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Remark on Non-Uniform Fundamental and Non Smooth Solutions of Some Classes of Differential Operators with Double Characteristics

By Nguyen Minh Tri

Abstract. We construct explicit formulas for fundamental solutions and non-smooth solutions at degenerate points of a model of Grushin's type operator. The results give all the discrete values of parameters where the operator is not hypoelliptic (nor analytic hypoelliptic).

The aim of this paper is to give explicit formulas of fundamental solutions and non-smooth solutions at degenerate points of the following operator

$$G_{k,\lambda} = \frac{\partial^2}{\partial x^2} + x^{2k}\frac{\partial^2}{\partial y^2} + i\lambda x^{k-1}\frac{\partial}{\partial y}, \tag{1}$$

where $(x, y) \in \mathbb{R}^2, \lambda \in \mathbb{C}, i = \sqrt{-1}$ and k is a positive integer. The above operator is the simplest model of Grushin's type operator [1]. According to a theorem of Grushin the operator $G_{k,\lambda}$ is hypoelliptic (or analytic hypoelliptic) if and only if the equations

$$L^+_{k,\lambda} w_1(x) = \frac{d^2 w_1(x)}{dx^2} - x^{2k} w_1(x) - \lambda x^{k-1} w_1(x) = 0$$

and

$$L^-_{k,\lambda} w_2(x) = \frac{d^2 w_2(x)}{dx^2} - x^{2k} w_2(x) + \lambda x^{k-1} w_2(x) = 0$$

do not have non-trivial solutions in $S(\mathbb{R})$. In [2], [3], [4], [5], [6], [7], [8] there are given explicit formulas of all k, λ for which $G_{k,\lambda}$ is hypoelliptic. We would like to mention a theorem (see, for example [8]).

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Theorem 1. *$G_{k,\lambda}$ is hypoelliptic (or analytic hypoelliptic) if and only if*

in the case k odd $\lambda \neq \pm[2N(k+1)+k]$ or else $\lambda \neq \pm[2N(k+1)+(k+2)]$ for some non-negative integer N.

in the case k even $\lambda \neq (2N+1)(k+1)$ for some $N \in \mathbb{Z}$.

Because of the non-symmetry of $G_{k,\lambda}$ there is a little hope to have an explicit formula for fundamental solutions of (1) even at one single point. While studying the analyticity of solutions of semilinear Kohn-Laplacian $\Box_b$ on the Heisenberg group, see [9], we found a lot of such solutions. We would like to mention that such kind of fundamental solutions was used to obtain the C^∞ regularity of solutions at boundary isolated characteristic points. It is the fact that for boundary value problems the C^∞ regularity of solutions up to boundary may fail at characteristic points [10]. The expression for fundamental solutions is suggested by the one of [11] and [12]. We try to find them in the following form

$$F_k^{\alpha,\beta,\gamma}(x,y) = \big(x^{k+1} - i(k+1)y\big)^{\alpha}\big(x^{k+1} + i(k+1)y\big)^{\beta} x^{\gamma}.$$

Here we take $z_1^{z_2} = e^{z_2 \ln z_1}$ for $z_1, z_2 \in \mathbb{C}$ and if $z_1 = re^{i\varphi}, -\pi < \varphi \leq \pi$ then $\ln z_1 = \ln r + i\varphi$. Let us rewrite $G_{k,\lambda} = X_2X_1 + i(\lambda+k)x^{k-1}\frac{\partial}{\partial y}$ where $X_1 = \frac{\partial}{\partial x} - ix^k\frac{\partial}{\partial y}, X_2 = \frac{\partial}{\partial x} + ix^k\frac{\partial}{\partial y}$. We would like to find all $\alpha, \beta, \gamma, \lambda \in \mathbb{C}$ such that $G_{k,\lambda}F_k^{\alpha,\beta,\gamma}(x,y) = 0$ except (possibly) the points where $F_k^{\alpha,\beta,\gamma}$ is not smooth. Note that

$$\begin{aligned} &X_1\big(x^{k+1} + i(k+1)y\big) = 2(k+1)x^k, X_1\big(x^{k+1} - i(k+1)y\big) = 0,\\ &X_2\big(x^{k+1} + i(k+1)y\big) = 0, X_2\big(x^{k+1} - i(k+1)y\big) = 2(k+1)x^k. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} G_{k,\lambda}F_k^{\alpha,\beta,\gamma}(x,y) &= \big(x^{k+1} - i(k+1)y\big)^{\alpha-1}\big(x^{k+1} + i(k+1)y\big)^{\beta-1}x^{\gamma-2} \\ \times \Big\{ &\big[4(k+1)^2\alpha\beta + 2(k+1)(k+\gamma)\beta + 2(k+1)\alpha\gamma + \gamma(\gamma-1) + (k+1)(k+\lambda)\alpha \\ &- (k+1)(k+\lambda)\beta\big]x^{2k+2} + (k+1)^2\gamma(\gamma-1)y^2 + i\big[-2(k+1)^2(k+\gamma)\beta \\ &+ (k+1)^2(k+\lambda)\alpha + (k+1)^2(k+\lambda)\beta + 2(k+1)^2\alpha\gamma\big]x^{k+1}y\Big\}. \end{aligned}$$

Hence formally we have $G_{k,\lambda}F_k^{\alpha,\beta,\gamma}(x,y)=0$ if

$\gamma=0,\ \lambda$ arbitrary, $\alpha=0,\ \beta=0$, the solution is a constant.
$\gamma=0,\ \lambda=k,\alpha=0,\ \beta$ arbitrary $\neq 0$.
$\gamma=0,\ \lambda=-k,\ \ \alpha$ arbitrary $\neq 0,\ \beta=0$.
$\gamma=0,\ \lambda$ arbitrary, $\alpha=\frac{\lambda-k}{2(k+1)}=:\alpha_1(k,\lambda),\ \beta=-\frac{\lambda+k}{2(k+1)}=:\beta_1(k,\lambda)$.
$\gamma=1,\ \lambda$ arbitrary, $\alpha=0,\ \beta=0$, the solution is the linear function x.
$\gamma=1,\ \lambda=k+2,\ \alpha=0,\ \beta$ arbitrary $\neq 0$.
$\gamma=1,\ \lambda=-(k+2),\ \alpha$ arbitrary $\neq 0,\ \beta=0$.
$\gamma=1,\ \lambda$ arbitrary, $\alpha=\frac{\lambda-k-2}{2(k+1)}=:\alpha_2(k,\lambda),\ \beta=-\frac{\lambda+k+2}{2(k+1)}=:\beta_2(k,\lambda)$.

THEOREM 2. *Assume that k is odd. Then*
I) $G_{k,\lambda}F_k^{\alpha_1(k,\lambda),\beta_1(k,\lambda),0}(x,y)=-\frac{2^{2+\frac{1}{k+1}}\pi\Gamma(\frac{k}{k+1})}{\Gamma(\frac{k+\lambda}{2k+2})\Gamma(\frac{k-\lambda}{2k+2})}\delta(x,y)=:a_{k,\lambda}\delta(x,y)$.
II) $G_{k,k}F_k^{0,\beta,0}(x,y)=0$ *if* $Re\,\beta>-\frac{k}{k+1}$.
III) $G_{k,-k}F_k^{\alpha,0,0}(x,y)=0$ *if* $Re\,\alpha>-\frac{k}{k+1}$.

PROOF.
I) We begin by noting that if k is odd then $\big(x^{k+1}-i(k+1)y\big)^{\alpha}$ and $\big(x^{k+1}+i(k+1)y\big)^{\beta}\in C^{\infty}\big(\mathbb{R}^2\backslash(0,0)\big)$ for every α and β. Let us introduce the following "polar coordinate"

$$x=\rho(\sin\theta)_{\pm}^{\frac{1}{k+1}},y=\frac{\rho^{k+1}}{k+1}\cos\theta,dxdy=\frac{\rho^{k+1}}{k+1}|\sin\theta|^{-\frac{k}{k+1}}d\rho d\theta.$$

Here we use the following notation $(\sin\theta)_{\pm}^{r}=\mathrm{sign}(\sin\theta)|\sin\theta|^{r}$ for every $r\in\mathbb{R}$. Note that the map $(x,y)\longrightarrow(\rho,\theta)$ is not a differmorphism along the line $x=0$. But it is good enough for us because in the future we will use it only for integration, and if necessary we can take integrals as a limit. Now it is easy to verify that $\rho^{2k+2}=x^{2k+2}+(k+1)^2y^2$. Let us write $F_k^1(x,y)=F_k^{\alpha_1(k,\lambda),\beta_1(k,\lambda),0}(x,y)$. First we prove that $F_k^1(x,y)\in L_{loc}^{\frac{k+2}{k}-\tau}(\mathbb{R}^2)$ for any small positive τ. Indeed, since $F_k^1(x,y)\in C^{\infty}\big(\mathbb{R}^2\backslash(0,0)\big)$ it suffices to prove that $F_k^1(x,y)\in L^{\frac{k+2}{k}-\tau}(B_{\varepsilon})$, where $B_{\varepsilon}=\{(x,y)|\rho(x,y)<\varepsilon\}$. We have

$$\int_{B_{\varepsilon}}\big|F_k^1(x,y)\big|^{\frac{k+2}{k}-\tau}dxdy\leq C\int_{-\pi}^{\pi}|\sin\theta|^{-\frac{k}{k+1}}d\theta\int_0^{\varepsilon}\rho^{k+1}(\rho^{-k})^{\frac{k+2}{k}-\tau}d\rho$$
$$\leq C\int_0^{\varepsilon}\rho^{-1+\tau k}d\rho<\infty.$$

Note that $F_k^1(x,y) \notin L_{loc}^{\frac{k+2}{k}}(\mathbb{R}^2)$. Let $\mathbb{R}^2_\varepsilon = \{(x,y) \in \mathbb{R}^2 | \rho(x,y) \geq \varepsilon\}$. By applying Green's formula we have

$$
\begin{aligned}
(2) \qquad & \int_{\mathbb{R}^2_\varepsilon} f(x,y) G_{k,-\lambda} v(x,y)\, dxdy \\
& = \int_{\mathbb{R}^2_\varepsilon} v(x,y) G_{k,\lambda} f(x,y)\, dxdy \\
& - \int_{\rho=\varepsilon} v(x,y) \Big\{ \nu_1.\ \frac{\partial f(x,y)}{\partial x} + \nu_2.\ x^{2k} \frac{\partial f(x,y)}{\partial y} \\
& \qquad\qquad + i\lambda.\ \nu_2.\ x^{k-1}\ f(x,y) \Big\}\, ds \\
& + \int_{\rho=\varepsilon} f(x,y) \Big\{ \nu_1.\ \frac{\partial v(x,y)}{\partial x} + \ \nu_2.\ x^{2k} \frac{\partial v(x,y)}{\partial y} \Big\}\, ds \\
& =: \int_{\mathbb{R}^2_\varepsilon} V(f,v,k,\lambda)\, dxdy \\
& - \int_{\rho=\varepsilon} v(x,y) B_1(f,k,\lambda)\, ds + \int_{\rho=\varepsilon} f(x,y) B_2(v,k)\, ds
\end{aligned}
$$

for every $v(x,y) \in C_0^\infty(\mathbb{R}^2), f(x,y) \in C^\infty\big(\mathbb{R}^2 \backslash (0,0)\big)$, where $\nu = (\nu_1, \nu_2)$ is the unit outward normal to $\partial\mathbb{R}^2_\varepsilon$. Replace $f(x,y)$ in (2) by $F_k^1(x,y)$ we obtain

$$
\begin{aligned}
(3) \qquad \int_{\mathbb{R}^2_\varepsilon} F_k^1(x,y) G_{k,-\lambda} v(x,y)\, dxdy = & \int_{\mathbb{R}^2_\varepsilon} V(F_k^1, v, k, \lambda)\, dxdy \\
& - \int_{\rho=\varepsilon} v(x,y) B_1(F_k^1, k, \lambda)\, ds \\
& + \int_{\rho=\varepsilon} F_k^1(x,y) B_2(v,k)\, ds.
\end{aligned}
$$

The first integral in the right side of (3) vanishes. We now compute the third integral in the right side of (3). It is easy to check that

$$
ds\Big|_{\partial B_\varepsilon} = \frac{1}{k+1} \left(\varepsilon^2 |\sin\theta|^{-\frac{2k}{k+1}} \cos^2\theta + \varepsilon^{2k+2} \sin^2\theta \right)^{\frac{1}{2}}\ d\theta \qquad \text{and}
$$

$$\begin{aligned}\nu\big|_{\partial B_\varepsilon} &= (\nu_1,\nu_2)\big|_{\partial B_\varepsilon}\\ &= -\left(\frac{x^{2k+1}}{\left(x^{4k+2}+(k+1)^2y^2\right)^{\frac{1}{2}}},\frac{(k+1)y}{\left(x^{4k+2}+(k+1)^2y^2\right)^{\frac{1}{2}}}\right)\Bigg|_{\partial B_\varepsilon}.\end{aligned}$$

Hence

$$\left(x^{4k+2}+(k+1)^2y^2\right)^{-\frac{1}{2}}ds\Big|_{\partial B_\varepsilon} = \frac{1}{k+1}\varepsilon^{-k}|\sin\theta|^{-\frac{k}{k+1}}d\theta.$$

It follows that

$$\begin{aligned}(4)\qquad &\left|\int_{\partial B_\varepsilon} F_k^1(x,y)B_2(v,k)\,ds\right|\\ &\le C\int_{\rho=\varepsilon}|F_k^1(x,y)|.\ \left(|\nu_1|+|\nu_2.\ x^{2k}|\right)\,ds\\ &\le C\int_{\rho=\varepsilon}\varepsilon^{-k}\frac{|x|^{2k+1}+(k+1)x^{2k}|y|}{\left(x^{4k+2}+(k+1)^2y^2\right)^{\frac{1}{2}}}\,ds\\ &\le C\int_{-\pi}^{\pi}|\sin\theta|^{\frac{k}{k+1}}\left(\varepsilon|\sin\theta|^{\frac{1}{k+1}}+\varepsilon^{k+1}|\cos\theta|\right)\,d\theta\to 0\\ &\qquad\qquad \text{as}\qquad \varepsilon\to 0.\end{aligned}$$

Next we evaluate $B_1(F_k^1,k,\lambda)$. We have

$$\begin{aligned}B_1(F_k^1,k,\lambda)\Big|_{\rho=\varepsilon} &= \left(x^{k+1}-i(k+1)y\right)^{\alpha_1(k,\lambda)}\left(x^{k+1}+i(k+1)y\right)^{\beta_1(k,\lambda)}\\ &\quad\times\left(kx^{2k}-i(k+1)\lambda x^{k-1}y\right)\\ &\quad\times\left(x^{4k+2}+(k+1)^2y^2\right)^{-\frac{1}{2}}\Big|_{\rho=\varepsilon}\\ &= \varepsilon^k\left(|\sin\theta|+i\cos\theta\right)^{-\frac{\lambda}{k+1}}\\ &\quad\times\left(k|\sin\theta|^{\frac{2k}{k+1}}-i\lambda|\sin\theta|^{\frac{k-1}{k+1}}\cos\theta\right)\\ &\quad\times\left(\varepsilon^{4k+2}|\sin\theta|^{\frac{4k+2}{k+1}}+\varepsilon^{2k+2}\cos^2\theta\right)^{-\frac{1}{2}}.\end{aligned}$$

Therefore

$$
\begin{aligned}
&-\int_{\rho=\varepsilon} v(x,y)B_1(F_k^1,k,\lambda)\,ds \\
&= -\frac{1}{k+1}\int_{-\pi}^{\pi}\Big(k|\sin\theta|^{\frac{k}{k+1}} - i\lambda|\sin\theta|^{-\frac{1}{k+1}}\cos\theta\Big) \\
&\times \big(|\sin\theta| + i\cos\theta\big)^{-\frac{\lambda}{k+1}} v(\varepsilon,\theta)\,d\theta \\
&= -\frac{1}{k+1}\int_{-\pi}^{\pi}\big(v(0,0)+\bar{o}(1)\big)\,(k|\sin\theta| - i\lambda\cos\theta)\,|\sin\theta|^{-\frac{1}{k+1}} \\
&\times \big(|\sin\theta| + i\cos\theta\big)^{-\frac{\lambda}{k+1}}\,d\theta \\
&\to -\frac{2v(0,0)}{k+1}\int_0^{\pi}\big(k\sin^{\frac{k}{k+1}}\theta - i\lambda\sin^{-\frac{1}{k+1}}\theta\cos\theta\big)\big(\sin\theta + i\cos\theta\big)^{-\frac{\lambda}{k+1}}\,d\theta
\end{aligned}
$$

as $\varepsilon \to 0$. By integrating by parts we have the following identity

$$
\begin{aligned}
&\int_0^{\pi}\big(\sin\theta + i\cos\theta\big)^{-\frac{\lambda}{k+1}}\sin^{-\frac{1}{k+1}}\theta\cos\theta\,d\theta \\
&\qquad = -\frac{\lambda i}{k}\int_0^{\pi}\big(\sin\theta + i\cos\theta\big)^{-\frac{\lambda}{k+1}}\sin^{\frac{k}{k+1}}\theta\,d\theta.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&-\int_{\rho=\varepsilon} v(x,y)B_1(F_k^1,k,\lambda)\,ds \\
&\quad\longrightarrow \frac{2(\lambda^2-k^2)v(0,0)}{k(k+1)}\int_0^{\pi}\big(\sin\theta + i\cos\theta\big)^{-\frac{\lambda}{k+1}}\sin^{\frac{k}{k+1}}\theta\,d\theta \quad \text{as} \quad \varepsilon \to 0.
\end{aligned}
$$

Lemma. *Assume that* $\omega_1, \omega_2 \in \mathbb{C}, \operatorname{Re}\omega_1 > -1$. *Then we have*

$$
\int_0^{\pi}\sin^{\omega_1}\theta(\sin\theta + i\cos\theta)^{\omega_2}\,d\theta = \frac{2^{-\omega_1}\pi\Gamma(\omega_1+1)}{\Gamma(1+\frac{\omega_1-\omega_2}{2})\Gamma(1+\frac{\omega_1+\omega_2}{2})}. \tag{5}
$$

Proof. We begin by noting that the integral in the left side of (5) makes sense if $\operatorname{Re}\omega_1 > -1$. Since $\operatorname{Re}\omega_1 > -1$, we have

$$
\int_0^{\pi}\sin^{\omega_1}\theta(\sin\theta + i\cos\theta)^{\omega_2}\,d\theta
$$

$$= -\int_0^\pi \sin^{\omega_1}\theta\Big(\frac{i}{\cos\theta+i\sin\theta}\Big)^{\omega_2+1} d(\cos\theta+i\sin\theta)$$
$$= -\int_{S_1^+}\Big(\frac{z^2-1}{2iz}\Big)^{\omega_1}\Big(\frac{i}{z}\Big)^{\omega_2+1}dz = \int_{-1}^{-\varepsilon}\Big(\frac{z^2-1}{2iz}\Big)^{\omega_1}\Big(\frac{i}{z}\Big)^{\omega_2+1}dz$$
$$-\int_{S_\varepsilon^+}\Big(\frac{z^2-1}{2iz}\Big)^{\omega_1}\Big(\frac{i}{z}\Big)^{\omega_2+1}dz + \int_\varepsilon^1\Big(\frac{z^2-1}{2iz}\Big)^{\omega_1}\Big(\frac{i}{z}\Big)^{\omega_2+1}dz$$

where S_r^+ is the upper circle of radius r in $\mathbb{C}$. First we consider the case $\operatorname{Re}\omega_2 < -\operatorname{Re}\omega_1$. We note the following inequality $|z^\mu| \le C|z|^{\operatorname{Re}\mu}$ where $0 < |z| \le 1$ and μ is some complex constant. Therefore we see that $\int_{S_\varepsilon^+}\big(\frac{z^2-1}{2iz}\big)^{\omega_1}\big(\frac{i}{z}\big)^{\omega_2+1}dz$ tends to 0 when ε tends to 0. Next we have

$$\int_{-1}^0\Big(\frac{z^2-1}{2iz}\Big)^{\omega_1}\Big(\frac{i}{z}\Big)^{\omega_2+1}dz = \int_0^1\Big(-\frac{z^2-1}{2iz}\Big)^{\omega_1}\Big(-\frac{i}{z}\Big)^{\omega_2+1}dz$$
$$= (-1)^{-1-\omega_1-\omega_2}\int_0^1\Big(\frac{z^2-1}{2iz}\Big)^{\omega_1}\Big(\frac{i}{z}\Big)^{\omega_2+1}dz.$$

Hence letting ε tend to 0 we deduce that

$$\int_0^\pi \sin^{\omega_1}\theta(\sin\theta+i\cos\theta)^{\omega_2}\,d\theta$$
$$= \big[1+(-1)^{-1-\omega_1-\omega_2}\big]\int_0^1\Big(\frac{z^2-1}{2iz}\Big)^{\omega_1}\Big(\frac{i}{z}\Big)^{\omega_2+1}dz$$
$$= \frac{\cos\frac{\pi}{2}(\omega_1+\omega_2+1)}{2^{\omega_1}}\int_0^1(1-z)^{\omega_1}z^{-\frac{\omega_1+\omega_2}{2}-1}dz$$
$$= \frac{2^{-\omega_1}\pi\Gamma(\omega_1+1)}{\Gamma(1+\frac{\omega_1-\omega_2}{2})\Gamma(1+\frac{\omega_1+\omega_2}{2})}.$$

Next we note that for every fixed $\omega_1 \in \mathbb{C}, \operatorname{Re}\omega_1 > -1$ the integral in the left side of (5) is an entire analytic function of ω_2 and $\Gamma(.)$ is a meromorphic function in $\mathbb{C}$. It follows that (5) is true for every $\omega_1, \omega_2, \operatorname{Re}\omega_1 > -1$. Therefore the proof of Lemma is completed. □

(continuing the proof of part I) Substituting $\omega_1 = \frac{k}{k+1}, \omega_2 = -\frac{\lambda}{k+1}$ in (5) we obtain

$$-\int_{\rho=\varepsilon} v(x,y)B_1(F_k^1,k,\lambda)\,ds \tag{6}$$

$$\to \frac{2^{\frac{1}{k+1}}\pi(\lambda^2-k^2)\Gamma(\frac{2k+1}{k+1})}{k(k+1)\Gamma(\frac{3k+2+\lambda}{2k+2})\Gamma(\frac{3k+2-\lambda}{2k+2})}v(0,0)$$
$$= -\frac{2^{2+\frac{1}{k+1}}\pi\Gamma(\frac{k}{k+1})}{\Gamma(\frac{k+\lambda}{2k+2})\Gamma(\frac{k-\lambda}{2k+2})}v(0,0) \quad \text{as} \quad \varepsilon\to 0.$$

Now from (3), (4), (6) we have

$$\begin{aligned}\big(G_{k,\lambda}F_k^1(x,y),v(x,y)\big) &= \big(F_k^1(x,y),G_{k,-\lambda}v(x,y)\big)\\ &= \lim_{\varepsilon\to 0}\int_{\rho\geq\varepsilon} F_k^1(x,y)G_{k,-\lambda}v(x,y)dxdy\\ &= -\frac{2^{2+\frac{1}{k+1}}\pi\Gamma(\frac{k}{k+1})}{\Gamma(\frac{k+\lambda}{2k+2})\Gamma(\frac{k-\lambda}{2k+2})}v(0,0).\end{aligned}$$

Hence $G_{k,\lambda}F_k^{\alpha_1(k,\lambda),\beta_1(k,\lambda),0}(x,y) = a_{k,\lambda}\delta(x,y)$.
II) Assume that $\operatorname{Re}\beta > -\frac{k}{k+1}$. We then have $F_k^{0,\beta,0}(x,y) = \big(x^{k+1}+i(k+1)y\big)^\beta \in L^p_{loc}(\mathbb{R}^2)$ for every $1\leq p < -\frac{k+2}{(k+1)\operatorname{Re}\beta}$ if $\operatorname{Re}\beta<0$ and $F_k^{0,\beta,0}(x,y)\in L^\infty_{loc}(\mathbb{R}^2)$ if $\operatorname{Re}\beta\geq 0$. It is easy to compute that

$$\begin{aligned}&B_1(F_k^{0,\beta,0},k,k)\Big|_{\rho=\varepsilon}\\ &= -(k+1)x^{k-1}\Big\{\beta x^{2k+2}+i\big(k+(k+1)\beta\big)x^{k+1}y-k(k+1)y^2\Big\}\\ &\times \big(x^{k+1}+i(k+1)y\big)^{\beta-1}\Big(x^{4k+2}+(k+1)^2y^2\Big)^{-\frac{1}{2}}\Big|_{\rho=\varepsilon}\\ &= -\varepsilon^{2k+(k+1)\beta}\Big(\beta(k+1)\sin^2\theta+i(k+k\beta+\beta)\cos\theta|\sin\theta|-k\cos^2\theta\Big)\\ &\times|\sin\theta|^{\frac{k-1}{k+1}}\big(|\sin\theta|+i\cos\theta\big)^{\beta-1}\Big(\varepsilon^{4k+2}|\sin\theta|^{\frac{4k+2}{k+1}}+\varepsilon^{2k+2}\cos^2\theta\Big)^{-\frac{1}{2}}\end{aligned}$$

and

$$\begin{aligned}&\Big|\big(|\nu_1|+|\nu_2.\ x^{2k}|\big)F_k^{0,\beta,0}\Big|ds\Big|_{\rho=\varepsilon}\\ &\leq \Big|\big(|\sin\theta|+i\cos\theta\big)^\beta\Big|\\ &\times\Big(\varepsilon^{(k+1)(1+\operatorname{Re}\beta)}|\sin\theta|+\varepsilon^{2k+1+(k+1)\operatorname{Re}\beta}|\cos\theta||\sin\theta|^{\frac{k}{k+1}}\Big)d\theta.\end{aligned}$$

Using the assumption that $\operatorname{Re}\beta > -\frac{k}{k+1}$ we deduce that

$$-\int_{\rho=\varepsilon} v(x,y)B_1(F_k^{0,\beta,0},k,k)\,ds + \int_{\rho=\varepsilon} F_k^{0,\beta,0}(x,y)B_2(v,k)\,ds \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Hence $G_{k,k}F_k^{0,\beta,0}(x,y) = 0$.
III) The proof of this part is the same as the proof of part II) with β replaced by α. This concludes the proof of Theorem 2. □

COROLLARY 1. *If $\lambda = \pm[2N(k+1)+k]$, where N is a non-negative integer, then $G_{k,\lambda}F_k^{\alpha_1(k,\lambda),\beta_1(k,\lambda),0}(x,y) = 0$. Hence $G_{k,\lambda}$ is not hypoelliptic (nor analytic hypoelliptic) at these points.*

PROOF. Indeed, if $\lambda = \pm[2N(k+1)+k]$ then $\Gamma(\frac{k-\lambda}{2k+2}) = \infty$ or $\Gamma(\frac{k+\lambda}{2k+2}) = \infty \Longrightarrow a_{k,\lambda} = 0 \Longrightarrow G_{k,\lambda}F_k^{\alpha_1(k,\lambda),\beta_1(k,\lambda),0}(x,y) = 0$. □

REMARK 1. The constant $-\frac{2k}{k+1}\sqrt{\pi}\frac{\Gamma(\frac{k+1}{2k})}{\Gamma(\frac{2k+1}{2k})}$ in Theorem 6 of [11] is not accurate. It should be replaced by $-\frac{2k}{k+1}\sqrt{\pi}\frac{\Gamma(\frac{2k+1}{2k+2})}{\Gamma(\frac{3k+2}{2k+2})}$.

THEOREM 3. *Assume that k is odd. Then*
I) $G_{k,\lambda}F_k^{\alpha_2(k,\lambda),\beta_2(k,\lambda),1}(x,y) = \frac{2^{2-\frac{1}{k+1}}\pi\Gamma(\frac{k+2}{k+1})}{\Gamma(\frac{k+2+\lambda}{2k+2})\Gamma(\frac{k+2-\lambda}{2k+2})}\frac{\partial\delta(x,y)}{\partial x} =: b_{k,\lambda}\frac{\partial\delta(x,y)}{\partial x}$.
II) $G_{k,k+2}F_k^{0,\beta,1}(x,y) = 0$ if $\operatorname{Re}\beta > -\frac{k+2}{k+1}$.
III) $G_{k,-k-2}F_k^{\alpha,0,1}(x,y) = 0$ if $\operatorname{Re}\alpha > -\frac{k+2}{k+1}$.

PROOF.
I) Let us write $F_k^2(x,y) = F_k^{\alpha_2(k,\lambda),\beta_2(k,\lambda),1}(x,y)$. As in the proof of Theorem 2 it is easy to check that $F_k^2(x,y) \in L_{loc}^{\frac{k+2}{k+1}-\tau}(\mathbb{R}^2)$ for any small positive τ. We see that

$$\int_{\rho=\varepsilon} \nu_2.\ x^{2k}.\ F_k^2(x,y).\ \frac{\partial v(x,y)}{\partial y}\,ds$$
$$= \frac{\varepsilon^k}{k+1}\int_{-\pi}^{\pi} \frac{\partial v(x,y)}{\partial y}(|\sin\theta| + i\cos\theta)^{-\frac{\lambda}{k+1}}\sin\theta\cos\theta\,d\theta \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Next we have

$$\begin{aligned} B_1(F_k^2,k,\lambda)\Big|_{\rho=\varepsilon} &= (k+1)x^k\big(x^{k+1}+i(k+1)y\big)^{\beta_2(k,\lambda)} \\ &\quad\times\big(x^{k+1}-i(k+1)y\big)^{\alpha_2(k,\lambda)} \\ &\quad\times\left(x^{k+1}-i\lambda y\right)\left(x^{4k+2}+(k+1)^2y^2\right)^{-\frac{1}{2}}\Big|_{\rho=\varepsilon} \\ &= \varepsilon^{k-1}(\sin\theta)_{\pm}^{\frac{k}{k+1}}\left(|\sin\theta|+i\cos\theta\right)^{-\frac{\lambda}{k+1}} \\ &\quad\times\Big((k+1)|\sin\theta|-i\lambda\cos\theta\Big) \\ &\quad\times\left(\varepsilon^{4k+2}|\sin\theta|^{\frac{4k+2}{k+1}}+\varepsilon^{2k+2}\cos^2\theta\right)^{-\frac{1}{2}}. \end{aligned}$$

We note that $v(\varepsilon,\theta)=v(0,0)+\varepsilon(\sin\theta)_{\pm}^{\frac{1}{k+1}}\frac{\partial v(0,0)}{\partial x}+\bar{o}(\varepsilon)$. It follows that

$$\begin{aligned} (7)\qquad & -\int_{\rho=\varepsilon} v(x,y)B_1(F_k^2,k,\lambda)\,ds \\ &= -\frac{1}{\varepsilon}\int_{-\pi}^{\pi} v(0,0)\mathrm{sign}(\sin\theta)\left(|\sin\theta|-\frac{i\lambda\cos\theta}{k+1}\right) \\ &\times\left(|\sin\theta|+i\cos\theta\right)^{-\frac{\lambda}{k+1}}\,d\theta \\ &-2\int_0^{\pi}\frac{\partial v(0,0)}{\partial x}\left(\sin^{\frac{k+2}{k+1}}\theta-\frac{i\lambda\sin^{\frac{1}{k+1}}\theta\cos\theta}{k+1}\right) \\ &\times\left(\sin\theta+i\cos\theta\right)^{-\frac{\lambda}{k+1}}\,d\theta+\bar{o}(1). \end{aligned}$$

Without a computation we can deduce that the first integral in the right side of (7) vanishes by applying a theorem of Schwartz. Alternatively, we can see this by noting that the integrand is an odd function of θ. To estimate the second integral in the right side of (7) we have again the following identity, which is easily obtained by integrating by parts

$$\begin{aligned} &\int_0^{\pi}\left(\sin\theta+i\cos\theta\right)^{-\frac{\lambda}{k+1}}\sin^{\frac{1}{k+1}}\theta\cos\theta\,d\theta \\ &\quad= -\frac{\lambda i}{k+2}\int_0^{\pi}\left(\sin\theta+i\cos\theta\right)^{-\frac{\lambda}{k+1}}\sin^{\frac{k+2}{k+1}}\theta\,d\theta. \end{aligned}$$

Therefore we deduce that

$$-\int_{\rho=\varepsilon} v(x,y)B_1(F_k^2,k,\lambda)\,ds \\ \to 2\frac{\partial v(0,0)}{\partial x}\frac{\lambda^2-(k+1)(k+2)}{(k+1)(k+2)}\int_0^\pi \big(\sin\theta+i\cos\theta\big)^{-\frac{\lambda}{k+1}}\sin^{\frac{k+2}{k+1}}\theta\,d\theta \\ \text{as}\quad \varepsilon\to 0.$$

Next we evaluate the remainder term in (3) with $F_k^1(x,y)$ replaced by $F_k^2(x,y)$

$$\int_{\rho=\varepsilon} F_k^2(x,y).\ \nu_1.\ \frac{\partial v(x,y)}{\partial x}\,ds \\ = -\frac{1}{k+1}\int_{-\pi}^{\pi}\frac{\partial v(\varepsilon,\theta)}{\partial x}\big(|\sin\theta|+i\cos\theta\big)^{-\frac{\lambda}{k+1}}|\sin\theta|^{\frac{k+2}{k+1}}\,d\theta \\ \to -\frac{2\frac{\partial v(0,0)}{\partial x}}{k+1}\int_0^\pi\big(\sin\theta+i\cos\theta\big)^{-\frac{\lambda}{k+1}}\sin^{\frac{k+2}{k+1}}\theta\,d\theta \quad \text{as}\quad \varepsilon\to 0.$$

Now applying Lemma with $\omega_1=\frac{k+2}{k+1},\omega_2=-\frac{\lambda}{k+1}$ we obtain

$$\begin{aligned}\big(G_{k,\lambda}F_k^2(x,y),v(x,y)\big) &= \big(F_k^2(x,y),G_{k,-\lambda}v(x,y)\big)\\ &= \lim_{\varepsilon\to 0}\int_{\rho\geq\varepsilon}F_k^2(x,y)G_{k,-\lambda}v(x,y)dxdy\\ &= -\frac{2^{-\frac{1}{k+1}}\pi\big((k+2)^2-\lambda^2\big)\Gamma(\frac{2k+3}{k+1})}{(k+1)(k+2)\Gamma(\frac{3k+4+\lambda}{2k+2})\Gamma(\frac{3k+4-\lambda}{2k+2})}\frac{\partial v(0,0)}{\partial x}\\ &= -\frac{2^{2-\frac{1}{k+1}}\pi\Gamma(\frac{k+2}{k+1})}{\Gamma(\frac{k+2+\lambda}{2k+2})\Gamma(\frac{k+2-\lambda}{2k+2})}\frac{\partial v(0,0)}{\partial x}.\end{aligned}$$

It follows that $G_{k,\lambda}F_k^{\alpha_2(k,\lambda),\beta_2(k,\lambda),1}(x,y)=b_{k,\lambda}\frac{\partial\delta(x,y)}{\partial x}$.
II) Assume that $\operatorname{Re}\beta>-\frac{k+2}{k+1}$. We then have $F_k^{0,\beta,1}(x,y)=x\big(x^{k+1}+i(k+1)y\big)^\beta\in L^p_{loc}(\mathbb{R}^2)$ for every $1\leq p<-\frac{k+2}{1+(k+1)\operatorname{Re}\beta}$ if $\operatorname{Re}\beta<-\frac{1}{k+1}$ and

$F_k^{0,\beta,1}(x,y) \in L^\infty_{loc}(\mathbb{R}^2)$ if $\operatorname{Re}\beta \geq -\frac{1}{k+1}$. It is easy to compute that

$$\begin{aligned}
&B_1(F_k^{0,\beta,1}, k, k+2)\Big|_{\rho=\varepsilon} \\
&\quad = -\Big\{(1+k\beta+\beta)x^{2k+2} + i(k+1)(k+3+k\beta+\beta)x^{k+1}y \\
&\quad - (k+1)^2(k+2)y^2\Big\}x^k \\
&\quad \times \left(x^{k+1} + i(k+1)y\right)^{\beta-1}\left(x^{4k+2} + (k+1)^2 y^2\right)^{-\frac{1}{2}}\Big|_{\rho=\varepsilon} \\
&\quad = -\varepsilon^{2k+1+(k+1)\beta} \\
&\quad \times \Big((1+k\beta+\beta)\sin^2\theta + i(k+3+k\beta+\beta)\cos\theta|\sin\theta| - (k+2)\cos^2\theta\Big) \\
&\quad \times (\sin\theta)_\pm^{\frac{k}{k+1}}\left(|\sin\theta| + i\cos\theta\right)^{\beta-1}\left(\varepsilon^{4k+2}|\sin\theta|^{\frac{4k+2}{k+1}} + \varepsilon^{2k+2}\cos^2\theta\right)^{-\frac{1}{2}}
\end{aligned}$$

and

$$\begin{aligned}
&F_k^{0,\beta,1}B_2(v,k)\,ds\Big|_{\rho=\varepsilon} \\
&\quad = \frac{1}{k+1}\Big(-\varepsilon^{k+2+(k+1)\beta}(|\sin\theta| + i\cos\theta)^\beta|\sin\theta|^{\frac{k+2}{k+1}}\frac{\partial v(x,y)}{\partial x} \\
&\qquad - \varepsilon^{(k+1)(2+\beta)}(|\sin\theta| + i\cos\theta)^\beta\cos\theta(\sin\theta)_\pm\frac{\partial v(x,y)}{\partial y}\Big)d\theta.
\end{aligned}$$

Therefore we deduce that

$$\begin{aligned}
(8)\quad & -\int_{\rho=\varepsilon} v(x,y)B_1(F_k^{0,\beta,1}, k, k+2)\,ds + \int_{\rho=\varepsilon} F_k^{0,\beta,1}(x,y)B_2(v,k)\,ds \\
&= \frac{\varepsilon^{(k+1)(\beta+1)}v(0,0)}{k+1} \\
&\times \int_{-\pi}^{\pi}\Big((1+k\beta+\beta)\sin^2\theta + i(k+3+k\beta+\beta)\cos\theta|\sin\theta| \\
&\qquad - (k+2)\cos^2\theta\Big) \\
&\times \operatorname{sign}(\sin\theta)\left(|\sin\theta| + i\cos\theta\right)^{\beta-1}d\theta + O(\varepsilon^{k+2+(k+1)\operatorname{Re}\beta}).
\end{aligned}$$

The integral in the right side of (8) vanishes for every ε since its integrand is an odd function of θ. Therefore using the assumption that $\operatorname{Re}\beta > -\frac{k+2}{k+1}$

we see that the expression in the left side of (8) tends to 0 as ε tends to 0. Hence $G_{k,k+2}F_k^{0,\beta,1}(x,y) = 0$.
III) The proof of this part is the same as the proof of part II) with β replaced by α. This concludes the proof of Theorem 3. □

Corollary 2. *If $\lambda = \pm[2N(k+1)+k+2]$, where N is a non-negative integer, then $G_{k,\lambda}F_k^{\alpha_2(k,\lambda),\beta_2(k,\lambda),1}(x,y) = 0$. Hence $G_{k,\lambda}$ is not hypoelliptic (nor analytic hypoelliptic) at these points.*

Proof. Indeed, if $\lambda = \pm[2N(k+1)+k+2]$ then $\Gamma(\frac{k+2-\lambda}{2k+2}) = \infty$ or $\Gamma(\frac{k+2+\lambda}{2k+2}) = \infty \Longrightarrow b_{k,\lambda} = 0 \Longrightarrow G_{k,\lambda}F_k^{\alpha_2(k,\lambda),\beta_2(k,\lambda),1}(x,y) = 0$. □

Theorem 4. *Assume that k is even, and $\lambda = (2N+1)(k+1)$, where N is an integer. Then*
$G_{k,\lambda}F_k^{\alpha_1(k,\lambda),\beta_1(k,\lambda),0}(x,y) = 0, G_{k,\lambda}F_k^{\alpha_2(\lambda),\beta_2(k,\lambda),1}(x,y) = 0.$
If $\lambda = 2N(k+1)$, where N is an integer, then
$G_{k,\lambda}F_k^{\alpha_1(k,\lambda),\beta_1(k,\lambda),0}(x,y) = a_{k,\lambda}\delta(x,y), G_{k,\lambda}F_k^{\alpha_2(k,\lambda),\beta_2(k,\lambda),1}(x,y) = b_{k,\lambda}\frac{\partial\delta(x,y)}{\partial x}$.

Proof. If $\lambda = (2N+1)(k+1)$ or $\lambda = 2N(k+1)$ then $\frac{k}{k+1} + 2\beta_1(k,\lambda)$ and $\frac{k+2}{k+1} + 2\beta_2(k,\lambda)$ are integers. Therefore $F_k^{\alpha_1(k,\lambda),\beta_1(k,\lambda),0}(x,y)$, $F_k^{\alpha_2(k,\lambda),\beta_2(k,\lambda),1}(x,y) \in C^\infty(\mathbb{R}^2\backslash(0,0))$. Again we have $F_k^{\alpha_1(k,\lambda),\beta_1(k,\lambda),0}(x,y) \in L_{loc}^{\frac{k+2}{k}-\tau}(\mathbb{R}^2)$ and $F_k^{\alpha_2(k,\lambda),\beta_2(k,\lambda),1}(x,y) \in L_{loc}^{\frac{k+2}{k+1}-\tau}(\mathbb{R}^2)$ for any small positive τ. First we prove the theorem for $F_k^{\alpha_1(k,\lambda),\beta_1(k,\lambda),0}$. As in the proof of Theorem 2 we can show that

$$\int_{\rho=\varepsilon} F_k^{\alpha_1(k,\lambda),\beta_1(k,\lambda),0}(x,y)B_2(v,k)\,ds \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Next we have

$$\begin{aligned} (9) \qquad & -\int_{\rho=\varepsilon} v(x,y)B_1(F_k^{\alpha_1(k,\lambda),\beta_1(k,\lambda),0},k,\lambda)\,ds \\ & = -\frac{1}{k+1}\int_{-\pi}^{\pi} v(\varepsilon,\theta)\Big(k|\sin\theta|^{\frac{k}{k+1}} - i\lambda(\sin\theta)_\pm^{-\frac{1}{k+1}}\cos\theta\Big) \\ & \times \big(\sin\theta + i\cos\theta\big)^{-\frac{\lambda}{k+1}}\,d\theta \end{aligned}$$

$$\to -\frac{v(0,0)}{k+1}\int_{-\pi}^{\pi}\Big(k|\sin\theta|^{\frac{k}{k+1}} - i\lambda(\sin\theta)_{\pm}^{-\frac{1}{k+1}}\cos\theta\Big)$$
$$\times\big(\sin\theta + i\cos\theta\big)^{-\frac{\lambda}{k+1}}\,d\theta \text{ as } \varepsilon\to 0.$$

If $\lambda = (2N+1)(k+1)$ (i. e. $-\frac{\lambda}{k+1} = -(2N+1)$) then the integrand in the right side of (9) changes sign when we replace θ by $\theta-\pi$. Therefore the integral vanishes. Hence $G_{k,\lambda}F_k^{\alpha_1(k,\lambda),\beta_1(k,\lambda),0}(x,y) = 0$.
If $\lambda = 2N(k+1)$ (i. e. $-\frac{\lambda}{k+1} = -2N$) then it follows that

$$-\frac{v(0,0)}{k+1}\int_{-\pi}^{\pi}\Big(k|\sin^{\frac{k}{k+1}}\theta| - i\lambda(\sin\theta)_{\pm}^{-\frac{1}{k+1}}\cos\theta\Big)\big(\sin\theta+i\cos\theta\big)^{-\frac{\lambda}{k+1}}\,d\theta$$
$$= \frac{2(\lambda^2-k^2)v(0,0)}{k(k+1)}\int_0^{\pi}\big(\sin\theta+i\cos\theta\big)^{-\frac{\lambda}{k+1}}\sin^{\frac{k}{k+1}}\theta\,d\theta = a_{k,\lambda}v(0,0).$$

Therefore $G_{k,\lambda}F_k^{\alpha_1(k,\lambda),\beta_1(k,\lambda),0}(x,y) = a_{k,\lambda}\delta(x,y)$.
Next we prove the theorem for $F_k^{\alpha_2(k,\lambda),\beta_2(k,\lambda),1}(x,y)$. As in Theorem 3 we have

$$(10)\quad -\int_{\rho=\varepsilon} v(x,y)B_1(F_k^{\alpha_2(k,\lambda),\beta_2(k,\lambda),1},k,\lambda)\,ds$$
$$+\int_{\rho=\varepsilon} F_k^{\alpha_2(k,\lambda),\beta_2(k,\lambda),1}(x,y)B_2(v,k)\,ds$$
$$\to -\int_{-\pi}^{\pi}\frac{\partial v(0,0)}{\partial x}\left(\frac{(k+2)|\sin\theta|^{\frac{k+2}{k+1}}}{k+1} - \frac{i\lambda(\sin\theta)_{\pm}^{\frac{1}{k+1}}\cos\theta}{k+1}\right)$$
$$\times\big(\sin\theta+i\cos\theta\big)^{-\frac{\lambda}{k+1}}\,d\theta$$

as $\varepsilon\to 0$. If $\lambda = (2N+1)(k+1)$ (i. e. $-\frac{\lambda}{k+1} = -(2N+1)$) then the integrand in the right side of (10) changes sign when we replace θ by $\theta-\pi$, therefore the integral vanishes. Hence $G_{k,\lambda}F_k^{\alpha_2(k,\lambda),\beta_2(k,\lambda),1}(x,y) = 0$.
If $\lambda = 2N(k+1)$ (i. e. $-\frac{\lambda}{k+1} = -2N$) then we deduce that

$$-\frac{\partial v(0,0)}{\partial x}\int_{-\pi}^{\pi}\Big(\frac{(k+2)|\sin\theta|^{\frac{k+2}{k+1}}}{k+1} - \frac{i\lambda(\sin\theta)_{\pm}^{-\frac{1}{k+1}}\cos\theta}{k+1}\Big)$$

$$\times \left(\sin\theta + i\cos\theta\right)^{-\frac{\lambda}{k+1}}\,d\theta$$
$$= \frac{2\left(\lambda^2 - (k+2)^2\right)\frac{\partial v(0,0)}{\partial x}}{(k+1)(k+2)} \int_0^{\pi} \left(\sin\theta + i\cos\theta\right)^{-\frac{\lambda}{k+1}} \sin^{\frac{k+2}{k+1}}\theta\,d\theta$$
$$= -b_{k,\lambda}\frac{\partial v(0,0)}{\partial x}.$$

It follows that $G_{k,\lambda}F_k^{\alpha_2(k,\lambda),\beta_2(k,\lambda),1}(x,y) = b_{k,\lambda}\frac{\partial\delta(x,y)}{\partial x}$. □

Corollary 3. *If k is even and $\lambda = (2N+1)(k+1)$, where N is an integer, then $G_{k,\lambda}$ is not hypoelliptic (nor analytic hypoelliptic).*

Remark 2. Altogether Corollary 1, Corollary 2 and Corollary 3 give all the values k, λ as stated in Theorem 1.

Remark 3. Since $G_{k,\lambda}$ is invariant under the translation $(x,y) \longrightarrow (x, y+c)$ it is easy to have the fundamental solutions or singular solutions at points $(0,c)$ in all cases considered above.

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Institute of Mathematics
P. O. Box 631, Boho 10000
Hanoi, Vietnam
and
Institute of Mathematics
University of Tsukuba
Tsukuba, Ibaraki 305-8571
Japan