J. Math. Sci. Univ. Tokyo
6 (1999), 217–227.

Siegel Modular Forms Having the Same L-Functions

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Abstract. We show how one can use theta liftings to generate pairs of Siegel modular forms having the same Hecke eigenvalues but different weights. The construction uses the disconnectedness of the orthogonal group. The concrete examples obtained here have arbitrary square free level; although the method works in principle for level one as well we have not yet been able to prove nonvanishing of both forms of a pair in a case of level 1.

Introduction

It is well-known that for the Hecke eigenvalues of Siegel modular forms a strong multiplicity one theorem does not hold. Examples for this can e. g. be obtained via the Yoshida-liftings investigated in [3, 5] in the following way: For a pair f_1, f_2 of newforms for $\Gamma_0(N)$ of squarefreee level N with the same Atkin-Lehner eigenvalues consider the Yoshida liftings of the associated pairs φ_1, φ_2 of automorphic forms on definite quaternion algebras with discriminant $N_1 \mid N$. A related representation theoretic construction has been given in [9].

In this note we show how one can use theta liftings to generate examples of Siegel modular forms having the same Hecke eigenvalues but different weights. The construction uses the disconnectedness of the orthogonal group. In a representation-theoretic context the possibility of such a construction has been observed and utilized in [7]. Our concrete examples have arbitrary square free level; although the method works in principle for level one as well we have not yet been able to prove nonvanishing of both forms in a case of level 1.

1. Forms on the Orthogonal Group

Let (V, q) be a positive definite quadratic space over $\mathbf{Q}$ of even dimension $m = 2m'$ with associated symmetric bilinear form $B(x, y) = q(x + y) -$

1991 *Mathematics Subject Classification.* Primary 11F46; Secondary 11F27.

$q(x) - q(y)$ and let L be a $\mathbf{Z}$-lattice of rank m on V such that $q(L) \subseteq \mathbf{Z}$. We denote by $O(V)$ (resp. $SO(V)$) the (special) orthogonal group of (V, q), by $O_{\mathbf{A}}(V)$, $SO_{\mathbf{A}}(V)$, $O(V_p)$, $SO(V_p)$ (for $p = \infty$ or p a prime) the respective adelizations and orthogonal groups of completions and by $O(L)$ ($SO(L)$) the (proper) group of units of L with adelization $O_{\mathbf{A}}(L)$, $SO_{\mathbf{A}}(L)$.

For an irreducible representation $\lambda : SO(V_\infty) \longrightarrow \mathrm{GL}(U_\lambda)$ of the compact Lie group $SO(V_\infty)$ we consider the space $\mathcal{A}(SO_{\mathbf{A}}(V), SO_{\mathbf{A}}(L), \lambda)$ of U_λ-valued functions φ on $SO_{\mathbf{A}}(V)$ such that

$$\varphi(\gamma x u) = \lambda(u_\infty^{-1})\varphi(x)$$

for $x \in SO_{\mathbf{A}}(V)$, $\gamma \in SO(V)$, $u = (u_p) \in SO_{\mathbf{A}}(L)$; such functions are of course determined by their values at the elements of a set of representatives of the (finite) double coset decomposition of $SO_{\mathbf{A}}(V)$ with respect to $SO(V)$ and $SO_{\mathbf{A}}(L)$.

For an irreducible representation $\tilde{\lambda}$ of $O(V_\infty)$ we define $\mathcal{A}(O_{\mathbf{A}}(V), O_{\mathbf{A}}(L), \tilde{\lambda})$ analogously. We assume in the sequel that λ has highest weight $(n_1, \ldots, n_{m'-1}, 0)$. It is well-known [10] that λ can be extended to two inequivalent irreducible representations λ_+, $\lambda_- = \lambda_+ \otimes \det$ of $O(V_\infty)$, both acting on the space U_λ.

We will also assume that there is an element ι of $O(L) \setminus SO(L)$.

LEMMA 1. *Let $\varphi \in A(SO_{\mathbf{A}}(V), SO_{\mathbf{A}}(L), \lambda)$ be such that*

$$\varphi(\iota g \iota^{-1}) = \lambda_*(\iota)\varphi(g) \tag{1.1}$$

(with $* = +$ or $* = -$) holds for all $g \in SO_{\mathbf{A}}(V)$.

Then there is a unique function $\tilde{\varphi} \in \mathcal{A}(O_{\mathbf{A}}(V), O_{\mathbf{A}}(L), \lambda_*)$ with $\tilde{\varphi}|_{SO_{\mathbf{A}}(V)} = \varphi$. Conversely any $\tilde{\varphi} \in \mathcal{A}(O_{\mathbf{A}}(V), O_{\mathbf{A}}(L), \lambda_*)$ can be obtained from its restriction to $SO_{\mathbf{A}}(V)$ in this way.

PROOF. For $g = (g_p) \in O_{\mathbf{A}}(V)$ define $g' = (g'_p) \in SO_{\mathbf{A}}(V)$ by

$$g'_p = \begin{cases} g_p & \det\ g_p = 1 \\ g_p \iota^{-1} & \det\ g_p = -1 \end{cases}$$

and denote by $*$ either + or -, depending on the validity of (1.1) for the respective sign.

Then the function $\tilde{\varphi}$ on $O_{\mathbf{A}}(V)$ defined by

$$\tilde{\varphi}(g) := \lambda_*(g_\infty^{-1} g'_\infty)\varphi(g') = \begin{cases} \varphi(g') & \det\ g_\infty = 1 \\ \lambda_*(\iota)^{-1}\varphi(g') & \det\ g_\infty = -1 \end{cases} \tag{1.2}$$

has the required properties. Conversely any function $\tilde{\varphi} \in \mathcal{A}(O_{\mathbf{A}}(V), O_{\mathbf{A}}(L), \lambda_*)$ extending φ satisfies (1.2).

The last statement of the lemma is obvious. □

LEMMA 2. *Let $\varphi \in \mathcal{A}(SO_{\mathbf{A}}(V), SO_{\mathbf{A}}(L), \lambda)$ and define for $*$ denoting $+$ or $-$:*

$$\varphi_*(g) = \varphi(g) + \lambda_*(\iota)^{-1}\varphi(\iota g \iota^{-1}). \tag{1.3}$$

Then φ_* satisfies (1.1).

PROOF. This is an easy computation. □

REMARK. In general it is not clear whether only one or both of φ_+, φ_- (and hence $\tilde{\varphi}_+$, $\tilde{\varphi}_-$) is nonzero. As an example let L be a lattice in the genus of the Leech lattice but not isometric to the Leech lattice. It is well-known that the genus of L (consisting of the even unimodular lattices of rank 24) contains 24 classes and 25 proper classes of lattices, the Leech lattice representing the only class with no automorphism of determinant -1. Moreover, for ι as above we can take the reflection in a root of L. Consequently, for $\lambda^{(0)} = 1$ the trivial representation the mappings $\varphi \longmapsto \tilde{\varphi}_+$, $\varphi \longmapsto \tilde{\varphi}_-$ from the 25-dimensional space $\mathcal{A}(SO_{\mathbf{A}}(V), SO_{\mathbf{A}}(L), 1)$ to $\mathcal{A}(O_{\mathbf{A}}(V), O_{\mathbf{A}}(L), 1)$ resp. $\mathcal{A}(O_{\mathbf{A}}(V), O_{\mathbf{A}}(L), \det)$ are not injective, but we can not describe the kernel and hence cannot decide whether the Hecke eigenvalues of the function spanning the 1-dimensional space $\mathcal{A}(O_{\mathbf{A}}(V), O_{\mathbf{A}}(L), \det)$ also occurs in the decomposition of the 24-dimensional space $\mathcal{A}(O_{\mathbf{A}}(V), O_{\mathbf{A}}(L), 1)$.

We will see below that the situation for certain 4-dimensional quadratic spaces is easier.

In order to obtain the examples mentioned in the introduction we have to relate the action of the Hecke algebra on φ and on $\tilde{\varphi}_*$.

We notice first that each double coset $O(L_p)g\,O(L_p)$ with g in the group $GO(V_p)$ of orthogonal similitudes of V_p has a representative in the group $GO^+(V_p)$ of proper similitudes.

We assume in the sequel that $\varphi \in \mathcal{A}(SO_{\mathbf{A}}(V), SO_{\mathbf{A}}(L), \lambda)$ is the restriction of a function (also denoted by φ) in $\mathcal{A}(GO^+_{\mathbf{A}}(V), GO^+_{\mathbf{A}}(L), \lambda)$, where λ is extended to $GO^+(V_\infty)$ by trivial action on the center and functions in $\mathcal{A}(GO^+_{\mathbf{A}}(V), GO^+_{\mathbf{A}}(L), \lambda)$ are also assumed to be invariant under the action of the center $\mathbf{Q}^\times_{\mathbf{A}}$ of $GO^+_{\mathbf{A}}(V)$. If the group of similitude norms of transformations in $GO^+_{\mathbf{A}}(V)$ is generated by the similitude norms of $GO^+(V)$ and of $GO^+_{\mathbf{A}}(L)$, each function in $\mathcal{A}(SO_{\mathbf{A}}(V), SO_{\mathbf{A}}(L), \lambda)$ can be extended in a unique way to a function in $\mathcal{A}(GO^+_{\mathbf{A}}(V), GO^+_{\mathbf{A}}(L), \lambda)$ [18]; this condition is satisfied if (e.g) L is even unimodular or if L is an Eichler order in a definite quaternion algebra.

For p prime we consider the Hecke algebra $\mathcal{H}_p$ of $GO^+(L_p)$-bi-invariant compactly supported locally constant functions on $GO^+(V_p)$ (or equivalently of formal sums of double cosets $GO^+(L_p)g\ GO^+(L_p)$ with $g \in GO^+(V_p)$), with the usual action of $GO^+(L_p)g\ GO^+(L_p)$ (or its characteristic function) on $GO^+(L_p)$-right invariant functions on $GO^+(V_p)$ given by

$$(T_g\varphi)(x) = \int_{GO^+(L_p)g\ GO^+(L_p)} \varphi(xy)dy.$$

A decomposition of $GO^+(L_p)gGO^+(L_p)$ into left cosets $g_iGO^+(L_p)$ with g_i of the same similitude norm as g gives rise to a decomposition of $SO(L_p)gSO(L_p)$ into left cosets with the same representatives g_i. Hence, writing out the integral as a summation over left coset representatives one can write the action of this Hecke algebra also as the action of double cosets $SO(L_p)gSO(L_p)$ with $g \in GO^+(V_p)$ in the same way in which one classically writes the action of Hecke operators $T(p)$ on (Siegel) modular forms. As in [18] the subalgebra generated by double cosets of elements of $SO(V_p)$ can be identified with the Hecke algebra of $SO(V_p)$, with the integral over $GO^+(L_p)g\ GO^+(L_p)$ above replaced by the integral over $SO(L_p)g\ SO(L_p)$.

On $\mathcal{H}_p$ we have an involution $T_g \longmapsto (T_g)^\iota = T_{\iota g \iota^{-1}}$ given by conjugation of the double cosets (or of the argument of the associated function) by ι.

Obviously the ι-invariant elements form a subalgebra $\mathcal{H}^\iota_p$ that is generated by the $T_g + T_{\iota g \iota^{-1}}$. Denoting in the same way by $\tilde{\mathcal{H}}_p$ the Hecke algebra for $GO(V_p)$ we have a natural surjective mapping $\mathcal{H}_p \longrightarrow \tilde{\mathcal{H}}_p$ (written $T \longmapsto \tilde{T}$) whose restriction to H^ι_p is an isomorphism (notice that $GO^+(V_p) \cap GO(L_p)g\ GO(L_p) = GO^+(L_p)g\ GO^+(L_p) \cup GO^+(L_p)\iota g \iota^{-1}\ GO^+(L_p))$.

LEMMA 3. *For* $\varphi :\ SO_{\mathbf{A}}(V) \longrightarrow U_\lambda$ *define* $\varphi^\iota(x) = \varphi(\iota x \iota^{-1})$. *Then*

a) *For all φ on $SO_{\mathbf{A}}(V)$ that are right invariant under $SO(L_p)$ we have*

$$\begin{array}{lcl} T_g\varphi^\iota(x) & = & (T_{\iota g\iota^{-1}}\varphi)^\iota(x) \quad \textit{and} \\ T_{\iota g\iota^{-1}}\varphi^\iota(x) & = & (T_g\varphi)^\iota(x). \end{array}$$

b) *For $\varphi \in \mathcal{A}(SO_{\mathbf{A}}(V), SO_{\mathbf{A}}(L), \lambda)$ we have (with $*$ denoting either $+$ or $-$)*

$$(T_g + T_{\iota g\iota^{-1}}(\varphi_*)(x) = ((T_g + T_{\iota g\iota^{-1}})\varphi)_*(x)$$

Proof.

a)

$$\begin{array}{lcl} T_g\varphi'(x) & = & \displaystyle\int_{GO^+(L_p)g\ GO^+(L_p)} \varphi(\iota xy\iota^{-1})dy \\ & = & \displaystyle\int_{GO^+(L_p)\iota g\iota^{-1}\ GO^+(L_p)} \varphi(\iota x\iota^{-1}y)dy \\ & = & (T_{\iota g\iota^{-1}}\varphi)(\iota x\iota^{-1}) = (T_{\iota g\iota^{-1}}\varphi)^\iota(x). \end{array}$$

Here the integrals over double cosets of $GO^+(L_p)$ can be replaced by integrals over cosets of $SO(L_p)$ if g is in $SO(V_p)$. The second equality follows because of $T_{\iota^2 g\iota^{-2}} = T_g$.

b)

$$\begin{array}{l} (T_g + T_{\iota g\iota^{-1}})(\varphi^*)(x) \\ = (T_g\varphi)(x) + (T_{\iota g\iota^{-1}}\varphi)(x) \\ \quad +\lambda_*(\iota)^{-1}(T_g\varphi^\iota(x) + \lambda_*(\iota)^{-1}(T_{\iota g\iota^{-1}}\varphi^\iota)(x) \\ = (T_g\varphi)(x) + \lambda_*(\iota)^{-1}(T_g\varphi)^\iota(x) \\ \quad +(T_{\iota g\iota^{-1}}\varphi)(x) + \lambda_*(\iota)^{-1}(T_{\iota g\iota^{-1}}\varphi)^\iota(x) \end{array}$$

(using part a)), which proves the assertion. □

Lemma 4. *Let $\varphi \in \mathcal{A}(SO_{\mathbf{A}}(V), SO_{\mathbf{A}}(L), \lambda)$ be an eigenfunction of $T \in \mathcal{H}_p^\iota$ with eigenvalue $\kappa(\varphi, T)$. Then for $*$ denoting $+$ or $-$ one has*

$$\tilde{T}\tilde{\varphi}_* = \kappa(\varphi, T)\tilde{\varphi}_*.$$

Proof. This follows from Lemma 3. □

2. Theta Liftings

In order to apply our lemmas to theta liftings we have to fix some more notations.

Let $O_{\mathbf{A}}(V) = \bigcup_{i=1}^{r} O(V)h_i\ O_{\mathbf{A}}(L)$ be a disjoint double coset decomposition with $(h_i)_\infty = \mathrm{Id}$; the lattices $h_iL =: L_i$ form then a set of representatives of the isometry classes of lattices in the genus of L.

We recall from [10, 17] that for each irreducible representation (τ, U_τ) of $O(V_\infty)$ the space $\mathcal{H}_n(\tau)$ of pluriharmonic polynomials $P: M_{m,n}(\mathbf{C}) \longrightarrow U_\tau$ such that $P(h^{-1}x) = \tau(h^t)P(x)$ for all $h \in O_m(\mathbf{R})$ is zero or (under the right action of $\mathrm{GL}_n(\mathbf{C})$ on the variable) isomorphic to an irreducible representation $(\rho_n(\tau), W_{\rho_n}(\tau))$ of $\mathrm{GL}_n(\mathbf{C})$ (here we identify $O(V_\infty)$ with its group of matrices $O_m(\mathbf{R})$ with respect to some fixed orthonormal basis of V_∞). In the latter case the space $H_q(\rho_n(\lambda))$ consisting of all q-pluriharmonic polynomials $P: M_{m,n}(\mathbf{C}) \longrightarrow W_{\rho_n(\tau)}$ such that $P(xg) = (\rho_n(\tau)(g^t))P(x)$ for all $g \in \mathrm{GL}_n(\mathbf{C})$ is isomorphic to (τ, U_τ) as a representation space of $O(V_\infty)$.

We denote by $\mathcal{P}_{n,*}$ the (essentially unique) isomorphism from U_{λ_*} to $\mathcal{H}_q(\rho_n(\lambda_*))$. Then, again for $*$ denoting $+$ or $-$, the n-th theta lifting of $\varphi \in \mathcal{A}(SO_{\mathbf{A}}(V), SO_{\mathbf{A}}(L), \lambda)$ is (whenever the representation $\rho_n(\lambda_*)$ is defined) (with $Z \in \mathbf{H}_n = \{X + iY \in M_n^{\mathrm{sym}}(\mathbf{C})\}$, Y positive definite):

$$
\begin{aligned}
&\theta_L^{(n,*)}(\varphi)(Z) = \\
&= \int_{O(V)\backslash O_{\mathbf{A}}(V)} \sum_{\mathbf{x} \in (hL)^n} \mathcal{P}_{n,*}(\tilde{\varphi}_*(h))(h_\infty^{-1}x_1, \ldots, h_\infty^{-1}x_n) \\
&\qquad\qquad \exp(2\pi i \mathrm{tr}(q(\mathbf{x})Z))dh \\
&= \sum_{i=1}^{r} \frac{1}{|O(L_i)|} \sum_{\mathbf{x} \in L_i^n} \mathcal{P}_{n,*}(\tilde{\varphi}_*(h_i))(x_1, \ldots, x_n) \\
&\qquad\qquad \exp(2\pi i \mathrm{tr}(q(\mathbf{x})Z)),
\end{aligned}
$$

where for $\mathbf{x} = (x_1, \ldots, x_n) \in L^n$ we denote by $q(\mathbf{x})$ the matrix $(\frac{1}{2}B(x_i, x_j)) \in M_n^{\mathrm{sym}}(\frac{1}{2}\mathbf{Z})$. We have then

THEOREM. *Let $\varphi \in \mathcal{A}(SO_{\mathbf{A}}(V), SO_{\mathbf{A}}(L), \lambda)$ (with λ as above) be such that $\varphi_+ \neq 0 \neq \varphi_-$ and let n be such that $\rho_n(\lambda_+), \rho_n(\lambda_-)$ both exist and that $\theta_L^{(n,+)}(\varphi)$ and $\theta_L^{(n,-)}(\varphi)$ are both nonzero. Assume that for all $p \not| \det(L)$*

the function φ is an eigenfunction of all Hecke operators in $\mathcal{H}_p^{\iota}$. Then the Siegel modular forms $\theta_L^{(n,+)}(\varphi)$ and $\theta_L^{(n,-)}(\varphi)$ are eigenfunctions of all Hecke operators from the p-components $\check{\mathcal{H}}_p$ of the Hecke algebra of $\Gamma_0^{(n)}(N)$ for $p \not| N$ (where N is the level of L) with the same eigenvalues occurring for $F_+ := \theta_L^{(n,+)}(\varphi)$ and $F_- := \theta_L^{(n,-)}(\varphi)$. In particular, the Satake parameters for $p \not| N$ of F_+ and F_- are the same and hence the N-free parts of the standard L-functions as well as of the spin L-functions of F_+ and of F_- agree.

Remark.

a) In the case that $\varphi_+ \neq 0 \neq \varphi_-$ let n_0 be the smallest n such that both $\theta_L^{(n,+)}(\varphi)$ and $\theta_L^{(n,-)}(\varphi)$ are defined and nonzero. Then using results of [13] and [12] it can be shown that at least one of F_+, F_- is cuspidal. For a proof of this in the case of square free level see [3].

b) In the case that λ is the trivial representation it can be seen from [10] and the almost trivial version for degree m of Lemma 1.1 of [5] that $n_0 = m$. Moreover, in that case $F_+^{(m)}$ and $F_-^{(m)}$ are scalar valued Siegel modular forms of weights $\frac{m}{2}$, $\frac{m}{2}+1$, with $F_-^{(m)}$ being cuspidal and $F_+^{(m)}$ not cuspidal. The N-free part $D^{(N)}(F_*^{(m)}, s)$ of the standard L-function of $F_*^{(m)}$ can then be completed by suitable factors at the $p|N$ and the Γ-factor

$$\gamma_{m,k}(s) = \pi^{(-2m+1)s/2}\Gamma(\frac{s}{2})\prod_{i=1}^{m}\Gamma(\frac{s+k-i}{2})\prod_{i=1}^{m}\Gamma(\frac{s+k-i+1}{2})$$

(k the weight of $F_*^{(m)}$) to an L-function $\Lambda(F_*^{(m)}, s)$ satisfying a functional equation for $s \longmapsto 1-s$, see [2]. Since $D^{(N)}(F_+, s) = D^{(N)}(F_-, s)$ this implies that $\dfrac{\gamma_{m,m/2}(s)}{\gamma_{m,(m/2+1)}(s)}$ is invariant under $s \longmapsto 1-s$.

Indeed a routine computation shows that $\dfrac{\gamma_{n,k}(s)}{\gamma_{n,k'}(s)}$ is invariant under $s \longmapsto 1-s$ if and only if $k + k' = n+1$. It should be noted that this is precisely the relation between the weights that implies equality of the infinitesimal characters of the corresponding representations of $Sp_n(\mathbf{R})$ (I thank the referee for pointing this out).

PROOF OF THEOREM 1. From [10] it can be seen that $n > \frac{m}{2}$ follows if both $\rho_n(\lambda_+)$ and $\rho_n(\lambda_-)$ are defined. The well-known commutation relation for Hecke operators ([1, 18]) (or the representation theoretic investigation of the local theta correspondence in [14, 11, 15] shows then that F_+, F_- are eigenfunctions of the N-free part of the symplectic Hecke algebra with the same eigenvalues, using of course that $\tilde{\varphi}_+$, $\tilde{\varphi}_-$ are eigenfunctions of the orthogonal Hecke algebra with the same Hecke eigenvalues by Lemma 4. □

We still have to show that there are examples in which both φ_+ and φ_- are nonzero. For this we use our investigation of Yoshida-liftings from [5, 4].

Again we have to recall some notation:

Let $V = D$ be a definite quaternion algebra over $\mathbf{Q}$ and R an Eichler order of square free level N in D, denote by $x \longmapsto \overline{x}$ the standard involution of D, by $\mathrm{tr}(x)$ and $n(x)$ the (reduced) trace resp. norm of $x \in D$. For $\nu \in \mathbb{N}$ let $U_\nu^{(0)}$ be the space of homogeneous harmonic polynomials of degree ν on $\mathbf{R}^3$ and view $P \in U_\nu^{(0)}$ as a polynomial on $D_\infty^{(0)} = \{x \in D_\infty \mid \mathrm{Tr}(x) = 0\}$ by putting $P(\sum_{i=1}^{3} x_i e_i) = P(x_1, x_2, x_3)$ for an orthonormal basis $\{e_i\}$ of $D_\infty^{(0)}$ with respect to the norm form n of D; denote by τ_ν the representation of $D_\infty^\times/\mathbf{R}^\times$ on $U_\nu^{(0)}$.

The group of proper similitudes of the quadratic form $q(x) = n(x)$ is isomorphic to $(D^\times \times D^\times)/Z(D^\times)$ (as algebraic group) via

$$(x_1, x_2) \longmapsto \sigma_{x_1,x_2} \quad \text{with } \sigma_{x_1,x_2}(y) = x_1 y x_2^{-1}$$

with the special orthogonal group being the image of $\{(x_1, x_2) \in D^\times \times D^\times \mid n(x_n) = n(x_2)\}$. The $SO(V_\infty)$-space $U_\nu^{(0)} \otimes U_\nu^{(0)}$ is isomorphic to the $SO(V_\infty)$-space $U_{(2\nu,0)}$ on D_∞^2 transforming according to the representation of $\mathrm{GL}_2(\mathbf{R})$ of highest weight $(2\nu, 0)$; an intertwining map Ψ has been given in [4, Section 3]. We let (U_λ, λ) be this representation space of $SO(V_\infty)$. Denoting by $\mathcal{A}(D_\mathbf{A}^\times, R_\mathbf{A}^\times, \tau_\nu)$ the space of functions $\varphi : D_\mathbf{A}^\times \longrightarrow U_\nu^{(0)}$ satisfying

$$\varphi(\gamma x u) = \tau_\nu(u_\infty^{-1})\varphi_\infty^{-1}\varphi(x) \quad \text{for } \gamma \in D_\mathbf{Q}^\times,$$

$u = u_\infty u_f \in R_\mathbf{A}^\times$ (where $R_\mathbf{A}^\times$ is the adelic group of units of R) we have for $\varphi_1, \varphi_2 \in \mathcal{A}(D_\mathbf{A}^\times, R_\mathbf{A}^\times, \tau_\nu)$ the function

$$\varphi := \Psi(\varphi_1 \otimes \varphi_2) \in \mathcal{A}(SO_\mathbf{A}(V), SO_\mathbf{A}(R), \lambda)$$

and can construct $\tilde{\varphi}_+, \tilde{\varphi}_-$ as above with respect to the standard involution ι of the quaternion algebra D. If φ_1 and φ_2 are in the cuspidal essential part (space of newforms) of $\mathcal{A}(D_{\mathbf{A}}^\times, R_{\mathbf{A}}^\times, \tau)$ [8], are eigenfunctions of all Hecke operators for $p \not| N$ and have the same eigenvalues under the involutions $\tilde{\omega}_p$ on $\mathcal{A}(D_{\mathbf{A}}^\times, R_{\mathbf{A}}^\times, \tau)$ for $p|N$, it has been shown in [4] that both φ_+, φ_- (and hence $\tilde{\varphi}_+, \tilde{\varphi}_-$) are nonzero Hecke eigenfunctions in $\mathcal{A}(SO_{\mathbf{A}}(V), SO_{\mathbf{A}}(R), \lambda)$ unless φ_1 and φ_2 are proportional. Moreover $\Psi(\varphi_1 \otimes \varphi_2)$ and $\Psi(\varphi_2 \otimes \varphi_1)$ have the same Hecke eigenvalues with respect to the symmetric Hecke algebra of $SO_{\mathbf{A}}(V)$. From [5] we have the following results:

For $\nu = 0$, the degree 4 theta lifting of $\Psi(\widetilde{\varphi_1 \otimes \varphi_2})_-$ is cuspidal (and nonzero), whereas for $\Psi(\widetilde{\varphi_1 \otimes \varphi_2})_+$ already the degree 2 theta lifting defines a nonzero cuspform. For $\nu > 0$, again the degree 2 theta lifting of $\Psi(\widetilde{\varphi_1 \otimes \varphi_2})_+$ is a nonzero cuspform, whereas the degree theta lifting of $\Psi(\widetilde{\varphi_1 \otimes \varphi_2})$ is a nonzero cuspform.

We therefore have the following corollary:

Corollary. *Let $\varphi_1, \varphi_2 \in \mathcal{A}(D_{\mathbf{A}}^\times, R_{\mathbf{A}}^\times, \tau_\nu)$ be eigenforms in the cuspidal essential part (space of newforms) that are not proportional to each other. Let $n = 3$ if $\nu > 0$, $n = 4$ if $\nu = 0$. Then the theta liftings of degree n of $\Psi(\widetilde{\varphi_1 \otimes \varphi_2})_\pm$ are nonzero Siegel modular forms of degree n having the same Satake parameters for all $p \not| N$, with one of the forms being cuspidal and the other one being in the orthogonal complement of the space of cusp forms.*

Proof. The orthogonality to the space of cusp forms of the n-th theta lifting of the $+$-form on the orthogonal group follows from the results of [3, 5]. The rest of the statement has been proved above. □

Remark. It has been proved in [3, 5] that for $p|N$ the theta lifting of both the plus-form and the minus-form are eigenfunctions of the Hecke operators associated to matrices

$$\begin{pmatrix} 0 & -M^{-1} \\ M & 0 \end{pmatrix} \quad \text{with } M \equiv 0 \mod N.$$

Moreover, it can be seen from [3, Corollary 6.1] or from the integral representation [3, Theorem 4.1] together with the functional equation [6, Lemma4.4] that the completion of the N-free part $D_F^{(N)}(S)$ of the standard L-function of the theta lifting by the factor

$$\Lambda_N(s) = \prod_{p|N} \prod_{i=1}^{n} (\frac{1}{1-\beta_{i,q} q^{-s}})$$

gives a smooth form of the functional equation, indicating that these factors are "the right ones" for bad places (here the $\beta_{i,q}$ are the Satake parameters associated to the Hecke operators for the $p|N$ considered above, see [3, section 2] for details). It seems not to be clear whether these parameters already suffice to characterize the p-adic representation of Sp_n generated by F up to isomorphy. However, from [7] it follows that there is a unique extension of the irreducible representation of $SO(V_p)$ generated by $\Psi(\varphi_1 \otimes \varphi_2)$ to an irreducible representation of $O(V_p)$ having an $SO(L_p)$-fixed vector. Hence the $O(V_p)$-representations generated by $\Psi(\widetilde{\varphi_1 \otimes \varphi_2})_+$ and $\Psi(\widetilde{\varphi_1 \otimes \varphi_2})_-$ are isomorphic, and for $p \neq 2$ the local theta correspondence [16] implies that the p-adic representations of Sp_n are isomorphic as well. If N is odd we see therefore that the automorphic representations generated by F_+ resp. F_- have the same local factors at all finite primes.

References

[1] Andrianov, A. N., Quadratic forms and Hecke operators, Grundlehren d. math. Wiss. **286**, Berlin-Heidelberg-New York, 1987.

[2] Böcherer, S., Über die Funktionalgleichung automorpher L-Funktionen zur Siegelschen Modulgruppe, J. f. d. reine und angew. Math. **362** (1985), 146–168.

[3] Böcherer, S. and R. Schulze-Pillot, Siegel modular forms and theta series attached to quaternion algebras, Nagoya Math. J. **121** (1991), 35–96.

[4] Böcherer, S. and R. Schulze-Pillot, Mellin transforms of vector valued theta series attached to quaternion algebras, Math. Nachr. **169** (1994), 31–57.

[5] Böcherer, S. and R. Schulze-Pillot, Siegel modular forms and theta series attached to quaternion algebras II, Nagoya Math. J. **147** (1997), 71–106.

[6] Böcherer, S. and R. Schulze-Pillot, On the central critical value of the triple product L-function, In: Number Theory 1993–94, 1–46, Cambridge University Press, 1996.

[7] Harris, M., Soudry, D. and R. Taylor, ℓ-adic representations associated to modular forms over imaginary quadratic fields, Invent. math. **112** (1993), 377–411.

[8] Hijikata, H. and H. Saito, On the representability of modular forms by theta series, p. 13–21 in Number Theory, Algebraic Geometry and Commutative Algebra, in honor of Y. Akizuki, Tokyo, 1973.

[9] Howe, R. and I. I. Piatetski-Shapiro, Some examples of automorphic forms on Sp_4, Duke Math. J. **50** (1983), 55–106.

[10] Kashiwara, M. and M. Vergne, On the Segal-Shale-Weil representations and harmonic polynomials, Invent. Math. **44** (1987), 1–44.

[11] Kudla, S., On the local theta correspondence, Invent. Math. **83** (1986), 229–255.

[12] Moeglin, C., Quelques propriétes de base des series theta, Preprint, 1995.

[13] Rallis, S., Injectivity properties of liftings associated to Weil representations, Comp. Math. **52** (1984), 139–169.

[14] Rallis, S., Langlands functoriality and the Weil representation, Amer. J. of Math. **104** (1982), 469–515.

[15] Roberts, B., The theta correspondence for similitudes, Isr. J. math. **94** (1996), 285–317.

[16] Waldspurger, J.-L., Demonstration d'une conjecture de dualite de Howe dans le cas p-adique, p$\neq$ 2. Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Pt. I: Papers in representation theory, Pap. Workshop L-Functions, Number Theory, Harmonic Anal., Tel-Aviv/Isr. 1989, Isr. Math. Conf. Proc. **2** (1990), 267–324.

[17] Weissauer, R., Vektorwertige Siegelsche Modulformen kleinen Gewichts, J. f. d. reine u. angew. Math. **343** (1983), 184–202.

[18] Yoshida, H., The action of Hecke operators on theta series, p. 197–238 in: Algebraic and topological theories-to the memory of Dr. T. Miyata-, 1985.

(Received April 13, 1998)

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