J. Math. Sci. Univ. Tokyo
6 (1999), 335–369.

Zonal Spherical Functions on Quantum Grassmann Manifolds

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Abstract. We give an explicit description of the zonal spherical functions on quantum Grassmann manifolds of orthogonal and symplectic type. A unified parametrization of their zonal spherical functions is given in terms of Macdonald polynomials and Koornwinder's multivariable Askey-Wilson polynomials.

§0. Introduction

In this paper we study two series of *quantum Grassmann manifolds*, of orthogonal and of symplectic type, and show that, for each Grassmann manifold of these series, the zonal spherical functions are expressed by Weyl group invariant q-orthogonal polynomials with respect to the Haar measure on the torus $\mathbb{T} = \{x = (x_1, \cdots, x_l) \in (\mathbb{C}^\times)^l \ ; \ |x_1| = \cdots = |x_l| = 1\}$ with weight functions $\Delta(x) := \Delta^+(x)\overline{\Delta^+(x)}$ where

$$\begin{aligned}
&\Delta^+(x)\\
&= \prod_{k=1}^{l} \frac{(x_k^2; q^2)_\infty}{(-q^{k_{2\epsilon_k}} x_k; q^2)_\infty (q^{k_{2\epsilon_k}+\frac{1}{2}k_{\epsilon_k}} x_k; q^2)_\infty (-q x_k; q^2)_\infty (q^{1+\frac{1}{2}k_{\epsilon_k}} x_k; q^2)_\infty}\\
&\qquad \times \prod_{i<j} \frac{(x_i/x_j; q^2)_\infty (x_i x_j; q^2)_\infty}{(q^{k_{\epsilon_i-\epsilon_j}} x_i/x_j; q^2)_\infty (q^{k_{\epsilon_i-\epsilon_j}} x_i x_j; q^2)_\infty}, \quad (0 < q < 1).
\end{aligned}$$

Here the constants $\{k_\alpha\}$ are nonnegative real numbers corresponding to the root multiplicities of the restricted root system and $(a;q)_\infty = \prod_{i=0}^\infty (1 - aq^i)$. Such polynomials coincide with so called Macdonald polynomials ([M]) or Koornwinder's multivariable Askey-Wilson polynomials ([K]). (See Theorem 4.1.)

1991 *Mathematics Subject Classification.* Primary 17B37, 33D80.

We consider the quantum Grassmann manifolds of the following series:

$$(BDI)\ ;\ (SO(N)/SO(l)\times SO(N-l))_q \quad (l\leq[N/2],\ N\geq 3, N\neq 4),$$
$$(CII)\ ;\ (Sp(2n)/Sp(2l)\times Sp(2n-2l))_q \quad (l\leq[n/2]).$$

They belong to a class of q-analogues of the series of irreducible Riemannian symmetric spaces of rank l, introduced in [N] (A_l type) and [NS1, 2] (B_l, C_l, D_l and BC_l types). Especially in the above series (BDI) and (CII) the restricted root systems of types B_l, C_l, D_l or BC_l appear according to the rank (see Subsection 4.1). And the above weight functions $\Delta(x)$ are interpreted as q-analogues of $\prod|1-x^{\alpha}|^{k_\alpha}$ where α runs through the positive roots of these restricted root systems with multiplicity k_α.

A class of quantized function algebras $A_q(G/K)$ of the Riemannian symmetric spaces G/K of compact type were introduced in [N] and [NS2] in a unified way. The $*$-algebra $A_q(G/K)$ is an *infinitesimal quantum subgroup* $\mathfrak{k}_{\mathbb{C}}, q$ invariant subalgebra of the quantized function algebra $A_q(G)$ of G. The infinitesimal quantum subgroup $\mathfrak{k}_{\mathbb{C}}, q$ is a coideal of the quantized universal enveloping algebra $U_q(\mathfrak{g})$ ($\mathfrak{g}$ =Lie(G)) such that $\mathfrak{k}_{\mathbb{C}}, q\to\mathfrak{k}_{\mathbb{C}}$ if q tends to 1 ($\mathfrak{k}_{\mathbb{C}}$ =Lie($K_{\mathbb{C}}$)). We call vectors of a *commutative* infinitesimal quantum subgroup-biinvariant $*-$subring $\mathcal{H}=A_q(K\backslash G/K)$ of $\mathcal{A}$ *zonal spherical functions* on *quantum symmetric space* $(G/K)_q$.

The zonal spherical functions of all the series of quantum Grassmann manifolds, however, were not determined in [NS1,2] because of some difficulties arising from the higher multiplicities of the restricted roots, although they occupy a greater part of the series of irreducible Riemannian symmetric spaces of classical type. Since the $*-$algebra $\mathcal{H}$ is identifed with a subring $\mathcal{H}|_{\mathbb{T}}$ of a Laurent polynomial ring $A(\mathbb{T})$ corresponding to the torus subgroup of G, to determine the zonal spherical functions we devote ourselves to compute the *radial part* of a central element of $\mathcal{U}$, that is, a q-difference operator on $\mathcal{H}|_{\mathbb{T}}$ which coincides with the action of the central element. Unlike the classical theory, even the Weyl group invariance must be derived from the Weyl group invariance of the radial part, because of the difficulties of algebraic structure of $\mathcal{H}$.

From the result of this paper, we conclude that the zonal spherical functions on all the quatum symmetric spaces of type A_l, B_l, C_l, D_l and BC_l, discussed in [N] and [NS2], are q-orhtogonal polynomials with respect to the measures of the class mentioned at the beginning of Introduction. Especially if the restricted root system Σ associated with G/K is of type A_l, B_l, C_l or D_l, the zonal spherical functions on $(G/K)_q$ coincide with Macdonald polynomials with specified parameters and if Σ is of type BC_l, then they are expressed in terms of Koornwinder's multivariable Askey-Wilson polynomials.

Contents

§1. Results from a Theory of the Quantum Symmetric Spaces

The statements (except Theorem 1.2) in this section is a summary of the results of [NS2] (see also [N]). For the detail descriptions, consult with the above paper.

Throughout this paper, for a Hopf algebra we denote the coproduct, the counit and the antipode by Δ, ε and S respectively. And q is a real number such that $0 < q < 1$.

1.1. Infinitesimal quantum subgroups

Let G/K be a compact Riemannian symmetric spaces of classical type (see [NS2]). In [NS2] the quantum symmetric spaces $(G/K)_q$ of all of such types were introduced. In this paper we restrict ourselves to the cases $(G\ ,\ K)$;

$$(BDI)\ ;\ (SO(N)\ ,\ SO(l)\times SO(N-l))\quad (l\leq[N/2],\ N\geq 3, N\neq 4),$$
$$(CII)\ ;\ (Sp(N)\ ,\ Sp(2l)\times Sp(N-2l))\quad (N=2n,\quad l\leq[n/2]).$$

We fix the root data of $G_{\mathbb{C}}$ (the complification of G) as follows.

Let $\mathcal{L}_n = \mathbb{Z}\epsilon_1\oplus\cdots\oplus\mathbb{Z}\epsilon_n$ be the $\mathbb{Z}$-free lattice of rank n in the n-dimensional Euclidean space $\mathcal{E}_n = \mathbb{R}\epsilon_1\oplus\cdots\oplus\mathbb{R}\epsilon_n$ with a nondegenerate inner product $\langle\ ,\ \rangle$ such that $\langle\epsilon_i,\epsilon_j\rangle=\delta_{ij}$ for any $1\leq i,\ j\leq n$. We realize the integral weight lattice P in $\mathcal{E}_n$ for the root system of type B_n, C_n and D_n corresponding to $\mathfrak{so}(2n+1;\mathbb{C})$, $\mathfrak{sp}(2n;\mathbb{C})$ and $\mathfrak{so}(2n;\mathbb{C})$. So we specify the set of simple roots $\{\alpha_i\}$ and the integral weight lattice $P=\sum_{k=1}^{n}\mathbb{Z}\Lambda_k$ with fundamental weights $\{\Lambda_k\}_{k=1}^n$ in the following table. We also give the condition for $\lambda=\sum_{i=1}^{n}\lambda_i\epsilon_i$ to the cone P_G^+ of dominant weights, in the integral weight lattice $P_G := P\cap\mathcal{L}_n$ corresponding to $G_{\mathbb{C}}$-rational representations.

(1.1)

	$SO(2n+1)$ $(n\geq 1)$	$Sp(2n)$ $(n\geq 1)$	$SO(2n)$ $(n\geq 3)$
simple roots for $G_{\mathbb{C}}$	$\alpha_k=\epsilon_k-\epsilon_{k+1}$ $1\leq k\leq n-1$ $\alpha_n=\epsilon_n$	$\alpha_k=\epsilon_k-\epsilon_{k+1}$ $1\leq k\leq n-1$ $\alpha_n=2\epsilon_n$	$\alpha_k=\epsilon_k-\epsilon_{k+1}$ $1\leq k\leq n-1$ $\alpha_n=\epsilon_{n-1}+\epsilon_n$
fundamental weights	$\Lambda_k=\epsilon_1+\cdots+\epsilon_k$ $1\leq k\leq n-1$ $\Lambda_n=\frac{1}{2}(\epsilon_1+\cdots+\epsilon_n)$	$\Lambda_k=\epsilon_1+\cdots+\epsilon_k$ $1\leq k\leq n$	$\Lambda_k=\epsilon_1+\cdots+\epsilon_k$ $1\leq k\leq n-2$ Λ_{n-1} $=\frac{1}{2}(\epsilon_1+\cdots+\epsilon_{n-1}-\epsilon_n)$ Λ_n $=\frac{1}{2}(\epsilon_1+\cdots+\epsilon_{n-1}+\epsilon_n)$
	$\lambda_1\geq\cdots\geq\lambda_n\geq 0$	$\lambda_1\geq\cdots\geq\lambda_n\geq 0$	$\lambda_1\geq\cdots\geq\lambda_{n-1}\geq\lvert\lambda_n\rvert$

To proceed to the notion of infinitesimal quantum subgroups we need a q-analogue of root vectors that behave nicely in the quantized universal enveloping algebra $\mathcal{U}_{\mathbb{C}} = U_q(\mathfrak{g}_{\mathbb{C}})$ ($\mathfrak{g}_{\mathbb{C}}$ =Lie$(G_{\mathbb{C}})$). L-operators $L^{\pm} = \sum_{1\leq i,j\leq N} e_{ij}L^{\pm}_{ij}\in\mathrm{End}_{\mathbb{C}}(V)\otimes_{\mathbb{C}}\mathcal{U}_{\mathbb{C}}$ give such q-analogues where V is the vector spaces of dimension N on which $\mathfrak{g}_{\mathbb{C}}$ naturally acts and e_{ij} are the matrix units with respect to a suitable basis of V. They are described through a suitable vector representation (π_V, V) of $\mathcal{U}_{\mathbb{C}}$ by

$$\pi_V(L^+)=R^+,\qquad \pi_V(L^-)=R^-$$

where $R^+ = PRP$ (P is the flip), $R^- = R^{-1}$ and

$$R = \sum_{1\le i,j\le N} e_{ii}\otimes e_{jj} q^{\delta_{ij}-\delta_{ij'}} + (q-q^{-1}) \times \sum_{1\le j<i\le N} (e_{ij}\otimes e_{ji} - e_{ij}\otimes e_{i'j'} q^{-\rho_i+\rho_j}\kappa_i\kappa_j).$$

Here we use the notation $j' := N-j+1$ for $1\le j\le N$ and the constants $\{\rho_k\}$ and $\{\kappa_i\}$ are given by

(1.2)

$\mathfrak{g}_{\mathbb{C}}$	$\mathfrak{so}(2n+1;\mathbb{C})$	$\mathfrak{sp}(2n;\mathbb{C})$	$\mathfrak{so}(2n;\mathbb{C})$
ρ_k	$\{n-\frac{1}{2},\cdots,\frac{3}{2},\frac{1}{2},0$ $-\frac{1}{2},-\frac{3}{2},\cdots,-n+\frac{1}{2}\}$	$\{n,,\cdots,2,1,$ $-1,-2,\cdots,-n\}$	$\{n-1,,\cdots,1,0,$ $0,-1,\cdots,-n+1\}$
κ_i	$\kappa_i = 1\ (1\le i\le N)$	$\kappa_i = 1\ (\le i\le n),$ $= -1\ (n+1\le i\le 2n)$	$\kappa_i = 1\ (1\le i\le 2n)$

Using these L-operators and the following constant solution J to the *reflection equation* (see [NS2]); (BI, DI): $SO(N)/SO(l)\times SO(N-l)$;

$$J = J^{[l]} := \sum_{1\le j,j'\le l} e_{jj} q^{\rho_j} + \sum_{l<j<l'} e_{jj'} q^{-\rho_j-\rho_l} + \sum_{j=1}^{l} e_{jj'}(1-q^{2\rho_l}) q^{-\rho_j-\rho_l},$$

$(CII):\ Sp(2n)/Sp(2l)\times Sp(2n-2l)$;

$$\begin{aligned} J = J^{[l]} := \sum_{k=1}^{l} (&-e_{2k,2k-1} q^{\rho_{2k-1}} + e_{2k-1,2k} q^{\rho_{2k}} \\ &+ e_{(2k-1)'(2k)'} q^{\rho_{(2k)'}} - e_{(2k)'(2k-1)'} q^{\rho_{(2k-1)'}}) \\ &+ \sum_{2l<j\le n} e_{jj'} q^{-\rho_j-\rho_{2l}+1} - \sum_{2l<j\le n} e_{j'j} q^{\rho_j-\rho_{2l}+1} \\ &+ \sum_{j=1}^{2l} e_{jj'}(1-q^{2\rho_{2l}-2}) q^{-\rho_j-\rho_{2l}+1}, \end{aligned}$$

we introduce the *infinitesimal quantum subgroups* as $\mathbb{C}$-vector space spanned by the matrix elements of

$$M := L^+ - JS(L^-)^t J^{-1}.$$

This $\mathbb{C}$-vector space is a q-analogue of the Lie subalgebra $\mathfrak{k}_{\mathbb{C}}$ =Lie($K_{\mathbb{C}}$) and this becomes a two-sided coideal of $\mathcal{U}_{\mathbb{C}}$. We denote this $\mathbb{C}$-vector space by $\mathfrak{k}_{\mathbb{C}}, q$.

When we say about the $*$-operation on $\mathcal{U}_{\mathbb{C}}$, we adopt the $*$-operation corresponding to the compact real form $\mathfrak{g}$ of $\mathfrak{g}_{\mathbb{C}}$. In this situation the infinitesimal quantum subgroup $\mathfrak{k}_{\mathbb{C}}, q$ is $\tau := *\circ S$-invariant and the "compact real form" $\mathfrak{k}_q$ can be defined by the (elementwise) τ-invariant subspace of $\mathfrak{k}_{\mathbb{C}}, q$, but we do not mention about the "real structure" itself. The $*$-operation of $\mathcal{U}_{\mathbb{C}}$ is given by

$$(L^{\pm})^* = S(L^{\mp})^t = CL^{\mp}C^{-1} \quad \text{where } C = \begin{pmatrix} 0 & & & q^{\rho_N}\kappa_N \\ & & \cdot & \\ & \cdot & & \\ q^{\rho_1}\kappa_1 & & & 0 \end{pmatrix}.$$

We denote by $\mathcal{U}_0$ the commutative $*-$algebra generated by group-like elements $\{L_{ii}^{\pm}\}_{1\le i\le N}$ with $*$-structure s.t. $(L_{ii}^{\pm})^*{=}L_{ii}^{\pm}$. For the description of the generators of $\mathcal{U}_0$, it is covenient to introduce the commutative Hopf $*$-algebra generated by the symbols q^{μ} ($\mu \in \mathcal{L}_n$) with multiplication and the Hopf algebra structure ; $q^{\lambda} \cdot q^{\mu} = q^{\lambda+\mu}$ (λ, $\mu \in \mathcal{L}_n$), $\Delta(q^{\mu}) = q^{\mu} \otimes q^{\mu}$, $\varepsilon(q^{\mu}) = 1$ and $S(q^{\mu}) = q^{-\mu}$. By setting $L_{ii}^{+} = q^{\epsilon_i}$, $L_{ii}^{-} = q^{-\epsilon_i}$ for $1 \le i \le N$ where $\epsilon_{i'} = -\epsilon_i$, we have the identification $\mathcal{U}_0{=}\mathbb{C}[q^{\mu}\ ;\ \mu \in \mathcal{L}_n]$. These are compatible with the notion of the weights in $\mathcal{U}$: For any $q^h \in \mathcal{U}_0{=}\mathbb{C}[q^{\mu}\ ;\ \mu \in \mathcal{L}_n]$, we have

$$q^h L^+ q^{-h} = H^{-1} L^+ H, \qquad q^h L^- q^{-h} = H^{-1} L^- H$$

where $H = \operatorname{diag}(q^{\langle h,\epsilon_1\rangle}, \cdots, q^{\langle h,\epsilon_n\rangle}, (1), q^{-\langle h,\epsilon_n\rangle}, \cdots, q^{-\langle h,\epsilon_1\rangle})$. Thus the elements L_{ij}^{+} and L_{ji}^{-} ($i < j$) have weights $-\epsilon_i + \epsilon_j$ and $\epsilon_i - \epsilon_j$ respectively.

1.2. Quantum symmeric spaces and zonal spherical functions

The quantized function algebra $\mathcal{M} := A_q(G/K)$ is the $\mathfrak{k}_{\mathbb{C}}, q$-invariant subspace of the quantized function algebra $\mathcal{A} := A_q(G)$.

The quantized function algebra $\mathcal{A}$ is a Hopf $*$-algebra spanned by the matrix elements of the finite dimensional P_G-weighted representaions of $\mathcal{U} = U_q(\mathfrak{g})$ =($\mathcal{U}_{\mathbb{C}}, *$). The $\mathcal{U}$-bimodule structure gives the Peter-Weyl decomposition with components $W(\lambda)$ parametrized by the dominant integral weight λ of the cone of dominant weights P_G^{+}.

We also have the quantized function algebra of the coset space $K\backslash G/K$;

$$\mathcal{H} = A_q(K\backslash G/K) := \{\phi \in \mathcal{A}\ ;\ \mathfrak{k}_{\mathbb{C}}, q.\phi = 0, \quad \phi.\overline{\mathfrak{k}_{\mathbb{C}}, q} = 0\}$$

where $\overline{}$ denotes an involutive anti-automorphism of $\mathcal{U}$ defined by

$$\overline{a} = q^{\rho} a^{*} q^{-\rho} \text{ for } a \in \mathcal{U} \quad (\rho = \frac{1}{2}\sum_{\alpha>0}\alpha = \sum_{k=1}^{n}\rho_k\epsilon_k)$$

and $\overline{\mathfrak{k}_{\mathbb{C}}, q} = \{\overline{a}\ ;\ a \in \mathfrak{k}_{\mathbb{C}}, q\}$.

Remark that the quantized function algebra $\mathcal{M}$ and $\mathcal{H}$ are both $*$-algebras by the τ-invariance of the coideal $\mathfrak{k}_{\mathbb{C}}, q$.

This $*$-algebra $\mathcal{H}$ becomes a commutative algebra and has the simultaneous eigen space decomposion of the center $\mathcal{Z}$ of $\mathcal{U}$ with components $\mathcal{H}(\lambda) := \mathcal{H} \cap W(\lambda)$ parametized by the dominant weights λ in $P^{+}_{G,\mathfrak{k}} := \bigoplus_{r=1}^{l} \mathbb{N}\widetilde{\Lambda}_r$ (spanned by the "fundamental spherical weights" $\widetilde{\Lambda}_r$ (see Definition 1.1)).

Moreover the $*$-algebra $\mathcal{H}$ can be identified with a subring of the Laurent polynomial ring $A(\mathbb{T}) = \mathbb{C}[z_1^{\pm 1}, \cdots, z_n^{\pm 1}]$ regarded as the quotient Hopf algebra corresponding to the torus group of G. We denote the restriction mapping $\mathcal{A} \to A(\mathbb{T})$ by $|_{\mathbb{T}}$. Remark that there is a nondegenerate Hopf pairing between $\mathcal{U}_0$ and $A(\mathbb{T})$ such that $(q^{\mu}, z^{\nu}) = q^{\langle \mu,\nu\rangle}$ for $\mu, \nu \in \mathcal{L}_n$ and $z^{\nu} = z_1^{\nu_1} \cdots z_n^{\nu_n}$ ($\nu_i = \langle \nu, \epsilon_i\rangle$, $1 \le i \le n$)

We say the nonzero elements of $\mathcal{H}(\lambda)$ the *zonal spherical functions* for $\lambda \in P^{+}_{G,\mathfrak{k}}$. The zonal spherical functions $\phi(\lambda)$ are uniquely determined up to constant since $\dim_{\mathbb{C}}\mathcal{H}(\lambda) = 1$ for each $\lambda \in P^{+}_{G,\mathfrak{k}}$. For more detail description of the zonal spherical functions we need to explain about the *restricted root system* of G/K.

Let $\mathfrak{g} = \mathfrak{k} \oplus \sqrt{-1}\mathfrak{p}$ be the decomposition corresponding to the Cartan decomposition of a real form $\mathfrak{g}_0$ of $\mathfrak{g}_{\mathbb{C}}$; $\mathfrak{g}_0 = \mathfrak{k} \oplus \mathfrak{p}$. In that decomposition choose a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$. Let $\Sigma = \Sigma(\mathfrak{g}; \mathfrak{a})$ be the restricted root system of rank l for G/K associated with the pair $(\mathfrak{g}, \mathfrak{a})$ and let us denote by $W(\Sigma)$ the associated Weyl group. For the precise description of Σ, see the next section.

Here we see that we can embed the restricted root system Σ into the root lattice of G in $\mathcal{E}_n = \mathbb{R}\epsilon_1 \oplus \cdots \oplus \mathbb{R}\epsilon_n$: By putting

$$\tilde{\epsilon}_i = \begin{cases} 2\epsilon_i & (1 \le i \le l) \text{ for the cases BI and DI,} \\ \epsilon_{2i-1} + \epsilon_{2i} & (1 \le i \le l) \text{ for the case CII,} \end{cases}$$

we can realize the root system Σ in $\widetilde{\mathcal{E}}_l := \mathbb{R}\tilde{\epsilon}_1 \oplus \cdots \oplus \mathbb{R}\tilde{\epsilon}_l$ with the nondegenerate inner product $\langle\ ,\ \rangle$ derived from that of $\mathcal{E}_n = \mathbb{R}\epsilon_1 \oplus \cdots \oplus \mathbb{R}\epsilon_n$.

Then we can take the standard simple roots $\{\tilde{\alpha}_i\}_{1\leq i\leq l}$ and fundamental weights $\{\widetilde{\Lambda}_i\}_{1\leq i\leq l}\}$ as in the table (1.1). For example, for the case BI ($l < n$), $\tilde{\alpha}_i = \tilde{\epsilon}_i - \tilde{\epsilon}_{i+1} = 2\alpha_i$ ($1 \leq i \leq l-1$), $\tilde{\alpha}_l = \tilde{\epsilon}_l = 2\alpha_l$, $\widetilde{\Lambda}_k = \tilde{\epsilon}_1 + \cdots + \tilde{\epsilon}_k = 2\Lambda_k$ ($1 \leq k \leq l-1$) and $\widetilde{\Lambda}_l = \frac{1}{2}(\tilde{\epsilon}_1 + \cdots + \tilde{\epsilon}_l) = \Lambda_l$.

Moreover corresponding to this embedding of root system, the integral weight lattice $P(\Sigma)$ is also embedded into $P(\Delta)$ and the Weyl group $W(\Sigma)$ is obtained as the restriction of a subgroup of $W(\Delta)$ onto $\widetilde{\mathcal{E}}_l$.

DEFINITION 1.1. We introduce a new inner product $\langle\ ,\ \rangle_\Sigma$ on $\widetilde{\mathcal{E}}_l$ such that $\langle \tilde{\epsilon}_i, \tilde{\epsilon}_j \rangle_\Sigma = \delta_{ij}$ for $1 \leq i, j \leq l$. When we emphasize the inner product on $P(\Sigma)$, we will write it as $P(\Sigma, \langle , \rangle)$ and $P(\Sigma, \langle , \rangle_\Sigma)$ respectively. We give the identification map

$$\pi\ :\ P(\Sigma, \langle , \rangle) \longrightarrow P(\Sigma, \langle , \rangle_\Sigma)$$

by rewriting $\mu = \mu_1\epsilon_1 + \cdots + \mu_n\epsilon_n \in P(\Sigma)$ into the form $(\mu =)\pi(\mu) = \tilde{\mu}_1\tilde{\epsilon}_1 + \cdots + \tilde{\mu}_l\tilde{\epsilon}_l$ such that $\tilde{\mu}_i = \langle \mu, \tilde{\epsilon}_i \rangle_\Sigma$ for $1 \leq i \leq l$, and denote by $x^{\pi(\mu)}$ the monomial $x_1^{\tilde{\mu}_1} \cdots x_l^{\tilde{\mu}_l}$ for $\mu \in P(\Sigma)$ setting $x_j = z_j^{\tilde{\epsilon}_j}$ for $1 \leq j \leq l$. Then all the zonal spherical functions $\phi(\lambda)|_{\mathbb{T}}$ ($\lambda \in P^+_{G,\mathfrak{k}}$) belong to the commutative algebra

$$A(\Sigma) := \bigoplus_{\nu \in P(\Sigma)} \mathbb{C}x^{\pi(\nu)}$$

where in the Theorem 1.2 we understand $x_j^{\frac{1}{2}}$ as z_j for the cases BDI and the $*-$operation is described as $x_j^* = x_j^{-1}$ for $1 \leq j \leq l$. Note that the correspondence between μ and $\pi(\mu)$ is given by $z^\mu = x^{\pi(\mu)}$.

From [NS2] we have

$$\phi(\lambda)|_{\mathbb{T}} = x^{\pi(\lambda)} + \sum_{\substack{\nu \in P(\Sigma) \\ \nu <_\Sigma \lambda}} a_{\lambda\nu} x^{\pi(\nu)} \qquad \text{for some } a_{\lambda\nu} \in \mathbb{C} \tag{1.3}$$

Especially, for the fundamental spherical weights $\widetilde{\Lambda}_r$, we have

$$\phi(\widetilde{\Lambda}_r)|_{\mathbb{T}} = m_{\widetilde{\Lambda}_r}(x) + \sum_{\substack{\nu \in P(\Sigma) \\ \nu <_\Sigma \widetilde{\Lambda}_r \\ \nu \notin W(\Sigma)\cdot\widetilde{\Lambda}_r}} a_\nu x^{\pi(\nu)} \tag{1.4}$$

Here $\lambda >_\Sigma \mu$ implies the dominance order in $P(\Sigma)$, that is, $\lambda - \mu = \sum_{i=1}^{l} m_i\tilde{\alpha}_i$ for some nonnegative integers m_i.

The results in this paper (see Section 3) give the following statement.

THEOREM 1.2. *The image of the restriction mapping $\mathcal{H} \to \mathbb{T}$ is precisely the subring of $W(\Sigma)$-invariants in $A(\Sigma)$;*

$$\mathcal{H}|_{\mathbb{T}} = A(\Sigma)^{W(\Sigma)}$$

and the zonal spherical functions $\phi(\lambda)|_{\mathbb{T}}$ ($\lambda \in P^+_{G,\mathfrak{k}}$) form a basis of this algebra.

We can prove Theorem 1.2 by using (1.3), Lemma 2.2, 2.3 and the results in the Section 3 that the zonal spherical functions $\phi(\lambda)|_{\mathbb{T}}$ are eigen functions of Macdonald's or Koornwinder's q-difference operators D_σ with specified parameters. However the proof has been already contained in [NS2], so we omit it here.

§2. Macdonald Polynomials and Koornwinder's Multivariable Askey-Wilson Polynomials

We explain one of the most important class of q-special functions.

2.1. Macdonald polynomials

Macdonald polynomials are belonging to a class of q-orthogonal polynomials associated with root systems. More precisely Macdonald polynomials are associated with a pair of root systems.

In this paper, for simplicity, let (R, S) be one of the pairs of root systems (B_l, $\mathrm{B}_l^\vee$), (C_l, $\mathrm{C}_l^\vee$), (D_l, $\mathrm{D}_l^\vee$) and (BC_l, $\mathrm{C}_l^\vee$) where B_l ($l \geq 1$), C_l ($l \geq 1$), D_l ($l \geq 3$) and BC_l ($l \geq 1$) stand for the root systems of rank l of type B, C, D and BC in $\mathcal{E}_l = \mathbb{R}\epsilon_1 \oplus \cdots \oplus \mathbb{R}\epsilon_l$ with standard inner product $\langle\ ,\ \rangle$ and $R^\vee$ denote the coroot system of R; $R^\vee = \{\alpha^\vee = \frac{2\alpha}{\langle\alpha\ ,\ \alpha\rangle}; \alpha \in R\}$.

Let W=$W(R)$ be the Weyl group associated with the root system R and $P(R)$ be the integral weight lattice in $\mathcal{E}_l$. For a pair of the above root systems (R, S), there exists a unique set of W-invariant positive real numbers $\{u_\alpha\}_{\alpha\in R}$ (W acts as $w.u_\alpha = u_{w\alpha}$ for $w \in W$) such that $\alpha^\vee = u_\alpha^{-1}\alpha \in S$.

We assume $0 < q < 1$ and introduce the following quantities for a set of W-invariant nonnegative real numbers $\{k_\alpha\}_{\alpha\in R}$; $q_\alpha = q^{u_\alpha}$ and $t_\alpha = q_\alpha^{k_\alpha/2}$ for each $\alpha \in R$.

Let A=$A(R)$ be a commutative subring of $\mathbb{C}[x_1^{\pm 1/2}, \cdots, x_l^{\pm 1/2}]$ spanned by the monomials $x^\mu = x_1^{\mu_1} \cdots x_l^{\mu_l}$ indexed by the integral weights $\mu = \mu_1\epsilon_1 + \cdots + \mu_l\epsilon_l$ in $P(R)$;

$$A = A(R) := \sum_{\mu\in P(R)} \mathbb{C}x^\nu$$

with $*$-operation s.t. $x_i^* = x_i^{-1}$ for $1 \le i \le l$.

We will introduce an inner product on A^W, the Weyl invariant subring of A. For f, $g \in A^W$ we define a hermitian inner product on A^W by

$$\langle f|g\rangle := |W|^{-1} \int_{\mathbb{T}} f(x)^* g(x)\Delta(x)dx$$

where $\int_{\mathbb{T}} dx$ denotes the normarized Haar measure on the torus

$$\mathbb{T} = \{x = (x_1, \cdots, x_l) \in (\mathbb{C}^\times)^l \ ; \quad |x_1| = \cdots |x_l| = 1\}.$$

And the weight function $\Delta(x)$ on $(\mathbb{C}^\times)^l$ is given by

$$\Delta(x) := \Delta^+(x)^*\Delta^+(x) \text{ and } \Delta^+(x) = \prod_{\alpha\in R^+} \frac{(t_{2\alpha}^{\frac{1}{2}} x^\alpha; q_\alpha)_\infty}{(t_\alpha t_{2\alpha}^{\frac{1}{2}} x^\alpha; q_\alpha)_\infty} \tag{2.1}$$

where $(a;q)_\infty = \prod_{i=0}^\infty (1-aq^i)$ and R^+ denotes the set of positive roots of$R$.

The Macdonald polynomials $\{P_\mu(x;q)\}_{\mu\in P^+(R)}$ associated with root systems (R, S), indexed by the dominant integral weights, are characterized by the following two conditions;

$$\begin{aligned} &(1) \quad P_\mu(x;q) = m_\mu(x) + \sum_{\mu>\nu} a_{\mu\nu} m_\nu(x) \qquad (a_{\mu\nu} \in \mathbb{C}) \\ &(2) \quad \langle P_\mu(x;q)|m_\nu(x)\rangle = 0 \qquad \text{if } \mu > \nu \end{aligned}$$

where $m_\mu(x)$ is the W-orbit sum of the monomial x^μ;

$$m_\mu(x) = \sum_{w\in W\cdot\mu} x^\nu \qquad (W \cdot \mu \ : \ \text{the orbit of } \mu \text{ in } W)$$

and $\nu \leq \mu$ denotes the *dominance order* of weights;

$$\nu \leq \mu \Longleftrightarrow \mu - \nu \in \mathbb{Z}_{\geq 0} - \text{span of } R^+.$$

Note that $\{P_\mu(x;q)\}_{\mu \in P^+(R)}$ form a basis of A^W. Moreover Macdonald polynomials $P_\mu(x;q)$ $(\mu \in P^+(R))$ are also eigenfunctions of the following q-difference operator D_σ:

We set $\sigma = \epsilon_1$ and define

$$\begin{aligned}
D_\sigma :=& |W_\sigma|^{-1} \sum_{w \in W} w\Phi_\sigma(x)(T_{w\sigma} - 1), \qquad (2.2)\\
T_\mu(x^\nu) :=& q^{\langle \mu, \nu \rangle} x^\nu \text{ for } \mu,\ \nu \in P(R),\\
\Phi_\sigma(x) :=& \frac{T_\sigma(\Delta^+(x))}{\Delta^+(x)},\\
a_\mu :=& q^{\frac{1}{2}\langle \sigma, \tilde{\rho} \rangle} \sum_{j=1}^{l} (q^{\frac{1}{2}\tilde{\rho}_j}(q^{\mu_j} - 1)) + q^{-\frac{1}{2}\tilde{\rho}_j}(q^{-\mu_j} - 1),\\
\tilde{\rho} :=& \frac{1}{2} \sum_{\alpha \in R^+} k_\alpha \alpha = \sum_{j=1}^{l} \tilde{\rho}_j \epsilon_j,\\
E_\sigma :=& |W_\sigma|^{-1} \sum_{w \in W} w\Phi_\sigma(x) T_{w\sigma},\\
b_\mu :=& q^{\frac{1}{2}\langle \sigma, \tilde{\rho} \rangle} \sum_{j=1}^{l} (q^{\frac{1}{2}\tilde{\rho}_j} q^{\mu_j} + q^{-\frac{1}{2}\tilde{\rho}_j} q^{-\mu_j})
\end{aligned}$$

where W_σ is the stabilizer of σ in W.

THEOREM 2.1 (Macdonald). *The q-difference operators D_σ is self ajoint operators with respect to the inner product on A^W. Moreover the $P_\mu(x;q)$ are eigenfunctions of D_σ with eigen values a_μ.*

2.2. Koornwinder's multivariable Askey-Wilson polynomials for root system BC

We will also recall an extension of Macdonald polynomials. Let R be the root system of type BC_l. Define a hermitian inner product on A^W by replacing the function $\Delta^+(x)$ of (2.1) by

$$\Delta^+(x) = \prod_{k=1}^{l} \frac{(x_k^2;q)_\infty}{(ax_k,bx_k,cx_k,dx_k;q)_\infty} \prod_{1\le i<j\le l} \frac{(x_i/x_j,x_ix_j;q)_\infty}{(tx_i/x_j,tx_ix_j;q)_\infty} \tag{2.3}$$

where a, b, c, d and t are real numbers such that $|a|$, $|b|$, $|c|$, $|d| \le 1$, but the pairwise product of a, b, c, d are not ≥ 1 and $|t| < 1$. Here we used the notation $(a_1,\cdots,a_r;q)_\infty = \prod_{i=1}^r (a_i;q)_\infty$. So we have the following forms of the q-difference operator D_σ;

$$\begin{aligned} D_\sigma &:= |W_\sigma|^{-1} \sum_{w\in W} w.\Phi_\sigma(x)(T_{w\sigma}-1) \\ &= \sum_{k=1}^{l} (\Phi_k^+(x)T_{\epsilon_k} + \Phi_k^-(x)T_{-\epsilon_k}) - \Phi_0(x) \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} \Phi_k^+(x) = &\frac{(1-ax_k)(1-bx_k)(1-cx_k)(1-dx_k)}{(1-x_k^2)(1-qx_k^2)} \\ &\times \prod_{a\ne k} \frac{tx_k-x_a}{x_k-x_a} \times \prod_{\substack{a=1\\a\ne k}}^{l} \frac{1-tx_ax_k}{1-x_ax_k}, \end{aligned}$$

$$\Phi_k^-(x) = \Phi_k^+(x^{-1}) \text{ and } \Phi_0(x) = \sum_{k=1}^{l}(\Phi_k^+(x)+\Phi_k^-(x)).$$

In [K] Koornwinder showed the existence of a basis $\{P_\mu(x|a,b,c,d,t;q)\}$ ($\mu \in P^+(R)$) of A^W, which the same statements of Theorem 2.1 holds for D_σ with eigenvalues

$$a_\mu = \sum_{k=1}^{l} (q^{-1}abcdt^{2l-k-1}(q^{\mu_k}-1) + t^{k-1}(q^{-\mu_k}-1)). \tag{2.5}$$

These polynomials $\{P_\mu(x|a,b,c,d,t;q)\}$ are generalizations of one-variable Askey-Wilson polynomials (see [GR]).

For later convenience we also introduce the following quantity;

$$b_\mu := \sum_{k=1}^{l}(q^{-1}abcdt^{2l-k-1}q^{\mu_k} + t^{k-1}q^{-\mu_k}) \tag{2.6}$$

such that $a_\mu = b_\mu - b_0$ for $\mu \in P^+(R)$.

Note that the operator D_σ defined above recovers Macdonald's operators in the previous subsection, as special limitting cases (see [K], [NS2]).

We will finish this section with some remarks on Macdonald's and Koornwinder's polynomials, which will be used to prove the Weyl group invariance of zonal spherical functions (see Theorem 1.2). For the proofs, see [NS2].

LEMMA 2.2. *We assume $0 < abcd < 1$ and $0 < t < 1$ for Koornwinder's D_σ and the condition of the parameters $\{k_\alpha\}$ for Macdonald's D_σ. For dominant integral weights λ, $\mu \in P^+(R)$ such that $\lambda > \mu$, we have*

$$a_\lambda - a_\mu > 0.$$

LEMMA 2.3. *Fix a dominant weight $\mu \in P^+(R)$. If $\phi \in A$ satisfies*

(i) $\phi = \sum_{\substack{\nu<\mu \\ \nu\notin W\cdot\mu}} b_\nu x^\nu$ *for certain constants* $b_\nu \in \mathbb{C}$

(ii) $D_\sigma\phi = a_\mu\phi$,

then $\phi = 0$, assuming the same conditions in Lemma 2.2.

For Lemma 2.2, see [SK], [NS2] and [M].

§3. Explicit Formulas for Zonal Spherical Functions on Quantum Grassmann Manifolds

3.1. Zonal spherical functions as q-orthogonal polynomials

We will show the zonal spherical functions $\phi(\lambda)|_{\mathbb{T}}$ are described in terms of Macdonald polynomials or Koornwinder's multivariable Askey-Wilson polynomials. They are controled only by the information of the restricted root system Σ associated with G/K. Let us recall the data of the restricted root system $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$. We denote by $k_{\tilde{\alpha}}$ the multiplicity of the root $\tilde{\alpha} \in \Sigma$.

G/K	type of Σ	$k_{\tilde{\epsilon_1} \pm \tilde{\epsilon_2}}$	$k_{\tilde{\epsilon_l}}$	$k_{2\tilde{\epsilon_l}}$
$BI\ :\ SO(2n+1)/SO(l) \times SO(2n+1-l)$	B_l	1	$2(n-l)+1$	0
$DI\ :\ SO(2n)/SO(l) \times SO(2n-l)$	$B_l\ (l < n)$	1	$2(n-l)$	0
	$D_l\ (l = n)$	1	0	0
$CII\ :\ Sp(2n)/Sp(2l) \times Sp(2n-2l)$	$BC_l\ (2l < n)$	4	$4(n-2l)$	3
	$C_l\ (2l = n)$	4	0	3

THEOREM 3.1. *For each $\lambda \in P^+_{G,\mathfrak{k}}$, the restriction of the zonal spherical function $\phi(\lambda)$ on the quantum Grassmann manifolds $(G/K)_q$ into $A = A(\Sigma) \subset A(\mathbb{T})$ associated with the representation $V(\lambda)$ is written in terms of Macdonald polynomials $P_{\pi(\lambda)}(x;q)$ or Koornwinder's Askey-Wilson polynomials $P_{\pi(\lambda)}(x|a,b,c,d,t;q)$.*

$$\begin{aligned} &\textit{Case } BI:\ DI,\ SO(N)/SO(l) \times SO(N-l);\ \phi(\lambda)|_{\mathbb{T}} = P_{\pi(\lambda)}(x;q^4) \\ &\textit{Case } CII:\ Sp(2n)/Sp(2l) \times Sp(2n-2l); \\ &\qquad\qquad \phi(\lambda)|_{\mathbb{T}} = P_{\pi(\lambda)}(x|-s, su, -q, qu, t; q^2) \end{aligned}$$

where the Macdonald polynomials $P_{\pi(\lambda)}(x;q^4)$ of the cases BDI are based on the data of the root system Σ with the inner product $\langle\ ,\ \rangle_\Sigma$, and $t = q^{k_{\tilde{\epsilon_1} \pm \tilde{\epsilon_2}}}$, $s = q^{k_{2\tilde{\epsilon_l}}}$ and $u = q^{\frac{1}{2}k_{\tilde{\epsilon_l}}}$.

REMARK. For the cases $SO(2n)/SO(n) \times SO(n)$ and $Sp(4l)/Sp(2l) \times Sp(2l)$, we already have the above results in [NS1, 2]. Especially the restricted root system for the cases are of type D_n and C_l respectively, and the zonal spherical functions turn out to be Macdonald polynomials.

To characterize the zonal spherical functions above, we will compute the *radial part* of a central element of $\mathcal{U}$. For a central element C of $\mathcal{U}$, the radial part D of C on $\mathcal{H}|_{\mathbb{T}}$ is defined as a q-difference operator such that

$$(C.\phi)|_{\mathbb{T}} = D.\phi|_{\mathbb{T}}$$

for any $\phi \in \mathcal{H}|_{\mathbb{T}}$.

Recall that the decomposition $\mathcal{H} = \bigoplus_{\lambda \in P^+_{G,\mathfrak{k}}} \mathcal{H}(\lambda)$ is the simultaneous eigenspace decomposition of the center of $\mathcal{U}$. Hence we have

$$D.\phi(\lambda)|_{\mathbb{T}} = \chi_\lambda(C)\phi(\lambda)|_{\mathbb{T}}$$

for $\lambda \in P^+_{G,\mathfrak{k}}$ where χ_λ is the central character on $V(\lambda)$ (the irreducible representation of highest weight λ).

The practical method for the computation of the radial part of a central element of $\mathcal{U}$ is given as follows.

By the $\mathcal{U}$-bimodule structure of $\mathcal{A}$, we have

$$(abc, \phi) = (b, c.\phi.a)$$

for any $a, b, c \in \mathcal{U}$ and $\phi \in \mathcal{A}$. Therefore by the nondegeneracy of the Hopf pairing we have

$$\phi \in \mathcal{H} \Leftrightarrow (\mathcal{J}, \phi) = 0 \text{ and } (\overline{\mathcal{J}}, \phi) = 0$$

where $\mathcal{J} = \mathcal{U}\mathfrak{k}_{\mathbb{C}}, q$ and $\overline{\mathcal{J}} = \overline{\mathfrak{k}_{\mathbb{C}}, q}\mathcal{U}$. Thus we have the following commutatve diagram in which all the arrows are injective mappings:

$$\begin{array}{ccc} \mathcal{H} & \longrightarrow & A(\mathbb{T}) \\ \big\downarrow & & \big\downarrow \\ (\mathcal{U}/\mathcal{J} + \overline{\mathcal{J}})^\vee & \longrightarrow & {\mathcal{U}_0}^\vee. \end{array}$$

Lemma 3.2 (see [N]). *We set* $\zeta_k = q^{\epsilon_k} \in \mathcal{U}_0$ $(1 \le k \le n)$ *and* $\zeta^\lambda = \zeta_1^{\lambda_1} \cdots \zeta_n^{\lambda_n} = q^\lambda$ *for* $\lambda = \lambda_1\epsilon_1 \cdots \lambda_n\epsilon_n \in P$. *Define q-difference operators* $T^{\pm 1}_{q,\zeta_k}$ $(1 \le k \le n)$ *on* $\mathcal{U}_0$ *by* $T^{\pm 1}_{q,\zeta_k}(\zeta^\lambda) = q^{\pm\langle \epsilon_k, \lambda\rangle}\zeta^\lambda$. *Suppose for a central element* C *of* $\mathcal{U}$ *there exists a nonzero Laurent polynomial* $a(z) \in \mathbb{C}[z_1^{\pm 1}, \cdots, z_n^{\pm n}]$ *and a q-difference operator* $Q(\zeta; T_{q,\zeta}) \in \mathbb{C}[\zeta_1^{\pm 1}, \cdots, \zeta_n^{\pm n}, T^{\pm 1}_{q,\zeta_1}, \cdots, T^{\pm 1}_{q,\zeta_n}]$ *such that*

$$(a(T_{q.\zeta})f)C \equiv Q(\zeta; T_{q,\zeta})f \mod \mathcal{J} + \overline{\mathcal{J}}$$

for any Laurent polynomial $f = f(\zeta)$ *in* $\mathcal{U}_0$. *Then the radial part* $D = C|_{\mathbb{T}} : \mathcal{H}|_{\mathbb{T}} \longrightarrow \mathcal{H}|_{\mathbb{T}}$ *is given by*

$$D = a(z)^{-1}Q\widehat{(\zeta; T_{q,\zeta})}$$

where $\widehat{\quad}$ *means the multiplicative Fourier Transform (anti-algebra isomorphism);*

$$\widehat{\quad} : \mathbb{C}[\zeta^{\pm 1}, T_{q,\zeta}^{\pm 1}] \longrightarrow \mathbb{C}[z^{\pm 1}, T_{q,z}^{\pm 1}]$$

such that $\widehat{\zeta_k} = T_{q,z_k}$, $\widehat{T_{q,\zeta_k}} = z_k$ *for* $1 \leq k \leq n$.

Applying this method, we will compute the radial part of a central element of $\mathcal{U}$ in the next subsection (Theorem 3.3) and we will see that the zonal spherical functions $\phi(\lambda)|_{\mathbb{T}}$ are eigen functions of Macdonald's D_σ or Koornwinder's D_σ. From this result we can prove Theorem 1.2 and Theorem 3.1 at the same time. The detail of such an argument is given in [NS2].

3.2. Explicit formula for the radial part of a central element in $\mathcal{U}$

We consider the following central elements $\{C_r\}$ $(1 \leq r \leq \mathrm{rank}\mathfrak{g})$ defined by [RTF];

$$C_r = \mathrm{tr}(D^2(L^+S(L^-))^r) = \sum_{i=1}^{N} q^{2\rho_i}(L^+S(L^-))^r_{ii}$$

where $D = \mathrm{diag}(q^{2\rho_1}, \cdots, q^{2\rho_n}, q^{-2\rho_n}, \cdots, q^{-2\rho_1})$. In particular we will compute the radial part of

$$C_1 = \sum_{i,j=1}^{N} q^{2\rho_i} L^+_{ij} S(L^-_{ji}).$$

Note that the central character $\chi_\lambda(C_1)$ is given by

$$\chi_\lambda(C_1) = \sum_{i=1}^{N} q^{2\rho_i} q^{2\lambda_i}$$

for the dominant integral weight $\lambda \in P^+$ where we set $\lambda_{i'} = -\lambda_i$ for $1 \leq i \leq N$.

THEOREM 3.3. *The radial part* D_1 *of* C_1 *on* $\mathcal{H}|_{\mathbb{T}}$ *is given as follows.*

Case BI, DI: $SO(N)/SO(l) \times SO(N-l)$;

$$D_1 = q^{-2\tilde{\rho}_1} D_\sigma(x; q^4) + q^{-2\tilde{\rho}_1} b_0 + \sum_{l<i<l'} q^{2\rho_i},$$

Case CII: $Sp(2n)/Sp(2l) \times Sp(2n-2l)$;

$$D_1 = (q+q^{-1}) q^{-\tilde{\rho}_1} D_\sigma(x|-s, su, -q, qu, t; q^2) + (q+q^{-1}) q^{-\tilde{\rho}_1} b_0 + \sum_{2l<i<(2l)'} q^{2\rho_i}$$

where each Macdonald's operator D_σ, in cases BI and DI, is the one corresponding to the restricted root system Σ with the inner product $\langle\ ,\ \rangle_\Sigma$ and the parameters (s, t, u) in the Koornwinder's operator D_σ are the same as in Theorem 3.1, moreover $\{\widetilde{\rho}_k\}$ are given by $\widetilde{\rho} = \frac{1}{2}\sum_{\widetilde{\alpha}\in P^+(\Sigma)} k_{\widetilde{\alpha}}\widetilde{\alpha} = \sum_{k=1}^{l} \widetilde{\rho}_k \widetilde{\epsilon}_k$ (see also the notation (2.6)).

For convenience we list the table of $\{\widetilde{\rho}_k\}$.

	BI	*DI*	*CII*
$\widetilde{\rho}_k$	$n-k+\frac{1}{2}$	$n-k$	$2n-4k+3$

Remark. In the Theorem above we have

$$D_1 - \chi_\lambda(C_1) = \text{const.} \times (D_\sigma - a_{\pi(\lambda)}) \text{ on } \mathcal{H}(\lambda)|_{\mathbb{T}}$$

for each $\lambda \in P^+_{G,\mathfrak{k}}$.

3.3. Computation of the radial part of the central element C_1

Lemma 3.4.

$$C_1 = tr(D^2 L^+ S(L^-)) = tr(D^{-2} S(L^-) L^+)$$

Proof. From the commutation relation $R^+_{12} L^+_1 L^-_2 = L^-_2 L^+_1 R^+_{12}$, we have $L^+_1 S(L^-_2)^t {R^+_{12}}^{t_2} = {R^+_{12}}^{t_2} S(L^-_2)^t L^+_1$ ($\because$ $L^\pm S(L^\pm) = S(L^\pm) L^\pm = I$).

Hence we have ${R^+_{12}}^{t_2 -1} L^+_1 S(L^-_2)^t = S(L^-_2)^t L^+_1 {R^+_{12}}^{t_2 -1}$. Moreover we have, with $D = \mathrm{diag}(q^{2\rho_1}, \cdots, q^{2\rho_{1'}})$,

$$
\begin{aligned}
{R^+_{12}}^{t_2 -1} = D_1^{-2} R_{12}^{-t_1} D_1^2 = & \sum_{i,j=1}^{N} e_{ii} \otimes e_{jj} q^{-\delta_{ij}+\delta_{ij'}} \\
& - (q - q^{-1}) \sum_{i<j} \left(e_{ij} \otimes e_{ij} q^{-2\rho_i + 2\rho_j} - e_{ij} \otimes e_{j'i'} q^{-\rho_i + \rho_j} \kappa_i \kappa_j \right).
\end{aligned}
$$

Let $\langle v_i \otimes v_j, L_{12} v_k \otimes v_l \rangle$ be the coefficient of $v_i \otimes v_j$ of $L_{12} v_k \otimes v_l$ for $L_{12} \in \mathrm{End}_{\mathbb{C}}(V \otimes_{\mathbb{C}} V) \otimes \mathcal{U}$. By combining the relations $\langle v_1 \otimes v_1, \left(L^+_1 S(L^-_2)^t {R^+_{12}}^{t_2} - {R^+_{12}}^{t_2} S(L^-_2)^t L^+_1 \right) .v_{1'} \otimes v_{1'} \rangle = 0$ and $\langle v_1 \otimes v_1, \left({R^+_{12}}^{t_2 -1} L^+_1 S(L^-_2)^t - S(L^-_2)^t L^+_1 {R^+_{12}}^{t_2 -1} \right) .v_{1'} \otimes v_{1'} \rangle = 0$, we can show the equation

$$
\begin{aligned}
& \sum_{1 \leq j \leq 1'} L^+_{1j} S(L^-_{j1}) q^{2\rho_1} + \sum_{1 < j \leq 1'} L^+_{j1'} S(L^-_{1'j}) q^{2\rho_j} \\
& \quad = \sum_{1 \leq j \leq 1'} S(L^-_{j1}) L^+_{1j} q^{-2\rho_j} + \sum_{1 < j \leq 1'} S(L^-_{1'j}) L^+_{j1'} q^{2\rho_1}.
\end{aligned}
$$

Then by the induction on the rank of $\mathfrak{g}$ we conclude the statement, noting the triangularity of the matrices $D^2 L^+ S(L^-) D^{-2} S(L^-) L^+$ with (i,j)-entries $L^+_{ij} S(L^-_{ji}) q^{2\rho_i}$ and $S(L^-_{ji}) L^+_{ij} q^{-2\rho_j}$ respectively. $\square$

REMARK. The above Lemma is also valid for the case $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}(n)$.

LEMMA 3.5. *When $G = SO(N)$ and $Sp(2n)$, for the involutive algebra automorphism $\tau = * \circ S$, we have*

$$\tau(C_1) = C_1.$$

PROOF. We have

$$
\begin{aligned}
\tau\left(D^2 L^+ S(L^-)\right) &= D^2 \tau(L^+) \tau\left(S(L^-)\right) = D^2 S(L^+)^* S^2(L^-)^* \\
&= D^2 L^{-t} \left(q^{-2\rho} L^- q^{2\rho}\right)^* \quad (\because\ S^2(a) = q^{-2\rho} a q^{2\rho} \text{ for } a \in \mathcal{U})
\end{aligned}
$$

$$= D^2L^{-t}\left(D^2L^-D^{-2}\right)^* = D^2L^{-t}D^2S(L^+)^tD^{-2}$$
$$= D^2\left(C^{-t}S(L^-)C^tD^2CL^+C^{-1}\right)D^{-2}$$
$$= D^2\left(\kappa C^{-t}S(L^-)L^+C^{-1}\right)D^{-2}$$

using $C^tD^2C = \kappa I$ where $\kappa = 1$ if $G = SO(N)$, -1 if $G = Sp(2n)$. Hence we have

$$\begin{aligned}\tau(C_1) &= \tau(\mathrm{tr}\left(D^2L^+S(L^-)\right) = \mathrm{tr}\left(D^2\kappa C^{-t}S(L^-)L^+C^{-1}D^{-2}\right)\\ &= \mathrm{tr}\left(\kappa C^{-t}C^{-1}S(L^-)L^+\right)\\ &= \mathrm{tr}\left(D^{-2}S(L^-)L^+\right) \quad (\because C^{-t}C^{-1} = \kappa D^{-2})\\ &= C_1 \quad (\because \text{Lemma 3.4}). \ \square\end{aligned}$$

LEMMA 3.6. *We set $\widetilde{\epsilon}_k = 2\epsilon_k$ in cases BI and DI and $\epsilon_{2k-1} + \epsilon_{2k}$ in case CII for $1 \le k \le l$ (l = the rank of Σ). For any $q^h \in \mathcal{U}_0$ ($h \in P$), if we have the expression*

$$q^hC_1 \equiv \left(\sum_{k=1}^{l}\Phi_k(x)\xi_k + \sum_{k=1}^{l}\Phi_{k'}(x)\xi_{k'} + \Phi_0(x)\right)q^h$$

modulo $\mathcal{J} + \overline{\mathcal{J}}$ where $\xi_k = \xi_{k'}^{-1} = q^{\widetilde{\epsilon}_k}$, and $\Phi_k(x)$, $\Phi_{k'}(x)$ and $\Phi_0(x)$ are rational functions in $x_1 = q^{\langle h,\tilde{\epsilon}_1\rangle}, \cdots, x_l = q^{\langle h,\tilde{\epsilon}_l\rangle}$, then we have the unique expression of $\Phi_k(x)$'s and $\Phi_{k'}(x) = \Phi_k(x^{-1})$ for any $1 \le k \le l$.

PROOF. We first prove the uniqueness of the expression of $\Phi_k(x)$'s. Suppose that ξ_k (resp. $\xi_{k'} = \xi_k^{-1}$) belongs to $\mathcal{J} + \overline{\mathcal{J}}$ for some $1 \le k \le l$ or $l' \le k \le 1'$. So from the arguments in Subsection 3.1 we have $(\xi_k.\phi(\lambda))|_{\mathbb{T}} = 0$ for any zonal spherical function $\phi(\lambda) \in \mathcal{H}$. On the other hand

$$(\xi_k.\phi(\lambda))|_{\mathbb{T}} = q^{\langle\widetilde{\epsilon}_k,\lambda\rangle}z^\lambda + \sum_{\mu<\lambda}a_{\lambda\mu}q^{\langle\widetilde{\epsilon}_k,\mu\rangle}z^\mu. \text{ (See (1.3).)}$$

Hence comparing the leading term we have $\langle\widetilde{\epsilon}_k, \lambda\rangle = 0$ for any $1 \le k \le l$ and $\lambda \in P^+_{G,\mathfrak{k}}$. This forces the contradictory conditions for spherical representations of $P^+_{G,\mathfrak{k}}$. Hence any ξ_k does not belong to $\mathcal{J} + \overline{\mathcal{J}}$.

Next suppose that $C := \sum_{k=1}^{l} C_k\xi_k + \sum_{k=1}^{l} C_{k'}\xi_{k'} \equiv 0$ modulo $\mathcal{J} + \overline{\mathcal{J}}$ for certain constants C_k's. By applying the same arguments above, we have

$$0 = (C.\phi(\lambda))|_{\mathbb{T}} = (\sum_{k=1}^{l} C_k q^{\langle\widetilde{\epsilon}_k,\lambda\rangle} + \sum_{k=1}^{l} C_{k'} q^{-\langle\widetilde{\epsilon}_k,\lambda\rangle})z^{\lambda} + \sum_{\mu<\lambda} b_{\lambda\mu} z^{\mu}$$

for certain constants $b_{\lambda\mu}$. Thus we have $\sum_{k=1}^{l} C_k q^{\langle\widetilde{\epsilon}_k,\lambda\rangle} + \sum_{k=1}^{l} C_{k'} q^{-\langle\widetilde{\epsilon}_k,\lambda\rangle} = 0$ for any $\lambda \in P_{G,\mathfrak{k}}^{+}$. Then for the equation $\sum_{k=1}^{l} C_k y_k + \sum_{k=1}^{l} C_{k'} y_k^{-1} = 0$ we consider a solution $(y_1, \cdots, y_l)$ $=(q^{\langle\widetilde{\epsilon}_1,\lambda\rangle}, \cdots, q^{\langle\widetilde{\epsilon}_l,\lambda\rangle})$. Hence if we choose an appropriate $\lambda \in P_{G,\mathfrak{k}}^{+}$, we can conclude $C_1 = C_{1'} = 0$ and $\sum_{k=2}^{l} C_k y_k + \sum_{k=2}^{l} C_{k'} y_{k'} = 0$. Thus the successive argument leads $C_k = 0$ for all k. This shows the uniquness of $\Phi_k(x)$'s since the above argument is true for $\xi_k q^h$'s.

Since $\tau(q^h) = q^{-h}$, $\tau(\xi_k) = \xi_{k'}$, we have, by the τ-invariance of $\mathcal{J}$ and $\overline{\mathcal{J}}$, $q^{-h}C_1 = \tau(q^h C_1) \equiv \sum_{k=1}^{l} (\Phi_k(x)\xi_{k'} + \Phi_{k'}(x)\xi_k + \Phi_0(x))\, q^{-h}$ modulo $\mathcal{J} + \overline{\mathcal{J}}$.

On the other hand $q^{-h}C_1 \equiv \sum_{k=1}^{l} \left(\Phi_k(x^{-1})\xi_k + \Phi_{k'}(x^{-1})\xi_{k'} + \Phi_0(x)\right) q^{-h}$ modulo $\mathcal{J} + \overline{\mathcal{J}}$. Thus we conclude $\Phi_{k'}(x) = \Phi_k(x^{-1})$ for $1 \leq k \leq l$. □

We will see that the modulo class $q^h C$ modulo $\mathcal{J} + \overline{\mathcal{J}}$ has the form in Lemma 3.6, hence by Lemma 3.2 we can get the radial part of C_1.

LEMMA 3.7 (see [N] and [NS]). *For any $q^h \in \mathcal{U}_0$ $(h \in P)$ we have*

$$q^h L_1^{+} S(L_2)^t \widetilde{R} \equiv \widetilde{R} q^h L_1^{+} S(L_2^{-})^t$$

modulo $\mathcal{J} + \overline{\mathcal{J}}$ where $\widetilde{R} = R_{12}^{+\,t_2} P_{12} J_2 H_1 J_1^{-1} H_1$, $H = diag(q^{\langle h,\epsilon_1\rangle}, \cdots, q^{\langle h,\epsilon_{1'}\rangle})$ and P is the flip in $\mathrm{End}_{\mathbb{C}}(V \otimes_{\mathbb{C}} V)$.

Let us consider a vector space

$$W = \bigoplus_{k=1}^{l} \mathbb{C}w_k \oplus \mathbb{C}w_{l+1} \oplus \bigoplus_{k=1}^{l} \mathbb{C}w_{k'} \quad (k' = 2l+1-k+1)$$

and the following linear maps

$$\pi_W^{(a)} \ : \ V \otimes_{\mathbb{C}} V \longrightarrow W \quad \text{and} \quad \imath_W^{(a)} \ : \ W \longrightarrow V \otimes_{\mathbb{C}} V \quad (a = 1, 2)$$

such that, for $1 \leq k \leq l$,

$$\begin{aligned}
&\pi_W^{(1)}(v_j \otimes v_j) = w_j \ (1 \leq j \leq l), \quad \pi_W^{(1)}(v_{j'} \otimes v_{j'}) = w_{j'} \ (1 \leq j \leq l),\\
&\pi_W^{(1)}(v_s \otimes v_s) = q^{2\rho_s} w_{l+1} \ (l < s < l'), \quad \pi_W^{(1)}(v_i \otimes v_j) = 0 \ (i \neq j),\\
&\imath_W^{(1)}(w_k) = v_k \otimes v_k \ (1 \leq k \leq l), \quad \imath_W^{(1)}(w_{k'}) = v_{k'} \otimes v_{k'} \ (1 \leq k \leq l),\\
&\imath_W^{(1)}(w_{l+1}) = \sum_{l < s < l'} v_s \otimes v_s,\\
&\pi_W^{(2)}(v_{2k-1} \otimes v_{2k-1}) = q w_k, \quad \pi_W^{(2)}(v_{2k} \otimes v_{2k}) = q^{-1} w_k,\\
&\pi_W^{(2)}(v_{(2k)'} \otimes v_{(2k)'}) = q w_{k'}, \quad \pi_W^{(2)}(v_{(2k-1)'} \otimes v_{(2k-1)'}) = q^{-1} w_{k'},\\
&\pi_W^{(2)}(v_s \otimes v_s) = q^{2\rho_s} w_{l+1} \ (2l < s < (2l)'), \quad \pi_W^{(2)}(v_i \otimes v_j) = 0 \ (i \neq j),\\
&\imath_W^{(2)}(w_k) = v_{2k-1} \otimes v_{2k-1} + v_{2k} \otimes v_{2k}, \imath_W^{(2)}(w_{k'})\\
&\qquad = v_{(2k)'} \otimes v_{(2k)'} + v_{(2k-1)'} \otimes v_{(2k-1)'},\\
&\imath_W^{(2)}(w_{l+1}) = \sum_{2l < s < (2l)'} v_s \otimes v_s.
\end{aligned}$$

Note here that in the notation of the vector space V we set $k' = N - k + 1$ for $1 \leq k \leq N$, but in the vector space W we set $k' = 2l + 1 - k + 1$ for $1 \leq k \leq 2l + 1$.

We put matrices $A = A(s, t, u)$ and $B = B(s, t, u) \in \mathrm{End}_{\mathbb{C}}(W)$ as follows with assuming $0 < t < 1$ and $0 < s, u \leq 1$ (e_{ij} denote the matrix units such that $e_{ij}.w_k = \delta_{jk} w_i$).

$$\begin{aligned}
A = A(s,t,u) = &\sum_{\substack{1 \leq j \leq l \\ l' \leq j' \leq 1'}} e_{jj} + e_{l+1,l+1} s t^{-1} + \sum_{\substack{1 \leq i,j \leq 2l+1 \\ i,j \neq l+1}} e_{ij}(1 - t^{-1})\\
&+ \sum_{1 \leq i \leq l} e_{i,l+1}(1 - t^{-1}) u^{-1}\\
&+ \sum_{l' \leq j' \leq 1'} e_{l+1,j'}(-1)(1 - u)(1 + s^2 t^{-1} u) s^{-1} u^{-1}
\end{aligned}$$

$$+\sum_{1\le j\le l} e_{jj'}(1-s^{-2}t)st^{-1}t^{-\check\rho_j},$$

$$B=B(s,t,u)=\sum_{\substack{1\le j\le l\\ l'\le j'\le 1'}} e_{jj}+e_{l+1,l+1}st^{-1}+\sum_{\substack{1\le i,j\le 2l+1\\ j\ne l+1}} e_{ij}(1-t^{-1})$$
$$+\sum_{1\le i\le l} e_{i,l+1}(-1)(1-u)(1+s^2t^{-1}u)s^{-1}u^{-2}$$
$$+\sum_{1\le j\le l} e_{jj'}(1-s^{-2}t)st^{-1}t^{-\check\rho_j}$$

where the constants $\check\rho=(\check\rho_1,\cdots,\check\rho_l)$ are nonnegative real numbers such that

$$\check\rho_j-\check\rho_{j+1}=1\ (1\le j\le l-1),\quad t^{\check\rho_1}=sut^{l-1}\text{ and }t^{\check\rho_l}=su.$$

Direct calculation shows the following two lemmas from the equation

$$\pi_W^{(a)}\circ\left(q^hL_1^+S(L_2)^t\widetilde R-\widetilde Rq^hL_1^+S(L_2^-)^t\right)\circ\imath_W^{(a)}\equiv 0\quad(\because\text{Lemma 3.7})$$

modulo $\mathcal J+\overline{\mathcal J}$ for $a=1$ or 2.

LEMMA 3.8. *We consider the cases BI and DI. Let us put* $(s,t,u)=(q^{k_{2\tilde\epsilon_l}},q^{k_{\tilde\epsilon_1\pm\tilde\epsilon_2}},q^{\frac12 k_{\tilde\epsilon_l}})$ *(s must be 1),* $A(x|s,t,u)=A(s,t,u)\overset{\vee}{H}$ *and* $B(x|s,t,u)=B(s,t,u)\overset{\vee}{H}$ *where* $\overset{\vee}{H}=diag(x_1,\cdots,x_l,1,x_l^{-1},\cdots,x_1^{-1})$ *with* $x_j=q^{2\langle h,\epsilon_j\rangle}$ *for* $1\le j\le l$. *We have*

$$q\left(A(x|s,t,u)q^hZ-q^hZB(x|s,t,u)\right)\equiv\pi_W^{(1)}q^hL_1^+S(L_2^-)^t\imath^{(1)}$$
$$-\pi^{(1)}q^hL_1^+S(L_2^-)^t\imath_W^{(1)}$$

modulo $\mathcal J+\overline{\mathcal J}$ *where* $Z:=\pi_W^{(1)}q^hL_1^+S(L_2^-)^t\imath_W^{(1)}$, $\pi^{(1)}:=\pi_W^{(1)}\circ\widetilde R-qA(x|s,t,u)\pi_W^{(1)}$ *and* $\imath^{(1)}:=\widetilde R\circ\imath_W^{(1)}-\imath_W^{(1)}qB(x|s,t,u)$.

LEMMA 3.9. *We next consider the case CII. Let us put* (s,t,u) *and the matrices* $A(x|s,t,u)$ *and* $B(x|s,t,u)$ *in the same way as above with* $x_k=q^{\langle h,\epsilon_{2k-1}+\epsilon_{2k}\rangle}$. *Then we have*

$$q^2\left(A(x|s,t,u)q^hZ-q^hZB(x|s,t,u)\right)\equiv\pi_W^{(2)}q^hL_1^+S(L_2^-)^t\imath^{(2)}\tag{3.1}$$
$$-\pi^{(2)}q^hL_1^+S(L_2^-)^t\imath_W^{(2)}$$

modulo $\mathcal{J}+\overline{\mathcal{J}}$ *where* $Z := \pi_W^{(2)} L_1^+ S(L_2^-)^t \imath_W^{(2)}$, $\pi^{(2)} := \pi_W^{(2)} \circ \widetilde{R} - q^2 A(x|s,t,u)\pi_W^{(2)}$ *and* $\imath^{(2)} := \widetilde{R}\circ \imath_W^{(2)} - \imath_W^{(2)} q^2 B(x|s,t,u)$.

3.4. Proof of Theorem 3.3

We will prove Theorem 3.3 only for the case CII. However we can get similar formula in cases BDI for the corresponding parameters (s,t,u). In what follows we simply denote by π_W, $\imath_W$, π, $\imath$ and $\mathbb{C}(x)$ the $\pi_W^{(2)}$, $\imath_W^{(2)}$, $\pi^{(2)}$, $\imath^{(2)}$ and $\mathbb{C}(x_1,\cdots,x_l)$ respectively and set

$$Z = \pi\circ L_1^+ S(L_2^-)^t \circ \imath = (Z_{ij})_{ij}.$$

For the case CII we have, for $1\le i,j\le l$,

$$\begin{aligned}
Z_{ij} =& qL^+_{2i-1,2j-1}S(L^-_{2j-1,2i-1}) + qL^+_{2i-1,2j}S(L^-_{2j,2i-1})\\
&+ q^{-1}L^+_{2i,2j-1}S(L^-_{2j-1,2i}) + q^{-1}L^+_{2i,2j}S(L^-_{2j,2i}),\\
Z_{i'j'} =& qL^+_{(2i)',(2j)'}S(L^-_{(2j)',(2i)'}) + qL^+_{(2i)',(2j-1)'}S(L^-_{(2j-1)',(2i)'})\\
&+q^{-1}L^+_{(2i-1)',(2j)'}S(L^-_{(2j)',(2i-1)'})\\
&+q^{-1}L^+_{(2i-1)',(2j-1)'}S(L^-_{(2j-1)',(2i-1)'}),\\
Z_{ij'} =& qL^+_{2i-1,(2j)'}S(L^-_{(2j)',2i-1}) + qL^+_{2i-1,(2j-1)'}S(L^-_{(2j-1)',2i-1})\\
&+ q^{-1}L^+_{2i,(2j)'}S(L^-_{(2j)',2i}) + q^{-1}L^+_{2i,(2j-1)'}S(L^-_{(2j-1)',2i}),\\
Z_{l+1,l+1} =& \sum_{2l<s\le t<(2l)'} L^+_{st}S(L^-_{ts})q^{2\rho_s},\\
Z_{i,l+1} =& \sum_{2l<s<(2l)'} \left(qL^+_{2i-1,s}S(L^-_{s,2i-1}) + q^{-1}L^+_{2i,s}S(L^-_{s,2i})\right),\\
Z_{l+1,j'} =& \sum_{2l<s<(2l)'} \left(L^+_{s,(2j)'}S(L^-_{(2j)',s}) + L^+_{s,(2j-1)'}S(L^-_{(2j-1)',s})\right) q^{2\rho_s}.
\end{aligned}$$

Hence we have

$$C_1 = \sum_{1\le i\le j\le 1'} Z_{ij}q^{\tilde\rho_i} = \sum_{1\le i\le j\le 1'} Z_{ij}t^{\check\rho_i} \tag{3.2}$$

where we set $\check\rho_{i'} = -\check\rho_i$ for $1\le i\le 1'$. From this expression and (3.1) in Lemma 3.9 we will derive the necessary reccurence relations for the modulo class of q^hZ_{ij} modulo $\mathcal{J}+\overline{\mathcal{J}}$.

Let us put a $(2l+1) \times (2l+1)$-matrix

$$E = (E_{ij})_{ij} := q^{-2}\left(\pi_W q^h L_1^+ S(L_2^-)^t \imath - \pi q^h L_1^+ S(L_2^-)^t \imath_W\right)$$

in (3.1).

LEMMA 3.10. *Let us fix a number k such that $1 \leq k \leq l$. For the matrix E above we have*

$$E_{ij'} \equiv \begin{cases} q^h Z_{ij} \beta_k t^{-\check{\rho}_j} & \text{if } 1 \leq i \leq k \leq j \leq l \\ 0 & \text{otherwise } (i,j \text{ run } 1 \leq i,j \leq 1') \end{cases}$$

modulo $\mathcal{I}_k + \mathcal{J} + \overline{\mathcal{J}}$ where $\beta_k = -\frac{(1+s^{-1}tx_k)}{1-s^{-1}tx_k}(1-u)^2u^{-1}$ and

$$\mathcal{I}_k := q^h\mathbb{C}(x) + \sum_{\substack{j=1 \\ j\neq k}}^{l} q^h q^{\tilde{\epsilon}_j}\mathbb{C}(x) + \sum_{j=1}^{l} q^h q^{-\tilde{\epsilon}_j}\mathbb{C}(x).$$

To conclude this lemma we need long computation, so we will give a sketch of the proof later (see subsectin 3.5).

Let us also introduce a matrix $\overset{\vee}{B}(x|s,t,u)$ by

$$\overset{\vee}{B}(x|s,t,u) := B(x|s,t,u) + \beta_k\left(\sum_{j=1}^{l} e_{jj'} t^{-\check{\rho}_j}\right)\operatorname{diag}(x_1,\cdots,x_l,1,x_l^{-1},\cdots,x_1^{-1}).$$

Thus we have from Lemma 3.10

$$A(x|s,t,u)q^hZ - q^hZ\overset{\vee}{B}(x|s,t,u) \equiv 0 \tag{3.3}$$

modulo $\mathcal{I}_k + \mathcal{J} + \overline{\mathcal{J}}$. Here note that the matrices Z, $A(x|s,t,u)$ and $\overset{\vee}{B}(x|s,t,u)$ are all upper triangular. So we see from (3.2) that the modulo reduction of the matrix q^hZ is recursively and uniquely determined by its diagonal parts modulo $\mathcal{I}_k + \mathcal{J} + \overline{\mathcal{J}}$.

The modulo class of the diagonal elements of $q^h Z$ are given by

$$\begin{aligned} q^h Z_{ii} &\equiv q^h(qq^{2\epsilon_{2i-1}} + q^{-1}q^{2\epsilon_{2i}}) \equiv (q+q^{-1})q^h q^{\tilde{\epsilon}_i} \quad (1 \le i \le l), \\ q^h Z_{l+1,l+1} &\equiv q^h \sum_{2l<s<(2l)'} q^{2\rho_s}, \\ q^h Z_{i'i'} &\equiv q^h(qq^{-2\epsilon_{2i}} + q^{-1}q^{-2\epsilon_{2i-1}}) \equiv (q+q^{-1})q^h q^{-\tilde{\epsilon}_i} \quad (1 \le i \le l) \end{aligned}$$

modulo $\mathcal{J} + \overline{\mathcal{J}}$, since $L^+_{2i-1,2i}$, $S(L^-_{2i,2i-1})$, $L^+_{(2i)'(2i-1)'}$, $S(L^-_{(2i-1)'(2i)'})$, L^+_{st}, $S(L^-_{ts})$, $q^{\epsilon_{2i-1}} - q^{\epsilon_{2i}}$ and $q^{\epsilon_s} - q^{-\epsilon_s}$ are all belonging to $\mathfrak{k}_{\mathbb{C},q}$ for $1 \le i \le l$ and $2l < s \le t < (2l)'$. Hence combining Lemma 3.10 and the consequent discussion, we see that that modulo class of $q^h C_1$ has the form in Lemma 3.6. Thus we can summarize the algebraic feature of the modulo reduction by $\mathcal{I}_k + \mathcal{J} + \overline{\mathcal{J}}$ in the next lemma.

LEMMA 3.11. *We fix the number k such that $1 \le k \le l$. For the matrix $A = A(x|s,t,u)$ and $\overset{\vee}{B}(x|s,t,u)$ above, we consider the same size upper triangular matrix $F = F(x,\xi|s,t,u) = (F_{ij})_{ij}$ with the entries in $\mathbb{C}(x)[\xi_1, \cdots, \xi_l, \xi_{l+1}, \xi_{l'} \cdot \cdots, \xi_{1'}]$ such that $F_{ii} = \xi_i$ $(1 \le i \le 1'$ and*

$$AF - F\overset{\vee}{B} \equiv 0$$

modulo $\sum_{\substack{i=1\\ i\ne k}}^{1'} \mathbb{C}(x)\xi_i$. Then we have

$$\begin{aligned} C := \sum_{1 \le i \le j \le 1'} F_{ij} t^{\check{\rho}_i} &\equiv t^{-\check{\rho}_k} \frac{(1+sx_k)(1-sux_k)(1-s^{-1}tux_k)}{(1-s^{-1}tx_k)(1-x_k^2)} \\ &\times \frac{1-t^{-1}x_{[1,k-1]}x_k^{-1}}{1-x_{[1,k-1]}x_k^{-1}} \cdot \frac{1-tx_k x^{-1}_{[k+1,l]}}{1-x_k x^{-1}_{[k+1,l]}} \cdot \frac{1-tx_k x_{[1,\hat{k},l]}}{1-x_k x_{[1,\hat{k},l]}} \cdot \xi_k \end{aligned}$$

modulo $\sum_{\substack{i=1\\ i\ne k}}^{1'} \mathbb{C}(x)\xi_i$. Here we adopt the notation like

$$\frac{1-tx_k x_{[1,\hat{k},l]}}{1-x_k x_{[1,\hat{k},l]}} := \prod_{\substack{i=1\\ i\ne k}}^{l} \frac{1-tx_k x_i}{1-x_k x_i}.$$

PROOF. There exist unique upper unitriangular matrices $G = G(x|s,t,u) = (G^{+}_{ij})_{ij}$ and $\check{G} = \check{G}(x|s,t,u) = (\check{G}^{+}_{ij})_{ij}$ with the entries in $\mathbb{C}(x)$ such that

$$A = G\mathrm{diag}(x_1, \cdots, x_l, st^{-1}, x_l^{-1}, \cdots, x_1^{-1})G^{-1},$$
$$\check{B} = \check{G}\mathrm{diag}(x_1, \cdots, x_l, st^{-1}, x_l^{-1}, \cdots, x_1^{-1})\check{G}^{-1}.$$

With these matrices G and $\check{G}$, the matrix F is uniquely determined by

$$F = G\mathrm{diag}(\xi_1, \cdots, \xi_{1'})\check{G}^{-1}.$$

Here we set $\check{G}^{-1} = (\check{G}^{-}_{ij})_{ij}$, and we have

$$F_{ij} = \sum_{i \le k \le j} G^{+}_{ik} \xi_k \check{G}^{-}_{kj}$$

for $1 \le i \le j \le 1'$. Moreover we have

$$C = \sum_{1 \le i \le j \le 1'} F_{ij} t^{\check{\rho}_i} \equiv \left(\sum_{1 \le i \le k} G^{+}_{ik} t^{\check{\rho}_i} \right) \left(\sum_{k \le j \le 1'} \check{G}^{-}_{kj} \right) \xi_k$$

modulo $\sum_{\substack{i=1 \\ i \ne k}}^{1'} \mathbb{C}(x)\xi_i$. Thus we need to compute the quantities $\sum_{1 \le i \le k} G^{+}_{ik} t^{\check{\rho}_i}$ and $\sum_{k \le j \le 1'} \check{G}^{-}_{kj}$.

CLAIM 1.
$$\sum_{1 \le i \le k} G^{+}_{ik} t^{\check{\rho}_i} = \frac{t^{\check{\rho}_1}(1 - t^{-1} x_{[1,k-1]} x_k^{-1})}{1 - x_{[1,k-1]} x_k^{-1}}$$

PROOF. For the matrix A we have $G\mathrm{diag}(x_1, \cdots, x_l, st^{-1}, x_l^{-1}, \cdots, x_1^{-1}) = AG$. So we have

$$G^{+}_{ik} x_k = G^{+}_{ik} x_i + \sum_{i < s \le k} G^{+}_{sk} (1 - t^{-1}) x_s \tag{3.4}$$

For $1 \leq i \leq k$. Compairing (3.4) with the one obtained from (3.4) replacing i for $i+1$, we have

$$G^+_{ik}(x_k - x_i) = G^+_{i+1k}(x_k - x_{i+1}) + G^+_{i+1k}(1-t^{-1})x_{i+1} = G^+_{i+1k}(x_k - t^{-1}x_{i+1}).$$

Hence we have

$$\begin{aligned}(3.5) \qquad G^+_{ik} &= \frac{1-t^{-1}x_{i+1}x_k^{-1}}{1-k_i x_k^{-1}} G^+_{i+1k} = \frac{1-t^{-1}x_{[i+1,k]}x_k^{-1}}{1-x_{[i,k-1]}x_k^{-1}} G^+_{kk} \\ &= \frac{(1-t^{-1})(1-t^{-1}x_{[i+1,k-1]}x_k^{-1})}{1-x_{[i,k-1]}x_k^{-1}}.\end{aligned}$$

On the other hand, the entries A^-_{ij} $(1 \leq i \leq j \leq l)$ of $A^{-1} = (A^-_{ij})_{ij}$ are given by $A^-_{ij} = (1-t)t^{\check{\rho}_j - \check{\rho}_i}x_i^{-1}$ if $1 \leq i < j \leq l$ and x_i^{-1} if $1 \leq i = j \leq l$. Thus we have another equations from $G\text{diag}(x_1^{-1}, \cdots, x_l^{-1}, s^{-1}t, x_l, \cdots, x_1) = A^{-1}G$ so that

$$G^+_{1k}x_k^{-1} = G^+_{1k}x_1^{-1} + \sum_{1<i\leq k} G^+_{ik}(1-t)t^{\check{\rho}_i - \check{\rho}_1}x_1^{-1}.$$

So we have

$$\begin{aligned}\sum_{1\leq i\leq k} G^+_{ik}t^{\check{\rho}_i} &= \frac{t^{\check{\rho}_1}x_1}{1-t}\left(G^+_{1k}(x_k^{-1} - x_1^{-1}) + G^+_{1k}(1-t)x_1^{-1}\right) \\ &= \frac{t^{\check{\rho}_1}}{1-t^{-1}} G^+_{1k}(1-t^{-1}x_1x_k^{-1}).\end{aligned}$$

Hence from (3.5) we have the expressin of Claim 1. □

CLAIM 2.

$$\begin{aligned}\sum_{k\leq j\leq 1'} G^-_{kj} &= \frac{t^{-\check{\rho}_1 - \check{\rho}_k}(1+sx_k)(1-sux_k)(1-s^{-1}tux_k)}{(1-s^{-1}tx_k)(1-x_k^2)} \\ &\cdot \frac{1-tx_kx_{[1,\hat{k},l]}}{1-x_kx_{[1,\hat{k},l]}} \cdot \frac{1-tx_kx^{-1}_{[k+1,l]}}{1-x_kx^{-1}_{[k+1,l]}}.\end{aligned}$$

PROOF. Similar discusion in Claim 1 shows

$$\overset{\vee}{G}^{-}_{ij} = \frac{t^{-1}(1 - tx_i x_{j-1}^{-1})}{1 - x_i x_j^{-1}} \overset{\vee}{G}^{-}_{ij-1} = \frac{t^{\check{\rho}_j - \check{\rho}_i}(1 - tx_i x_{[i,j-1]}^{-1})}{1 - x_i x_{[i+1,j]}^{-1}} \tag{3.6}$$

for $1 \leq i < j \leq l$, and

$$\overset{\vee}{G}^{-}_{il+1} = \frac{bs^{-1}(1 - tx_i x_l^{-1})}{(1 - t^{-1})(1 - s^{-1}tx_i)} \overset{\vee}{G}^{-}_{il} \tag{3.7}$$

for $1 \leq i \leq l$ where we set $b := -(1-u)(1+s^2t^{-1}u)s^{-1}u^{-2} (= B_{il+1})$. Moreover we have

$$\begin{aligned} \overset{\vee}{G}^{-}_{il'} &= \frac{(1 - t^{-1}st^{-1}(1 - s^{-1}tx_i)}{b(1 - x_i x_l)} \overset{\vee}{G}^{-}_{il+1} \\ &\quad - \frac{1 - t^{-1}}{1 - x_i x_l} \overset{\vee}{G}^{-}_{il+1} - \frac{\{-(1 - s^2t^{-1})s^{-1} + \beta_k\} t^{-\check{\rho}_l}}{1 - x_i x_l} \overset{\vee}{G}^{-}_{il} \\ &= \left\{ \frac{t^{-1}(1 - tx_i x_l^{-1})}{1 - x_i x_l} - \frac{bs^{-1}(1 - tx_i x_l^{-1})}{(1 - x_i x_l)(1 - s^{-1}tx_i)} \right. \\ &\qquad \left. - \frac{\{-(1 - s^2t^{-1})s^{-1} + \beta_k\} t^{-\check{\rho}_l}}{1 - x_i x_l} \right\} \overset{\vee}{G}^{-}_{il} \end{aligned} \tag{3.8}$$

for $1 \leq i \leq l$, and

$$\begin{aligned} \overset{\vee}{G}^{-}_{ij'} &= \frac{t^{-1}(1 - tx_i x_{j+1})}{1 - x_i x_j} \overset{\vee}{G}^{-}_{i\ (j+1)'} \\ &\quad + \frac{\left((1 - s^2t^{-1})s^{-1} - \beta_k\right) t^{-\check{\rho}_j}}{1 - x_i x_j} \cdot \frac{tx_i x_j (1 - t^{-1} x_j x_{j+1}^{-1})}{1 - x_i x_{j+1}^{-1}} \overset{\vee}{G}^{-}_{ij} \end{aligned} \tag{3.9}$$

for $1 \leq i, j \leq l$.

From (3.7), (3.8) and (3.9), for $1 \leq i, j \leq l$, if we put

$$\widetilde{G}^{-}_{ij'} := \overset{\vee}{G}^{-}_{ij'} + \frac{t}{1 - tx_i^2} \left\{ -(1 - s^2t^{-1})s^{-1} t^{-\check{\rho}_j} x_i x_j^{-1} + \beta_k t^{-\check{\rho}_j} x_i x_j^{-1} \right\} \overset{\vee}{G}^{-}_{ij},$$

then a long computation shows that

$$\widetilde{G}^-_{ij'} = \frac{t^{-1}(1 - tx_ix_{j+1})}{1 - x_ix_j}\widetilde{G}^-_{i(j+1)'} = \frac{t^{\check{\rho}_l-\check{\rho}_j}(1 - tx_ix_{[j+1,l]})}{1 - x_ix_{[j,l-1]}}\widetilde{G}^-_{il'} \tag{3.10}$$

and

$$\widetilde{G}^-_{il'} = \frac{u^{-2}s^{-2}(1 - tx_ix_l^{-1})(1 + sx_i)(1 - sux_i)(1 - s^{-1}tux_i)}{(1 - x_ix_l)(1 - s^{-1}tx_i)(1 - tx_i^2)}\overset{\vee}{G}{}^-_{il}. \tag{3.11}$$

Combining (3.6), (3.10), (3.11) and $u^{-2}s^{-2} = t^{-2\check{\rho}_l}$, we have

$$\begin{aligned}\overset{\vee}{G}{}^-_{i1'} = \widetilde{G}^-_{i1'} &= t^{-\check{\rho}_1-\check{\rho}_i}(1 - t)\frac{(1 + sx_i)(1 - sux_i)(1 - s^{-1}tux_i)}{(1 - s^{-1}tx_i)(1 - tx_i^2)(1 - x_i^2)} \\ &\times \frac{1 - tx_ix_{[2,l]}}{1 - x_ix_{[1,\hat{i},l]}} \cdot \frac{1 - tx_ix^{-1}_{[i+1,l]}}{1 - x_ix^{-1}_{[i+1,l]}}.\end{aligned} \tag{3.12}$$

On the other hand, from $(k, 1')$ component of the equation $\mathrm{diag}(x_1, \cdots, x_l, st^{-1}, x_l^{-1}, \cdots, x_1^{-1})\overset{\vee}{G}{}^{-1} = \overset{\vee}{G}{}^{-1}\overset{\vee}{B}$, we have

$$\sum_{k\le j\le 1'} \overset{\vee}{G}{}^-_{kj} = \frac{1}{1 - t}(1 - tx_1x_k)\overset{\vee}{G}{}^-_{k1'}.$$

Therefore by putting $i = k$ in (3.12) we complete the proofs of Claim 2 and Lemma 3.11. □

Now we return to the proof of Theorem 3.3 for the case CII.

By applying Lemma 3.11 putting $\xi_i = (q + q^{-1})q^hq^{\tilde{\epsilon}_i}$ $(1 \le i \le 1')$ and $\xi_{l+1} = \sum_{2l<s<(2l)'} q^hq^{2\rho_s}$, from (3.2) and (3.3) we have

$$q^hC_1 \equiv \sum_{k=1}^{l} C_k(x)\xi_k + \sum_{k=1}^{l} C_{k'}(x)\xi_{k'} + C_{l+1}(x)\xi_{l+1}$$

modulo $\mathcal{J} + \overline{\mathcal{J}}$ where

$$C_k(x) = t^{-\check{\rho}_k}\frac{(1 + sx_k)(1 - sux_k)(1 - s^{-1}tux_k)}{(1 - s^{-1}tx_k)(1 - x_k^2)}$$

$$\times \frac{1-t^{-1}x_{[1,k-1]}x_k^{-1}}{1-x_{[1,k-1]}x_k^{-1}} \cdot \frac{1-tx_kx_{[k+1,l]}^{-1}}{1-x_kx_{[k+1,l]}^{-1}} \cdot \frac{1-tx_kx_{[1,\hat{k},l]}}{1-x_kx_{[1,\hat{k},l]}}$$

for $1 \leq k \leq l$ and $C_i(x)$ $(l+1 \leq i \leq 1')$ are certain elements of $\mathbb{C}(x)$. But from Lemma 3.6 we have $C_{k'}(x) = C_k(x^{-1})$ for $1 \leq k \leq l$. Hence by Lemma 3.2 the radial part of C_1 is given by

$$D_1 = (q+q^{-1}) \sum_{k=1}^{l} \left(\Phi_k^+(x) T_{q^2,x_k} + \Phi_{k'}^+(x) T_{q^2,x_k}^{-1} \right) + \Phi_{l+1}^+(x)$$

where $\Phi_i^+(x)$ are rational functions in $\mathbb{C}(x) = \mathbb{C}(x_1, \cdots, x_l)$ with $x_i = z_{2i-1}z_{2i}$ such that $\Phi_i^+(x) = C_i(x)$, $1 \leq i \leq 1'$ and $i \neq l+1$.

The constant term $\Phi_{l+1}^+(x)$ of D_1 is uniquely determined by

$$\Phi_{l+1}^+(x) = -(q+q^{-1}) \sum_{k=1}^{l} \left(\Phi_k^+(x) + \Phi_{k'}^+(x) \right) + \chi_0(C_1)$$

where χ_0 is the central character χ_λ with $\lambda = 0$, since D_1 has the eigenvector 1. For the dominant integral weight $\lambda \in P_{G,\mathfrak{k}}^+$, the central character has the value

$$\begin{aligned} \chi_\lambda(C_1) &= \sum_{i=1}^{2l} \left(q^{2\rho_i} q^{2\lambda_i} + q^{-2\rho_i} q^{-2\lambda_i} \right) + \sum_{2l<i<(2l)'} q^{2\rho_i} \\ &= (q+q^{-1}) \sum_{i=1}^{l} \left(t^{\check{\rho}_i} q^{\tilde{\lambda}_i} + t^{-\check{\rho}_i} q^{-\tilde{\lambda}_i} \right) + \sum_{2l<i<(2l)'} q^{2\rho_i} \end{aligned}$$

where $\tilde{\lambda}_i := \lambda_{2i-1} = \lambda_{2i}$ $(1 \leq i \leq l)$. So using the Koornwinder's q-difference operator D_σ, we have

$$D_1 = (q+q^{-1}) q^{-\tilde{\rho}_1} D_\sigma + (q+q^{-1}) q^{-\tilde{\rho}_1} b_0 + \sum_{2l<i<(2l)'} q^{2\rho_i}.$$

Thus we complete the proof of Theorem 3.3 for the case CII. In the proof for the cases BDI we can proceed completely in the same way with corresponding parameters (s,t,u,q). □

3.5. Remaining computations

In the previous section there remains a lemma unproven, that is, Lemma 3.10. We will finish this paper with giving a sketch of computation in Lemma 3.10.

We recall that we treat the case CII.

We put, for $1 \leq i, k \leq l$ and $1 \leq j, j' \leq 2l$,

$$\begin{aligned}
Z_i^{2k} &:= qL^+_{2i-1,2k}S(L^-_{(2k-1)'\ 2i-1}) + q^{-1}L^+_{2i,2k}S(L^-_{(2k-1)'\ 2i}),\\
Z_i^{2k-1} &:= qL^+_{2i-1,2k-1}S(L^-_{(2k)'\ 2i-1}) + q^{-1}L^+_{2i,2k-1}S(L^-_{(2k)'\ 2i}),\\
Z_i^{(2k)'} &:= qL^+_{2i-1\ (2k)'}S(L^-_{2k-1,2i-1}) + q^{-1}L^+_{2i,(2k)'}S(L^-_{2k-1,2i}),\\
Z_i^{(2k-1)'} &:= qL^+_{2i-1\ (2k-1)'}S(L^-_{2k,2i-1}) + q^{-1}L^+_{2i\ (2k-1)'}S(L^-_{2k,2i}),\\
Y_i^j &:= qL^+_{2i-1,j}S(L^-_{j\ (2i)'}) - q^{-1}L^+_{2i,j}S(L^-_{j\ (2i-1)'}),\\
Y_{i'}^j &:= qL^+_{(2i)'\ j}S(L^-_{j,2i-1}) - q^{-1}L^+_{(2i-1)'\ j}S(L^-_{j,2i}).
\end{aligned}$$

Let us denote by $\langle w_i, L.w_j\rangle$ the coefficient of w_i in $L.w_j$ for $L \in \mathrm{End}_{\mathbb{C}}(W) \otimes \mathcal{U}$.

We state some claims.

Claim 1. If we put, for $1 \leq k \leq l$,

$$P_i^k := q^h \left\{ Z_i^{2k}x_k^{-1} - Z_i^{2k-1}x_k^{-1} - Z_i^{(2k)'}q^{-2\rho_{2k}-1} + Z_i^{(2k-1)'}q^{-2\rho_{2k}-1} \right\},$$

then we have, for $1 \leq i \leq k \leq l$,

$$\begin{aligned}
&\langle w_i, \pi_W \circ q^h L_1^+ S(L_2^-)^t \circ \imath(w_{k'})\rangle\\
&= q^2 \Bigg\{ P_i^k (1 - q^{2\rho_{2l}-2}) q^{-2\rho_{2l}}\\
&\qquad + \left(q^h Z_{ik} + \sum_{i \leq s < k} q^h Z_{is}(1 - q^{-4}) \right) (1 - q^{2\rho_{2l}-2})^2 q^{-2\rho_{2k}-2\rho_{2l}+1} \Bigg\}.
\end{aligned}$$

Moreover the other matrix elements of $\pi_W \circ q^h L_1^+ S(L_2^-)^t \circ \imath$ are all zero.

CLAIM 2.

$$P_k^k \equiv -\frac{2(1-q^{2\rho_{2l}-2})q^{-2\rho_{2k}-\rho_{2l}+1}}{1-qx_k}(q+q^{-1})q^h q^{\tilde{\epsilon}_k}$$

modulo $q^h\mathbb{C}(x)+\mathcal{J}+\overline{\mathcal{J}}$.

CLAIM 3. For $1 \le i \le k \le l$, we have

$$\begin{aligned} &P_i^k x_i + \sum_{i<j\le k} P_j^k x_j(1-q^{-4}) \\ &\quad \equiv P_i^k q^{-1} + \left\{ q^h Z_{ik} + \sum_{i\le j<k} q^h Z_{ij}(1-q^{-4}) \right\} 2(1-q^{2\rho_{2l}-2})q^{-2\rho_{2k}-\rho_{2l}} \end{aligned}$$

modulo $q^h\mathbb{C}(x)+\mathcal{J}+\overline{\mathcal{J}}$.

CLAIM 4. For $1 \le i \le k \le l$ we have

$$P_i^k \equiv -2(1-q^{2\rho_{2l}-2})q^{-2\rho_{2k}-\rho_{2l}+1}(q+q^{-1}) \times \left\{ \sum_{i\le s\le k} q^h q^{\tilde{\epsilon}_s} G_{is}^+ \overset{\vee}{G}{}_{sk}^- \frac{x_s x_k^{-1}}{1-qx_s} \right\}$$

modulo $q^h\mathbb{C}(x)+\mathcal{J}+\overline{\mathcal{J}}$.

CLAIM 5.

$$\begin{aligned} &\langle w_i, \pi_W \circ q^h L_1^+ S(L_2^-)^t \circ \imath(w_{k'}) \rangle \\ &\quad \equiv q^2(1-q^{2\rho_{2l}-2})^2(q+q^{-1})q^{-2\rho_{2k}-2\rho_{2l}+1} \\ &\qquad \times \sum_{i\le s\le k} G_{is}^+ \overset{\vee}{G}{}_{sk}^- q^h q^{\tilde{\epsilon}_s} \cdot \frac{(1+qx_s)x_s x_k^{-1}}{1-qx_s} \end{aligned}$$

modulo $q^h\mathbb{C}(x)+\mathcal{J}+\overline{\mathcal{J}}$.

We get Claim 1 directly and Claim 2 by direct modulo reduction modulo $\mathcal{J}+\overline{\mathcal{J}}$. We get Claim 3 by the computation of the equations $\langle v_i \otimes$

$v_j, \left(\widetilde{R} q^h L_1^+ S(L_2^-)^t - q^h L_1^+ S(L_2^-)^t \widetilde{R}\right) v_s \otimes v_t\rangle \equiv 0$ modulo $\mathcal{J} + \overline{\mathcal{J}}$ for the pairs

$$\begin{aligned}
&(v_i \otimes v_j, v_s \otimes v_t) \\
&\quad = (v_{2i-1} \otimes v_{2i-1}, v_{2k} \otimes v_{(2k-1)'}), \quad (v_{2i} \otimes v_{2i}, v_{2k} \otimes v_{(2k-1)'}), \\
&\quad\quad (v_{2i-1} \otimes v_{2i-1}, v_{2k-1} \otimes v_{(2k)'}), \quad (v_{2i} \otimes v_{2i}, v_{2k-1} \otimes v_{(2k)'}), \\
&\quad\quad (v_{2i-1} \otimes v_{2i-1}, v_{(2k)'} \otimes v_{2k-1}), \quad (v_{2i} \otimes v_{2i}, v_{(2k)'} \otimes v_{2k-1}), \\
&\quad\quad (v_{2i-1} \otimes v_{2i-1}, v_{(2k-1)'} \otimes v_{2k}), \quad (v_{2i} \otimes v_{2i}, v_{(2k-1)'} \otimes v_{2k}).
\end{aligned}$$

We get Claim 4 by solving the recurrence formula of Claim 3 using the results of the modulo reduction of $q^h Z_{ij}$ for $1 \leq i \leq j \leq l$ in the previous section (remark Claim 6 later). Note here that we can compute the coefficient of $q^h q^{\tilde{\epsilon}_s}$ in the modulo class of $q^h Z_{ik} + \sum_{i \leq j < k} q^h Z_{ij}(1 - q^{-4})$ modulo $\mathcal{J} + \overline{\mathcal{J}}$ for $i \leq s \leq k$ as follows;

$$\begin{aligned}
&q^h Z_{ik} + \sum_{i \leq j < k} q^h Z_{ij}(1 - q^{-4}) \\
&\qquad \equiv (q + q^{-1}) \left\{ G_{is}^+ \overset{\vee}{G}{}_{sk}^- + \sum_{s \leq j < k} G_{is}^+ \overset{\vee}{G}{}_{sj}^- (1 - q^{-4}) \right\} q^h \\
&\qquad \equiv G_{sk}^+ \left\{ \overset{\vee}{G}{}_{sk}^- (x_s x_k^{-1} - 1) + \overset{\vee}{G}{}_{sk}^- \right\} = G_{is}^+ \overset{\vee}{G}{}_{sk}^- x_s x_k^{-1}
\end{aligned}$$

modulo $\sum_{j \neq s} q^h q^{\tilde{\epsilon}_j} \mathbb{C}(x) + \mathcal{J} + \overline{\mathcal{J}}$. The second equality follows from $\overset{\vee}{G}{}^{-1} \overset{\vee}{B} = \mathrm{diag}(x_1, \cdots, x_l, st^{-1}, x_l^{-1}, \cdots, x_1^{-1}) \overset{\vee}{G}{}^{-1}$. Using these results we get Claim 5. Similar arguments show the next claim.

Claim 6. If we put

$$Q_i^k := q^h \left\{ Y_i^{(2k)'} x_i q^{-2\rho_{2i}} + Y_i^{(2k-1)'} x_i q^{-2\rho_{2i}} + Y_{i'}^{(2k)'} q + Y_{i'}^{(2k-1)'} q \right\}$$

for $1 \leq i, k \leq l$, then we have for $1 \leq k \leq i \leq l$

$$\langle w_i, \pi \circ L_1^+ S(L_2^-)^t \circ \imath_W(w_{k'}) \rangle$$

$$= q^2 \left(- \left\{ Q_i^k + (1-q^{-4}) \sum_{i<s\le l} Q_s^k \right\} (1-q^{2\rho_{2l}-2}) q^{-\rho_{2l}} \right.$$

$$+ \left\{ q^h Z_{i'k'} q^{-2\rho_{2i}} + (1-q^{-4}) \sum_{i<s\le l} q^h Z_{s'k'} q^{-2\rho_{2s}} \right\}$$

$$\left. \times (1-q^{2\rho_{2l}-2})^2 q^{-2\rho_{2l}+1} \right).$$

And the other matrix elements of $\pi \circ L_1^+ S(L_2^-)^t \circ \imath_W$ are all zero. Moreover we have for $1 \le k \le i \le l$

$$Q_i^k \equiv -2(1-q^{2\rho_{2l}-2})^2 q^{-2\rho_{2i}-\rho_{2l}} (q+q^{-1}) \times \left\{ \sum_{i'\le s'\le k'} \frac{x_s}{1-q^{-1}x_s} G^+_{i's'} \overset{\vee}{G}{}^-_{s'k'} q^h q^{-\tilde{\epsilon}_s} \right\}$$

modulo $q^h\mathbb{C}(x) + \mathcal{J} + \overline{\mathcal{J}}$. Especially we have

$$\langle w_i, \pi \circ L_1^+ S(L_2^-)^t \circ \imath_W(w_{k'}) \rangle \equiv 0$$

modulo $\sum_{s=1}^l q^h q^{-\tilde{\epsilon}_s}\mathbb{C}(x) + q^h\mathbb{C}(x) + \mathcal{J} + \overline{\mathcal{J}}$.

Thus from Claim 1, Claim 5 and Claim 6 we get the Lemma 3.10. □

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(Received February 6, 1996)
(Revised January 19, 1999)

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