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On the Isomorphism Classes of Iwasawa Modules Associated to Imaginary Quadratic Fields with $\boldsymbol{\lambda} = 2$

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Abstract. Let p be an odd prime number. Let $\Lambda = \mathbb{Z}_p[[T]]$. We determine the Λ-isomorphism classes of finitely generated Λ-torsion Λ-modules with $\lambda = 2$ and $\mu = 0$ which have no non-trivial finite Λ-submodule. We apply this classification to Iwasawa modules $X = \varprojlim A_n$ associated to the cyclotomic $\mathbb{Z}_p$-extensions of imaginary quadratic fields and give some numerical examples.

1. Introduction

Let p be an odd prime number. Let K be an imaginary abelian field and K_∞ the cyclotomic $\mathbb{Z}_p$-extension of K, namely K_∞ is the maximal p-extension of K in $K(\zeta_{p^\infty})$. For each $n \geq 0$, let K_n be the intermediate field of K_∞/K such that K_n is a cyclic extension of degree p^n over K. Let A_n be the p-Sylow subgroup of the ideal class group of K_n. Set $X = \varprojlim A_n$, where the inverse limit is with respect to the relative norms. Then X becomes a $\Lambda = \mathbb{Z}_p[[T]]$-module by fixing a topological generator of $\mathrm{Gal}(K_\infty/K)$. Furthermore X is a finitely generated Λ-torsion Λ-module. It is known that the odd part X^- has no non-trivial finite Λ-submodule.

For a distinguished polynomial $f(T) \in \mathbb{Z}_p[T]$, let $\mathcal{M}_{f(T)}$ be the set of Λ-isomorphism classes of finitely generated Λ-torsion Λ-modules N such that

$$\begin{cases} N \text{ has no non-trivial finite } \Lambda\text{-submodule}, \\ \mathrm{char} N = f(T), \end{cases}$$

where $\mathrm{char}N$ is the characteristic polynomial of N. Then $[X^-] \in \mathcal{M}_{\mathrm{char}X^-}$. Here $[X^-]$ represents the Λ-isomorphism class of X^-.

Sumida [Su] showed that $\mathcal{M}_{f(T)}$ is a finite set if and only if $f(T)$ is separable. He determined the set $\mathcal{M}_{f(T)}$ when $f(T) = (T - \alpha)(T - \beta)$, where $\alpha, \beta \in p\mathbb{Z}_p$ (Proposition 2.1). In this paper, we extend this fact and

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determine the set $\mathcal{M}_{f(T)}$ completely when $\deg(f(T)) = 2$ (Theorem 2.1). In particular we can find that $\mathcal{M}_{f(T)}$ is a finite set if and only if $f(T)$ is separable when $\deg(f(T)) = 2$.

Next we deal with the adjoint module $\alpha(M)$ of a finitely generated Λ-torsion Λ-module M ([Fe], [Wa, §15]). It is known that $[\alpha(M)] \in \mathcal{M}_{f(T)}$, where $f(T) = \mathrm{char}M$. We consider a relation between $[\alpha(M)]$ and $[M]$ when M has no non-trivial finite Λ-submodule. By using Theorem 2.1, we shall show that $[\alpha(M)] = [M]$, i.e., $\alpha(M) \cong M$ when $\deg(f(T)) \leq 2$ (Theorem 3.1).

We apply Theorem 2.1 to the above X^-. We let K be an imaginary quadratic field. Then $X = X^-$ and $\mathrm{char}X$ can be approximately calculated by the Iwasawa main conjecture. We shall determine the Λ-isomorphism classes of X when $\deg(\mathrm{char}X) = 2$.

As numerical examples, we deal with the case of $p = 3, K = \mathbb{Q}(\sqrt{-m})$, where $1 < m < 10^5$ and $m \not\equiv 2 \mod 3$. The total number of such fields with $\deg(\mathrm{char}X) = 2$ is 3286. We give two methods to determine the Λ-isomorphism classes of X. One uses the ideal class groups A_n and the other the unit group of the real quadratic field $\mathbb{Q}(\sqrt{3m})$. By using the former method, we determine the Λ-isomorphism classes of X for 3260 fields among the 3286. By using the latter method, we determine 7 fields among the remaining 26 fields.

An outline of this paper is as follows. In §2 we give the proof of Theorem 2.1. In §3 we show Theorem 3.1. In §6 we give two methods to determine the Λ-isomorphism classes of X associated to imaginary quadratic fields K and give numerical examples. Both §4 and §5 are preparation for §6.

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2. Isomorphism Classes of Finitely Generated Λ-Torsion Λ-Modules

Let p be an odd prime number. Let E be a finite extension of the p-adic number field $\mathbb{Q}_p$. Let $\mathcal{O}_E$, π_E and ord_E denote the integer ring, a prime element and a normalized additive valuation of E, respectively. We denote the formal power series ring $\mathcal{O}_E[[T]]$ by Λ_E.

Let both M and M' be Λ_E-modules. A Λ_E-homomorphism

$$\varphi : M \to M'$$

is called a pseudo-isomorphism if the kernel and the cokernel of φ are both finite Λ_E-modules. When there exists a pseudo-isomorphism, we write

$$M \sim M'.$$

For a finitely generated Λ_E-torsion Λ_E-module M, there is a pseudo-isomorphism

$$M \to (\bigoplus_{i=1}^{s} \Lambda_E/(\pi_E^{m_i})) \oplus (\bigoplus_{j=1}^{t} \Lambda_E/(f_j(T)^{n_j})),$$

where m_i and n_j are non-negative integers and $f_j(T)$ is an irreducible distinguished polynomial in $\mathcal{O}_E[T]$. ([Wa, Theorem 13.12]). We call a Λ_E-module of the right hand side an elementary Λ_E-module associated to M and denote it by $\mathcal{E}(M)$. We put

$$\lambda(M) = \sum_{j=1}^{t} n_j \mathrm{deg}(f_j(T)),\ \mu(M) = \sum_{i=1}^{s} m_i,$$

and call the λ-invariant and the μ-invariant of M, respectively. Furthermore we put

$$\mathrm{char} M = \pi_E^{\mu(M)} \prod_{j=1}^{t} f_j(T)^{n_j}.$$

It is called the characteristic polynomial of M.

For a distinguished polynomial $f(T) \in \mathcal{O}_E[T]$, let $\mathcal{M}_{f(T)}^E$ be the set of Λ_E-isomorphism classes of finitely generated Λ_E-torsion Λ_E-modules N such that

$$\begin{cases} N \text{ has no non-trivial finite } \Lambda_E\text{-submodule}, \\ \mathrm{char} N = f(T). \end{cases}$$

We denote the Λ_E-isomorphism class of N by $[N]_E$. The set $\mathcal{M}^E_{f(T)}$ has been introduced by Sumida [Su].

It is easy to see

$$\mathcal{M}^E_{f(T)} = \{[\Lambda_E/(f(T))]_E\}$$

when $\deg(f(T)) = 1$.

Now we assume that $f(T)$ is a distinguished polynomial of degree 2. Let F be the splitting field of $f(T)$ over E. Then we can write

$$f(T) = (T-\alpha)(T-\beta),$$

where $\alpha, \beta \in (\pi_F)$.

For every $[N]_E \in \mathcal{M}^E_{f(T)}$, we may assume that N is a Λ_E-submodule of $\mathcal{E}(N)$ of finite index. Here $\mathcal{E}(N) = \Lambda_E/(T-\alpha) \oplus \Lambda_E/(T-\beta)$ or $\Lambda_E/(f(T))$. Since $\mathcal{E}(N) \cong \mathcal{O}_E \oplus \mathcal{O}_E$, N is a free $\mathcal{O}_E$-module of rank 2. Therefore N is written in the form

$$N = \langle (a,b), (c,d)\rangle_E \ \subset \Lambda_E/(T-\alpha) \oplus \Lambda_E/(T-\beta)$$

or

$$N = \langle aT+b,\ cT+d\rangle_E \ \subset \Lambda_E/(f(T)),$$

where $a,b,c,d \in \mathcal{O}_E,\ (a,b) \in \mathcal{O}_E \oplus \mathcal{O}_E \cong \Lambda_E/(T-\alpha) \oplus \Lambda_E/(T-\beta)$ and $N = \langle e,\ f\rangle_E$ means that N is generated by e and f over $\mathcal{O}_E$.

Lemma 2.1. *Assume that* $\mathrm{ord}_E(a) \le \mathrm{ord}_E(c)$. *Then*
(i) *An $\mathcal{O}_E$-module $\langle (a,b),(c,d)\rangle_E$ is a Λ_E-module if and only if* $\mathrm{ord}_E(d - a^{-1}bc) - \mathrm{ord}_E(b) \le \mathrm{ord}_E(\beta - \alpha)$.
(ii) *An $\mathcal{O}_E$-module $\langle aT+b,\ cT+d\rangle_E$ is a Λ_E-module if and only if* $\mathrm{ord}_E(a) \le \mathrm{ord}_E(b)$ *and* $\mathrm{ord}_E(a) \le \mathrm{ord}_E(d - a^{-1}bc) \le \mathrm{ord}_E(a) + \mathrm{ord}_E(f(-\frac{b}{a}))$. *In particular $\langle T+b,\ d\rangle_E$ is a Λ_E-module if and only if* $0 \le \mathrm{ord}_E(d) \le \mathrm{ord}_E(f(-b))$.

Proof. We show only (i) since we can show (ii) in similar way.

First $\langle (a,b),(c,d)\rangle_E = \langle (a,b),(0,d-a^{-1}bc)\rangle_E$. Because T acts on (a,b) by

$$\begin{aligned} T(a,\ b) &= (\alpha a,\ \beta b) \\ &= \alpha(a,\ b) + (\beta-\alpha)(0,\ b), \end{aligned}$$

it follows that $\langle (a,b),(c,d)\rangle_E$ is a Λ_E-module if and only if $\mathrm{ord}_E(d-a^{-1}bc) \le \mathrm{ord}_E(\beta-\alpha)+\mathrm{ord}_E(b)$. $\square$

THEOREM 2.1. *Let* $f(T) = T^2 + c_1 T + c_0$. *Then*

$$\mathcal{M}^E_{f(T)} = \{ \ [N]_E \mid N = \langle T + \frac{c_1}{2}, \ \pi_E^x \rangle_E, \ 0 \le x \le \frac{1}{2}\mathrm{ord}_E(c_1^2 - 4c_0) \ \}.$$

DEFINITION 2.1. We put

$$N_x = \langle T + \frac{c_1}{2}, \ \pi_E^x \rangle_E$$

for $0 \le x \le \frac{1}{2}\mathrm{ord}_E(c_1^2 - 4c_0)$.

If $f(T) = (T-\alpha)^2$, then we put

$$N_\infty = \Lambda_E/(T-\alpha) \oplus \Lambda_E/(T-\alpha)$$

for convenience.

PROOF OF THEOREM 2.1. We shall divide the proof of this theorem into the following three cases:

1. $f(T)$ is separable and reducible over E.
2. $f(T)$ is irreducible over E.
3. $f(T)$ is inseparable.

Here we call $f(T)$ separable when $f(T)$ has no multiple root.

1. When $f(T)$ is separable and reducible over E.

In this case Sumida proved the following.

PROPOSITION 2.1 ([Su] Proposition 10). *Let* $N = \langle (a,b),(c,d)\rangle_E$ *be a* Λ_E*-module such that* $[N]_E \in \mathcal{M}^E_{f(T)}$. *Assume that* $\mathrm{ord}_E(a) \le \mathrm{ord}_E(c)$. *Then*

$$N \cong \langle (1,1),(0,\pi_E^k)\rangle_E,$$

where $k = \max\{0, \ \mathrm{ord}_E(d-a^{-1}bc) - \mathrm{ord}_E(b)\}$. *Moreover if* $0 \le k \ne k' \le \mathrm{ord}_E(\beta-\alpha)$, *then* $\langle (1,1),(0,\pi_E^k)\rangle_E \not\cong \langle (1,1),(0,\pi_E^{k'})\rangle_E$. *In other words,*

$$\mathcal{M}^E_{f(T)} = \{ \ [\langle (1,1),(0,\pi_E^k)\rangle_E]_E \mid 0 \le k \le \mathrm{ord}_E(\beta-\alpha) \ \}.$$

We shall show

$$\langle (1,1),(0,\pi_E^k)\rangle_E \cong N_x \subset \Lambda_E/(T-\alpha)(T-\beta),$$

where $x = \mathrm{ord}_E(\beta-\alpha)-k$. We have an Λ_E-isomorphism $N_x \cong \langle (\frac{\alpha-\beta}{2}, \frac{\beta-\alpha}{2}),(\pi_E^x,\pi_E^x)\rangle_E$ under the canonical injective Λ_E-homomorphism with finite cokernel $\Lambda_E/(T-\alpha)(T-\beta) \to \Lambda_E/(T-\alpha)\oplus\Lambda_E/(T-\beta)$. By Proposition 2.1 and $\mathrm{ord}_E(\beta-\alpha) = \frac{1}{2}\mathrm{ord}_E(c_1^2-4c_0) \geq x$, $\langle (\frac{\alpha-\beta}{2}, \frac{\beta-\alpha}{2}),(\pi_E^x,\pi_E^x)\rangle_E \cong \langle (1,1),(0,\pi_E^{\mathrm{ord}_E(\beta-\alpha)-x})\rangle_E$ as required.

2. When $f(T)$ is irreducible over E.

For every $[N]_E \in \mathcal{M}_{f(T)}^E$, we might choose $N \not\subset (\pi_E, f(T))/(f(T))$ because the multiplication by $\pi_E : N \to \pi_E N$ is a Λ_E-isomorphism. Then N is written in the form

$$N = \langle T-a,\ \pi_E^x\rangle_E,$$

where $a \in \mathcal{O}_E$ and $x \geq 0$. Since N is a Λ_E-module, we find $0 \leq x \leq \mathrm{ord}_E(f(a))$ by Lemma 2.1.

We need several lemmas which are proved easily. The first lemma is as follows.

LEMMA 2.2. *There is a Λ_E-isomorphism $\Lambda_F \to \Lambda_E \oplus \Lambda_E$ induced by an $\mathcal{O}_E$-isomorphism $\mathcal{O}_F \to \mathcal{O}_E \oplus \mathcal{O}_E$. Therefore Λ_F is a faithfully flat Λ_E-module.*

Next lemma is an immediate consequence of this.

LEMMA 2.3. *Let N and N' be arbitrary Λ_E-modules. Then $N \cong N'$ as Λ_E-modules if and only if $N \otimes_{\Lambda_E} \Lambda_F \cong N' \otimes_{\Lambda_E} \Lambda_F$ as Λ_F-modules.*

The following lemma follows from Lemma 2.2 and the fact that an elementary Λ_E-module has no non-trivial finite Λ_E-submodule.

LEMMA 2.4. *Let N be a Λ_E-module such that $[N]_E \in \mathcal{M}_{f(T)}^E$. Then $[N \otimes_{\Lambda_E} \Lambda_F]_F \in \mathcal{M}_{f(T)}^F$.*

By these lemmas we may consider the Λ_F-isomorphism class of $N\otimes_{\Lambda_E}\Lambda_F$ instead of the Λ_E-isomorphism class of N. If $N = \langle T-a,\ \pi_E^x\rangle_E$, then

$N \otimes_{\Lambda_E} \Lambda_F = \langle T-a,\ \pi_E^x\rangle_F$. Moreover by Proposition 2.1, as in the case when $f(T)$ is reducible, $\langle T-a,\ \pi_E^x\rangle_F \cong \langle(1,1),(0,\pi_F^k)\rangle_F$, where

$$k = \begin{cases} \mathrm{ord}_F(\beta-\alpha) + \mathrm{ord}_F(\pi_E^x) - 2\mathrm{ord}_F(\alpha-a) & \\ & \text{if } \mathrm{ord}_F(\alpha-a) \le \mathrm{ord}_F(\pi_E^x) \\ \mathrm{ord}_F(\beta-\alpha) - \mathrm{ord}_F(\pi_E^x) & \text{if } \mathrm{ord}_F(\alpha-a) \ge \mathrm{ord}_F(\pi_E^x). \end{cases}$$

From this and Lemma 2.1, k can take any value in $0 \le k \le \mathrm{ord}_F(\beta-\alpha)$ if F/E is an unramified extension, while in $0 \le k \le \mathrm{ord}_F(\beta-\alpha)$ and in the form $k = \mathrm{ord}_F(\beta-\alpha) - 2m,\ m \in \mathbb{N}$ if F/E is a ramified extension.

Now we shall show that Λ_E-modules N_x make a system of representatives of Λ_E-isomorphism classes. Since $\mathrm{ord}_F(\alpha+\frac{c_1}{2}) = \mathrm{ord}_F(\beta-\alpha) = \frac{1}{2}\mathrm{ord}_F(c_1^2-4c_0) \ge \mathrm{ord}_F(\pi_E^x)$, $N_x \otimes_{\Lambda_E} \Lambda_F \cong \langle(1,1),(0,\pi_F^k)\rangle_F$, where $k = \mathrm{ord}_F(\beta-\alpha) - \mathrm{ord}_F(\pi_E^x)$ which takes any value in $0 \le k \le \mathrm{ord}_F(\beta-\alpha)$ if F/E is an unramified extension, while in $0 \le k \le \mathrm{ord}_F(\beta-\alpha)$ and in the form $k = \mathrm{ord}_F(\beta-\alpha) - 2m,\ m \in \mathbb{N}$ if F/E is a ramified extension. Hence k takes all possible values when x runs the range $0 \le x \le \frac{1}{2}\mathrm{ord}_E(c_1^2-4c_0)$. Finally if $x \ne x'$, then $k \ne k' = \mathrm{ord}_F(\beta-\alpha) - \mathrm{ord}_F(\pi_E^{x'})$, therefore $N_x \otimes_{\Lambda_E} \Lambda_F \not\cong N_{x'} \otimes_{\Lambda_E} \Lambda_F$ by Proposition 2.1. Hence $N_x \not\cong N_{x'}$ by Lemma 2.3.

3. When $f(T)$ is inseparable.

In this case $\mathcal{E}(N) = \Lambda_E/(T-\alpha) \oplus \Lambda_E/(T-\alpha)$ or $\Lambda_E/(T-\alpha)^2$ for $[N]_E \in \mathcal{M}_{f(T)}^E$ and $\alpha = -\frac{c_1}{2}$. If $\mathcal{E}(N) = \Lambda_E/(T-\alpha) \oplus \Lambda_E/(T-\alpha)$, then we easily see $N \cong \Lambda_E/(T-\alpha) \oplus \Lambda_E/(T-\alpha) = N_\infty$. If $\mathcal{E}(N) = \Lambda_E/(T-\alpha)^2$, then $N = \langle T-a,\ \pi_E^x\rangle_E$ with $0 \le x \le 2\mathrm{ord}_E(\alpha-a)$ in the same way as the irreducible case.

We shall classify $\langle T-a,\ \pi_E^x\rangle_E$ by Λ_E-isomorphisms.

(I) When $a \ne \alpha$. If $0 \le x \le \mathrm{ord}_E(\alpha-a)$, then $\langle T-a,\ \pi_E^x\rangle_E = \langle T-\alpha,\ \pi_E^x\rangle_E = \langle T+\frac{c_1}{2},\ \pi_E^x\rangle_E$. Suppose $\mathrm{ord}_E(\alpha-a) < x \le 2\mathrm{ord}_E(\alpha-a)$. Set $A = \mathrm{ord}_E(\alpha-a)$ and $\alpha - a = w\pi_E^A$, where $w \in \mathcal{O}_E^\times$. Then

$$\begin{pmatrix} \pi_E^{x-A} & -w \\ (1-\pi_E^{x-A})w^{-1} & 1 \end{pmatrix} \in GL(2,\mathcal{O}_E)$$

gives a Λ_E-isomorphism $\langle T-a,\ \pi_E^x\rangle_E \to \langle T-\alpha,\ \pi_E^{2A-x}\rangle_E$ with respect to these $\mathcal{O}_E$-bases, i.e., $\langle T-a,\ \pi_E^x\rangle_E \cong \langle T-\alpha,\ \pi_E^{2A-x}\rangle_E$.

(II) When $a = \alpha$. Suppose $\langle T-\alpha,\ \pi_E^x\rangle_E \cong \langle T-\alpha,\ \pi_E^y\rangle_E$, where $x, y \geq 0$. Then there exists $\begin{pmatrix} s & t \\ u & v \end{pmatrix} \in GL(2, \mathcal{O}_E)$ such that

$$\begin{pmatrix} s & t \\ u & v \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ \pi_E^x & \alpha \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \pi_E^y & \alpha \end{pmatrix}\begin{pmatrix} s & t \\ u & v \end{pmatrix}.$$

By comparing (1,1)-entries, we get $t = 0$. Then $s, v \in \mathcal{O}_E^\times$ since $\det\begin{pmatrix} s & t \\ u & v \end{pmatrix} = sv \in \mathcal{O}_E^\times$. On the other hand we get $v\pi_E^x = s\pi_E^y$ by comparing (2,1)-entries. Therefore $x = \mathrm{ord}_E(v\pi_E^x) = \mathrm{ord}_E(s\pi_E^y) = y$.

Finally since $(T-\alpha)N_\infty = 0$ and $(T-\alpha)N_x = \langle \pi_E^x(T-\alpha)\rangle_E \neq 0$, $N_\infty \ncong N_x$ for all $x \geq 0$.

The proof of Theorem 2.1 is completed. $\square$

REMARK 2.1. When $f(T) = T^2 + c_1T + c_0$, the set $\mathcal{M}_{f(T)}^E$ is a finite set if and only if $f(T)$ is separable. When $\mathcal{M}_{f(T)}^E$ is a finite set, $\#\mathcal{M}_{f(T)}^E = [\frac{1}{2}\mathrm{ord}_E(c_1^2 - 4c_0)] + 1$, where $[z]$ denotes the maximal integer less than or equal to the real number z. Sumida proved that for a distinguished polynomial $f(T)$ of any degree, the set $\mathcal{M}_{f(T)}^E$ is a finite set if and only if $f(T)$ is separable ([Su, Theorem 2]).

From now on we assume that the base field is $E = \mathbb{Q}_p$ and omit the subscript E of Λ_E, $[\ \ ]_E$, etc.

Let $\omega_n = \omega_n(T) = (1+T)^{p^n} - 1$ and $\dot\omega_n = \omega_n(\dot T)$, $\dot T = (1+p)(1+T)^{-1} - 1$. Here we take $1+p$ as a topological generator of $1 + p\mathbb{Z}_p$.

We show the following two propositions. We will use these propositions in §6 to determine the Λ-isomorphism classes of some Λ-modules associated to imaginary quadratic fields. We shall give two methods to determine them. Proposition 2.2 is used for first method, Proposition 2.3 for second method.

PROPOSITION 2.2. *Put* $\mathrm{char}N_x = T^2 + c_1T + c_0$. *Assume that* $\mathrm{char}N_x$ *and* ω_n *are relatively prime.*

(a) *When* $p \geq 5$, *or* $p = 3$ *and* $\mathrm{ord}_3(c_0) \geq 2$.

For $n \geq 0$,

(i) *if* $0 \leq x \leq \frac{1}{2}\mathrm{ord}_p(c_0)$, *then*

$$N_x/\omega_n N_x \cong \mathbb{Z}/p^{\mathrm{ord}_p(c_0)+n-x}\mathbb{Z} \oplus \mathbb{Z}/p^{n+x}\mathbb{Z}.$$

(ii) *if* $\frac{1}{2}\mathrm{ord}_p(c_0) < x \leq \frac{1}{2}\mathrm{ord}_p(c_1^2 - 4c_0)$, *then*

$$N_x/\omega_n N_x \cong \mathbb{Z}/p^{\frac{1}{2}\mathrm{ord}_p(c_0)+n}\mathbb{Z} \oplus \mathbb{Z}/p^{\frac{1}{2}\mathrm{ord}_p(c_0)+n}\mathbb{Z}.$$

Moreover

$$(T + \frac{c_1}{2})(N_x/\omega_n N_x)\begin{cases} = 0 & if\ x \geq \frac{1}{2}\mathrm{ord}_p(c_0) + n \\ \neq 0 & if\ x < \frac{1}{2}\mathrm{ord}_p(c_0) + n. \end{cases}$$

(b) *When* $p = 3$, $\mathrm{ord}_3(c_0) = 1$ *and* $(c_1,\ c_0) \neq (3,\ 3)$.
For $n = 0$,

$$N_0/TN_0 \cong \mathbb{Z}/3\mathbb{Z}.$$

For $n \geq 1$,
(iii) *if* $\mathrm{ord}_3(c_0 - 3) > \mathrm{ord}_3(c_1 - 3)$, *then*

$$N_0/\omega_n N_0 \cong \mathbb{Z}/3^{\mathrm{ord}_3(c_1-3)+n}\mathbb{Z} \oplus \mathbb{Z}/3^{\mathrm{ord}_3(c_1-3)+n}\mathbb{Z}.$$

(iv) *if* $\mathrm{ord}_3(c_0 - 3) \leq \mathrm{ord}_3(c_1 - 3)$, *then*

$$N_0/\omega_n N_0 \cong \mathbb{Z}/3^{\mathrm{ord}_3(c_0-3)+n}\mathbb{Z} \oplus \mathbb{Z}/3^{\mathrm{ord}_3(c_0-3)+n-1}\mathbb{Z}.$$

Proposition 2.3. *Put* $\mathrm{char}N_x = T^2 + c_1T + c_0$. *Assume that* $\mathrm{char}N_x$ *and* ω_n *are relatively prime. Moreover we assume that* $p^2 + c_1p + c_0 \neq 0$ *and that* $\mathrm{ord}_p(c_0) \geq 2$. *Then*

(i) *If* $\mathrm{ord}_p(p + \frac{c_1}{2}) < x$, *then*

$$N_x/\dot{\omega}_n N_x \cong \mathbb{Z}/p^{\mathrm{ord}_p(p+\frac{c_1}{2})+n}\mathbb{Z} \oplus \mathbb{Z}/p^{\mathrm{ord}_p(p+\frac{c_1}{2})+n}\mathbb{Z}.$$

(ii) *If* $\mathrm{ord}_p(p + \frac{c_1}{2}) \geq x$, *then*

$$N_x/\dot{\omega}_n N_x \cong \mathbb{Z}/p^{\mathrm{ord}_p(p^2+c_1p+c_0)+n-x}\mathbb{Z} \oplus \mathbb{Z}/p^{x+n}\mathbb{Z}.$$

PROOF. We omit the proof of Proposition 2.3 since we can show it in the same way of that of Proposition 2.2.

To show Proposition 2.2 we need the following lemma.

LEMMA 2.5. *Let F be the splitting field of $T^2 + c_1 T + c_0$ over $\mathbb{Q}_p$. Let α, β be the roots of $T^2 + c_1 T + c_0 = 0$ in F. Then*

(a) *When $p \geq 5$, or $p = 3$ and $\mathrm{ord}_3(c_0) \geq 2$.*

For $n \geq 0$,

$$\mathrm{ord}_F(\omega_{n+1}(\beta) - \omega_{n+1}(\alpha)) = \mathrm{ord}_F(\omega_n(\beta) - \omega_n(\alpha)) + \mathrm{ord}_F(p).$$

In particular,

$$\mathrm{ord}_F(\omega_n(\beta) - \omega_n(\alpha)) = \mathrm{ord}_F(\beta - \alpha) + n\mathrm{ord}_F(p).$$

(b) *When $p = 3$ and $\mathrm{ord}_3(c_0) = 1$.*

For $n \geq 1$,

$$\mathrm{ord}_F(\omega_{n+1}(\beta) - \omega_{n+1}(\alpha)) = \mathrm{ord}_F(\omega_n(\beta) - \omega_n(\alpha)) + \mathrm{ord}_F(p).$$

In particular,

$$\mathrm{ord}_F(\omega_n(\beta) - \omega_n(\alpha)) = \mathrm{ord}_F(\omega_1(\beta) - \omega_1(\alpha)) + (n-1)\mathrm{ord}_F(p).$$

For $n = 0$,

$$\begin{aligned} &\mathrm{ord}_F(\omega_1(\beta) - \omega_1(\alpha)) \\ &\begin{cases} = \mathrm{ord}_F(\omega_1(\alpha)) = 2\mathrm{ord}_3(c_0 - 3) + 1 & \textit{if } \mathrm{ord}_3(c_0 - 3) \leq \mathrm{ord}_3(c_1 - 3) \\ > \mathrm{ord}_F(\omega_1(\alpha)) = 2\mathrm{ord}_3(c_1 - 3) + 2 & \textit{if } \mathrm{ord}_3(c_0 - 3) > \mathrm{ord}_3(c_1 - 3). \end{cases} \end{aligned}$$

We postpone the proof of Lemma 2.5.

Because

$$\omega_n \equiv \frac{\omega_n(\beta) - \omega_n(\alpha)}{\beta - \alpha} T + \frac{\beta\omega_n(\alpha) - \alpha\omega_n(\beta)}{\beta - \alpha} \mod T^2 + c_1 T + c_0$$

and $c_1 = -(\alpha+\beta)$, we have

$$\begin{aligned}\omega_n N_x = \langle & \frac{\omega_n(\alpha)+\omega_n(\beta)}{2}(T+\frac{c_1}{2}) + \frac{p^{-x}(\beta-\alpha)(\omega_n(\beta)-\omega_n(\alpha))}{4}p^x, \\ & \frac{p^x(\omega_n(\beta)-\omega_n(\alpha))}{\beta-\alpha}(T+\frac{c_1}{2}) + \frac{\omega_n(\alpha)+\omega_n(\beta)}{2}p^x\rangle.\end{aligned}$$

We change the generators of $\omega_n N_x$ suitably.
(a) When $p \geq 5$, or $p=3$ and $\mathrm{ord}_3(c_0) \geq 2$.
(i) If $0 \leq x \leq \frac{1}{2}\mathrm{ord}_p(c_0)$, then

$$\begin{aligned}\omega_n N_x = \langle & \frac{(\beta-\alpha)\omega_n(\alpha)\omega_n(\beta)}{p^x(\omega_n(\beta)-\omega_n(\alpha))}\ p^x, \\ & \frac{p^x(\omega_n(\beta)-\omega_n(\alpha))}{\beta-\alpha} \\ & \times((T+\frac{c_1}{2}) + \frac{\omega_n(\alpha)+\omega_n(\beta)}{2}\frac{\beta-\alpha}{p^x(\omega_n(\beta)-\omega_n(\alpha))}\ p^x)\rangle.\end{aligned}$$

These coefficients are contained in $\mathbb{Z}_p$ by Lemma 2.5. Hence

$$N_x/\omega_n N_x \cong \mathbb{Z}/p^{\mathrm{ord}_p(c_0)+n-x}\mathbb{Z} \oplus \mathbb{Z}/p^{n+x}\mathbb{Z}.$$

(ii) If $\frac{1}{2}\mathrm{ord}_p(c_0) < x \leq \frac{1}{2}\mathrm{ord}_p(c_1^2-4c_0)$, then

$$\begin{aligned}\omega_n N_x &= \langle \frac{\omega_n(\alpha)+\omega_n(\beta)}{2}((T+\frac{c_1}{2}) + \frac{(\beta-\alpha)(\omega_n(\beta)-\omega_n(\alpha))}{2p^x(\omega_n(\alpha)+\omega_n(\beta))}p^x), \\ &\qquad \frac{\omega_n(\alpha)+\omega_n(\beta)}{2}(\frac{2p^x(\omega_n(\beta)-\omega_n(\alpha))}{(\beta-\alpha)(\omega_n(\alpha)+\omega_n(\beta))}(T+\frac{c_1}{2}) + p^x)\rangle \\ &= \frac{\omega_n(\alpha)+\omega_n(\beta)}{2}\langle T+\frac{c_1}{2},\ p^x\rangle.\end{aligned}$$

Since $\mathrm{ord}_F(\alpha) = \mathrm{ord}_F(\beta) = \frac{1}{2}\mathrm{ord}_p(c_0) < \frac{1}{2}\mathrm{ord}_p(c_1^2-4c_0) = \mathrm{ord}_F(\beta-\alpha)$ and $\omega_n(\alpha)+\omega_n(\beta) = (\omega_n(\beta)-\omega_n(\alpha)) + 2\omega_n(\alpha)$, we get

$$\mathrm{ord}_p(\frac{\omega_n(\alpha)+\omega_n(\beta)}{2}) = \frac{1}{2}\mathrm{ord}_p(c_0) + n$$

by Lemma 2.5. Hence

$$N_x/\omega_n N_x \cong \mathbb{Z}/p^{\frac{1}{2}\mathrm{ord}_p(c_0)+n}\mathbb{Z} \oplus \mathbb{Z}/p^{\frac{1}{2}\mathrm{ord}_p(c_0)+n}\mathbb{Z}.$$

Here $\mathrm{ord}_p(c_0)$ is an even number because $\mathrm{ord}_p(c_0) < \mathrm{ord}_p(c_1^2 - 4c_0)$. Moreover

$$(T + \frac{c_1}{2})N_x = \langle p^x(T + \frac{c_1}{2}), \frac{c_1^2 - 4c_0}{4p^x}p^x\rangle.$$

Therefore

$$\begin{aligned}
&(T + \frac{c_1}{2})N_x \subset \omega_n N_x \\
&\quad \Leftrightarrow \mathrm{ord}_F(p^x) \geq \mathrm{ord}_F(\frac{\omega_n(\alpha) + \omega_n(\beta)}{2}) = \mathrm{ord}_F(\alpha) + n\mathrm{ord}_F(p) \\
&\quad\quad \text{and } \mathrm{ord}_F(\frac{c_1^2 - 4c_0}{4p^x}) \geq \mathrm{ord}_F(\frac{\omega_n(\alpha) + \omega_n(\beta)}{2}) \\
&\quad \Leftrightarrow x \geq \frac{1}{2}\mathrm{ord}_p(c_0) + n,
\end{aligned}$$

i.e.,

$$(T + \frac{c_1}{2})(N_x/\omega_n N_x)\begin{cases} = 0 & if\ x \geq \frac{1}{2}\mathrm{ord}_p(c_0) + n \\ \neq 0 & if\ x < \frac{1}{2}\mathrm{ord}_p(c_0) + n. \end{cases}$$

We can show the case (b) similarly. □

Proof of Lemma 2.5. We have

$$\omega_{n+1}(\beta) - \omega_{n+1}(\alpha) = (\omega_n(\beta) - \omega_n(\alpha))\Phi(\alpha,\ \beta),$$

where

$$\Phi(\alpha,\ \beta) = (1+\beta)^{p^n(p-1)} + (1+\beta)^{p^n(p-2)}(1+\alpha)^{p^n} + \cdots + (1+\alpha)^{p^n(p-1)}.$$

Short calculations show $\mathrm{ord}_F(\Phi(\alpha,\ \beta)) = \mathrm{ord}_F(p)$ for $n \geq 0$ if $p \geq 5$ or if $p = 3$ and $\mathrm{ord}_3(c_0) \geq 2$, and for $n \geq 1$ if $p = 3$ and $\mathrm{ord}_3(c_0) = 1$. If $p = 3$ and $\mathrm{ord}_3(c_0) = 1$, then $F/\mathbb{Q}_p$ is a ramified extension,

$$\omega_1(\alpha) = \alpha((3 - c_1)\alpha + (3 - c_0))$$

and

$$\omega_1(\beta) - \omega_1(\alpha) = (\beta - \alpha)(c_1(c_1 - 3) - (c_0 - 3)).$$

By comparing an order of each term, we get Lemma 2.5. □

3. Adjoint Modules

Let M be a finitely generated Λ-torsion Λ-module. Let $\alpha(M)$ be the adjoint module of M. For the definition and some properties of $\alpha(M)$, see [Fe], [Wa, §15.5], etc.

Now we consider the following question.

QUESTION. If M is a finitely generated Λ-torsion Λ-module which has no non-trivial finite Λ-submodule, then $\alpha(M) \cong M$?

This answer is not known in general. But we shall show this is true when $\lambda(M) \leq 2$ and $\mu(M) = 0$.

THEOREM 3.1. *Let M be a finitely generated Λ-torsion Λ-module which has no non-trivial finite Λ-submodule. If $\lambda(M) \leq 2$ and $\mu(M) = 0$, then $\alpha(M) \cong M$.*

PROOF. If $\lambda(M) = 1$ and $\mu(M) = 0$, then M is isomorphic to an elementary Λ-module. Hence $\alpha(M) \cong M$.

Next we consider the case when $\lambda(M) = 2$ and $\mu(M) = 0$. First

$$\alpha(M) \cong \varprojlim \operatorname{Hom}(M/p^{n+1}M, \mathbb{Q}_p/\mathbb{Z}_p),$$

where the inverse limit is with respect to the maps induced by the maps

$$\begin{array}{ccc} M/p^{n+1}M & \to & M/p^{m+1}M \\ z & \mapsto & p^{m-n}z \qquad m \geq n \geq 0 \end{array}$$

([Fe, Theorem 2.7]). Since $\lambda(M) = 2$ and $\mu(M) = 0$, M is a free $\mathbb{Z}_p$-module of rank 2. Hence $M = \langle b_1,\ b_2\rangle$, $M/p^{n+1}M = \langle b_1 \mod p^{n+1}M,\ b_2 \mod p^{n+1}M\rangle$. Let $g_{1n},\ g_{2n}$ be the dual bases of $M/p^{n+1}M$. Then $g_1 = (g_{1n})_n,\ g_2 = (g_{2n})_n \in \varprojlim \operatorname{Hom}(M/p^{n+1}M, \mathbb{Q}_p/\mathbb{Z}_p)$ with respect to the above maps. Clearly $g_1,\ g_2$ are linearly independent over $\mathbb{Z}_p$. Therefore $\alpha(M) \cong \langle g_1,\ g_2\rangle$. Let A be the transformation matrix associated to the multiplication by T map

$$M = \langle b_1,\ b_2\rangle \xrightarrow{\times T} M = \langle b_1,\ b_2\rangle.$$

Then that of $\alpha(M) = \langle g_1,\ g_2\rangle$ is ${}^tA$ by the observation on each $M/p^{n+1}M$ and the definition of the action of T on $\alpha(M)$. Therefore $\alpha(M) \cong M$ if and only if there exists some $S \in GL(2, \mathbb{Z}_p)$ such that $SA = {}^tAS$. Since $\lambda(M) = 2$ and $\mu(M) = 0$, we can write

$$\mathrm{char} M = T^2 + c_1 T + c_0, \quad c_0, c_1 \in p\mathbb{Z}_p.$$

By Theorem 2.1, $M \cong N_x$ with some $0 \le x \le \frac{1}{2}\mathrm{ord}_p(c_1^2 - 4c_0) < \infty$ or $M \cong N_\infty$. When $M \cong N_x$ with $x < \infty$,

$$A = \begin{pmatrix} -\frac{c_1}{2} & \frac{c_1^2-4c_0}{4p^x} \\ p^x & -\frac{c_1}{2} \end{pmatrix}.$$

Therefore if we put

$$S = \begin{cases} \begin{pmatrix} 1 & 1 \\ 1 & \frac{c_1^2-4c_0}{4p^{2x}} \end{pmatrix} & \text{if } x \ne \frac{1}{2}\mathrm{ord}_p(c_1^2 - 4c_0) \\ \begin{pmatrix} 1 & 0 \\ 0 & \frac{c_1^2-4c_0}{4p^{2x}} \end{pmatrix} & \text{if } x = \frac{1}{2}\mathrm{ord}_p(c_1^2 - 4c_0), \end{cases}$$

then $SA = {}^tAS$, that is, $\alpha(M) \cong M$.

If $M \cong N_\infty$, then $\alpha(M) \cong M$ since the adjoint preserves direct sums. □

4. Λ-Module X

4.1. Let K be an imaginary abelian field. We assume that K does not contain p^2-th roots of unity. Let K_∞ be the cyclotomic $\mathbb{Z}_p$-extension of K. Let γ_0 be a topological generator of $\mathrm{Gal}(K_\infty/K)$. In this paper we define γ_0 such that

$$\zeta^{\gamma_0} = \zeta^{1+p}$$

for any p-th power root of unity ζ. Let K_n be the n-th layer of K_∞/K and A_n the p-Sylow subgroup of the ideal class group of K_n.

Iwasawa proved that there exist three integers $\lambda = \lambda_p(K) \ge 0, \mu = \mu_p(K) \ge 0$ and $\nu = \nu_p(K)$, all independent of n, such that

$$\#A_n = p^{\lambda n + \mu p^n + \nu}$$

for all sufficiently large n. We call $\lambda_p(K), \mu_p(K)$ and $\nu_p(K)$ the Iwasawa invariants of K.

Set

$$X = \varprojlim A_n,$$

where the inverse limit is with respect to the relative norms. Then X becomes a Λ-module since there is an isomorphism

$$\varprojlim \mathbb{Z}_p[\mathrm{Gal}(K_n/K)] \cong \Lambda$$

induced by $\gamma_0 \mapsto 1 + T$. It is known that

$$\mu(X) = 0$$

(Ferrero-Washington [FW]) and

$$[X^-] \in \mathcal{M}_{\mathrm{char} X^-},$$

where $X^{\pm} = \{ a \in X \mid Ja = \pm a \}$ and J is the complex conjugation ([Wa, Proposition 13.28]).

Our goal is to determine the Λ-isomorphism classes of X^- when $\lambda(X^-) = 2$. It is important to determine the Λ-isomorphism classes of X^- because of the following fact. We assume that exactly one prime is ramified in K_∞/K and it is totally ramified. Then there are Λ-isomorphisms

$$X^-/\omega_n X^- \cong A_n^-, \quad \textit{for all } n \geq 0.$$

([Wa, Proposition 13.22]). Therefore we get the structures of A_n^- for all $n \geq 0$.

4.2. The following Proposition 4.1, which asserts the Iwasawa ν-invariants, follows from Proposition 2.2. We, however, do not need this proposition later.

PROPOSITION 4.1. *Let K be an imaginary abelian field. Assume that exactly one prime is ramified in K_∞/K and that it is totally ramified. Moreover we assume that $A_0^+ = 0$ and that $\lambda_p(K) = 2$. Then*
(a) *When $p \geq 5$, or $p = 3$ and $\#A_0 \geq 3^2$, for $n \geq 0$, we have*

$$\#A_n = p^{2n} \#A_0,$$

i.e.,

$$\nu_p(K) = \mathrm{ord}_p(\#A_0).$$

(b) *When $p = 3$ and $\#A_0 = 3$, for $n \geq 1$, we have*

$$\#A_n = \begin{cases} p^{2n+2\mathrm{ord}_3(c_1-3)} & \textit{if } \mathrm{ord}_3(c_0-3) > \mathrm{ord}_3(c_1-3) \\ p^{2n+2\mathrm{ord}_3(c_0-3)-1} & \textit{if } \mathrm{ord}_3(c_0-3) \leq \mathrm{ord}_3(c_1-3), \end{cases}$$

i.e.,

$$\nu_p(K) = \begin{cases} 2\mathrm{ord}_3(c_1-3) & \textit{if } \mathrm{ord}_3(c_0-3) > \mathrm{ord}_3(c_1-3) \\ 2\mathrm{ord}_3(c_0-3)-1 & \textit{if } \mathrm{ord}_3(c_0-3) \leq \mathrm{ord}_3(c_1-3), \end{cases}$$

where $\mathrm{char}X = T^2 + c_1T + c_0$.

Proof. Because $A_0^+ = 0$, we have $X = X^-$, hence $[X] \in \mathcal{M}_{\mathrm{char}X}$. When $\lambda_p(K) = 2$, i.e., $\lambda(X) = 2$, we can write $\mathrm{char}X = T^2 + c_1T + c_0$. Then, by Theorem 2.1, $X \cong N_x$ for some x.

It is sufficient to show that $\#A_0 = p^{\mathrm{ord}_p(c_0)}$ since $\#A_n = \#N_x/\omega_n N_x$ and Proposition 2.2. There is the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & \Lambda/(T^2+c_1T+c_0) & \longrightarrow & B & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \times T} & & \big\downarrow{\scriptstyle \times T} & & \big\downarrow{\scriptstyle \times T} & & \\ 0 & \longrightarrow & X & \longrightarrow & \Lambda/(T^2+c_1T+c_0) & \longrightarrow & B & \longrightarrow & 0, \end{array}$$

where B is a finite Λ-module. By snake lemma we have an exact sequence

$$0 \to B^\Gamma \to X/TX \to \Lambda/(T,\ T^2+c_1T+c_0) \to B/TB \to 0,$$

where B^Γ is the kernel of the multiplication by $T : B \to B$. Therefore $\#X/TX = p^{\mathrm{ord}_p(c_0)}$, i.e., $\#A_0 = p^{\mathrm{ord}_p(c_0)}$. $\square$

Remark 4.1. When $\lambda < p-1$, or $\lambda = p-1$ and $\#A_0 \geq p^2$, Sands proved

$$\#A_n = p^{\lambda n}\#A_0 \quad \textit{for } n \geq 0$$

([Sa, Theorem 3.1]). Case (a) of Proposition 4.1 also follows from this fact when $\lambda = 2$.

5. X_- and $\mathfrak{X}_+$

5.1. In this section, let $p=3$ and $K=\mathbb{Q}(\sqrt{-m},\ \sqrt{-3})$, where $m>0$ and $m\neq 3$. Let K_∞, K_n, A_n, X be the same meaning as in 4.1. Let M_∞ be the maximal abelian p-extension of K_∞ unramified outside p, L_∞ the maximal unramified abelian p-extension of K_∞. By class field theory, $X\cong \mathrm{Gal}(L_\infty/K_\infty)$. Put $\mathfrak{X}=\mathrm{Gal}(M_\infty/K_\infty)$. Then $\mathfrak{X}$ is a finitely generated Λ-module with no non-trivial finite Λ-submodule ([Iw, Theorem 18]).

There are three intermediate fields of $K/\mathbb{Q}$. For two of them, we put $K_-=\mathbb{Q}(\sqrt{-m})$ and $K_+=\mathbb{Q}(\sqrt{3m})$. We will modify the notation suitably. Thus, $K_{-,\infty}, K_{-,n}$, etc. (resp. $K_{+,\infty}, K_{+,n}$, etc.) will denote the corresponding objects of K_- (resp. K_+).

Let

$$\begin{aligned} \Delta\ &= \mathrm{Gal}(K/\mathbb{Q}) = \{id,\sigma,\tau,\sigma\tau\} \\ \Delta_- &= \mathrm{Gal}(K_-/\mathbb{Q}) \cong \mathrm{Gal}(K/K_+) = \{id,\sigma\} \\ \Delta_+ &= \mathrm{Gal}(K_+/\mathbb{Q}) \cong \mathrm{Gal}(K/K_-) = \{id,\tau\} \end{aligned}$$

and their character groups

$$\begin{aligned} \Delta^\wedge\ &= \{1,\chi,\omega,\chi\omega\} \\ (\Delta_-)^\wedge &= \{1,\chi\} \\ (\Delta_+)^\wedge &= \{1,\chi\omega\}, \end{aligned}$$

where ω is the Teichmüller character. Furthermore let

$$e_\psi = \frac{1}{\#\Delta}\sum_{\delta\in\Delta}\psi(\delta)\delta^{-1} \quad \in \mathbb{Z}_p[\Delta], \quad \textit{for } \psi\in\Delta^\wedge,$$

and

$$M(\psi) = e_\psi M, \quad \textit{for a } \mathbb{Z}_p[\Delta]\textit{-module } M.$$

5.2. Let L_0 be the maximal unramified abelian p-extension of K, and let $Y=\mathrm{Gal}(L_\infty/K_\infty L_0)$. Then Y is a Λ-submodule of X of finite index. It is known that

$$\begin{aligned} \mathfrak{X}(\chi\omega)^\bullet &\cong \alpha(Y(\chi)) \\ &\sim Y(\chi) \\ &\sim X(\chi), \end{aligned}$$

where $\mathfrak{X}(\chi\omega)^\bullet$ is equal to $\mathfrak{X}(\chi\omega)$ as a $\mathbb{Z}_p$-module with new Λ-structure defined by

$$T \cdot x = \dot{T}x, \quad for\ x \in \mathfrak{X}(\chi\omega).$$

(See [Iw, Theorem 11], [Ts, pp. 200].)

Since the order of $\mathrm{Gal}(K_\infty/K_{-,\infty}) \cong <\tau>$ is prime to p,

$$X_- \cong X/(\tau - 1)X.$$

Therefore we have

$$X_- \cong X(\chi)$$

because $X \cong X(1) \oplus X(\chi) \oplus X(\omega) \oplus X(\chi\omega)$, $(\tau - 1)X \cong X(\omega) \oplus X(\chi\omega)$ and $X(1) = 0$. Similarly we have

$$Y_- \cong Y(\chi),$$

$$\mathfrak{X}_+ \cong \mathfrak{X}(\chi\omega).$$

Hence

$$(\mathfrak{X}_+)^\bullet \cong \alpha(Y_-) \sim Y_- \sim X_-.$$

Theorem 5.1. *Assume that $p = 3$ does not split in K_- and that $\lambda(X_-) \leq 2$. Then $(\mathfrak{X}_+)^\bullet \cong X_-$.*

Proof. By Theorem 3.1, we have

$$\alpha(Y_-) \cong Y_-.$$

Because p does not split in K_-, we get $Y_- = TX_-$. Since $\mathrm{char}X_-$ and T are relatively prime, the kernel of the multiplication by $T : X_- \to TX_-$ is finite, hence 0. Therefore

$$Y_- \cong X_-.$$

We get $(\mathfrak{X}_+)^\bullet \cong X_-$ by the above arguments. □

5.3. Let M_n be the maximal abelian extension of K_n in M_∞ and L_n the maximal unramified abelian p-extension of K_n. Then $\mathrm{Gal}(M_n/K_\infty) \cong \mathfrak{X}/\omega_n\mathfrak{X}$ and $\mathrm{Gal}(L_n/K_n) \cong A_n$. Moreover, the structure of $\mathrm{Gal}(M_n/L_n)$ is known.

For each prime divisor v of K_n lying above p, let $U_{n,v}$ be the group of local units in the v-completion $K_{n,v}$ which are congruent to 1 modulo the maximal ideal, and let $\mathcal{U}_n = \prod_{v|p} U_{n,v}$. Let E_n be the group of all units in K_n. We identify E_n with the image of the diagonal embedding $K_n \hookrightarrow \prod_{v|p} K_{n,v}$. Let $\overline{E_n}$ be the closure of $E_n \cap \mathcal{U}_n$ in $\mathcal{U}_n$. By class field theory, $\mathrm{Gal}(M_n/L_n) \cong \mathcal{U}_n/\overline{E_n}$ ([Co, Theorem 1.1], [Wa, Corollary 13.6]). Hence we get the structures of a subgroup and a quotient of $\mathrm{Gal}(M_n/K_n)$ by the unit group and the ideal class group of K_n.

6. Numerical Examples

Let $p = 3$ and let K_-, K_+, etc. be same as in §5. Let χ be the non-trivial primitive Dirichlet character which is associated to K_-. Let f_0 be the least common multiple of p and the conductor of χ. There exists a power series $g_{\chi^{-1}\omega}(T) \in \Lambda$ such that $L_p(s, \chi^{-1}\omega) = g_{\chi^{-1}\omega}((1+p)^s - 1)$ for all $s \in \mathbb{Z}_p$ ([Wa, §7.2]). By p-adic Weierstrass preparation theorem ([Wa, Theorem 7.3]), we can uniquely express $g_{\chi^{-1}\omega}(T) = P_{\chi^{-1}\omega}(T) U_{\chi^{-1}\omega}(T)$, where $P_{\chi^{-1}\omega}(T)$ is a distinguished polynomial and $U_{\chi^{-1}\omega}(T) \in \Lambda^{\times}$. The Iwasawa main conjecture proved by Mazur-Wiles [MW] asserts $\mathrm{char} X_- = P_{\chi^{-1}\omega}(T)$.

Though we cannot get $g_{\chi^{-1}\omega}(T)$ exactly, we can approximate $g_{\chi^{-1}\omega}(T)$ with arbitrary accuracy. An approximation of $g_{\chi^{-1}\omega}(T)$ is as follows.

$$g_{\chi^{-1}\omega}(T) \equiv -\frac{1}{2 f_0 p^n} \sum_{a=1,(a,f_0)=1}^{f_0 p^n} a\chi(a)(1+T)^{-l_n(a)} \mod \omega_n$$

for $n \geq 0$, where $l_n(a)$ is the unique integer such that $a \equiv \omega(a)(1+p)^{l_n(a)}$ mod p^{n+1} and $0 \leq l_n(a) < p^n$. Therefore we can obtain $\mathrm{char} X_-$ approximately ([IS, Lemma 5]) and determine $\lambda(X_-)$ and $w = \frac{1}{2}\mathrm{ord}_p(c_1^2 - 4c_0)$. For details about computation of $g_{\chi^{-1}\omega}(T)$, see, for example, [EM].

Let $K_- = \mathbb{Q}(\sqrt{-m})$, where $1 < m < 10^5$, $m \not\equiv 2 \mod 3$ and m is a square-free integer. We computed $\mathrm{char} X_-$ by the above method with Pari [Pa] and see the total number of such fields with $\lambda(X_-) = 2$ is 3286. We also referred [Fu] for the λ-invariants of imaginary quadratic fields. Table 1 is the distribution of X_-. Here # represents the number of fields.

The purpose of this section is to determine the Λ-isomorphism classes of such X_-. First, by "Nakayama's Lemma" ([Wa, Lemma 13.16]), X_- is a

Table 1. The distribution of X_-

w	$1/2$	1	$3/2$	2	$5/2$	3	$7/2$	4	$\geq 9/2$	total
#	2204	720	244	79	24	10	4	1	0	3286

cyclic Λ-module if and only if $A_{-,0}$ is a cyclic group. There are 3081 fields whose $A_{-,0}$ are cyclic groups, hence $X_- \cong N_0$.

Example 1. Let $K_- = \mathbb{Q}(\sqrt{-1306})$. By computation, $\mathrm{char} X_- \equiv T^2 + 18T + 18 \mod 3^3$ and $w = 1$. On the other hand, we have $A_{-,0} \cong \mathbb{Z}/9\mathbb{Z}$. Therefore $X_- \cong N_0$.

From now on we assume that $A_{-,0}$ is not a cyclic group. There are 205 such fields. In the last of this paper, we find tables of these 205 fields. We give two methods to determine the Λ-isomorphism classes of X_-.

The first method uses $A_{-,n}$ which are isomorphic to $N_x/\omega_n N_x$ as $\mathbb{Z}_p[\mathrm{Gal}(K_{-,n}/K_-)]$-modules. Because $\mathrm{char} X_-$ and ω_n are relatively prime by finiteness of class number, Proposition 2.2 tells us that we can determine the Λ-isomorphism class of X_- by the structures of $A_{-,n}$ for some $n \geq 0$. We use Proposition 2.2 as in the following example. It is the easiest case because we can determine the Λ-isomorphism class of X_- based on the data for $n = 0$. We referred [SW] for the structures of $A_{-,0}$.

Example 2. Let $K_- = \mathbb{Q}(\sqrt{-89269})$. By computation, $\mathrm{char} X_- \equiv T^2 + 1521T + 81 \mod 3^7$ and $w = 3$. By Theorem 2.1,

$$X_- \cong N_1 \text{ or } N_2 \text{ or } N_3.$$

By Proposition 2.2,

$$N_x/\omega_0 N_x \cong \begin{cases} (3^{4-x},\ 3^x) & (x = 1, 2) \\ (9,\ 9) & (x = 3). \end{cases}$$

On the other hand, we have $A_{-,0} \cong \mathbb{Z}/27\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. Therefore we get $x = 1$, i.e., $X_- \cong N_1$.

We can determine the Λ-isomorphism classes of X_- for 179 fields by the structures of $A_{-,0}$ as above example. The remaining 26 fields are in Case (ii) of Proposition 2.2. Therefore we must get the structures of $A_{-,n}$ for $n \geq 1$.

But it is difficult to compute $A_{-,n}$ for $n \geq 1$ because the discriminants of $K_{-,n}$ for $n \geq 1$ are too large.

We give the second method for these 26 fields. By Theorem 5.1, we have $(\mathfrak{X}_+)^\bullet \cong X_-$. Hence if $X_- \cong N_x$ for some x, then $\mathrm{Gal}(M_{+,0}/K_{+,\infty}) \cong \mathfrak{X}_+/T\mathfrak{X}_+ \cong N_x/\dot{T}N_x$. In this case, the assumptions of Proposition 2.3 on $\mathrm{char}X_- = T^2 + c_1T + c_0$ are valid; we have $p^2 + c_1p + c_0 \neq 0$ because of the Iwasawa main conjecture and Leopoldt's conjecture, and we have $\mathrm{ord}_p(c_0) \geq 2$ because $A_{-,0}$ is not a cyclic group. Therefore we get the structure of $\mathrm{Gal}(M_{+,0}/K_{+,\infty})$ by Proposition 2.3, hence that of X_-.

On the other hand, $\mathrm{Gal}(L_{+,0}/K_+) \cong A_{+,0}$ and

$$\begin{aligned}\mathrm{Gal}(M_{+,0}/L_{+,0}) &\cong \mathcal{U}_{+,0}/\overline{E_{+,0}} \\ &\cong (\mathcal{U}_{+,0}/\overline{E_{+,0}})(1) \oplus (\mathcal{U}_{+,0}/\overline{E_{+,0}})(\chi\omega) \\ &\cong \mathbb{Z}_p \oplus (\mathcal{U}_{+,0}(\chi\omega)/\overline{E_{+,0}}(\chi\omega)).\end{aligned}$$

Since p does not split in K_-,

$$\mathcal{U}_{+,0}(\chi\omega) \cong \mathbb{Z}_p$$

([Gi, Proposition 1,2]). Therefore we can get the structure of $\mathrm{Gal}(M_{+,0}/L_{+,0})$ by investigating $\overline{E_{+,0}}(\chi\omega)$. Hence we can determine the Λ-isomorphism classes of X_- by the structures of $A_{+,0}$ and $\mathcal{U}_{+,0}/\overline{E_{+,0}}$. We computed $A_{+,0}$ and $\mathcal{U}_{+,0}/\overline{E_{+,0}}$ with KANT [KA].

Example 3. Let $K_- = \mathbb{Q}(\sqrt{-10173})$. By computation, $\mathrm{char}X_- \equiv T^2 + 102T + 9 \mod 3^5$ and $w = 2$. Hence it follows from Theorem 2.1 that

$$X_- \cong N_1 \text{ or } N_2.$$

By Proposition 2.2,

$$N_x/\omega_0 N_x \cong (3,\ 3) \quad (x = 1, 2)$$

and $A_{-,0} \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. Therefore we cannot determine the Λ-isomorphism class of X_- by first method.

Next we consider $K_+ = \mathbb{Q}(\sqrt{3391})$. By another computation, $\mathrm{Gal}(M_{+,0}/L_{+,0}) \cong \mathbb{Z}_3 \oplus \mathbb{Z}/27\mathbb{Z}$ and $A_{+,0} \cong \mathbb{Z}/3\mathbb{Z}$. On the other hand, by Proposition 2.3 (ii),

$$N_x/\dot{\omega}_0 N_x \cong (3^{4-x},\ 3^x) \quad (x = 1, 2).$$

Table 2.

m	c_1	c_0	N	w	$A_{-,0}$	x	m	c_1	c_0	N	w	$A_{-,0}$	x
2437	9	9	3	1	(3, 3)	1	27649	3	0	3	1	(9, 3)	1
3886	18	18	3	1	(3, 3)	1	28279	48	171	5	2	(3, 3)	1*
4027	0	18	3	1	(3, 3)	1	28734	0	18	3	1	(3, 3)	1
5703	63	54	4	3/2	(9, 3)	1	28759	3	18	3	1	(3, 3)	1
5857	3	36	4	3/2	(3, 3)	1	28902	15	0	3	1	(9, 3)	1
6085	21	63	4	3/2	(3, 3)	1	28945	168	171	5	2	(3, 3)	
6226	1212	549	7	3	(3, 3)		30466	21	0	3	1	(9, 3)	1
6690	12	18	3	1	(3, 3)	1	31081	204	36	5	2	(3, 3)	
6789	6	18	3	1	(3, 3)	1	31246	9	18	3	1	(3, 3)	1
6910	132	63	5	2	(3, 3)		31413	66	9	4	3/2	(3, 3)	1
7977	9	18	3	1	(3, 3)	1	31462	6	0	3	1	(9, 3)	1
8242	18	18	3	1	(3, 3)	1	31983	6	0	3	1	(9, 3)	1
9385	33	36	4	3/2	(3, 3)	1	32137	75	171	5	2	(3, 3)	2*
10015	21	0	3	1	(9, 3)	1	32826	15	0	3	1	(9, 3)	1
10173	102	9	5	2	(3, 3)	1*	33082	15	18	3	1	(3, 3)	1
10798	9	18	3	1	(3, 3)	1	33585	12	18	3	1	(3, 3)	1
11001	3	18	3	1	(3, 3)	1	33879	15	0	3	1	(9, 3)	1
12067	0	18	3	1	(3, 3)	1	34603	0	54	4	3/2	(9, 3)	1
12394	63	27	4	3/2	(9, 3)	1	34617	18	18	3	1	(3, 3)	1
12837	39	63	4	3/2	(3, 3)	1	34989	66	117	6	5/2	(3, 3)	2*
14334	6	0	3	1	(9, 3)	1	35331	6	0	3	1	(9, 3)	1
14730	33	63	4	3/2	(3, 3)	1	35353	9	18	3	1	(3, 3)	1
15049	18	18	3	1	(3, 3)	1	35367	24	18	3	1	(3, 3)	1
16870	24	18	3	1	(3, 3)	1	36021	0	18	3	1	(3, 3)	1
17146	18	18	3	1	(3, 3)	1	36678	24	0	3	1	(9, 3)	1
18555	0	9	3	1	(3, 3)	1	36807	3	18	3	1	(3, 3)	1
19545	21	18	3	1	(3, 3)	1	37219	6	18	3	1	(3, 3)	1
19677	18	18	3	1	(3, 3)	1	38278	12	0	3	1	(9, 3)	1
21418	12	0	3	1	(9, 3)	1	39802	30	9	4	3/2	(3, 3)	1
22443	66	198	5	2	(3, 3)	1*	39819	24	18	3	1	(3, 3)	1
22711	6	18	3	1	(3, 3)	1	40314	15	18	3	1	(3, 3)	1
22965	33	36	4	3/2	(3, 3)	1	41365	24	0	3	1	(9, 3)	1
23605	21	63	4	3/2	(3, 3)	1	41698	3	0	3	1	(9, 3)	1
23862	3	9	4	3/2	(3, 3)	1	41766	9	9	3	1	(3, 3)	1
25009	18	9	3	1	(3, 3)	1	42423	21	18	3	1	(3, 3)	1
25447	18	9	3	1	(3, 3)	1	42567	15	0	3	1	(9, 3)	1
26139	57	9	4	3/2	(3, 3)	1	42577	6	0	3	1	(9, 3)	1
26305	69	9	4	3/2	(3, 3)	1	42619	573	981	7	3	(3, 3)	
26962	75	36	4	3/2	(3, 3)	1	42901	18	18	3	1	(3, 3)	1
27186	24	18	3	1	(3, 3)	1	43198	51	9	4	3/2	(3, 3)	1
27355	0	18	3	1	(3, 3)	1	43827	24	18	3	1	(3, 3)	1

Table 2. (continued)

m	c_1	c_0	N	w	$A_{-,0}$	x	m	c_1	c_0	N	w	$A_{-,0}$	x
45397	0	9	3	1	(3, 3)	1	65686	21	36	4	3/2	(3, 3)	1
46290	3	0	3	1	(9, 3)	1	65977	0	18	3	1	(3, 3)	1
46587	18	18	3	1	(3, 3)	1	66981	18	18	3	1	(3, 3)	1
46753	33	9	5	2	(3, 3)		67255	12	0	3	1	(9, 3)	1
46929	18	18	3	1	(3, 3)	1	68406	15	0	3	1	(9, 3)	1
47017	6	0	3	1	(9, 3)	1	68626	15	0	3	1	(9, 3)	1
47482	177	198	5	2	(3, 3)		69070	12	0	3	1	(9, 3)	1
47878	27	54	4	3/2	(9, 3)	1	69366	12	0	3	1	(9, 3)	1
48039	63	27	4	3/2	(9, 3)	1	69402	24	18	3	1	(3, 3)	1
48153	15	18	3	1	(3, 3)	1	69721	6	63	4	3/2	(3, 3)	1
48634	24	0	3	1	(27, 3)	1	70330	3	18	3	1	(3, 3)	1
48918	9	9	3	1	(3, 3)	1	70606	21	18	3	1	(3, 3)	1
49837	18	9	3	1	(3, 3)	1	70930	150	198	5	2	(3, 3)	
50169	51	36	4	3/2	(3, 3)	1	70977	192	63	5	2	(3, 3)	1*
50281	18	18	3	1	(3, 3)	1	72034	231	198	5	2	(3, 3)	
50293	54	27	4	3/2	(9, 3)	1	72426	24	0	3	1	(9, 3)	1
50983	12	18	3	1	(3, 3)	1	72435	0	18	3	1	(3, 3)	1
52021	144	162	5	2	(9, 9)	2	72805	0	18	3	1	(3, 3)	1
53229	0	18	3	1	(3, 3)	1	72946	12	9	4	3/2	(3, 3)	1
53502	42	63	4	3/2	(3, 3)	1	73869	15	0	3	1	(9, 3)	1
54195	24	18	3	1	(3, 3)	1	74086	18	9	3	1	(3, 3)	1
54931	54	54	4	3/2	(9, 3)	1	75774	9	18	3	1	(3, 3)	1
55486	402	549	6	5/2	(3, 3)		75913	69	63	4	3/2	(3, 3)	1
55546	3	18	3	1	(3, 3)	1	77281	54	27	4	3/2	(9, 3)	1
56145	60	36	4	3/2	(3, 3)	1	77649	3	18	3	1	(3, 3)	1
56478	15	0	3	1	(9, 3)	1	77829	12	0	3	1	(9, 3)	1
56733	24	0	3	1	(9, 3)	1	78223	21	0	3	1	(9, 3)	1
57079	9	18	3	1	(3, 3)	1	79066	21	36	4	3/2	(3, 3)	1
57810	9	9	3	1	(3, 3)	1	79482	213	63	5	2	(3, 3)	
58105	87	171	5	2	(3, 3)		81309	18	18	3	1	(3, 3)	1
58213	24	18	3	1	(3, 3)	1	81867	15	18	3	1	(3, 3)	1
59182	2112	1224	7	3	(3, 3)		82077	9	18	3	1	(3, 3)	1
59221	21	18	3	1	(3, 3)	1	82183	3	0	3	1	(9, 3)	1
59293	24	0	3	1	(9, 3)	1	82702	6	0	3	1	(9, 3)	1
62121	6	18	3	1	(3, 3)	1	82834	4839	2115	8	7/2	(3, 3)	
63010	1689	1494	7	3	(3, 3)		83341	12	0	3	1	(27, 3)	1
63079	0	9	3	1	(3, 3)	1	83395	3	18	3	1	(3, 3)	1
63303	3	36	4	3/2	(3, 3)	1	83578	69	9	4	3/2	(3, 3)	1
64063	24	0	3	1	(9, 3)	1	84145	6	0	3	1	(9, 3)	1
65014	12	0	3	1	(27, 3)	1	84454	186	63	5	2	(3, 3)	
65203	15	18	3	1	(3, 3)	1	85489	9	18	3	1	(3, 3)	1

Table 2. (continued)

m	c_1	c_0	N	w	$A_{-,0}$	x
85741	18	18	3	1	(3, 3)	1
85845	6	18	3	1	(3, 3)	1
85858	21	0	3	1	(9, 3)	1
86542	6	0	3	1	(9, 3)	1
86551	69	9	4	3/2	(3, 3)	1
86694	18	9	3	1	(3, 3)	1
88447	15	18	3	1	(3, 3)	1
88558	3	0	3	1	(9, 3)	1
88762	0	18	3	1	(3, 3)	1
89269	1521	81	7	3	(27, 3)	1
89641	12	0	3	1	(9, 3)	1
89686	570	549	6	5/2	(3, 3)	2*
89818	30	36	4	3/2	(3, 3)	1
89923	21	0	3	1	(9, 3)	1
90163	9	9	3	1	(3, 3)	1
90313	15	18	3	1	(3, 3)	1
91402	0	9	3	1	(3, 3)	1
91471	30	36	4	3/2	(3, 3)	1
92685	6	36	4	3/2	(3, 3)	1
92827	18	18	3	1	(3, 3)	1
93154	6	0	3	1	(9, 3)	1
93445	60	36	4	3/2	(3, 3)	1
93714	18	54	4	3/2	(9, 3)	1
93823	0	18	3	1	(3, 3)	1
94498	18	18	3	1	(3, 3)	1
95155	51	36	4	3/2	(3, 3)	1
95869	0	18	3	1	(3, 3)	1
95977	15	18	3	1	(3, 3)	1
96693	87	171	5	2	(3, 3)	
96762	21	0	3	1	(9, 3)	1
96766	105	63	5	2	(3, 3)	
97063	42	9	4	3/2	(3, 3)	1
97687	12	18	3	1	(3, 3)	1
97801	72	54	4	3/2	(9, 3)	1
98281	63	27	4	3/2	(9, 3)	1
98347	21	18	3	1	(3, 3)	1
98443	54	27	4	3/2	(9, 3)	1
98605	57	36	4	3/2	(3, 3)	1
98746	24	0	3	1	(9, 3)	1
98773	321	792	7	3	(3, 3)	
98817	0	9	3	1	(3, 3)	1

Therefore we get $x = 1$, i.e., $X_- \cong N_1$.

We can determine the Λ-isomorphism classes of X_- for 7 fields among these 26 fields by the second method. We can not determine the Λ-isomorphism classes of X_- for the remaining 19 fields because $N_x/\dot{\omega}_0 N_x$ are isomorphic to $(3,\ 3)$ independent of x.

We explain about Table 2. The characters c_1, c_0, N and w represent $\mathrm{char} X_- \equiv T^2 + c_1 T + c_0 \mod p^N$ and $w = \frac{1}{2}\mathrm{ord}_p(c_1^2 - 4c_0)$. The character x represents $X_- \cong N_x$. When we determine x not by the first method but by the second method, then x is written as "1*". When we cannot determine x by these two methods, then we write no character.

Remark 6.1. Kurihara [Ku] developed another method to determine the Λ-isomorphism classes of X_-. Yamazaki [Ya] calculated with this method and determined $X_- \cong N_1$ when $m = 6226$ and 6910.

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