ} can be written as

$$\nabla_{\tau_\lambda}\varphi \ = \ 2C^2 \sum_{\beta=\beta_{13},\beta_{23}} R_{X_{-\beta}}\varphi \otimes X_\beta + 2C^2 \sum_{\beta=\beta_{13},\beta_{23}} R_{X_\beta}\varphi \otimes X_{-\beta}.$$

Here we note that $\{X_{\beta_{13}}, X_{\beta_{23}}\}$ is the set of root vectors corresponding to positive noncompact roots. The above description of ∇_{τ_λ} in two terms corresponds to the decomposition of $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$.

Now we define two differential operators

$$\nabla^{\pm}_{\tau_\lambda} \ : \ C^{\infty}_{\tau_\lambda}(G/K) \to C^{\infty}_{\tau_\lambda \otimes \mathrm{Ad}_{\mathfrak{p}_\pm}}(G/K)$$

as

$$\begin{aligned} \nabla^{+}_{\tau_\lambda}\varphi \ &:= \ R_{X_{-\beta_{13}}}\varphi \otimes X_{\beta_{13}} + R_{X_{-\beta_{23}}}\varphi \otimes X_{\beta_{23}}, \\ \nabla^{-}_{\tau_\lambda}\varphi \ &:= \ R_{X_{\beta_{13}}}\varphi \otimes X_{-\beta_{13}} + R_{X_{\beta_{23}}}\varphi \otimes X_{-\beta_{23}}. \end{aligned}$$

For later use, we prepare *the $\pm\beta$-shift operators* for every positive noncompact root $\beta \in \Sigma_{n,+}$ and $\lambda \in L^{+}_{T}$.

$$\mathcal{D}^{\pm\beta}_{\tau_\lambda} \ : \ C^{\infty}_{\tau_\lambda}(G/K) \to C^{\infty}_{\tau_{\lambda\pm\beta}}(G/K)$$

$$\mathcal{D}^{\pm\beta}_{\tau_\lambda}\varphi(g) \ := \ p^{\pm}_{\beta}\big(\nabla^{\pm}_{\tau_\lambda}\varphi(g)\big).$$

Here $p^{\pm}_{\beta}$ are the projectors $\tau_\lambda \otimes \mathrm{Ad}_{\mathfrak{p}_\pm} \twoheadrightarrow \tau_{\lambda\pm\beta}$ defined in subsection 2.2.

All the operators constructed above can be defined similarly for the space

$$C^{\infty}_{\eta,\tau_\lambda}(R\backslash G/K) := \left\{ \varphi : G \to \mathcal{S}(\mathbb{R}) \otimes_{\mathbb{C}} V_\lambda \ \middle| \ \begin{array}{l} \varphi \text{ is a } C^{\infty}\text{-function satisfying} \\ \varphi(rgk) = \eta(r)\tau_\lambda(k)^{-1}.\varphi(g), \\ \forall r \in R \ , \forall g \in G \ , \forall k \in K \end{array} \right\}.$$

We denote these by the same symbols:

$$\begin{array}{rcl} \nabla_{\eta,\tau_\lambda} & : & C^{\infty}_{\eta,\tau_\lambda}(R\backslash G/K) \ \to \ C^{\infty}_{\eta,\tau_\lambda \otimes \mathrm{Ad}_{\mathfrak{p}_{\mathbb{C}}}}(R\backslash G/K), \\ \nabla^{\pm}_{\eta,\tau_\lambda} & : & C^{\infty}_{\eta,\tau_\lambda}(R\backslash G/K) \ \to \ C^{\infty}_{\eta,\tau_\lambda \otimes \mathrm{Ad}_{\mathfrak{p}_\pm}}(R\backslash G/K), \\ \mathcal{D}^{\pm\beta}_{\eta,\tau_\lambda} & : & C^{\infty}_{\eta,\tau_\lambda}(R\backslash G/K) \ \to \ C^{\infty}_{\eta,\tau_{\lambda\pm\beta}}(R\backslash G/K). \end{array}$$

Choose a K-homomorphism $\iota_{\tau_\lambda} : V_\lambda \hookrightarrow \mathcal{H}_{\pi_\Lambda}$ and consider the restriction map

$$\mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\mathcal{H}^{*}_{\pi_\Lambda}, C^{\infty}_{\eta}(R\backslash G)) \to \mathrm{Hom}_{K}(V^{*}_{\lambda}, C^{\infty}_{\eta}(R\backslash G)),$$

induced from ι_{τ_λ}. By the canonical isomorphism

$$\mathrm{Hom}_{K}(V^{*}_{\lambda}, C^{\infty}_{\eta}(R\backslash G)) \ \cong \ \big(C^{\infty}_{\eta}(R\backslash G) \otimes_{\mathbb{C}} V_\lambda\big)^{K},$$

where $(\ )^K$ means the subspace fixed by K, the restriction homomorphism induces

$$\mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\mathcal{H}^*_{\pi_\Lambda}, C^\infty_\eta(R\backslash G)) \to C^\infty_{\eta,\tau_\lambda}(R\backslash G/K).$$

Proposition 3.1.1 ([Ya2] Theorem 2.4). *Let π_Λ be a discrete series representation of G of the Harish-Chandra parameter $\Lambda \in \Xi_J$, the Blattner parameter $\lambda = \Lambda + \rho_J - 2\rho_c$. Let η be the representation constructed in subsection 2.3. Assume Λ is far from walls, then the image of $\mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi^*_\Lambda, \mathrm{Ind}_R^G\eta)$ in $C^\infty_{\eta,\tau_\lambda}(R\backslash G/K)$ by the correspondence above is characterized by*

$$(D)\ :\qquad \mathcal{D}^{-\beta}_{\eta,\tau_\lambda}.F \ = \ 0 \qquad (\forall\beta \in \Sigma^+_J \cap \Sigma_n).$$

In short

$$I^{\tau_\lambda}_{\pi_\Lambda,\eta} \ \cong \ \bigcap_{\beta\in\Sigma^+_J\cap\Sigma_n} Ker\,\mathcal{D}^{-\beta}_{\eta,\tau_\lambda},$$

where $I^{\tau_\lambda}_{\pi_\Lambda,\eta}$ is the intertwining space $\mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\tau^*_\lambda, \mathrm{Ind}_R^G\eta)$.

Naturally our generalized Whittaker functions satisfies the above system of differential equations (D).

3.2. Difference-differential equations for coefficients

Radial part of Schmid operators. For the representation $(\eta, L^2(\mathbb{R}))$ constructed in subsection 2.3 and for any finite dimensional K-module W, we denote the space of the smooth $\mathcal{S}(\mathbb{R}) \otimes_{\mathbb{C}} W$-valued functions on A by

$$C^\infty(A\,;\,\mathcal{S}(\mathbb{R})\otimes_{\mathbb{C}} W) \ := \ \{\phi : A \to \mathcal{S}(\mathbb{R})\otimes_{\mathbb{C}} W \mid C^\infty\text{-function}\}.$$

Let

$$\begin{array}{rcllcl} \mathrm{res}_A & : & C^\infty_{\eta,\tau_\lambda}(R\backslash G/K) & \to & C^\infty(A\,;\,\mathcal{S}(\mathbb{R})\otimes_{\mathbb{C}} V_\lambda), \\ \mathrm{res}_{A,\pm} & : & C^\infty_{\eta,\tau_\lambda\otimes Ad_{\mathfrak{p}_\pm}}(R\backslash G/K) & \to & C^\infty(A;\mathcal{S}(\mathbb{R})\otimes_{\mathbb{C}} V_\lambda \otimes_{\mathbb{C}} \mathfrak{p}_\pm) \end{array}$$

be the restriction maps to A. Then we define *the radial part* $R(\nabla^\pm_{\eta,\tau_\lambda})$ of $\nabla^\pm_{\eta,\tau_\lambda}$ on the image of res_A by

$$R(\nabla^\pm_{\eta,\tau_\lambda}).(\mathrm{res}_A\varphi) \ = \ \mathrm{res}_{A,\pm}(\nabla^\pm_{\eta,\tau_\lambda}.\varphi).$$

Let us denote by ϕ and ∂ the restriction to A of $\varphi \in C^\infty_{\eta,\tau_\lambda}(R\backslash G/K)$ and of the generator H of $\mathfrak{a}$, respectively; $\partial\phi = (H.\varphi)|_A$. We remark $\partial = r\frac{d}{dr}$: the Euler operator in the variable r.

Proposition 3.2.1 ([K-O] Prop 4.1). *Let ϕ be the above element in $C^\infty(A\,;\mathcal{S}(\mathbb{R})\otimes_{\mathbb{C}} V_\lambda)$. Then the radial part $R(\nabla^+_{\eta,\tau_\lambda})$ of $\nabla^+_{\eta,\tau_\lambda}$ is given by*

(i) $R(\nabla^+_{\eta,\tau_\lambda}).\phi$

$$= \frac{1}{2}\{\partial - \sqrt{-1}r^2\eta(E_1) - 4\}.(\phi\otimes X_{\beta_{13}}) + \frac{1}{2}(\tau_\lambda\otimes Ad_{\mathfrak{p}_+})(H'_{13}).(\phi\otimes X_{\beta_{13}})$$
$$- \frac{1}{2}r\{\eta(E_{2,+}) - \sqrt{-1}\eta(E_{2,-})\}.(\phi\otimes X_{\beta_{23}}) + (\tau_\lambda\otimes Ad_{\mathfrak{p}_+})(X_{\beta_{12}}).(\phi\otimes X_{\beta_{23}}).$$

Similarly for the radial part $R(\nabla^-_{\eta,\tau_\lambda})$ of $\nabla^-_{\eta,\tau_\lambda}$, we have[†]

(ii) $R(\nabla^-_{\eta,\tau_\lambda}).\phi$

$$= \frac{1}{2}\{\partial + \sqrt{-1}r^2\eta(E_1) - 4\}.(\phi\otimes X_{\beta_{31}}) - \frac{1}{2}(\tau_\lambda\otimes Ad_{\mathfrak{p}_-})(H'_{13}).(\phi\otimes X_{\beta_{31}})$$
$$- \frac{1}{2}r\{\eta(E_{2,+}) + \sqrt{-1}\eta(E_{2,-})\}.(\phi\otimes X_{\beta_{32}}) + (\tau_\lambda\otimes Ad_{\mathfrak{p}_-})(X_{\beta_{21}}).(\phi\otimes X_{\beta_{32}}).$$

Compatibility of S-type and K-type. Here we investigate the compatibility of the action of S from left hand side and the action of K or M from right hand side for the function $\phi = \mathrm{res}_A\varphi$, $\varphi \in C^\infty_{\eta,\tau_\lambda}(R\backslash G/K)$.

If we write $\phi = \varphi|_A \in C^\infty(A\,;\mathcal{S}(\mathbb{R})\otimes_{\mathbb{C}} V_\lambda)$ as

$$\phi(a) \;=\; \sum_{j=0}^{\infty}\sum_{k=0}^{d_\lambda} c_{jk}(a)\big(h_j\otimes v_k^\lambda\big)$$

in terms of bases $\{h_j | j\in\mathbb{N}\}$ and $\{v_k^\lambda | k = 0,\dots,d_\lambda\}$ of $\mathcal{S}(\mathbb{R})$ and V_λ respectively, the compatibility of S-action and K-action implies the vanishing of many coefficients c_{jk}. Here is the precise statement.

Recall the representation $(\eta,\mathcal{S}(\mathbb{R}))$ of R is of the form $\widetilde{\chi}_\mu\otimes(\omega_\psi\times\rho_\psi)|_{\widetilde{R}}$. Here $\widetilde{\chi}_\mu$ is an odd character of $\widetilde{S}\cong\widetilde{SO}(2)$ parameterized by a half integer μ.

[†]In [K-O], there is a misprint in the formula (ii). The sign $-$ after ∂ is $+$ correctly.

Lemma 3.2.2. (1) *The image of* res_A *in* $C^\infty(A\,;\mathcal{S}(\mathbb{R})\otimes_{\mathbb{C}} V_\lambda)$ *is zero unless*

$$\frac{-\lambda_1+2\lambda_2}{3}\in\mathbb{Z} \qquad \text{and} \qquad \frac{-\lambda_1+2\lambda_2}{3}\leq\frac{1}{2}-\mu.$$

(2) *Assume the condition above in* (1) *holds, then the* A*-radial part* ϕ *of* $\varphi\in C^\infty_{\eta,\tau_\lambda}(R\backslash G/K)$ *is written as*

$$\phi(a_r) \;=\; \sum_{k=0}^{d_\lambda} c_k(a_r)\big(h_j\otimes v_k^\lambda\big),$$

where $c_k(a_r)$*'s are* C^∞*-functions on* A *and the index* j *is given by*

$$j \;=\; k-\frac{2\lambda_1-\lambda_2}{3}-\frac{1}{2}-\mu. \tag{3.2.1}$$

Proof. We calculate $\phi(mam^{-1})$, $m\in S=M, a\in A$ in two different ways.
First, $M=Z_K(A)$, $mam^{-1}=a$ for any $a\in A$ and $m\in M$, therefore

$$\phi(mam^{-1})=\phi(a).$$

Second, since $M=S\subset R$ and ϕ is a function which comes from $\varphi\in C^\infty_{\eta,\tau_\lambda}(R\backslash G/K)$,

$$\begin{aligned}\phi(mam^{-1})&=\eta(m)\tau_\lambda(m^{-1})^{-1}.\phi(a)\\ &=\sum_{j=0}^{\infty}\sum_{k=0}^{d_\lambda}c_{jk}(a)\Big\{\big(\eta(m).h_j\big)\otimes\big(\tau_\lambda(m).v_k^\lambda\big)\Big\}.\end{aligned}$$

Here we note the action η decomposes into two parts: $\eta\;=\;\widetilde{\chi}_\mu\otimes(\omega_\psi\times\rho_\psi)|_{\widetilde{R}}$, where

$$\begin{array}{rcccccc}\widetilde{\chi}_\mu & : & \widetilde{S} & \cong & \widetilde{SO}(2) & \to & U(1)\\ & & \mathrm{diag}(e^{i\theta},e^{-2i\theta},e^{i\theta}) & \mapsto & R(3\theta) & \mapsto & e^{-i\mu3\theta}\end{array}$$

and

$$(\omega_\psi\times\rho_\psi)\big(R(3\theta)\big).h_j \;=\; e^{-i3\theta(j+\frac{1}{2})}h_j\;.$$

For the τ_λ-action on v_k^λ, noting $m = \exp i\theta(2H'_{12} - H'_{13})$, we know

$$\tau_\lambda(m).v_k^\lambda \;=\; e^{i(3k-2\lambda_1+\lambda_2)\theta} v_k^\lambda.$$

Hence we have

$$\phi(mam^{-1}) \;=\; \sum_{j=0}^{\infty}\sum_{k=0}^{d_\lambda} c_{jk}(a) e^{i(-3\mu-3j-\frac{3}{2}+3k-2\lambda_1+\lambda_2)\theta}\big(h_j \otimes v_k^\lambda\big).$$

Since this is equal to $\phi(a)$,

$$c_{jk}(a) \;=\; c_{jk}(a) e^{i(-3\mu-3j-\frac{3}{2}+3k-2\lambda_1+\lambda_2)\theta}$$

for all $\theta \in \mathbb{R}$. Therefore the function c_{jk} is identically zero unless the equality

$$j \;=\; k - \tfrac{2\lambda_1-\lambda_2}{3} - \tfrac{1}{2} - \mu$$

holds. Here we note $j, k \in \mathbb{N}$, $0 \le k \le d_\lambda$, $\mu \in \frac{3}{2}\mathbb{Z}\backslash\mathbb{Z}$, above linear relation between j and k tells the assertion of Lemma. □

Difference-differential equations for coefficients. As we saw before, Yamashita's characterization tells that the function F in $C^\infty_{\eta,\tau_\lambda}(R\backslash G/K)$, which comes from $l \in I_{\pi_\Lambda,\eta}$, satisfies the system of differential equations (D) in Proposition 3.1.1. Since F is determined by the A-radial part $\phi = F|_A$, and ϕ is determined by the $d_\lambda + 1$ coefficient functions $c_k(a_r)$ $(k = 0, \ldots, d_\lambda)$ in Lemma 3.2.2, we first write down the A-radial part $R(\mathcal{D}^{-\beta}_{\eta,\tau_\lambda})$ of the β-shift operators $\mathcal{D}^{-\beta}_{\eta,\tau_\lambda}$ in terms of coefficient functions $c_k(a_r)$'s of ϕ.

Proposition 3.2.3. *Let ϕ be any function in $C^\infty(A; \mathcal{S}(\mathbb{R}) \otimes_{\mathbb{C}} V_\lambda)$ which is the A-radial part of $\varphi \in C^\infty_{\eta,\tau_\lambda}(R\backslash G/K)$. By using Lemma 3.2.2, we can express ϕ as*

$$\phi(a_r) \;=\; \sum_{k=0}^{d_\lambda} c_k(a_r)\big(h_{j_{\mu,\lambda}(k)} \otimes v_k^\lambda\big),$$

where we denote $k - \frac{2\lambda_1 - \lambda_2}{3} - \frac{1}{2} - \mu$ *by* $j_{\mu,\lambda}(k)$. *Then for an arbitrary noncompact root* β, *the action of the A-radial part* $R(\mathcal{D}^{-\beta}_{\eta,\tau_\lambda})$ *of the* β*-shift operator is given in terms of* c_k*'s as follows*:

$$R(\mathcal{D}^{-\beta}_{\eta,\tau_\lambda})\phi(a_r) \;=\; \sum_{k=0}^{d_{\lambda-\beta}} c_k^{-\beta}(a_r)\big(h_{j_{\mu,\lambda}(k)} \otimes v_k^{\lambda-\beta}\big)$$

with

$$\begin{aligned}
(3.2.2)\quad c_k^{-\beta_{23}}(a_r) &= \frac{1}{2}\{(d_\lambda - k + 1)(\partial + k - \lambda_2 - sr^2).c_k(a_r) \\
&\qquad - k\frac{1+2s}{2}r \cdot c_{k-1}(a_r)\}, \\
(3.2.3)\quad c_k^{-\beta_{13}}(a_r) &= \frac{1}{2}\{(\partial + k - 2d_\lambda - \lambda_2 - 1 - sr^2).c_{k+1}(a_r) \\
&\qquad + \frac{1+2s}{2}r \cdot c_k(a_r)\}, \\
(3.2.4)\quad c_k^{-\beta_{32}}(a_r) &= \frac{-1}{2}\{(\partial - k + \lambda_2 - 2 + sr^2).c_k(a_r) \\
&\qquad + (1+2s)(j_{\mu,\lambda}(k)+1)r \cdot c_{k+1}(a_r)\}, \\
(3.2.5)\quad c_k^{-\beta_{31}}(a_r) &= \frac{1}{2}\{k(\partial - k + \lambda_2 + 2d_\lambda + 1 + sr^2).c_{k-1}(a_r) \\
&\qquad - (d_\lambda - k + 1)(1+2s)(j_{\mu,\lambda}(k)+1)r \cdot c_k(a_r)\}.
\end{aligned}$$

Proof. We show here calculation only for $c_k^{-\beta_{32}}(a_r)$, since others are exactly similar.

$$\begin{aligned}
&R(\mathcal{D}^{-\beta_{32}}_{\eta,\tau_\lambda}).\phi(a_r) \;=\; (1_{\mathcal{S}(\mathbb{R})} \otimes p^+_{\beta_{23}}(\lambda))(R(\nabla^+_{\eta,\tau_\lambda}).\phi) \\
&= \frac{1}{2}\{\partial - \sqrt{-1}r^2\eta(E_1) - 4\}.\sum_{k=0}^{d_\lambda} c_k(a_r)h_{j_{\mu,\lambda}(k)} \otimes p^+_{\beta_{23}}(\lambda).(v_k^\lambda \otimes X_{\beta_{13}}) \\
&\quad + \frac{1}{2}p^+_{\beta_{23}}(\lambda)(\tau_{\lambda+\beta_{13}} \oplus \tau_{\lambda-\beta_{32}})(H'_{13}) \\
&\quad .\sum_{k=0}^{d_\lambda} c_k(a_r)h_{j_{\mu,\lambda}(k)} \otimes p^+_{\beta_{23}}(\lambda).(v_k^\lambda \otimes X_{\beta_{13}})
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2} r\{\eta(E_{2,+}) - \sqrt{-1}\eta(E_{2,-})\}. \sum_{k=0}^{d_\lambda} c_k(a_r) h_{j_\mu,\lambda(k)} \otimes p^{+}_{\beta_{23}}(\lambda).(v^{\lambda}_k \otimes X_{\beta_{23}}) \\
&+ \quad p^{+}_{\beta_{23}}(\lambda)(\tau_{\lambda+\beta_{13}} \oplus \tau_{\lambda-\beta_{32}})(X_{\beta_{12}}) \\
&. \sum_{k=0}^{d_\lambda} c_k(a_r) h_{j_\mu,\lambda(k)} \otimes p^{+}_{\beta_{23}}(\lambda).(v^{\lambda}_k \otimes X_{\beta_{23}}) \\
= &- \frac{1}{2}\{\partial - \sqrt{-1} r^2 \eta(E_1) - 4\}. \sum_{k=0}^{d_\lambda - 1} c_k(a_r)\big(h_{j_\mu,\lambda(k)} \otimes v^{\lambda-\beta_{32}}_k\big) \\
&- \frac{1}{2} \tau_{\lambda-\beta_{32}}(H'_{13}). \sum_{k=0}^{d_\lambda - 1} c_k(a_r)\big(h_{j_\mu,\lambda(k)} \otimes v^{\lambda-\beta_{32}}_k\big) \\
&- \frac{1}{2} r\{\eta(E_{2,+}) - \sqrt{-1}\eta(E_{2,-})\}. \sum_{k=1}^{d_\lambda} c_k(a_r)\big(h_{j_\mu,\lambda(k)} \otimes v^{\lambda-\beta_{32}}_{k-1}\big) \\
&+ \quad \tau_{\lambda-\beta_{12}}(X_{\beta_{12}}). \sum_{k=1}^{d_\lambda} c_k(a_r)\big(h_{j_\mu,\lambda(k)} \otimes v^{\lambda-\beta_{32}}_{k-1}\big).
\end{aligned}
$$

Abbreviate $\lambda - \beta_{32}$ as ν. Then, noting $d_\nu = d_\lambda - 1$,

(3.2.6) $\quad R(\mathcal{D}^{-\beta_{32}}_{\eta,\tau_\lambda}).\phi(a_r)$

$$
\begin{aligned}
= &- \frac{1}{2} \sum_{k=0}^{d_\nu} (\partial - 4).c_k(a_r)\big(h_{j_\mu,\lambda(k)} \otimes v^{\nu}_k\big) \\
&+ \frac{1}{2} \sum_{k=0}^{d_\nu} \sqrt{-1} r^2 \cdot c_k(a_r)\big(\eta(E_1).h_{j_\mu,\lambda(k)} \otimes v^{\nu}_k\big) \\
&- \frac{1}{2} \sum_{k=0}^{d_\nu} c_k(a_r)\big(h_{j_\mu,\lambda(k)} \otimes \tau_\nu(\frac{H'_{12} + Z}{2}).v^{\nu}_k\big) \\
&- \frac{1}{2} \sum_{k'=0}^{d_\nu} r \cdot c_{k'+1}(a_r)\big(\{\eta(E_{2,+}) - \sqrt{-1}\eta(E_{2,-})\}.h_{j_\mu,\lambda(k'+1)} \otimes v^{\nu}_{k'}\big) \\
&+ \quad \sum_{k=1}^{d_\nu + 1} c_k(a_r)\big(h_{j_\mu,\lambda(k)} \otimes \tau_\nu(X_{\beta_{12}}).v^{\nu}_{k-1}\big).
\end{aligned}
$$

Here we recall the τ_λ-action on standard basis in subsection 2.2

$$\begin{aligned}
\tau_\nu(H'_{12}).v_k^\nu &= (2k-(d_\lambda-1))\cdot v_k^\nu\\
\tau_\nu(Z).v_k^\nu &= (\nu_1+\nu_2)\cdot v_k^\nu = (d_\lambda+2\lambda_2+3)\cdot v_k^\nu\\
\tau_\nu(X_{\beta_{12}}).v_{k-1}^\nu &= k\cdot v_k^\nu.
\end{aligned}$$

Put these into the above formula (3.2.6), then we complete the calculation for $\mathfrak{k}$. The result is

$$\begin{aligned}
&R(\mathcal{D}_{\eta,\tau_\lambda}^{-\beta_{32}}).\phi(a_r)\\
=\ &-\frac{1}{2}\sum_{k=0}^{d_\nu}\{(\partial-4)+(k+\lambda_2+2)-2k\}.c_k(a_r)\big(h_{j_{\mu,\lambda}(k)}\otimes v_k^\nu\big)\\
&+\frac{1}{2}\sum_{k=0}^{d_\nu}\sqrt{-1}r^2\cdot c_k(a_r)\big(\eta(E_1).h_{j_{\mu,\lambda}(k)}\otimes v_k^\nu\big)\\
&-\frac{1}{2}\sum_{k'=0}^{d_\nu}r\cdot c_{k'+1}(a_r)\big(\{\eta(E_{2,+})-\sqrt{-1}\eta(E_{2,-})\}.h_{j_{\mu,\lambda}(k'+1)}\otimes v_{k'}^\nu\big).
\end{aligned}$$

Next we use the η-action on the basis $\{h_j\}$ of $\mathcal{S}(\mathbb{R})$ prepared in Lemma 2.3.3. Then

$$\begin{aligned}
&R(\mathcal{D}_{\eta,\tau_\lambda}^{-\beta_{32}}).\phi(a_r)\\
=\ &-\frac{1}{2}\sum_{k=0}^{d_\nu}(\partial-k+\lambda_2-2).c_k(a_r)\big(h_{j_{\mu,\lambda}(k)}\otimes v_k^\nu\big)\\
&+\frac{1}{2}\sum_{k=0}^{d_\nu}\sqrt{-1}r^2\cdot c_k(a_r)\big(\sqrt{-1}sh_{j_{\mu,\lambda}(k)}\otimes v_k^\nu\big)\\
&-\frac{1}{2}\sum_{k=0}^{d_\nu}r\cdot c_{k+1}(a_r)\left\{-\frac{1-2s}{2}h_{j_{\mu,\lambda}(k+1)+1}\otimes v_k^\nu\right\}\\
&-\frac{1}{2}\sum_{k=0}^{d_\nu}r\cdot c_{k+1}(a_r)\left\{+(1+2s)j_{\mu,\lambda}(k+1)h_{j_{\mu,\lambda}(k+1)-1}\otimes v_k^\nu\right\}\\
=\ &-\frac{1}{2}\sum_{k=0}^{d_\nu}(\partial-k+\lambda_2-2+sr^2).c_k(a_r)\big(h_{j_{\mu,\lambda}(k)}\otimes v_k^\nu\big)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\sum_{k=0}^{d_\nu} -\frac{1-2s}{2} r\cdot c_{k+1}(a_r)\big(h_{j_{\mu,\lambda}(k)+2}\otimes v_k^\nu\big)\\
&-\frac{1}{2}\sum_{k=0}^{d_\nu}(1+2s)r\ j_{\mu,\lambda}(k+1)\cdot c_{k+1}(a_r)\big(h_{j_{\mu,\lambda}(k)}\otimes v_k^\nu\big)\\
=\ &-\frac{1}{2}\sum_{k=0}^{d_\nu}\Big\{(\partial-k+\lambda_2-2+sr^2).c_k(a_r)\\
&\qquad\qquad +(1+2s)(j_{\mu,\lambda}(k)+1)r\cdot c_{k+1}(a_r)\Big\}\big(h_{j_{\mu,\lambda}(k)}\otimes v_k^\nu\big).
\end{aligned}
$$

In the last equality, we use the fact that unless the indices of bases h_j and v_k satisfy the linear relation (3.2.1) of Lemma 3.2.2, the coefficient c_k of $h_j \otimes v_k$ is identically zero. Now we accomplished the calculation of the action of the radial part $R(\mathcal{D}^{-\beta_{32}}_{\eta,\tau_\lambda})$ of β_{32}-shift operator $\mathcal{D}^{-\beta_{32}}_{\eta,\tau_\lambda}$. Rewrite it by the coefficient functions, then we have (3.2.4) described in Proposition. $\square$

Using the above Proposition 3.2.3, we can write the differential equations (D) in Proposition 3.1.1 in terms of the coefficient functions c_k of the A-radial part ϕ of the algebraic generalized Whittaker function $F \in C^\infty_{\eta,\tau_\lambda}(R\backslash G/K)$, which comes from $l \in I_{\pi_\Lambda,\eta}$.

First we investigate the case of the large discrete series representation which is most interesting for us among the discrete series. The Fourier-Jacobi expansion of automorphic forms belonging to the representations was not classically unknown, whereas holomorphic ones are investigated in [PS2].

Proposition 3.2.4. *Assume that the Harish-Chandra parameter Λ of π belongs to Ξ_{II} and let $F \in C^\infty_{\eta,\tau_\lambda}(R\backslash G/K)$ be in the image of an element l of the intertwining space $I_{\pi_\Lambda,\eta}$ by the correspondence of subsection 1.2. Then the system of differential equations in Proposition 3.1.1 is equivalent to the following system of difference-differential equations for the coefficient functions c_k's of the A-radial part of F.*

$$
\begin{aligned}
(A^+_{\mu,\lambda})_k:\qquad &\big(\partial + A^+_{\lambda k}(r)\big).c_{j_{\mu\lambda}(k),k}(a_r)\\
&\qquad = -(1+2s)(j_{\mu,\lambda}(k)+1)r\cdot c_{j_{\mu\lambda}(k+1),k+1}(a_r),\\
(B^-_{\mu,\lambda})_k:\qquad &\big(\partial + B^-_{\lambda k}(r)\big).c_{j_{\mu\lambda}(k),k+1}(a_r) = -\frac{1+2s}{2} r\cdot c_{j_{\mu\lambda}(k-1),k}(a_r),
\end{aligned}
$$

where

$$\begin{aligned} A^{+}_{\lambda k}(r) &= sr^2 + (\lambda_2 - 2 - k) \\ B^{-}_{\lambda k}(r) &= -sr^2 - (\lambda_2 + 2d_\lambda - k) - 1. \end{aligned}$$

PROOF. Recall that we fixed Σ^+_{II} as $\{\beta_{12}, \beta_{32}, \beta_{13}\}$, so $\Sigma^+_{II} \cap \Sigma_n = \{\beta_{32}, \beta_{13}\}$. Hence combining the investigation of subsection 3.2, the system (D) is equivalent to two differential equations

$$\begin{aligned} R(\mathcal{D}^{-\beta_{32}}_{\eta,\tau_\lambda}).\phi &= 0, \\ R(\mathcal{D}^{-\beta_{13}}_{\eta,\tau_\lambda}).\phi &= 0, \end{aligned}$$

where ϕ is the A-radial part of F. These are equivalent to the fact that all coefficient functions $c^{-\beta}_k$ ($\beta = \beta_{32}, \beta_{13}$) in the recurrence relations (3.2.3), (3.2.4) are zero, which is the statement of Proposition. □

In the same manner we can derive the system of difference-differential equations satisfied by the coefficient functions of the A-radial part of elements F's of $C^\infty_{\eta,\tau_\lambda}(R\backslash G/K)$ when π_Λ is a holomorphic or an antiholomorphic discrete series representation ($i.e. \Lambda \in \Xi_I$ or $\in \Xi_{III}$).

PROPOSITION 3.2.5. *Assume that the Harish-Chandra parameter Λ of π belongs to Ξ_I, then the system of differential equation in Proposition 3.1.1 for F which is in the image of an element of $I_{\pi_\Lambda,\eta}$ is equivalent to the following system for the coefficient functions c_k's of $F|_A$.*

$$\begin{aligned} (A^{-}_{\mu,\lambda})_k: \quad & (d_\lambda - k + 1)\big(\partial + A^{-}_{\lambda k}(r)\big).c_{j_{\mu\lambda}(k),k}(a_r) \\ & \qquad = k\frac{1+2s}{2} r \cdot c_{j_{\mu\lambda}(k-1),k-1}(a_r), \\ (B^{-}_{\mu,\lambda})_k: \quad & \big(\partial + B^{-}_{\lambda k}(r)\big).c_{j_{\mu\lambda}(k),k+1}(a_r) = -\frac{1+2s}{2} r \cdot c_{j_{\mu\lambda}(k-1),k}(a_r), \end{aligned}$$

where

$$A^{-}_{\lambda k}(r) = -sr^2 - (\lambda_2 - k)$$

$$B^-_{\lambda k}(r) = -sr^2 - (\lambda_2 + 2d_\lambda - k) - 1.$$

Proposition 3.2.6. *When the Harish-Chandra parameter Λ of π belongs to Ξ_{III}, the system of differential equations in Proposition 3.1.1 is equivalent to the following system of difference-differential equations for the coefficient functions c_k's of $F|_A$.*

$$\begin{aligned}
(A^+_{\mu,\lambda})_k : \qquad & \big(\partial + A^+_{\lambda k}(r)\big).c_{j_{\mu\lambda}(k),k}(a_r) \\
& \qquad = -(1+2s)(j_{\mu,\lambda}(k)+1)r \cdot c_{j_{\mu\lambda}(k+1),k+1}(a_r), \\
(B^+_{\mu,\lambda})_k : \qquad & k\big(\partial + B^+_{\lambda k}(r)\big).c_{j_{\mu\lambda}(k),k-1}(a_r) \\
& \qquad = (d_\lambda - k + 1)(1+2s)(j_{\mu,\lambda}(k)+1)r \cdot c_{j_{\mu\lambda}(k+1),k}(a_r),
\end{aligned}$$

where

$$\begin{aligned}
A^+_{\lambda k}(r) &= sr^2 + (\lambda_2 - 2 - k) \\
B^+_{\lambda k}(r) &= sr^2 + (\lambda_2 + 2d_\lambda - k + 1).
\end{aligned}$$

The details of computations of the above two proposition are omitted. They are direct consequence of Proposition 3.2.3.

3.3. An explicit formula and the multiplicity one theorem

In the previous subsection we obtain the system of difference-differential equations satisfied by the coefficient functions c_k's of the algebraic generalized Whittaker function $F \in C^\infty_{\eta,\tau_\lambda}(R\backslash G/K)$ which comes from $l \in I_{\pi_\Lambda,\eta}$. In this subsection we solve this. Firstly we eliminate the difference term from the system and get the differential equation for each coefficient function c_k. After some calculation, we find that differential equations in question is equivalent to the classical Whittaker equations. The moderate growth condition for F transfered to the moderate growth condition for the classical Whittaker function. Hence we have an explicit formula of the generalized Whittaker functions and the multiplicity one theorem for the discrete series representation of $SU(2,1)$.

An explicit formula for coefficients. Now we are in a position to formulate the generalized Whittaker functions with analytic condition. Let us define *the generalized Whittaker model for the representation* π *of* G *with* K*-type* τ as follow

$$
\begin{aligned}
&Wh_{\eta}^{\tau}(\pi)\\
&:= \{F \in C_{\eta,\tau}^{\infty}(R\backslash G/K) \mid F|_A(a_r) \text{ is of moderate growth when } r \to \infty,\\
&\qquad\qquad l(v^*) = \langle v^*, F(\cdot)\rangle_K,\ l \in I_{\pi,\eta},\ v^* \in V_\tau^*\}.
\end{aligned}
$$

We call the elements in the space above *the generalized Whittaker functions associated to the representation* π *with* K*-type* τ. Here moderate growth means that the coefficient functions c_k's of $F|_A$ satisfy $|c_k(a_r)| < Cr^N$ for some constants $C, N > 0$.

We first work in the case of the large discrete series representation for the same reason with previous subsection.

Proposition 3.3.1. *Let* $\phi = F|_A$ *be a function in* $C^\infty(A\,; \mathcal{S}(\mathbb{R})\otimes_{\mathbb{C}} V_\lambda)$ *which comes from* $l \in \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi_\Lambda, \mathrm{Ind}_R^G\eta)$, $\Lambda \in \Sigma_{II}$. *Functions* c_k*'s are the coefficient functions of* ϕ *expanded with respect to the bases* $\{h_j\}$ *and* $\{v_k^\lambda\}$. *Then each* c_k $(0 \le k \le d_\lambda - 1)$ *satisfies the following differential equation.*

$$
(\Gamma_{\mu\lambda})_k^{II} \ :\ \{\partial^2 - (2d_\lambda + 4)\partial + G_k(r)\}.c_k(a_r) \ = \ 0,
$$

where

$$
\begin{aligned}
G_k(r) = &-s^2r^4 - \{2(\lambda_2 - k + d_\lambda - 1)s + (j_k+1)(1+2s)^2/2\}r^2\\
&- (k - 2d_\lambda - \lambda_2 - 2)(k - \lambda_2 + 2).
\end{aligned}
$$

Here we abbreviated $j_{\mu\lambda}(k)$ *as* j_k.

Proof. The task is elimination of the difference term from the system of difference-differential equations $(A_{\mu\lambda}^+)_k$ and $(B_{\mu\lambda}^-)_k$ $(k = 0, \ldots, d_\lambda - 1)$ obtained in Proposition 3.2.4. We neglect the suffix $j_{\mu\lambda}(k)$ since it does not contribute to this proof. Differentiate the equation $(A_{\mu\lambda}^+)_k$ by the Euler operator ∂, then we get

$$
(3.3.1)\quad \partial^2.c_k + \partial.A_k^+ \cdot c_k + A_k^+ \cdot \partial c_k = -(1+2s)(j_k+1)r(c_{k+1} + \partial.c_{k+1}).
$$

In order to cancel the term containing c_{k+1}, we add $(A^+_{\mu\lambda})_k$ multiplied by $(B^-_k - 1)$ and $(B^-_{\mu\lambda})_k$ multiplied by $-(1+2s)(j_k+1)r$ to the above formula (3.3.1). Then we have a differential equation for c_k of second order:

$$\partial^2.c_k+(A^+_k+B^-_k-1)\partial.c_k+(\partial.A^+_k-A^+_k+A^+_kB^-_k-(1+2s)^2(j_k+1)r^2/2)c_k \;=\; 0.$$

Compute the coefficient of $\partial.c_k$ and c_k using the form of A^+_k and B^-_k described in Proposition 3.2.4, we have the differential equation of Proposition. $\square$

As a result, we obtain an explicit formula of the coefficient functions c_k's of the minimal K-type generalized Whittaker functions for the large discrete series representations.

THEOREM 3.3.2. *The coefficient functions c_k's of the A-radial part of the minimal K-type generalized Whittaker functions F's $\in Wh^{\tau_\lambda}_\eta(\pi_\Lambda)$ for the large discrete series representations π_Λ's $(\Lambda \in \Sigma_{II})$ of $SU(2,1)$ are of the form*

$$c_k(a_r) \;=\; \gamma^{II}_k \times r^{d_\lambda+1} W_{\kappa,\frac{k-\lambda_1}{2}}(|s|r^2)$$

with parameters

$$\kappa = \{-(\lambda_2 - k + d_\lambda - 1)s - (j_k+1)(2s+1)^2/4\}/2|s|,$$

$k = 0,\ldots,d_\lambda$. Here variable a_r is an element of A, γ^{II}_k is a constant, $\lambda = (\lambda_1,\lambda_2)$ is the Blattner parameter which coincides with Harish-Chandra parameter $\Lambda \in \Sigma_{II} = \{(\lambda_1,\lambda_2) \in \mathbb{Z}^{\oplus 2} \mid \lambda_1 > 0 > \lambda_2\}$ in this case. For $j_k = j_{\mu\lambda}(k)$, see Proposition 3.2.3. $W_{\kappa,m}(x)$ is the classical Whittaker function which can be expressed by the integral

$$W_{\kappa,m}(x) \;=\; \frac{e^{-\frac{x}{2}}x^\kappa}{\Gamma(m-\kappa+\frac{1}{2})}\int_0^\infty t^{m-\kappa+\frac{1}{2}}(1+\frac{t}{x})^{m+\kappa-\frac{1}{2}}e^{-t}\frac{dt}{t}$$

when $\mathrm{Re}(m-\kappa+\frac{1}{2}) > 0$ *and* $x > 0$.

PROOF. Change the variable r by $\sqrt{x/|s|}$, and set

$$c_k(a_r) \;=\; \left\{\frac{x}{|s|}\right\}^{\frac{d_\lambda+1}{2}} u_k(x),$$

$k = 0, \ldots, d_\lambda$, then noting $\partial = r\frac{d}{dr}$, we find that the differential equations $(\Gamma_{\mu\lambda})_k^{II}$ turn into the classical Whittaker differential equations

$$\left\{\frac{d^2}{dx^2} + \left(-\frac{1}{4} + \frac{\{-(\lambda_2 - k + d_\lambda - 1)s - (j_k+1)(2s+1)^2/4\}/2|s|}{x} \right.\right.$$
$$\left.\left. + \frac{\frac{1}{4} - \{(d_\lambda+2)^2/4 + (k - 2d_\lambda - \lambda_2 - 2)(k - \lambda_2 + 2)/4\}}{x^2}\right)\right\} u_k(x) = 0.$$

Calculation shows that the coefficient of x^{-2} equals to $\frac{1}{4} - m^2$ with $m = (k - \lambda_1)/2$. We obtain the unique solution $W_{\kappa,m}(x)$ of moderate growth and the claimed formula for $c_k(a_r)$. $\square$

Here we normalize the constant multiples of c_k's. By the recurrence formula $(B^-_{\mu\lambda})_k$, we have

$$-\frac{1+2s}{2\gamma^{II}_{k+1}}c_k = \left(\frac{d}{dr} - sr - \frac{\lambda_2 + d_\lambda - k + 1}{r}\right).r^{d+1}W_{\kappa,\mu}(|s|r^2)$$
$$= 2sr^{d+2}W'_{\kappa,\mu}(|s|r^2) - sr^{d+2}W_{\kappa,\mu}(|s|r^2) + 2\mu r^d W_{\kappa,\mu}(|s|r^2).$$

Here $\mu = \frac{k-\lambda_1}{2}$. The differentiation formula

$$\frac{d}{dz}\left[e^{-z/2}z^{\mu-\frac{1}{2}}W_{\kappa,\mu}(|s|r^2)\right] = -e^{-z/2}z^{\mu-1}W_{\kappa+\frac{1}{2},\mu-\frac{1}{2}}(|s|r^2)$$

is satisfied by $W_{\kappa,\mu}(z)$ in general (*cf.* [M-O-S] p.302, l.1). From this we have

$$-\frac{1+2s}{2\gamma^{II}_{k+1}}c_k = -2\sqrt{s}\frac{c_k}{\gamma^{II}_k}.$$

Normalize the constants γ^{II}_k's by $\gamma^{II}_d = 1$, then we get

$$\gamma^{II}_k = \left(\frac{4\sqrt{s}}{1+2s}\right)^{d_\lambda - k}.$$

By using $(A^+_{\mu\lambda})_k$, we can get the same result after a little bit more complicated calculation.

Now let us discuss the cases of the holomorphic and the antiholomorphic discrete series representations. In these cases the differential equations satisfied by the coefficient functions are of the first order. Consequently the solutions are essentially exponential functions.

PROPOSITION 3.3.3. *Let $\phi = F|_A$ be a function in $C^\infty(A\,;\mathcal{S}(\mathbb{R})\otimes_{\mathbb{C}} V_\lambda)$ which comes from $l \in \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi_\Lambda, \mathrm{Ind}_R^G\eta)$. Functions c_k's are the coefficient functions of ϕ expanded with respect to the bases $\{h_n\}$ and $\{v_k^\lambda\}$. Then each c_k $(0 \le k \le d_\lambda + 1)$ satisfies the following differential equation*

$$(\Gamma_{\mu\lambda})_k^J \;:\; \{\partial + G_k(r)\}.c_k(a_r) \;=\; 0 \quad (J = I \text{ or } II),$$

where

$$G_k(r) = \begin{cases} -sr^2 - (\lambda_2 + k) & \text{when } \Lambda \in \Sigma_I; \\ sr^2 + (\lambda_2 + k) & \text{when } \Lambda \in \Sigma_{III}. \end{cases}$$

Hence we have an explicit formula of c_k's.

THEOREM 3.3.4. *The coefficient functions c_k's of the A-radial part of the minimal K-type generalized Whittaker functions F's $\in Wh_\eta^{\tau_\lambda}(\pi_\Lambda)$ for the holomorphic (resp. antiholomorphic) discrete series representations π_Λ's $(\Lambda \in \Sigma_I(resp.\Sigma_{III}))$ of $SU(2,1)$ are of the form*

$$c_k(a_r) \;=\; \gamma_k^I \times r^{\lambda_2+k} e^{sr^2/2},$$

$k = 0, \ldots, d_\lambda$ with $s < 0$, (resp.

$$c_k(a_r) \;=\; \gamma_k^{III} \times r^{-\lambda_2-k} e^{-sr^2/2},$$

$k = 0, \ldots, d_\lambda$ with $s > 0$). Here variable a_r is an element of A and γ_k^I, γ_k^{III} are constants.

By the same procedure in the large discrete series case, we have normalized constant multiples

$$\gamma_k^I = \left(\frac{4}{1+2s}\right)^{d_\lambda - k} (d_\lambda - k)!,$$

$$\gamma_k^{III} = \left(\frac{2}{1+2s}\right)^{k-1} \prod_{l=1}^{k} \frac{l}{j_l+1},$$

$k = 0, \ldots, d_\lambda$.

Remark. These explicit formulae of generalized Whittaker functions for the holomorphic and the antiholomolphic discrete series representations are compatible with the classical theory of Fourier-Jacobi expansion of holomorphic modular forms on $SU(2,1)$, or on the associated symmetric domain $SU(2,1)/K$. Since these results are well-known, we omitted the details of these cases in this paper. We just remark here that the conditions on the parameter s of the central character of ρ_{ψ_s} in Theorem 3.3.4 are the consequence of the moderate growth condition on $Wh_\eta^{\tau_\lambda}(\pi_\Lambda)$.

The multiplicity one theorem for the discrete series. Assemble the parts prepared in previous subsections, then we obtain simultaneously the multiplicity one theorem and an explicit form of elements in the minimal K-type generalized Whittaker model $Wh_\eta^{\tau_\lambda}(\pi)$ for the discrete series representations π's of $SU(2,1)$.

In order to formulate the multiplicity one theorem we have to introduce a $(\mathfrak{g}_{\mathbb{C}}, K)$-submodule $\mathcal{A}_\eta(R\backslash G)$ of $C_\eta^\infty(R\backslash G)$.

$$\mathcal{A}_\eta(R\backslash G) \ := \ \left\{ f \in C_\eta^\infty(R\backslash G) \ \middle| \ \begin{array}{l} c_{f,h} \text{ is right } K\text{-finite and} \\ c_{f,h}|_A(a_r) \text{ is of moderate growth} \\ \text{when } r \to \infty, \qquad \forall h \in (\eta, \mathcal{S}(\mathbb{R})) \end{array} \right\},$$

where $c_{f,h}$ is a $\mathbb{C}$-valued function on G defined as $c_{f,h}(g) := (f(g), h)_\eta$ and $c_{f,h}|_A$ is the A-radial part of $c_{f,h}$. Here $(\ ,\ )_\eta$ means the inner product on $L^2(\mathbb{R})$. It is easy to see that $\mathcal{A}_\eta(R\backslash G)$ is a $(\mathfrak{g}_{\mathbb{C}}, K)$-submodule of $C_\eta^\infty(R\backslash G)$. In fact, clearly it is a continuous K-submodule, hence stable under the action of $\mathfrak{k} = \mathrm{Lie}K$. The action of any element of $\mathfrak{a}$ also stabilizes this submodule. The action of $X \in \mathfrak{n}$ is given by

$$c_{f,h}(a_r \exp tX) \ = \ c_{f,h}(a_r \exp tX a_r^{-1} \cdot a_r) \ = \ \eta(\mathrm{Ad}(a_r)X).c_{f,h}(a_r).$$

Recall

$$R(E_1).c_{f,h}|_A(a_r) = r^2(\eta(E_1).c_{f,h})|_A(a_r),$$

$$R(E_{2,\pm}).c_{f,h}|_A(a_r) = r(\eta(E_{2,\pm}).c_{f,h})|_A(a_r).$$

Therefore the action of generators $E_1, E_{2,\pm}$ of $\mathfrak{n}$ also stabilizes this submodule. Hence it is stable under the action of the whole $\mathfrak{g}$.

THEOREM 3.3.5. *The discrete series representation π_Λ of $SU(2,1)$ of the Harish-Chandra parameter $\Lambda \in \Xi$ and the Blattner parameter $\lambda = (\lambda_1, \lambda_2) \in L_T^+$ has the multiplicity one property i.e.*

$$\dim_{\mathbb{C}}\mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\mathcal{H}^*_{\pi_\Lambda}, \mathcal{A}_{\eta_{\mu,\psi}}(R\backslash G)) \;=\; 1$$

if and only if

$$\frac{-\lambda_1 + 2\lambda_2}{3} \in \mathbb{Z}, \qquad \frac{-\lambda_1 + 2\lambda_2}{3} \le \frac{1}{2} - \mu.$$

Under this condition, the minimal K-type generalized Whittaker model $Wh_\eta^{\tau_\lambda}(\pi_\Lambda)$ of π_Λ has a basis $F_\eta^{\tau_\lambda}$ whose A-radial part is given as follows.

1) *When $\Lambda \in \Xi_{II}$ (i.e. π_Λ is a large discrete series representation),*

$$F_\eta^{\tau_\lambda}(a_r) \;=\; \sum_{k=0}^{d_\lambda} \gamma_k^{II} r^{d_\lambda+1} W_{\kappa,\frac{k-\lambda_1}{2}}(|s|r^2) \cdot \Big(h_{j_{\mu\lambda}(k)} \otimes v_k^\lambda\Big),$$

where

$$\kappa = \{-(\lambda_2 - k + d_\lambda - 1)s - (j_{\mu\lambda}(k) + 1)(2s+1)^2/4\}/2|s|.$$

2) *When $\Lambda \in \Xi_I$ (i.e. π_Λ is a holomorphic discrete series representation),*

$$F_\eta^{\tau_\lambda}(a_r) \;=\; \sum_{k=0}^{d_\lambda} \gamma_k^{I} r^{\lambda_2+k} e^{sr^2/2} \cdot \Big(h_{j_{\mu\lambda}(k)} \otimes v_k^\lambda\Big),$$

where $s < 0$.

3) *When $\Lambda \in \Xi_{III}$ (i.e. π_Λ is an antiholomorphic discrete series representation),*

$$F_\eta^{\tau_\lambda}(a_r) \;=\; \sum_{k=0}^{d_\lambda} \gamma_k^{III} r^{-\lambda_2-k} e^{-sr^2/2} \cdot \Big(h_{j_{\mu\lambda}(k)} \otimes v_k^\lambda\Big),$$

where $s > 0$.
Here $r \in \mathbb{R}_{>0}$, *and the index of each basis* h_j *of* η *is*

$$j_{\mu\lambda}(k) = k - \frac{2\lambda_1 - \lambda_2}{3} - \frac{1}{2} - \mu.$$

Proof. The generalized Whittaker functions $F \in Wh_\eta^{\tau_\lambda}(\pi_\Lambda)$ correspond to the generalized Whittaker functionals $l \in \mathrm{Hom}_{(\mathfrak{g}_\mathbb{C},K)}(\mathcal{H}_{\pi_\Lambda}^*, \mathcal{A}_{\eta_{\mu,\psi}}(R\backslash G))$ are characterized as the solutions of (D) in Proposition 3.1.1 which satisfy the moderate growth condition. As we saw before the system of differential equations (D) turn into differential equations $(\Gamma_{\mu\lambda})_k^J$ $(J = I,\ II,\ III)$ for coefficient functions c_k's of F in Proposition 3.3.1 and Proposition 3.3.3. The moderate growth condition on F is translated into that the coefficient functions are of moderate growth when r tends to infinity. Recalling that the coefficient function is not a trivial function if and only if

$$\frac{-\lambda_1 + 2\lambda_2}{3} \in \mathbb{Z}, \qquad \frac{-\lambda_1 + 2\lambda_2}{3} \leq \frac{1}{2} - \mu, \tag{3.3.2}$$

we know that differential equation $(\Gamma_{\mu\lambda})_k^J$ has a non-trivial moderate growth solution exactly when (3.3.2) holds. Hence we have the first half of the assertion of Theorem. The rest is obvious. Indeed it is a direct consequence of Theorem 3.3.2 and Theorem 3.3.4. Just arrange c_k's into

$$F(a_r) \ = \ \sum_{k=0}^{d_\lambda} \gamma_k^J c_k(a_r)\big(h_j \otimes v_k^\lambda\big),$$

we have a basis of $Wh_\eta^{\tau_\lambda}(\pi_\Lambda)$ of the form above. □

4. The Case of Principal Series Representations

The preceding section was devoted to the discrete series case. We now study in this section the principal series case and give an explicit formula of the corner K-type generalized Whittaker functions. In this case, we can obtain the differential equation satisfied by the generalized Whittaker function in a much simpler way than the case of the discrete series representations. We have only to calculate the A-radial part of the Casimir operator explicitly in terms of coefficient functions.

4.1. Radial part of the Casimir operator

In this subsection we give the A-radial part of the Casimir operator. Take

$$\{\theta(E_1), \theta(E_{2,+}), \theta(E_{2,-}), H, M_1, E_1, E_{2,+}, E_{2,-}\}$$

as a basis of $\mathfrak{g}$ where $M_1 = \sqrt{-1}(H'_{13} - H'_{23})$. Then the Casimir operator can be defined by

$$\Omega \;=\; \frac{1}{2}H^2 - \frac{1}{6}M_1^2 - \frac{1}{2}\{E_1\theta(E_1) + E_{2,+}\theta(E_{2,+}) + E_{2,-}\theta(E_{2,-})\} - 2H_1$$

with appropriate normalization of the Killing form. The eigen-value of Ω can be calculated as $\frac{1}{2}\nu^2 + \frac{1}{6}{\lambda_0}^2 - 2$ (*cf.* [K-O] 7.3 p.997).
The Casimir operator $\Omega \in U(\mathfrak{g}_{\mathbb{C}})$ defines a unique differential operator $R(\Omega)$ on the image of the restriction map $\mathrm{res}_A : C^\infty_{\eta,\tau}(R\backslash G/K) \to C^\infty(A\,; \mathcal{S}(\mathbb{R})\otimes_{\mathbb{C}} V_\tau)$ by

$$R(\Omega).\phi \;=\; (\Omega.\varphi)|_A.$$

Here we used the notation in subsection 3.2; ϕ means the restriction of $\varphi \in C^\infty_{\eta,\tau}(R\backslash G/K)$ to A. We call $R(\Omega)$ *the A-radial part* of the Casimir operator Ω, which is given as follows.

Proposition 4.1.1. *Let $\phi \in C^\infty(A\,; \mathcal{S}(\mathbb{R}) \otimes_{\mathbb{C}} V_\tau)$ be the A-radial part of $\varphi \in C^\infty_{\eta,\tau}(R\backslash G/K)$. Then the radial part $R(\Omega)$ of Ω is given by*

$$\begin{aligned} R(\Omega).\phi =& \frac{1}{2}\{\partial^2 - 4\partial + \frac{1}{3}{\lambda_0}^2 + r^4\eta(E_1)^2 \\ & - 2\sqrt{-1}r^2\eta(E_1)\tau(H'_{13}) + r^2(\eta(E_{2,+})^2 + \eta(E_{2,-})^2) \\ & + 2r\eta(E_{2+})\tau(X_{\beta_{12}} + X_{\beta_{21}}) + 2\sqrt{-1}r\eta(E_{2-})\tau(X_{\beta_{12}} - X_{\beta_{21}})\}\phi. \end{aligned}$$

Proof. Decompose the opposite $\theta(E)$ of E as

$$\theta(E) \;=\; (\theta(E) + E) - E.$$

Here the symbol E represents either E_1 or $E_{2,\pm}$. Noting $\theta(E) + E \in \mathfrak{k}$, $E \in \mathfrak{n}$, we have

$$R(\theta(E)).\phi(a_r) \;=\; \tau(\theta(E) + E).\phi(a_r) - R(E).\phi(a_r).$$

The action of $\mathfrak{k}$ can be computed directly

$$\theta(E_1) + E_1 = 2iH'_{13},$$
$$\theta(E_{2,+}) + E_{2,+} = -2X_{\beta_{12}} - 2X_{\beta_{21}},$$
$$\theta(E_{2,-}) + E_{2,-} = -2iX_{\beta_{12}} + 2iX_{\beta_{21}}.$$

Recall that the A-radial parts of E, H and M_1 are given as

$$R(E_1).\phi(a_r) = r^2(\eta(E_1).\varphi)|_A(a_r), \qquad R(E_{2,\pm}).\phi(a_r) = r(\eta(E_{2,\pm}).\varphi)|_A(a_r),$$
$$R(H).\phi(a_r) = \partial.\phi|_A(a_r), \qquad R(M_1).\phi(a_r) = \sqrt{-1}\lambda_0 \cdot \phi|_A(a_r),$$

respectively. Then we get the assertion of Proposition. □

4.2. An explicit formula and the multiplicity one theorem

Let F be the generalized Whittaker function associated to the principal series representation $\pi_{\lambda_0,\nu}$ with the corner K-type $\tau_{(-\lambda_0,-\lambda_0)}$ corresponding to the generalized Whittaker functional $l \in I_{\pi_{\lambda_0,\nu},\eta}^{\tau_{(-\lambda_0,-\lambda_0)}}$. Since the function $l(v^*) = \langle v^*, F(\ )\rangle_K \in \mathrm{Ind}_R^G\eta$, $v^* \in \tau^*_{(-\lambda_0,-\lambda_0)}$ is an eigen-vector for the Casimir operator Ω with eigen-value $\frac{1}{2}\nu^2 + \frac{1}{6}{\lambda_0}^2 - 2$, F satisfies the differential equation

$$\Omega.F \ = \ (\frac{1}{2}\nu^2 + \frac{1}{6}{\lambda_0}^2 - 2)F \tag{4.2.1}$$

accordingly.

Because of the one-dimensionality of the corner K-type of $\pi_{\lambda_0,\nu}$, a function F which comes from $l \in \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi_{\lambda_0,\nu}, \mathrm{Ind}_R^G\eta)$ can be expanded as

$$F(g) \ = \ c_0(g)\big(h_{j_0} \otimes v_0\big),$$

where v_0 is a fixed generator of $V_{\tau_{(-\lambda_0,-\lambda_0)}}$ and h_{j_0} is the basis of $\mathcal{S}(\mathbb{R})$ with $j_0 = -\frac{\lambda_0}{3} - \frac{1}{2} - \mu$ (*cf.* subsection 3.2). Then we have the differential equation for the A-radial part of c_0.

PROPOSITION 4.2.1. *Let c_0 be the coefficient function of the A-radial part of F which comes from $l \in \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\tau_{(-\lambda_0,-\lambda_0)}, \mathrm{Ind}_R^G\eta)$. Then c_0 satisfies the differential equation*

$$(\Gamma_\nu)_0^{PS} \ : \ \{\partial^2 - 4\partial + G(r)\}.c_0(a_r) \ = \ 0$$

with

$$G(r) = -s^2r^4 - \{2\lambda_0 s + (2j_0+1)(1+4s^2)/2\}r^2 - (\nu^2-4).$$

PROOF. The assertion of Proposition is an immediate consequence of (4.2.1) and Proposition 4.1.1. Indeed we first remark that the third line of the expression of $R(\Omega)$ in Proposition 4.1.1 does not contribute to the action on c_0, since $\tau_{(-\lambda_0,-\lambda_0)}(X_{\beta_{12}}) = \tau_{(-\lambda_0,-\lambda_0)}(X_{\beta_{21}}) = 0$ for one dimensional representation $\tau_{(-\lambda_0,-\lambda_0)}$. We need only

$$\tau(H'_{13}).v_0 \;=\; -\lambda_0 v_0.$$

Secondly using Lemma 2.3.3, we can see

$$\eta(E_1)^2.h_j = -s^2 h_j,$$

$$\eta(E_{2,+})^2.h_j = \frac{1}{4}h_{j+2} - \frac{2j+1}{2}h_j + j(j-1)h_{j-2}, \tag{4.2.2}$$

$$\eta(E_{2,-})^2.h_j = -s^2h_{j+2} - 2(2j+1)s^2h_j - 4j(j-1)s^2h_{j-2}, \tag{4.2.3}$$

by direct computation. Again the one-dimensionality of the corner K-type makes the η-action above much simpler. It forces all the terms but the middle ones of the right hand side of (4.2.2), (4.2.3) vanish. Therefore we have $(\Gamma_\nu)_0^{PS}$. $\square$

We solve $(\Gamma_\nu)_0^{PS}$ and obtain an explicit form of the coefficient function of the corner K-type generalized Whittaker function.

THEOREM 4.2.2. *The coefficient functions c_0 of the A-radial part of the corner K-type generalized Whittaker functions F's $\in Wh_\eta^{\tau_{(-\lambda_0,-\lambda_0)}}(\pi_{\lambda_0,\nu})$ for the principal series representations $\pi_{\lambda_0,\nu}$'s of $SU(2,1)$ are of the form*

$$c_0(a_r) \;=\; (\text{const.}) \times rW_{\kappa,\frac{\nu}{2}}(|s|r^2)$$

with parameters

$$\kappa = \{-\lambda_0 s - (2j_0+1)(4s^2+1)/4\}/2|s|.$$

Here a_r is an element of A, ν is the infinitesimal character of $\pi_{\lambda_0,\nu}$, $j_0 = -\frac{\lambda_0}{3} - \frac{1}{2} - \mu$, and $W_{\kappa,m}(x)$ is the classical Whittaker function.

PROOF. The procedure is quite similar to the case of the large discrete series representation (Theorem 3.3.2). We just change the variable r by $\sqrt{x/|s|}$, and set

$$c_0(a_r) \ = \ \left\{ \frac{x}{|s|} \right\}^{\frac{1}{2}} u_0(x)$$

in order to transform $(\Gamma_\nu)_0^{PS}$ into the standard form

$$\left\{ \frac{d^2}{dx^2} + \left(-\frac{1}{4} + \frac{\{-\lambda_0 s - (2j_0+1)(4s^2+1)/4\}/2|s|}{x} \right.\right. \\ \left.\left. + \frac{\frac{1}{4} - (\frac{\nu}{2})^2}{x^2} \right) \right\} u_0(x) \ = \ 0. \ \square$$

Here is a variant of the recent result of Tsuzuki which can be considered to be an analogue of Proposition 3.1.1.

PROPOSITION 4.2.3 ([Tsu] Theorem 9.2.1). *Let $\pi_{\lambda_0,\nu}$ be an irreducible principal series representation of G with the corner K-type $\tau_{(-\lambda_0,-\lambda_0)}$, the infinitesimal character ν and η be the representation in subsection 2.3. Then the image of $\mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi^*_{\lambda_0,\nu}, \mathrm{Ind}_R^G \eta)$ by the correspondence of subsection 1.2 in $C^\infty_{\eta,\tau_{(-\lambda_0,-\lambda_0)}}(R\backslash G/K)$ is characterized by*

$$R(\Omega).F \ = \ (\frac{1}{2}\nu^2 + \frac{1}{6}{\lambda_0}^2 - 2) \cdot F.$$

In short

$$I_{\pi_{\lambda_0,\nu}}^{\tau_{(-\lambda_0,-\lambda_0)}} \ \cong \ Ker\,(R(\Omega) - \frac{1}{2}\nu^2 - \frac{1}{6}{\lambda_0}^2 + 2),$$

where $I_{\pi_{\lambda_0,\nu}}^{\tau_{(-\lambda_0,-\lambda_0)}}$ is the intertwining space $\mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\tau^*_{(-\lambda_0,-\lambda_0)}, \mathrm{Ind}_R^G \eta)$.

The multiplicity one theorem for the principal series. By virtue of Proposition 4.2.3, we have the next multiplicity one result for the principal series representations.

THEOREM 4.2.4. *The irreducible principal series representation* $\pi_{\lambda_0,\nu}$ *of* $SU(2,1)$ *has the multiplicity one property i.e.*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi^*_{\lambda_0,\nu}, \mathcal{A}_{\eta_{\mu,\psi}}(R\backslash G)) \;=\; 1$$

if and only if

$$-\frac{\lambda_0}{3} - \frac{1}{2} - \mu \in \mathbb{Z}_{\geq 0}. \tag{4.2.4}$$

Under this condition, the corner K*-type generalized Whittaker model* $Wh_\eta^{\tau(-\lambda_0,-\lambda_0)}(\pi_{\lambda_0,\nu})$ *has a basis* $F_\eta^{\tau(-\lambda_0,-\lambda_0)}$ *whose* A*-radial part is given by*

$$F_\eta^{\tau(-\lambda_0,-\lambda_0)}(a_r) \;=\; rW_{\kappa,\frac{\nu}{2}}(|s|r^2)\cdot\Big(h_{j_0}\otimes v_0\Big),$$

where

$$\kappa = \{-\lambda_0 s - (2j_0+1)(4s^2+1)/4\}/2|s|.$$

Here $r \in \mathbb{R}_{>0}$*, and the index of the basis* h_{j_0} *of* η *is*

$$j_0 = -\frac{\lambda_0}{3} - \frac{1}{2} - \mu. \tag{4.2.5}$$

PROOF. The linear relation (3.2.1) of indices j and k turns into (4.2.5). From the fact that $j_0 \in \mathbb{Z}_{\geq 0}$, we have (4.2.4) as the necessary condition. Conversely when (4.2.5) is valid, we can built up F of Proposition 4.2.3 by defining its A-radial part via

$$F(a_r) \;=\; rW_{\kappa,m}(|s|r^2)\cdot\Big(h_{j_0}\otimes v_0\Big),$$

which satisfies the moderate growth condition. □

5. The Fourier Expansion

Having explicit formulae of Whittaker and generalized Whittaker functions for the standard representations of $SU(2,1)$ at our disposal, we are ready to develop the theory of Fourier expansion of automorphic forms on $SU(2,1)$ belonging to arbitrary standard representations.

5.1. The formulation of the Fourier expansion

We first formulate the Fourier expansion of automorphic forms by using the spectral decomposition theory. The following argument works for an arbitrary non-cocompact discrete subgroup Γ of G. But, for the sake of simplicity, we assume that Γ is a congruence subgroup such that $N_\Gamma = N \cap \Gamma$ contains $(1,0;0)$ and $(0,1;0)$ in the exponential coordinate of N. For example group $\Gamma = G \cap SL_3(\mathbb{Z}[i])$ satisfies the condition. In fact, for this Γ, N_Γ contains elements

$$(x,y;z) \;=\; \begin{pmatrix} 1-\frac{1}{2}|w|^2+(xy+z)i & -w & \frac{1}{2}|w|^2-(xy+z)i \\ \overline{w} & 1 & -\overline{w} \\ -\frac{1}{2}|w|^2+(xy+z)i & -w & 1+\frac{1}{2}|w|^2-(xy+z)i \end{pmatrix},$$

$w = x+yi \in (1+i)\mathbb{Z}[i]$, $z \in \mathbb{Z}$. Here i means $\sqrt{-1}$ and $|w|^2 = x^2+y^2$. Let Φ be an automorphic form on G with respect to Γ belonging to π with K-type τ.

As $N_\Gamma \backslash N$ is compact, the irreducible decomposition of the right regular representation Reg_N of N on $L^2(N_\Gamma\backslash N)$ is given by

$$L^2(N_\Gamma\backslash N) \;=\; \bigoplus_{\sigma\in\widehat{N}} m_\sigma \cdot \mathcal{S}_\sigma,$$

where $(\sigma, \mathcal{S}_\sigma)$ is an N_Γ-invariant unitary representation of N and m_σ is the multiplicity of the representation σ in Reg_N. This reads that a naive Fourier expansion of Φ along N should be

$$\Phi(ng) \;=\; \sum_{\sigma\in\widehat{N}} \sum_{i=1}^{m_\sigma} F^{\pi,\tau}_{\sigma,(i)}(ng),$$

where σ runs through the N_Γ-invariant unitary representations of N. Here $F^{\pi,\tau}_{\sigma,(i)}$ is a smooth function in $n \in N$ belonging to the i-th copy $\mathcal{S}^{(i)}_\sigma$ of $\mathcal{S}_\sigma$ $(i = 1,\ldots,m_\sigma)$. As we saw in Proposition 2.3.1, the unitary dual $\widehat{N}$ of N is exhausted by unitary characters $\psi_{u,v}$'s parameterized by $(u,v) \in \mathbb{R}^2$ and infinite-dimensional irreducible unitary representations ρ_{ψ_s}'s determined by their nontrivial central characters ψ_s's: Stone von Neumann representations. When σ is a unitary character, its multiplicity $m_{\psi_{u,v}}$ is one. Hence we have

$$\Phi(ng) \;=\; \sum_{\psi_{u,v}} F^{\pi,\tau}_{\psi_{u,v}}(ng) + \sum_{\rho_{\psi_s}} \sum_{i=1}^{m_\rho} F^{\;\pi,\tau}_{\rho_{\psi_s},(i)}(ng).$$

Expanding this with respect to the standard basis $\{v_k^\tau\}_{k=0}^{d_\tau}$ of V_τ, we have

$$\Phi(ng) \;=\; \sum_{(u,v)} \Big(\sum_{k=0}^{d_\tau} f_{\psi_{u,v}}^{\pi,\tau;k}(ng) v_k^\tau \Big) + \sum_{s} \sum_{i=1}^{m_\rho} \Big(\sum_{k=0}^{d_\tau} f_{\rho_{\psi_s},(i)}^{\pi,\tau;k}(ng) v_k^\tau \Big).$$

Parameters (u,v) and s run through discrete subsets of $\mathbb{R}^2$ and $\mathbb{R}\backslash\{0\}$ respectively, which are determined by N_Γ-invariantness.

But this formulation fails in general as we mentioned in introduction. When σ is a Stone von Neumann representation ρ_{ψ_s}, the generalized Gelfant-Graev representation $\mathrm{Ind}_N^G \rho_{\psi_s}$, where the function $f_{\rho_{\psi_s},(i)}^{\pi,\tau;k}$ belongs, is huge in the meaning below. To investigate the function $f_{\rho_{\psi_s},(i)}^{\pi,\tau;k}$ is to study the intertwining space $\mathrm{Hom}_N(\pi^*|_N, \rho_{\psi_s})$ which is isomorphic to $\mathrm{Hom}_G(\pi^*, \mathrm{Ind}_N^G \rho_{\psi_s})$ by Frobenius reciprocity. However this space is infinite-dimensional and uncontrollable.

Now we remove the difficulty above and give a correct formulation of the Fourier expansion of automorphic forms. Because of the next identifications (see Lemma 2.3.2)

$$\begin{aligned}
\mathrm{Ind}_N^G \rho_{\psi_s} &\cong \mathrm{Ind}_R^G \Big(\mathrm{Ind}_N^{\widetilde{R}} \rho_{\psi_s} \Big)^{\ker(\widetilde{R} \to R)} \\
&\cong \mathrm{Ind}_R^G \Big(\mathrm{Reg}_{\widetilde{S}} \otimes (\omega_{\psi_s} \times \rho_{\psi_s})|_{\widetilde{R}} \Big)^{\ker(\widetilde{R} \to R)} \\
&\cong \mathrm{Ind}_R^G \Big(\oplus_{\mu \in \frac{1}{2}\mathbb{Z}\backslash\mathbb{Z}} \widetilde{\chi}_\mu \otimes (\omega_{\psi_s} \times \rho_{\psi_s})|_{\widetilde{R}} \Big) \\
&\cong \bigoplus_{\mu \in \frac{1}{2}\mathbb{Z}\backslash\mathbb{Z}} \mathrm{Ind}_R^G \Big(\widetilde{\chi}_\mu \otimes (\omega_{\psi_s} \times \rho_{\psi_s})|_{\widetilde{R}} \Big),
\end{aligned}$$

the intertwining space in question decomposes as

$$\mathrm{Hom}_G(\pi^*, \mathrm{Ind}_N^G \rho_{\psi_s}) \;\cong\; \bigoplus_{\mu \in \frac{1}{2}\mathbb{Z}\backslash\mathbb{Z}} \mathrm{Hom}_G(\pi^*, \mathrm{Ind}_R^G \eta_{\tilde{\chi}_\mu \psi_s}).$$

We abbreviated $\widetilde{\chi}_\mu \otimes (\omega_{\psi_s} \times \rho_{\psi_s})|_{\widetilde{R}}$ as $\eta_{\tilde{\chi}_\mu \psi_s}$. Accordingly, the function $f_{\rho_{\psi_s},(i)}^{\pi,\tau;k}$ can be expressed as

$$f_{\rho_{\psi_s},(i)}^{\pi,\tau;k} \;=\; \sum_{\mu \in \frac{1}{2}\mathbb{Z}\backslash\mathbb{Z}} f_{\eta_{\tilde{\chi}_\mu \psi_s},(i)}^{\pi,\,\tau;k},$$

where $f^{\pi,\tau;k}_{\eta_{\tilde{\chi}_\mu\psi_s},(i)}$'s are functions in the space of $\mathrm{Ind}_R^G\eta_{\tilde{\chi}_\mu,\psi_s}$. Note that the space

$$\mathrm{Hom}_G(\pi^*, \mathrm{Ind}_R^G\eta_{\tilde{\chi}_\mu,\psi_s})$$

is the space of the generalized Whittaker functionals and the dimension of it is always at most one. Hence we have a fine expansion:

$$\Phi(ng) \;=\; \sum_{(u,v)}\Big(\sum_{k=0}^{d_\tau} f^{\pi,\tau;k}_{\psi_{u,v}}(ng)v_k^\tau\Big) + \sum_s\sum_{i=1}^{m_\rho}\Big(\sum_{k=0}^{d_\tau}\sum_{\mu\in\frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}} f^{\pi,\tau;k}_{\eta_{\tilde{\chi}_\mu\psi_s},(i)}(ng)v_k^\tau\Big).$$

Here is *the Fourier expansion of a V_τ-valued automorphic form Φ on G along N*:

$$\begin{aligned}\Phi(ng) = &\sum_{(u,v)}\Big(\sum_{k=0}^{d_\tau} c^{\Phi,\tau;k}_{\psi_{u,v}}(g)\cdot\psi_{u,v}(n)v_k^\tau\Big)\\ &+\sum_s\sum_{i=1}^{m_\rho}\Big(\sum_{k=0}^{d_\tau}\sum_{\mu\in\frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}}\sum_{j\in\mathbb{N}} c^{\Phi,\tau;k}_{\eta_{\tilde{\chi}_\mu\psi_s}(i);j}(g)\cdot\theta_j^{\rho_{\psi_s}(i)}(n)v_k^\tau\Big),\end{aligned}$$

where $\{\theta_j^{\rho_{\psi_s}(i)}\}_{j\in\mathbb{N}}$ is a basis of the i-th copy $\mathcal{S}^{(i)}_{\rho_{\psi_s}}\subset L^2(N_\Gamma\backslash N)$ of $\mathcal{S}_{\rho_{\psi_s}}$.

5.2. Generalized theta functions

In the explicit formulae of generalized Whittaker functions obtained in subsection 3.3 and subsection 4.2, the Hermitian functions h_j appeared, as the consequence of our choice of a basis of the Shrödinger model $(\rho_{\psi_s}, \mathcal{S}(\mathbb{R}))$ of a Stone von Neumann representation σ of N. In this subsection, we construct $\theta_j^{\rho_{\psi_s}(i)}$ in $L^2(N_\Gamma\backslash N)$ as the image of h_j by an N-intertwiner

$$T:\ \mathcal{S}(\mathbb{R})\to L^2(N_\Gamma\backslash N).$$

In other word, we realize a Stone von Neumann representation σ in $L^2(N_\Gamma\backslash N)$. For the purpose, we write down the Hermite descending operator by elements of $U(\mathfrak{n})$ and translate it by T. Then we have the differential equation which is satisfied by the image $\theta_0^{\rho_{\psi_s}(i)} = T(h_0)$ of h_0. Using quasi-periodicity of $\theta_0^{\rho_{\psi_s}(i)}$, which comes from N_Γ-invariantness, we can solve the differential equation and obtain an explicit form of $\theta_0^{\rho_{\psi_s}(i)}$. Finally, by using the raising operator recursively, we obtain the following form of $\theta_j^{\rho_{\psi_s}(i)} = T(h_j)$. Essentially the same argument can be found in [Mum].

THEOREM 5.2.1. (1) *The image of N-intertwiner T in $L^2(N_\Gamma\backslash N)$ is zero unless the central character's parameter s of the Schrödinger model ρ_{ψ_s} is of the form*

$$s = 2\pi\ell,$$

where ℓ is a non-zero integer.
(2) *Assume $s = 2\pi\ell$ ($\ell \in \mathbb{Z}\backslash\{0\}$), then there are $2|\ell|$ non-trivial N-intertwiners $T^{(i)}: \mathcal{S}(\mathbb{R}) \to L^2(N_\Gamma\backslash N)$, where $i = 1,\ldots,2|\ell|$. Moreover we can take*

$$\theta_0^{\ell;(i)}(x,y;z) = \sum_{k\in\mathbb{Z}} e^{-(x+\frac{i+2|\ell|k}{2\ell})^2/2} \cdot \mathrm{e}[(i+2|\ell|k)y + \ell xy + \ell z] \tag{5.2.1}$$

as the image $\theta_0^{\ell;(i)}$ of h_0 by $T^{(i)}$. As for the image $\theta_j^{\ell;(i)}$ of h_j by $T^{(i)}$,

$$\theta_j^{\ell;(i)}(x,y;z) = \sum_{k\in\mathbb{Z}} h_j\Big(x + \frac{i+2|\ell|k}{2\ell}\Big) \cdot \mathrm{e}[(i+2|\ell|k)y + \ell xy + \ell z] \tag{5.2.2}$$

can be taken. Here we denote $\mathrm{e}^{2\pi\sqrt{-1}X}$ by $\mathrm{e}[X]$ and $\exp(xE_{2,+}+yE_{2,-}+zE_1)$ by $(x,y;z)$ (the exponential coordinate on N; see subsection 2.1).

PROOF. The generator E_1 of the center of $\mathfrak{n}$ acts on $\theta \in L^2(N_\Gamma\backslash N)$ by multiplying $\sqrt{-1}s$; the derivative of ψ_s at 0. By the relation (2.1.1),

$$\begin{aligned}&\Big(\mathrm{Reg}_N(\exp tE_{2,+}).\theta\Big)\Big(\exp(xE_{2,+} + yE_{2,-} + zE_1)\Big)\\ &\quad = \theta\Big(\exp((x+t)E_{2,+} + yE_{2,-} + (z-yt)E_1)\Big),\end{aligned}$$

hence we have

$$\mathrm{Reg}_N(E_{2,+}) = \frac{\partial}{\partial x} - \sqrt{-1}sy.$$

Similarly

$$\mathrm{Reg}_N(E_{2,-}) = \frac{\partial}{\partial y} + \sqrt{-1}sx.$$

Note that the zero-th Hermite function $h_0(\xi)$ is annihilated by the descending operator $a = \xi + \frac{d}{d\xi}$. This operator is written as

$$a \;=\; \frac{1}{2\sqrt{-1}s}\rho_{\psi_s}(E_{2,-}) + \rho_{\psi_s}(E_{2,+})$$

on the Srödinger model $(\rho_{\psi_s}, \mathcal{S}(\mathbb{R}))$, accordingly

$$a \;=\; \frac{1}{2\sqrt{-1}s}(\frac{\partial}{\partial y} + \sqrt{-1}sx) + (\frac{\partial}{\partial x} - \sqrt{-1}sy)$$

on the realization of σ in $L^2(N_\Gamma \backslash N)$. Therefore $\theta_0^{\rho_{\psi_s},(i)}$ satisfies the differential equation

$$\text{(5.2.3)} \quad \Big\{(\frac{\partial}{\partial x} - \sqrt{-1}sy) + \frac{1}{2\sqrt{-1}s}(\frac{\partial}{\partial y} + \sqrt{-1}sx)\Big\}.\theta_0^{\rho_{\psi_s},(i)}\Big)(x, y; z) \;=\; 0.$$

On the coordinate E_1, we can separate the variable z as

$$\theta_0^{\rho_{\psi_s},(i)}(x, y; z) \;=\; G_0(x, y) \cdot e^{\sqrt{-1}sz}.$$

The N_Γ-invariantness of $\theta_0^{\rho_{\psi_s},(i)}$ requires that the parameter s should be of the form $s = 2\pi\ell$, $\ell \in \mathbb{Z}$, and that the function G_0 should satisfy a quasi-periodicity;

$$\text{(5.2.4)} \qquad G_0(x + m, y + n) \;=\; G_0(x, y) \cdot e^{\sqrt{-1}s(nx - my)},$$

for $n, m \in \mathbb{Z}$ (*cf.* 2.1.1). Set $\widetilde{G}_0(x, y)e^{2\pi\sqrt{-1}\ell xy} = G_0(x, y)$, then the differential equation (5.2.3) and the quasi-periodicity (5.2.4) turn into

$$\text{(5.2.5)} \qquad \Big\{\frac{\partial}{\partial x} + x + \frac{1}{2\sqrt{-1}s}\frac{\partial}{\partial y}\Big\}\widetilde{G}_0(x, y) \;=\; 0,$$

$$\text{(5.2.6)} \qquad \widetilde{G}_0(x + m, y + n) \;=\; \widetilde{G}_0(x, y) \cdot e^{-4\pi\sqrt{-1}\ell my},$$

for $n, m \in \mathbb{Z}$. By the assumption on Γ, $\widetilde{G}_0(x, y)$ is periodic in variable y with period 1. Partial Fourier expansion of $\widetilde{G}_0$ in y tells

$$\widetilde{G}_0(x, y) \;=\; \sum_{k' \in \mathbb{Z}} g_{0,k'}(x)e^{2\pi\sqrt{-1}k'y}.$$

In terms of $g_{0,k'}$, (5.2.5) and (5.2.6) fall into

$$\begin{aligned} \Big\{\frac{d}{dx}+x+\frac{k'}{2\ell}\Big\}g_{0,k'}(x) &= 0, \\ g_{0,k'}(x+m) &= g_{0,k'+2\ell m}(x), \end{aligned}$$

for all $m \in \mathbb{Z}$. By taking $m = 1$, which is possible because of our choice of Γ, we know that the solution space for (5.2.5) is of dimension $2|\ell|$ and has a basis consisting of

$$\sum_{k\in\mathbb{Z}} e^{-(x+\frac{i+2|\ell|k}{2\ell})^2/2} \cdot \mathrm{e}[(i+2|\ell|k)y],$$

$i = 1, \ldots, 2|\ell|$. This gives (5.2.1) and the claim that there are $2|\ell|$ non-trivial N-intertwiners $T^{(i)}$'s.

The raising operator $a^\dagger = \xi - \frac{d}{d\xi}$ is written as

$$a^\dagger = \frac{1}{2\sqrt{-1}s}\Big(\frac{\partial}{\partial y}+\sqrt{-1}sx\Big) - \Big(\frac{\partial}{\partial x}-\sqrt{-1}sy\Big).$$

The action of $a^\dagger$ on the k'-th term ($k' = i + 2|\ell|k$)

$$e^{-\frac{(x+k'/2\ell)^2}{2}} \cdot \mathrm{e}[k'y + \ell xy + \ell z]$$

of $\theta_0^{\ell,(i)}$ becomes the action on $e^{-\frac{(x+k'/2\ell)^2}{2}} = h_0(x + k'/2\ell)$:

$$a^\dagger.h_0(\xi) = -\frac{d}{d\xi}h_0(\xi) + \xi \cdot h_0(\xi),$$

where $\xi = x + k'/2\ell$. This is equal to $h_1(\xi) = h_1(x + k'/2\ell)$ by definition of the Hermite function. Recursively, we have the expression (5.2.2) of $\theta_j^{\ell,(i)}$. $\square$

Proposition 5.2.2. *Theta functions constructed above satisfy the orthogonality;*

$$(\theta_j^{\ell,(i)}, \theta_{j'}^{\ell,(i')})_{L^2} = 2^j j!\sqrt{\pi}\delta_{jj'}\delta_{ii'}.$$

Here δ_{mn} denotes the Kronecker symbol and $(\ ,\ )_{L^2}$ means the inner product on $L^2(N_\Gamma \backslash N)$ normalized as

$$(f,g)_{L^2} := \int_N f(n)\overline{g(n)}dn,$$

where $f,g \in L^2(N_\Gamma \backslash N)$, dn is the N_Γ-invariant Haar measure on N carried from the Lebesgue measure on $\mathbb{R}^3$.

Proof. Easily seen by the orthogonality of the Hermite functions. □

5.3. An explicit form of the Fourier expansion

Now we can give an explicit form of the Fourier expansion of an automorphic form belonging to an arbitrary standard representation π with a special K-type.

Theorem 5.3.1. *Let Φ be an automorphic form on $SU(2,1)$ belonging to π with K-type τ. Then the Fourier expansion of Φ is given as follows.*
i) *When π is a discrete series representation π_Λ with Blattner parameter $\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^{\oplus 2}$, put $j_k = k - \frac{2\lambda_1 - \lambda_2}{3} - \frac{1}{2} - \mu$ and take the minimal K-type τ_λ as τ.*
i-1) *The case of large discrete series i.e. $\pi = \pi_\Lambda,\ \Lambda \in \Xi_{II}^+$*

$$\begin{aligned}\Phi(na_r) &= C_{0,0}^{\Phi} \cdot r^{d_\lambda+2} \cdot 1_N(n) v_{\lambda_1}^{\lambda} \\ &\quad + \sum_{(\ell,\ell') \in \mathbb{Z}^2 \backslash (0,0)} C_{\ell,\ell'}^{\Phi} \Big(\sum_{k=0}^{d_\lambda} \gamma_k r^{d_\lambda + \frac{3}{2}} W_{0,k-\lambda_1}(4\pi\sqrt{\ell^2+\ell'^2}r) \\ &\qquad\qquad \cdot \psi_{2\pi\ell,2\pi\ell'}(n) v_k^{\lambda} \Big) \\ &\quad + \sum_{\ell \in \mathbb{Z}\backslash\{0\}} \sum_{i=1}^{2|\ell|} \sum_{\mu \in \frac{1}{2}\mathbb{Z}\backslash\mathbb{Z}} C_{\mu,\ell,(i)}^{\Phi} \Big(\sum_{k=0}^{d_\lambda} \gamma_k^{II} r^{d_\lambda+1} W_{\kappa,\frac{k-\lambda_1}{2}}(2\pi|\ell|r^2) \\ &\qquad\qquad \cdot \theta_{j_k}^{\ell,(i)}(n) v_k^{\lambda} \Big),\end{aligned}$$

where

$$\kappa = \{-(\lambda_2 - k + d_\lambda - 1)2\pi\ell - (j_k+1)(4\pi\ell+1)^2/4\}/4\pi|\ell|.$$

i-2) *The case of holomorphic discrete series i.e.* $\pi = \pi_\Lambda$, $\Lambda \in \Xi_I^+$

$$\Phi(na_r) = \sum_{-\ell=0}^{\infty} \sum_{i=1}^{2|\ell|} \sum_{\mu \in \frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}} C^{\Phi}_{\mu,\ell,(i)} \Big(\sum_{k=0}^{d_\lambda} \gamma_k^I r^{\lambda_2+k} e^{\pi\ell r^2} \cdot \theta^{\ell,(i)}_{j_k}(n) v_k^\lambda \Big).$$

i-3) *The case of antiholomorphic discrete series i.e.* $\pi = \pi_\Lambda$, $\Lambda \in \Xi_{III}^+$

$$\Phi(na_r) = \sum_{\ell=0}^{\infty} \sum_{i=1}^{2\ell} \sum_{\mu \in \frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}} C^{\Phi}_{\mu,\ell,(i)} \Big(\sum_{k=0}^{d_\lambda} \gamma_k^{III} r^{-\lambda_2-k} e^{-\pi\ell r^2} \cdot \theta^{\ell,(i)}_{j_k}(n) v_k^\lambda \Big).$$

In these cases, the index μ *runs over half integers which satisfy* $j_k \geq 0$.
ii) *When* π *is a principal series representation* $\pi_{\lambda_0,\nu} = \mathrm{Ind}_P^G(1_N \otimes e^\nu \otimes \chi_{\lambda_0})$, *put* $j_0 = -\frac{\lambda_0}{3} - \frac{1}{2} - \mu$ *and take the corner* K*-type* $\tau_{(-\lambda_0,-\lambda_0)}$ *as* τ.

$$\begin{aligned}\Phi(na_r) = {} & C^{\Phi}_{0,0} \cdot r^{\nu+2} \cdot 1_N(n) v_0 \\ & + \sum_{(\ell,\ell') \in \mathbb{Z}^2\setminus(0,0)} C^{\Phi}_{\ell,\ell'} \cdot r^{\frac{3}{2}} W_{0,\nu}(4\pi\sqrt{\ell^2+\ell'^2} r) \cdot \psi_{2\pi\ell,2\pi\ell'}(n) v_0 \\ & + \sum_{\ell \in \mathbb{Z}\setminus\{0\}} \sum_{i=1}^{2|\ell|} \sum_{\mu \in \frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}} C^{\Phi}_{\mu,\ell,(i)} \cdot r W_{\kappa,\frac{\nu}{2}}(2\pi|\ell| r^2) \cdot \theta^{\ell,(i)}_{j_0}(n) v_0,\end{aligned}$$

where

$$\kappa = \{-\lambda_0 2\pi\ell - (2j_0+1)(16\pi^2\ell^2+1)/4\}/4\pi|\ell|.$$

In this case, the index μ *runs over half integers such that* $-\mu \geq \frac{\lambda_0}{3} + \frac{1}{2}$. *Generalized theta functions* $\theta^{\ell,(i)}_j$*'s are given in* (5.2.1), (5, 2, 2). *We call* $C^{\Phi}_{\ell,\ell'}$*'s,* $C^{\Phi}_{\mu,\ell,(i)}$*'s the Fourier coefficients of* Φ.

PROOF. This is an immediate consequence of our previous argument. Here we just note that parameters appearing in explicit formulae (Theorem 3.3.5, Theorem 4.2.4 and [K-O] Theorem 4.5, l.12 p.998) s and $\eta_+\eta_-$ should be of the form $2\pi\ell$, $\ell \in \mathbb{Z}\setminus\{0\}$ and $-4\pi^2(\ell^2+\ell'^2)$, $(\ell,\ell') \in \mathbb{Z}^2\setminus\{(0,0)\}$ by the same reason in proof of Theorem 5.2.1. $\square$

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