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Examples of Self-Dual Hopf Algebras

By Nicolás ANDRUSKIEWITSCH and Sonia NATALE

Abstract. We present a construction of non-trivial self-dual Hopf algebras over a field $\mathbf{k}$ obtained by a double extension process. The construction is not symmetric: it involves a 2-cocycle which, under a certain coboundary condition, disappears in the dual structure. We present several explicit examples in low dimensions; we recover families of examples introduced by S. Gelaki and R. Williams.

§0. Introduction

A finite dimensional Hopf algebra H over a field $\mathbf{k}$ is called *self-dual* if $H \simeq H^*$ as Hopf algebras. Self-dual Hopf algebras are quite common in nature: when $\mathbf{k}$ is algebraically closed, the group algebra of an abelian group is a self-dual Hopf algebra; given a finite dimensional Hopf algebra A, the tensor product Hopf algebra $A \otimes A^*$ is self-dual. More generally, if A, B are finite dimensional Hopf algebras, and $(\rightharpoonup, \sigma, \rho, \tau)$ is a Hopf data for the pair (A, B) (see [AD, Def. 2.26]), the dual Hopf algebra of the extension $A^\tau \#_\sigma B$ is the Hopf algebra

$$(B^*)^{\sigma^*} \#_{\tau^*} A^*.$$

Hence, if $A \simeq B^*$, the action $\rightharpoonup$ is the dual of the coaction ρ and $\sigma = \tau^*$, then whenever $(\rightharpoonup, \sigma, \rho, \tau)$ is a Hopf data, $A^\tau \#_\sigma B$ is a self-dual Hopf algebra. It is however not so evident how to obtain Hopf data like this; we discuss conditions in section 5.

We present in this paper a less direct construction of self-dual Hopf algebras. This construction was suggested by the work in [W, Ch. IV]. There, an infinite family of semisimple non-cocommutative self-dual Hopf algebras is built. Self-duality does not arise, as we suspected naively at the

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begining of our research, from an extension as in the preceeding paragraph. Rather, there is a non-symmetry given by the presence of a Hopf 2-cocycle which disappears in the dual structure. It turns out that other known examples of self-dual Hopf algebras [G] fit into our scheme.

The article is organized as follows: In §1 we present a basic construction of a bialgebra over $\mathbf{k}$ which arises as a double extension from what we call a *basic data.* This consists of a bialgebra R, two Hopf algebras H and K, together with a coaction and an action $R \to K \otimes R$, and $H \otimes R \to R$, and a 2-cocycle $\sigma : H \otimes H \to K$ satisfying certain natural requirements (see (1.3)). The process consists of first taking the smash coproduct of R by K, $R \rtimes K$, and then the twisted smash product of the resulting bialgebra with H by means of σ, $R \rtimes K \#_\sigma H$. We also give in Proposition (1.5.1) an alternative presentation of this bialgebra as a bosonization or biproduct (see [R], [M1]). We also describe the dual bialgebra of $R \rtimes K \#_\sigma H$, provided that R, K and H are finite dimensional; see (1.6).

In section 2, we present non-trivial examples of Hopf algebras arising from the construction in §1. A very simple pattern producing basic datum is described in Proposition (2.1.5): it is enough to have two finite groups G and Γ acting by Hopf algebra automorphisms on a Hopf algebra R whose images in $\mathrm{Aut}(R)$ commute with each other; and a normalized cocycle $\sigma : G \times G \to \widehat{\Gamma}$. This Proposition provides an ample source of examples of Hopf algebras. For $R = \mathbf{k}N$, N a finite group, it is enough to have two subgroups G and Γ of $\mathrm{Aut}\, N$ commuting with each other, plus the cocycle. We study in detail the situation when all N, G and Γ are cyclic. We recover, in particular, the examples in [G, §3] from the case when $N \simeq \mathbb{Z}/p$ and $G \simeq \Gamma \simeq \mathbb{Z}/q$, where p and q are prime numbers such that q divides $p - 1$. We show in fact that there are exactly q non-isomorphic self dual Hopf algebras of dimension pq^2 arising from this construction. Taking instead $N \simeq \mathbb{Z}/p$, $G \simeq \mathbb{Z}/q$ and $\Gamma \simeq \mathbb{Z}/r$, where p, q and r are distinct prime numbers such that q and r divide $p - 1$, we obtain two non-isomorphic examples of non-trivial, non self-dual, semisimple Hopf algebras A of dimension pqr. In the lowest case, for instance, this gives two Hopf algebras of dimension $42 = 2.\ 3.\ 7$, which were apparently not known (see [Mo1]).

More examples arise for instance from the theory of dual pairs. If V and W are finite dimensional vector spaces over a finite field $\mathbb{F}_q$ and $\rho : G \to \mathrm{End}\, V$, $\iota : \Gamma \to \mathrm{End}\, W$ are respectively, representations of G and Γ, then we can take N as (the underlying abelian group of) $V \otimes_{\mathbb{F}_q} W$ and let G, resp. Γ, act by $\rho \otimes \mathrm{id}$, resp. $\mathrm{id} \otimes \iota$. A standard example is $G = \Gamma = GL(n, \mathbb{F}_q)$ acting on $N = M(n, \mathbb{F}_q)$ by left and right multiplication. Another example is $G = GL(n, \mathbb{F}_q)$ acting diagonally on the tensor product of m copies of $\mathbb{F}_q^n$ and $\Gamma = \mathbb{S}_m$ acting by permutation of the factors. Taking $N = \mathbb{F}_q$ and $G = \Gamma = (\mathbb{F}_q)^\times$ acting by multiplication, we recover the family of examples in [W, Ch. IV].

In section 3 we study the dual structure of the bialgebra constructed in §1 from a different point of view. We define in (3.2.5) a bilinear form on $R \rtimes K\#_\sigma H \times R' \rtimes K'\#_{\sigma'} H'$, for two given basic datum (H, K, R, σ) and (H', K', R', σ'), from pairings between respectively R and R', K and H' and H and K'; but we need also a further ingredient- a map $\phi : H \times H' \to \mathbf{k}$. In our main result Theorem (3.3.1) we give sufficient conditions for this bilinear form to be a *duality* (see (3.2.6)). Roughly speaking, one of the conditions says that σ should be the coboundary of a map related to ϕ. In particular, we obtain sufficient conditions for $R \rtimes K\#_\sigma H$ to be self-dual even when the cocycle σ is allowed to be non-trivial in $H^2(G, \widehat{\Gamma})$.

In section 4, we pursue the study of the examples in §2, with the results of §3 in mind. In the setting of Proposition (2.1.5), to obtain self-dual Hopf algebras, we need: G should equal Γ and admit a "symmetric" cocycle $\sigma : G \times G \to \widehat{G}$; also, N should be abelian and carry a G-"invariant" bilinear form. See Proposition (4.1.5). If the cocycle is trivial, one does not need Theorem 3.2.1 to treat self duality; hence we focus the case where σ is non-trivial. For this, $\widehat{G}$ should in particular be non-trivial. As a consequence, we recover for instance, the self-duality property of the Hopf algebras in [G, §3], [W, Ch. IV].

Conventions. We shall work over a field $\mathbf{k}$. The multiplicative group of non-zero elements in $\mathbf{k}$ will be denoted by $\mathbf{k}^\times$. The notation for Hopf algebras is standard: m, 1, Δ, ϵ, $\mathcal{S}$, denote respectively the multiplication, the unit, the comultiplication, the counit, and the antipode; we add a subscript to indicate the Hopf algebra when necessary. For instance, Δ_R denotes the comultiplication of the bialgebra R. We use this version of Sweedler notation for comultiplications and coactions: If H is a bialgebra and $\rho : M \to H \otimes M$ is a left coaction, for $h \in H$, $m \in M$, we write $h_1 \otimes h_2$ and $m_{-1} \otimes m_0$, for $\Delta(h)$ and $\rho(m)$, respectively. However, if a bialgebra R is equipped with a coaction $R \to H \otimes R$ of another bialgebra H, then to avoid confussions we denote $\Delta_R(r) = r^1 \otimes r^2$, for $r \in R$.

For a finite dimensional Hopf algebra H, H^* will denote the linear dual of H with the Hopf algebra structure obtained by transposing that of H.

For a finite group Γ, we denote by $\widehat{\Gamma} := \mathrm{Hom}_{\mathrm{groups}}(\Gamma, \mathbf{k}^\times)$ the group of one-dimensional $\mathbf{k}$-representations of Γ. Then $\widehat{\Gamma} \subseteq \mathbf{k}^\Gamma$ coincides with $G(\mathbf{k}^\Gamma)$. Also, if Γ is abelian and $\mathbf{k}$ is algebraically closed of characteristic zero, then $\mathbf{k}^\Gamma = \mathbf{k}\widehat{\Gamma} \simeq \mathbf{k}\Gamma$, although the isomorphism is not canonical. We denote as usual $\delta_\gamma \in \mathbf{k}^\Gamma$ for the primitive idempotent corresponding to $\gamma \in \Gamma$, i.e., $\delta_\gamma(h) = \delta_{\gamma,h}$, $\forall h \in \Gamma$.

Our reference for the theory of Hopf algebras is [Mo]. We follow [A] for the basic theory of extensions of Hopf algebras. See also [AD], [M2], [Ma], [Sch].

§1. Basic Construction

1.1 Smash coproduct bialgebra

Let K be a Hopf algebra, and R a bialgebra over $\mathbf{k}$. Let $\rho : R \to K \otimes R$ be a left coaction of K on R.

Assume that the following conditions hold:

(1.1.1). $\Delta_R : R \to R \otimes R$, $\epsilon_R : R \to \mathbf{k}$, are left K-comodule maps.

(1.1.2). $m_R : R \otimes R \to R$, $1_R : \mathbf{k} \to R$, are left K-comodule maps.

(1.1.3). $r_{-1}k \otimes r_0 = kr_{-1} \otimes r_0$, for all $r \in R$, $k \in K$.

LEMMA (1.1.4). *The vector space $R \otimes K$ becomes a bialgebra with the tensor product algebra structure and the smash coproduct coalgebra structure. Explicitly,*

$$(r\#k)(s\#g) = rs\#kg, \qquad 1 = 1\#1,$$
$$\Delta(r\#k) = r^1\#(r^2)_{-1}k_1 \otimes (r^2)_0\#k_2, \quad \epsilon(r\#k) = \epsilon(r)\epsilon(k),$$

for all $r, s \in R$, $k, g \in K$; where $r\#k$ denotes the element $r \otimes k$ of $R \otimes K$. We denote this bialgebra structure by $R \rtimes K$.

If R is a Hopf algebra, then $R \rtimes K$ also is and its antipode is given by

$$\mathcal{S}(r\#k) = \mathcal{S}_R(r_0)\#\mathcal{S}(r_{-1}k),$$

for all $r \in R$, $k \in K$.

We have in this case a short exact sequence of Hopf algebras:

$$1 \to K \xrightarrow{\iota} R \rtimes K \xrightarrow{\pi} R \to 1,$$

where ι *and* π *are defined respectively, by* $\iota(k) = 1\#k$, $\pi(r\#k) = \epsilon(k)r$, *for all* $r \in R$, $k \in K$.

Proof. Follows from [A, §3] taking $A = K^{\mathrm{cop}}$, $B = R^{\mathrm{op}}$, τ, σ and $\rightharpoonup$ trivial. Indeed, condition (1.1.1) is contained in [A, 2.3], observe that as τ is trivial [A, 2.1.6] implies that ρ must be a coaction; condition (1.1.2) corresponds to [A, 2.1.9], condition (1.1.3) corresponds to [A, 2.1.10]. □

Remark (1.1.5). Conditions (1.1.1)–(1.1.3) say that (R, Δ_R, m_R) is a bialgebra (respectively Hopf algebra) in the braided category of Yetter-Drinfeld modules over K with left coaction ρ and trivial left action. Also, the structure in Lemma (1.1.4) is the corresponding *biproduct* bialgebra structure (respectively Hopf algebra structure) on $R \otimes K$. See [R], [M1].

1.2 Twisted smash product bialgebra

Let H be a Hopf algebra, and T a bialgebra over $\mathbf{k}$. Let $. : H \otimes T \to T$, be a left *weak action* [BCM] of H on T, that is

$$\begin{aligned} h.(ts) &= (h_1.t)(h_2.s), \\ h.1 &= \epsilon(h)1, \\ 1.t &= t, \end{aligned} \tag{1.2.1}$$

for all $h \in H$, $t, s \in T$.

We say $. : H \otimes T \to T$ is an *algebra action* if in adition it is associative, i.e., if $h.(g.t) = hg.t$, for all $h, g \in H$, $t \in T$.

Let also $\sigma : H \otimes H \to T$ be a normalized 2-cocycle, that is,

$$\begin{aligned} \sigma(h, 1) =&\sigma(1, h) = \epsilon(h)1, \\ [h_1.\sigma(l_1, m_1)]\sigma(h_2, l_2m_2) &= \sigma(h_1, l_1)\sigma(h_2l_2, m), \end{aligned} \tag{1.2.2}$$

for all $h, l, m \in H$, such that

$$(h_1.(l_1.t))\sigma(h_2, l_2) = \sigma(h_1, l_1)(h_2l_2.t), \tag{1.2.3}$$

for all $t \in T$, $h, l, m \in H$.

Then the vector space $T \otimes H$ becomes an algebra with the multiplication

$$(t\#h)(u\#l) = t(h_1.u)\sigma(h_2, l_1)\#h_2l_2, \tag{1.2.4}$$

for all $t, u \in T$, $h, l \in H$, with unit element $1\#1$ [BCM], [DT]. As above, we denote by $t\#h$ the element $t \otimes h$ of $T \otimes H$.

Let us now also assume that

(1.2.5). $\epsilon \circ \sigma = \epsilon \otimes \epsilon$.

(1.2.6). $\Delta_T(h.t) = h_1.t^1 \otimes h_2.t^2$, and $\epsilon(h.t) = \epsilon(h)\epsilon(t)$, for all $h \in H$, $t \in T$.

(1.2.7). $h_2 \otimes h_1.t = h_1 \otimes h_2.t$, for all $t \in T$, $h \in H$.

(1.2.8). $h_2l_2 \otimes \sigma(h_1, l_1) = h_1l_1 \otimes \sigma(h_2, l_2)$, for all $h, l \in H$.

(1.2.9). $\Delta(\sigma(h, l)) = \sigma(h_1, l_1) \otimes \sigma(h_2, l_2)$, for all $h, l \in H$.

LEMMA (1.2.10). *The vector space $T \otimes H$ is a bialgebra with multiplication (1.2.4), tensor product comultiplication $\Delta(t\#h) = t^1\#h_1 \otimes t^2\#h_2$, and counit $\epsilon(t\#h) = \epsilon(t)\epsilon(h)$. We denote this bialgebra structure by $T\#_\sigma H$.*

If T is a Hopf algebra and σ is convolution invertible, then $T\#_\sigma H$ also is and its antipode is given by

$$\mathcal{S}(t\#h) = \sigma^{-1}(\mathcal{S}(h_3), h_4)(\mathcal{S}(h_2).\mathcal{S}_T(t))\#\mathcal{S}(h_1),$$

for all $t \in T$, $h \in H$.

We have in this case a short exact sequence of Hopf algebras

$$1 \to T \xrightarrow{\iota} T\#_\sigma H \xrightarrow{\pi} H \to 1,$$

where ι and π are defined respectively, by $\iota(t) = t\#1$, $\pi(t\#h) = \epsilon(t)h$, for all $t \in T$, $h \in H$.

PROOF. Follows from [A, §3] taking $A = T$, $B = H$, τ, ρ trivial. Note that (1.2.5) corresponds to [A, 2.1.7], (1.2.6) corresponds to [A, 2.1.8], (1.2.7) corresponds to [A, 2.1.10], (1.2.8) corresponds to [A, 2.1.9], and (1.2.9) corresponds to [A, 2.1.11]. □

1.3 Basic construction

We want to combine now the constructions in (1.1) and (1.2).

Definition. Let K, H be Hopf algebras, and let $\sigma : H \otimes H \to K$ be a normalized 2-cocycle of H on K with trivial H-action, such that σ satisfies (1.2.5), (1.2.8), (1.2.9). Let R be a bialgebra, $. : H \otimes R \to R$ a left algebra action, $\rho : R \to K \otimes R$ a left coaction such that (1.1.1)–(1.1.3) hold. We say that the collection $(K, H, R, ., \rho, \sigma)$ is a *basic data,* if in addition the following conditions are fulfilled:

(1.3.1). $\sigma(H \otimes H) \subseteq Z(K)$.

(1.3.2). $h_2 \otimes h_1.r = h_1 \otimes h_2.r$, for all $r \in R$, $h \in H$.

(1.3.3). Δ_R and ϵ_R are left H-module maps.

(1.3.4). $\rho(h.r) = r_{-1} \otimes h.r_0$, for all $r \in R$, $h \in H$.

Lemma (1.3.5). *Let $(K, H, R, ., \rho, \sigma)$ be a basic data. Then the vector space $R \otimes K \otimes H$ becomes a bialgebra, with multiplication and comultiplication determined respectively by*

$$\begin{aligned}(r\#k\#h)(s\#g\#l) &= r(h_1.s)\#kg\sigma(h_2, l_1)\#h_3 l_2,\\ \Delta(r\#k\#h) &= r^1\#(r^2)_{-1}k_1\#h_1 \otimes (r^2)_0\#k_2\#h_2,\end{aligned} \tag{1.3.6}$$

for all $r, s \in R$, $h, l \in H$, $k, g \in K$. The unit and the counit are the obvious ones. We denote this bialgebra by $R \rtimes K\#_\sigma H$.

If σ is convolution invertible, then $R \rtimes K\#_\sigma H$ is a Hopf algebra and its antipode is given by

$$\mathcal{S}(r\#k\#h) = \mathcal{S}(h_2).\mathcal{S}_R(r_0)\#\sigma^{-1}(\mathcal{S}(h_3), h_4)\mathcal{S}(r_{-1}k)\#\mathcal{S}(h_1), \tag{1.3.7}$$

for all $t \in T$, $h \in H$.

In this case the Hopf algebra $R \rtimes K\#_\sigma H$ is obtained from R by a double process of extension. In fact, we have two exact sequences of Hopf algebras

$$1 \to K \xrightarrow{\iota} R \rtimes K \xrightarrow{\pi} R \to 1,$$

and

$$1 \to R \rtimes K \xrightarrow{\iota} R \rtimes K\#_\sigma H \xrightarrow{\pi} H \to 1.$$

PROOF. Let $T := R \rtimes K$ be the smash coproduct bialgebra (see (1.1)). Let $. : H \otimes T \to T$, $h.(r\#k) = h.r\#k$, $\sigma : H \otimes H \to K \subseteq T$. We verify that the conditions in (1.2) hold. Conditions (1.2.1) and (1.2.2) are trivial. Since $. : H \otimes R \to R$ is an *algebra* action, (1.2.3) follows from (1.2.8) and (1.3.1). It remains to show (1.2.6) and (1.2.7). The first is a consequence of (1.3.3) and (1.3.4), while the second is inmediate from (1.3.2). $\square$

As a Corollary of Lemma (1.2.10) we obtain:

LEMMA (1.3.8). *Let H, K be Hopf algebras. Let $\sigma : H \otimes H \to K$ be a normalized 2-cocycle with trivial action of H on K. Suppose σ is convolution invertible and satisfies (1.2.5), (1.2.8), (1.2.9), (1.3.1). Then the vector space $K \otimes H$ becomes a Hopf algebra, denoted by $K\#_\sigma H$ with multiplication, comultiplication and antipode*

$$\begin{aligned} (k\#h)(g\#l) &= kg\sigma(h_1, l_1)\#h_2l_2, \\ \Delta(k\#h) &= k_1\#h_1 \otimes k_2\#h_2, \\ \mathcal{S}(k\#h) &= \sigma^{-1}(\mathcal{S}(h_3), h_3)\#\mathcal{S}(h_1), \end{aligned} \tag{1.3.9}$$

for all $k, g \in K$, $h, l \in H$. We have an extension of Hopf algebras

$$1 \to K \xrightarrow{\iota} K\#_\sigma H \xrightarrow{\pi} H \to 1,$$

where the maps are canonical.

PROOF. Follows directly from Lemma (1.2.10) taking $T = K$, with trivial H-action. $\square$

Let $(H, K, R, \rho, ., \sigma)$ be a basic data. Let $\psi_1 : H \to H$ and $\psi_2 : K \to K$ be Hopf algebra automorphisms, and define maps $.' : H \otimes R \to R$, $\rho' : R \to K \otimes R$ and $\sigma' : H \otimes H \to K$ by the formulas

$$\begin{aligned} h.'r &= \psi_1(h).r, \\ \rho'(r) &= \psi_2(r_{-1}) \otimes r_0, \\ \sigma'(h, l) &= \psi_2 \circ \sigma \circ (\psi_1 \otimes \psi_1)(h, l), \end{aligned}$$

for $h, l \in H$, $r \in R$. We state the following Lemma for future use (see §2). Its proof is straightforward and left to the reader.

LEMMA (1.3.10). i). *$(H, K, R, \rho', .', \sigma')$ is a basic data. If σ is convolution invertible, then so is σ'.*

ii). *Denote by $R \rtimes K' \#_{\sigma'} H'$ the bialgebra associated to $(H, K, R, \rho', .', \sigma')$. Then the map $F = \mathrm{id} \otimes {\psi_2}^{-1} \otimes \psi_1 : R \rtimes K' \#_{\sigma'} H' \to R \rtimes K \#_\sigma H$ is a bialgebra isomorphism.*

1.4 Presentation of $R \rtimes K \#_\sigma H$ as a biproduct

Let $(H, K, R, \rho, ., \sigma)$ be a basic data. Keep the notation in (1.3). Denote by $A := R \rtimes K \#_\sigma H$, and let $K \#_\sigma H$ be the Hopf algebra in (1.3.8).

PROPOSITION (1.4.1). *(R, Δ_R, m_R) is a braided bialgebra in the category of Yetter-Drinfeld modules over $K \#_\sigma H$, ${}^{K\#_\sigma H}_{K\#_\sigma H}\mathcal{YD}$, with left coaction $\delta : R \to K \#_\sigma H \otimes R$, $\delta(r) = (r_{-1} \# 1) \otimes r_0$, and left action $\circ : K \#_\sigma H \otimes R \to R$, $(k \# h) \circ r = \epsilon(k) h.r$.*

Moreover the correponding biproduct bialgebra $A' = R \times (K \#_\sigma H)$ is isomorphic to A via the map $F : A' \to A$, $F(r \times (k \# h)) = r \# k \# h$.

PROOF. It follows from (1.1.1), (1.1.2) that Δ_R, ϵ_R, m_R, 1_R are $K \#_\sigma H$-comodule maps.

On the other hand, using that $. : H \otimes R \to R$ is an algebra action, it follows that m_R, 1_R are $K \#_\sigma H$-module maps. Also, by (1.3.3), Δ_R is an H-module map, and thus a $K \#_\sigma H$-module map. That ϵ_R is a $K \#_\sigma H$-module map follows from the definition of an action (see (1.2.1)).

To show that R is a Yetter-Drinfeld module, we must prove that

$$\delta((k \# h) \circ r) = (k \# h)_1 (r_{-1} \# 1) \mathcal{S}((k \# h)_3) \otimes (k \# h)_2 \circ r_0, \tag{1.4.2}$$

for all $k \in K$, $h \in H$, $r \in R$. Now, we have

$$\delta((k \# h) \circ r) = \epsilon(k) \delta(h.r) = \epsilon(k)(r_{-1} \# 1) \otimes h.r_0,$$

where the last equality follows from (1.3.4). On the other hand, we have

$$\begin{aligned}
&(k \# h)_1 (r_{-1} \# 1) \mathcal{S}((k \# h)_3) \otimes (k \# h)_2 \circ r_0 \\
&\qquad = (k_1 \# h_1)(r_{-1} \# 1) \mathcal{S}(k_3 \# h_3) \otimes (k_2 \# h_2) \circ r_0 \quad \text{by (1.3.2)} \\
&\qquad = (k_1 \# h_2)(r_{-1} \# 1) \mathcal{S}(k_2 \# h_3) \otimes h_1.r_0
\end{aligned}$$

$$= (k_1 r_{-1} \# h_2)(\mathcal{S}(k_2)\#1)(\sigma^{-1}(\mathcal{S}(h_3), h_4)\#\mathcal{S}(h_2)) \otimes h_1.r_0$$
by (1.1.3)
$$= \epsilon(k)(r_{-1}\#1)(1\#h_2)(\sigma^{-1}(\mathcal{S}(h_3), h_4)\#\mathcal{S}(h_2)) \otimes h_1.r_0$$
by (1.3.9)
$$= \epsilon(k)(r_{-1}\#1)(1\#h_2)\mathcal{S}(1\#h_2) \otimes h_1.r_0$$
$$= \epsilon(k)(r_{-1}\#1) \otimes h.r_0.$$

So (1.4.2) follows.

Finally, we have to check that R is a braided bialgebra in ${}^{K\#_\sigma H}_{K\#_\sigma H}\mathcal{YD}$, i.e., that for all $r, s \in R$,

$$\Delta_R(rs) = r^1(((r^2)_{-1}\#1) \circ s^1) \otimes (r^2)_0 s^2. \tag{1.4.3}$$

Now, as R is a bialgebra in $\mathcal{M}_k$, $\Delta_R(rs) = r^1 s^1 \otimes r^2 s^2$. On the other hand,

$$\begin{aligned} r^1(((r^2)_{-1}\#1) \circ s^1) \otimes (r^2)_0 s^2 &= r^1 s^1 \otimes \epsilon((r^2)_{-1})(r^2)_0 s^2 \\ &= r^1 s^1 \otimes r^2 s^2, \end{aligned}$$

hence (1.4.3) follows.

Now, it is straightforward to verify that F is an isomorphism of bialgebras. □

REMARK (1.4.4). In general, let B be a finite dimensional Hopf algebra, and let (R, Δ_R, m_R) be a finite dimensional Hopf algebra in ${}^B_B\mathcal{YD}$ with coaction $\rho : R \to B \otimes R$ and action $. : B \otimes R \to R$. In particular, Δ_R should satisfy

$$\Delta_R(ab) = a^1((a^2)_{-1}.b^1) \otimes (a^2)_0 b^2,$$

for all $a, b \in R$.

Let us suppose that there exists a *normal* Hopf subalgebra K of B which acts trivially on R while $\rho(R) \subseteq K \otimes R$. Then R is a Hopf algebra in the category of Yetter-Drinfeld modules over K.

Now, it is straighforward to verify that $R \times K \subseteq R \times B$ coincides with the smash coproduct $R \rtimes K$ corresponding to the restricted coaction $\rho : R \to K \otimes R$, and is a normal Hopf subalgebra. Therefore, the biproduct

$R \times B$ may be obtained by a double process of extension; indeed we have two exact sequences of Hopf algebras

$$1 \to R \rtimes K \xrightarrow{\iota} R \times B \xrightarrow{\pi} B/BK^{+} \to 1, \tag{1.4.5}$$

and

$$1 \to K \to R \rtimes K \to R \to 1, \tag{1.4.6}$$

where the maps in (1.4.6) are the obvious ones, the map $\iota : R \rtimes K \to R \times B$ in (1.4.5), is defined by $\iota(r\#k) = r\#k$ and the map π is $\pi(r\#b) = \epsilon(r)\overline{b}$, where $\overline{b}$ is the class of b in B/BK^{+}.

Denoting $H := B/BK^{+}$, we are in the previous situation, except that the action of H on K and the dual cocycle $\tau : H \to K \otimes K$ are not necessarily trivial. (Here we use that an extension of finite dimensional Hopf algebras is always *cleft*). The analysis of this more general situation exceeds the purposes of this paper.

It would be also interesting to know the answer to the following reciproque question: given a (usual) Hopf algebra R which is also a braided Hopf algebra in ${}_{B}^{B}\mathcal{YD}$, does there exists a normal Hopf subalgebra K of B which acts trivially on R while $\rho(R) \subseteq K \otimes R$?

An affirmative answer would imply, in the case where B is a *simple* Hopf algebra, that if R is a braided Hopf algebra in ${}_{B}^{B}\mathcal{YD}$, which is also a usual Hopf algebra, then either the action or the coaction of B on R are trivial.

1.5 First properties of $R \rtimes K\#_{\sigma}H$

In this subsection we give sufficient conditions for $R \rtimes K\#_{\sigma}H$ to be non-trivial (i.e., not commutative and not cocommutative), and determine $G(R \rtimes K\#_{\sigma}H)$. Throughout $(R, K, H, \rho, ., \sigma)$ is a basic data. We keep the notation and conventions in (1.4).

Lemma (1.5.1). i). *The Hopf algebra $K\#_{\sigma}H$ is cocommutative iff K and H are cocommutative.*

ii). *The Hopf algebra $K\#_{\sigma}H$ is commutative iff K and H are commutative and $\sigma(h, l) = \sigma(l, h)$, for all $h, l \in H$.*

Proof. Follows directly from (1.3.9). □

Let G denote the group $G(H)$. Condition (1.2.9) implies that $\sigma(g\otimes h) \in G(K)$, for all $g,h\in G$. Also, by (1.3.1) $\sigma(G\times G)\subseteq Z(K)$. Now, conditions (1.2.2) imply that the restriction of σ to $G\times G$ is a normalized factor set of G with values in $G(K)\cap Z(K)$. So that σ belongs to $H^2(G, G(K)\cap Z(K))$: the second cohomology group of G with coefficients in the trivial G-module $G(K)\cap Z(K)$ [B, Ch. IV].

In particular we may form a "twisted" product of $G(K)$ by G, which we will denote by $G(K)\times_\sigma G$, as follows: as set $G(K)\times_\sigma G = G(K)\times G$, and multiplication is given by $(x,g)(x',g') = (xx'\sigma(g,g'), gg')$, for all $x,x'\in G(K)$, $g,g'\in G$. And we have a short exact sequence

$$1\to G(K)\to G(K)\times_\sigma G\to G\to 1,$$

where the maps are canonical.

LEMMA (1.5.2). $G(K\#_\sigma H) = G(K)\times_\sigma G(H)$.

PROOF. Follows from (1.3.9) and the preceeding considerations. □

As an application of Proposition (1.4.1) we have the following corollaries:

COROLLARY (1.5.3). i). *The bialgebra $R\rtimes K\#_\sigma H$ is cocommutative iff R, K and H are cocommutative and the coaction ρ is trivial.*

ii). *The bialgebra $R\rtimes K\#_\sigma H$ is commutative iff R, K and H are commutative, the action $. : H\otimes R\to R$ is trivial and $\sigma(h,l)=\sigma(l,h)$, for all $h,l\in H$.*

PROOF. Combine Proposition (1.4.1) with Lemma (1.5.1) and [R, Section 2, Prop. 1]. □

Call $G(R)^{\mathrm{co}\,K}$ the semigroup of $u\in G(R)$ such that $\rho(u)=1\otimes u$. If R is a Hopf algebra, then $G(R)^{\mathrm{co}\,K}$ is a subgroup of $G(R)$. The action $\circ$ of $K\#_\sigma H$ on R induces an action by Hopf algebra automorphisms $\circ : (G(K)\times_\sigma G)\times R\to R$, and it is clear that this action preserves $G(R)$. Moreover, because of the Yetter-Drinfeld condition (1.4.2), this in turn induces by restriction an action by group automorphisms $\circ : (G(K)\times_\sigma G)\times G(R)^{\mathrm{co}\,K}\to G(R)^{\mathrm{co}\,K}$. So that we may consider the semidirect product $G(R)^{\mathrm{co}\,K}\rtimes G(K)\times_\sigma G$.

COROLLARY (1.5.4). i). *$G(R \rtimes K\#_\sigma H) = G(R)^{\mathrm{co}\,K} \times G(K) \times_\sigma G$. If R is a Hopf algebra, then $G(R \rtimes K\#_\sigma H)$ coincides with the semidirect product group $G(R)^{\mathrm{co}\,K} \rtimes (G(K) \times_\sigma G)$.*

PROOF. Combine Proposition (1.4.1) with Lemma (1.5.2) and [R, 2.11]. $\square$

PROPOSITION (1.5.5). *$A = R \rtimes K\#_\sigma H$ is semisimple if and only if R, K and H are semisimple.*

PROOF. Follows from [BM]. $\square$

1.6 Structure of the dual bialgebra

Let $(K, H, R, ., \rho, \sigma)$ be a basic data. In this subsection we assume that K, H and R are finite dimensional and describe the structure of the dual bialgebra A^*, where $A = R \rtimes K\#_\sigma H$ is the bialgebra constructed in (1.3).

(1.6.1). The left action $. : H \otimes R \to R$ gives rise to a left coaction $\rho^* : R^* \to H^* \otimes R^*$ in the form

$$\rho^*(f) = f_{-1} \otimes f_0 \quad \text{iff} \quad \langle f, h.r\rangle = \langle f_{-1}, h\rangle\langle f_0, r\rangle, \quad \forall h \in H, r \in R.$$

Dualizing the conditions on $. : H \otimes R \to R$, it is not difficult to see that ρ^* satisfies (1.1.1)–(1.1.3). So that we may consider the smash coproduct bialgebra $R^* \rtimes H^*$.

(1.6.2). The left coaction $\rho : R \to K \otimes R$ gives rise to a left action $* : K^* \otimes R^* \to R^*$ in the form

$$\langle \alpha.f, r\rangle = \langle \alpha, f_{-1}\rangle\langle f_0, r\rangle,$$

for all $\alpha \in K^*$, $f \in R^*$, $r \in R$. Also, $\sigma : H \otimes H \to K$ gives rise to $\tau : K^* \to H^* \otimes H^*$, in the form

$$\langle \tau(\alpha), h \otimes l\rangle = \langle \alpha, \sigma(h, l)\rangle,$$

for all $\alpha \in K^*$, $h, l \in H$.

Letting K^* act trivially on H^*, we now extend the action of K^* on R^* to an action of K^* on $R^* \rtimes H^*$. We consider the trivial coaction of K^* on $R^* \rtimes H^*$, the trivial cocycle $K^* \otimes K^* \to R^* \rtimes H^*$ and the dual cocycle $K^* \to R^* \rtimes H^* \otimes R^* \rtimes H^*$ obtained from τ. Hence, we can form $(R^* \rtimes H^*)^\tau \# K^*$ as in [A, §3].

PROPOSITION (1.6.3). *If $A = R \rtimes K \#_\sigma H$, then $A^* \simeq (R^* \rtimes H^*)^\tau \# K^*$.*

PROOF. One identifies A^* with the vector space $R^* \otimes H^* \otimes K^*$, and verifies that the multiplication and comultiplication have the prescribed form. We leave the details to the reader. □

§2. Examples

We show in this section some examples of the basic construction in §1. We follow a simple pattern described in (2.1).

2.1 A simple pattern

Let G, Γ be finite groups, and let $K = \mathbf{k}^\Gamma = (\mathbf{k}\Gamma)^*$, $H = \mathbf{k}G$. Let R be a bialgebra. Then it is equivalent to provide the two data in each of (2.1.1-3).

(2.1.1). A left coaction $\rho : R \to K \otimes R$ satisfying (1.1.1)–(1.1.3), or a group morphism $\theta : \Gamma \to \mathrm{Aut}_{\mathrm{Bialg}}(R)$. Indeed the left coaction ρ gives rise to a right action of Γ on R bialgebra automorphisms; composing with the inversion of Γ, we have a left action and hence the morphism θ. Conversely, given θ, ρ is determined by

$$\rho(r) = \sum_{\gamma \in \Gamma} \delta_\gamma \otimes \theta(\gamma^{-1})(r),$$

for $r \in R$.

(2.1.2). A left action $. : H \otimes R \to R$ satisfying (1.3.2)–(1.3.4), or a group morphism $\mu : G \to \mathrm{Aut}_{\mathrm{Bialg}}(R)$, such that the commutator $[\mu(G), \theta(\Gamma)] = 1$ in $\mathrm{Aut}_{\mathrm{Bialg}}(R)$. Indeed, as H is cocommutative, condition (1.3.2) is void.

(2.1.3). A normalized 2-cocicle $\sigma : H \otimes H \to K$ with trivial H-action, satisfying (1.2.3), (1.2.5), (1.2.8), (1.2.9), (1.3.1), or a normalized factor set $\sigma : G \times G \to \widehat{\Gamma}$ [B], which *a fortiori* results invertible.

(2.1.4). Let $\sigma : G \times G \to \widehat{\Gamma}$, be a normalized factor set. By the results in (1.3), if the data and conditions in (2.1.1)-(2.1.2) are fullfilled, there is an associated bialgebra $R \rtimes \mathbf{k}^\Gamma \#_\sigma \mathbf{k}G$.

Consider the trivial action of G on $\widehat{\Gamma}$. Let σ' be another normalized factor set. Observe that if $[\sigma] = [\sigma']$ in $H^2(G, \widehat{\Gamma})$, i.e., if there exists $f : G \to \widehat{\Gamma}$, such that $\sigma = \sigma'\delta f$, then the associated Hopf algebra $R \rtimes \mathbf{k}^\Gamma \#_\sigma \mathbf{k}G$ is isomorphic to the Hopf algebra $R \rtimes \mathbf{k}^\Gamma \#_{\sigma'} \mathbf{k}G$. An explicit isomorphism is given by $r\#k\#g \mapsto r\#kf(g)\#g$. Indeed, by [AD, Lemma 3.1.6], in this case $(R \rtimes \mathbf{k}^\Gamma)\#_\sigma \mathbf{k}G$ is isomorphic to the Hopf algebra $(R \rtimes \mathbf{k}^\Gamma)^{\tau^{f^{-1}}} \#_{\sigma'} \mathbf{k}G$, where $\tau^{f^{-1}} : \mathbf{k}G \to R \rtimes \mathbf{k}^\Gamma \otimes R \rtimes \mathbf{k}^\Gamma$, is the *dual cocycle* given by

$$\begin{aligned}\tau^{f^{-1}}(g) &= \Delta(f(g))(f^{-1}(g) \otimes \mathrm{id})(g \otimes 1)(1 \otimes f^{-1}(g)) \\ &= \Delta(f(g))(f^{-1}(g) \otimes f^{-1}(g)),\end{aligned}$$

for all $g \in G$. Hence $\tau^{f^{-1}}$ coincides with the trivial cococycle $\tau(g) = 1$, $\forall g \in G$, because $f(G) \subseteq \widehat{\Gamma}$. So that in order to obtain non-isomorphic Hopf algebras $R \rtimes \mathbf{k}^\Gamma \#_\sigma \mathbf{k}G$, we may restrict ourselves to consider the class of σ, $[\sigma]$, in $H^2(G, \widehat{\Gamma})$.

We formally resume the previous discussion in the following proposition.

Proposition (2.1.5). *Let G, Γ be finite groups, and R a bialgebra. Let also*

$$\theta : \Gamma \to \mathrm{Aut}_{\mathrm{Bialg}}(R) \quad \textit{and} \quad \mu : G \to \mathrm{Aut}_{\mathrm{Bialg}}(R),$$

be group morphisms, such that

$$[\mu(G), \theta(\Gamma)] = 1.$$

For any $[\sigma] \in H^2(G, \widehat{\Gamma})$, there is an associated bialgebra $R \rtimes \mathbf{k}^\Gamma \#_\sigma \mathbf{k}G$, with multiplication and comultiplication determined respectively by

$$\begin{aligned}(r\#f\#g)(s\#f'\#g') &= r\mu(g)(s)\#ff'\sigma(g,g')\#gg', \\ \Delta(r\#\delta_\gamma\#g) &= \sum_{uv=\gamma} r^1\#\delta_u\#g \otimes \theta(u^{-1})(r^2)\#\delta_v\#g,\end{aligned}$$

for all $r, s \in R$, $f, f' \in \mathbf{k}^\Gamma$, $\gamma, u, v \in \Gamma$, $g, g' \in G$. If R is a Hopf algebra, then $R \rtimes \mathbf{k}^\Gamma \#_\sigma \mathbf{k}G$ also is and its antipode is given by

$$\mathcal{S}(r\#\delta_\gamma\#g) = \mu(g^{-1}) \circ \theta(\gamma^{-1})(\mathcal{S}(r))\#\sigma(g^{-1}, g)^{-1}\delta_{\gamma^{-1}}\#g^{-1},$$

for all $r \in R$, $\gamma \in \Gamma$, $g \in G$. We have $G(A) = G(R)^\Gamma \rtimes (\widehat{\Gamma} \times_\sigma G)$.

2.2 Examples with cyclic groups

We want to apply Proposition (2.1.5) to the case where $G = \{1, g, \dots, g^{a-1}\}$, $\Gamma = \{1, u, \dots, u^{b-1}\}$, are cyclic groups of orders a and b respectively, and $R = \mathbf{k}N$ for a finite group N. In this case $\mathrm{Aut}_{\mathrm{Bialg}}(R)$ coincides with the group of group automorphisms $\mathrm{Aut}(N)$.

(2.2.1). Let Π be an abelian group with trivial G-action, and denote by $\Pi^a := \{\xi^a : \xi \in \Pi\}$. Recall that there is an isomorphism $H^2(G, \Pi) \simeq \Pi/\Pi^a$, which assigns to each $\overline{\xi} \in \Pi/\Pi^a$, the 2-cocycle $[\sigma_\xi]$ given by

$$\sigma_\xi(g^i, g^j) = \xi^{q_{ij}},$$

for all $0 \leq i, j \leq a-1$, where q_{ij} is the quotient of the division of $i+j$ by a, i.e., $i + j = aq_{ij} + r_{ij}$, $q_{ij}, r_{ij} \in \mathbb{Z}$, $0 \leq r_{ij} \leq a-1$. See e.g. [B, pp. 100].

(2.2.2). Suppose $N = \{1, x, \dots, x^{n-1}\}$ is a cyclic group of order n. Then $\mathrm{Aut}(N) \simeq (\mathbb{Z}/(n))^\times$, is abelian of order $\phi(n)$, where ϕ denotes the Euler indicator. In particular, $\mathrm{Aut}(N)$ is cyclic when n is a prime number.

Assume there exist non-negative integers m, t, such that $(m; n) = (t; n) = 1$, and such that the orders of the classes of m and t in $(\mathbb{Z}/(n))^\times$ divide a and b, respectively. This amounts to say that $\mathrm{Aut}(N)$ contains elements μ and θ whose orders divide a and b, respectively. Explicitly,

$$\mu(x) = x^m, \quad \text{and} \quad \theta(x) = x^t. \tag{2.2.3}$$

Then there are group morphisms $G \to \mathrm{Aut}(N)$, and $\Gamma \to \mathrm{Aut}(N)$, given respectively by $g \mapsto \mu$, $u \mapsto \theta$. (This is a slight abuse of notation; the former μ is determined by $\mu(g)$ which is equal to the new μ and similarly for θ.)

So that if $[\sigma] \in H^2(G, \widehat{\Gamma})$, (2.1.5) gives a Hopf algebra $A := R \rtimes \mathbf{k}^\Gamma \#_\sigma \mathbf{k}G$, of dimension nab.

As a corollary of Proposition (2.1.5) we get the following:

PROPOSITION (2.2.4). *Let $N = \{1, x, \dots, x^{n-1}\}$, $G = \{1, g, \dots, g^{a-1}\}$, $\Gamma = \{1, u, \dots, u^{b-1}\}$ be cyclic groups of order n, a and b, respectively. Let m, t be units modulo n, and such that the orders of the classes of m and t in $(\mathbb{Z}/(n))^\times$ divide a and b, respectively. Let $\overline{\xi} \in \widehat{\Gamma}/(\widehat{\Gamma})^a$. Then there is* *an associated Hopf algebra* $A := \mathbf{k}N \rtimes \mathbf{k}^{\Gamma} \#_{\overline{\xi}} \mathbf{k}G$*, of dimension* nab*, whose multiplication and comultiplication are determined by*

$$(x^i \# \delta_{u^j} \# g^l)(x^k \# \delta_{u^s} \# g^h) = x^{i+km^l} \# \delta_{j,s} \delta_{u^j} \xi^{q_{lh}} \# g^{l+h},$$
$$\Delta(x^i \# \delta_{u^j} \# g^l) = \sum_{s+w=j(\mathrm{mod}\, b)} x^i \# \delta_{u^s} \# g^l \otimes x^{it^{-s}} \# \delta_{u^w} \# g^l,$$

for all $0 \leq i, k \leq n-1$*,* $0 \leq j, s, w \leq b-1$*,* $0 \leq l, h \leq a-1$*, where* t^{-s} *denotes the inverse of* t^s *in* $(\mathbb{Z}/(n))^{\times}$*. We also have:*

i)*. The Hopf algebra* A *is semisimple iff the characteristic of* $\mathbf{k}$ *does not divide the product* na*.*

ii)*. If* m *and* $t \neq 1(\mathrm{mod}\, n)$*,* A *is non-commutative and non-cocommutative; and conversely.*

iii)*.* $G(A) \backsimeq \mathbb{Z}/(n; t-1) \rtimes \widehat{\Gamma} \times_{\sigma} G$*, in particular* $G(A)$ *is abelian iff* $(n; t-1)$ *divides* $m-1$*.*

Proof. The construction of A is a particular case of (2.1.5). Assertion i) follows from Proposition (1.5.5), since k^{Γ} is always semisimple. As for ii), by (1.5.3), if m and $t \neq 1(\mathrm{mod}\, n)$, the action and coaction are non-trivial and hence A is non-commutative and non-cocommutative. Finally, it is not difficult to see that

$$G(\mathbf{k}N)^{\Gamma} = N^{\Gamma} = \langle x^{\frac{n}{(n;t-1)}} \rangle \backsimeq \mathbb{Z}/(n; t-1).$$

So that, by (1.5.4), the isomorphism in iii) follows; observe that $G(A)$ is abelian iff $\mu(g)(x^{\frac{n}{(n;t-1)}}) = x^{\frac{n}{(n;t-1)}}$, iff $(n; t-1)$ divides $m-1$ as claimed. □

2.3 Hopf algebras of dimension pqr

In this subsection we assume, for simplicity of the statements that k is an algebraically closed field of characteristic zero; we remark however that most of the results remain valid (with similar arguments for the proofs), under weaker hypotheses. Let q, r, p be prime numbers, such that q and r divide $p-1$.

(2.3.1). Let G, Γ, and N be cyclic groups of orders q, r and p, respectively. Let m and t be units modulo p such that $m, t \neq 1(\mathrm{mod}\, p)$, and

$m^q = t^r = 1 (\operatorname{mod} p)$. Let $[\sigma] \in H^2(G, \widehat{\Gamma})$. We shall denote the associated Hopf algebra in Proposition (2.2.4), by $\mathbf{k}N \rtimes_m^t \mathbf{k}^\Gamma \#_\sigma \mathbf{k}G$. It is a non-trivial Hopf algebra of dimension pqr, and satisfies $G(\mathbf{k}N \rtimes_m^t \mathbf{k}^\Gamma \#_\sigma \mathbf{k}G) \backsimeq \mathbb{Z}/(r) \times_\sigma \mathbb{Z}/(q)$.

PROPOSITION (2.3.2). *Assume that $q \neq r$. Then:*

i). *The Hopf algebras $\mathbf{k}N \rtimes_m^t \mathbf{k}^\Gamma \#_\sigma \mathbf{k}G$ fall into one isomorphism class.*

ii). *The dual Hopf algebra of $\mathbf{k}N \rtimes_m^t \mathbf{k}^\Gamma \# \mathbf{k}G$ is isomorphic to $\mathbf{k}N \rtimes_t^m \mathbf{k}^G \# \mathbf{k}\Gamma$. The Hopf algebras $\mathbf{k}N \rtimes_m^t \mathbf{k}^\Gamma \# \mathbf{k}G$ and $\mathbf{k}N \rtimes_t^m \mathbf{k}^G \# \mathbf{k}\Gamma$ are not isomorphic.*

iii). *For both $A = \mathbf{k}N \rtimes_m^t \mathbf{k}^\Gamma \# \mathbf{k}G$ or $A = \mathbf{k}N \rtimes_t^m \mathbf{k}^G \# \mathbf{k}\Gamma$, we have $G(A) \backsimeq \mathbb{Z}/(qr)$.*

PROOF. Write $G = \{1, g, \dots, g^{q-1}\}$, $\Gamma = \{1, u, \dots, u^{r-1}\}$, $N = \{1, x, \dots, x^{p-1}\}$, as above. Observe that, when $q \neq r$, $H^2(G, \widehat{\Gamma}) = 0$ by (2.2.1). In particular, the last assertion follows.

We next prove that the isomorphism class of $\mathbf{k}N \rtimes_m^t \mathbf{k}^\Gamma \# \mathbf{k}G$ does not depend on the choice of m and t. For this, let m' and t' be units modulo p such that $m', t' \neq 1 (\operatorname{mod} p)$, and $m'^q = t'^r = 1 (\operatorname{mod} p)$. Then there exist a unit modulo q, α, and a unit modulo r, β, such that $m' = m^\alpha$ and $t' = t^\beta$. Let $f_\alpha : \mathbf{k}G \to \mathbf{k}G$ and $f_\beta : \mathbf{k}^\Gamma \to \mathbf{k}^\Gamma$, be the Hopf algebra automorphisms given respectively by $f_\alpha(g^i) = g^{i\alpha}$, and $\langle f_\beta(k), u^j \rangle = \langle k, u^{j\overline{\beta}} \rangle$, for all $0 \leq i \leq q-1$, $0 \leq j \leq r-1$, $k \in \mathbf{k}^\Gamma$, where $\overline{\beta}$ denotes the inverse of β modulo r. Then, it is not difficult to show, using Lemma (1.3.10),

$$\operatorname{id} \otimes f_\beta \otimes f_\alpha : \mathbf{k}N \rtimes_{m'}^{t'} \mathbf{k}^\Gamma \# \mathbf{k}G \to \mathbf{k}N \rtimes_m^t \mathbf{k}^\Gamma \# \mathbf{k}G,$$

is a Hopf algebra isomorphism.

Observe that by Proposition (1.6.3),

$$(\mathbf{k}N \rtimes_m^t \mathbf{k}^\Gamma \# \mathbf{k}G)^* \backsimeq \mathbf{k}N \rtimes_{t'}^{m'} \mathbf{k}^G \# \mathbf{k}\Gamma,$$

for certain m', t' units modulo p such that $m', t' \neq 1 (\operatorname{mod} p)$, and $m'^q = t'^r = 1 (\operatorname{mod} p)$. So $(\mathbf{k}N \rtimes_m^t \mathbf{k}^\Gamma \# \mathbf{k}G)^* \backsimeq \mathbf{k}N \rtimes_t^m \mathbf{k}^G \# \mathbf{k}\Gamma$. It remains to prove that $A = \mathbf{k}N \rtimes_t^m \mathbf{k}^G \# \mathbf{k}\Gamma$ is not isomorphic to $A' = \mathbf{k}N \rtimes_m^t \mathbf{k}^\Gamma \# \mathbf{k}G$.

We claim now that A and A' are not isomorphic as coalgebras. To see this, consider the semidirect product group $F = N \rtimes G$ corresponding to the given action of G on N. This is, up to isomorphism, the only non-abelian group of order pq. Now, as coalgebras, $A \backsimeq \mathbf{k}^{N \rtimes G} \otimes \mathbf{k}\Gamma$. Observe that we may assume that $\mathbf{k}$ is algebraically closed. In this case, it is well-known that F has q irreducible representations of degree 1, and $\frac{p-1}{q}$ irreducible representations of degree q. Hence, as coalgebras

$$A \ \backsimeq \ \underbrace{\mathbf{k} \oplus \cdots \oplus \mathbf{k}}_{rq \text{ times}} \oplus \underbrace{M_q(\mathbf{k}) \oplus \cdots \oplus M_q(\mathbf{k})}_{r(\frac{p-1}{q}) \text{ times}},$$

where $M_q(\mathbf{k})$ denotes the full matrix coalgebra over $\mathbf{k}$ of dimension q^2. By the same argument,

$$A' \ \backsimeq \ \underbrace{\mathbf{k} \oplus \cdots \oplus \mathbf{k}}_{rq \text{ times}} \oplus \underbrace{M_r(\mathbf{k}) \oplus \cdots \oplus M_r(\mathbf{k})}_{q(\frac{p-1}{r}) \text{ times}},$$

as coalgebras. This shows that A and A' are not isomorphic as coalgebras and the claim follows. $\square$

The smallest application of Proposition (2.3.2) gives us two apparently new examples of non-trivial semisimple Hopf algebras of dimension $42 = 2.3.7$, with $G(A) = \mathbb{Z}/(6)$ (see [Mo1]).

2.4 Hopf algebras of dimension pq^2

We also assume here that k is an algebraically closed field of characteristic zero. Put in (2.3.1) $r = q$, where we recall $q < p$ are prime numbers such that $p = 1(\operatorname{mod} q)$.

Let $\eta \in \mathbf{k}$ be a primitive q-th root of unity . Then $\widehat{G} = \{1, y, \ldots, y^{q-1}\}$, where $y \in \widehat{G}$, is defined by $\langle y, g \rangle = \eta$.

For each $0 \leq l \leq q-1$, let $\sigma_l : G \times G \to \widehat{G}$, be the 2-cocycle given in (2.2.1) corresponding to $y^l \in \widehat{G}/(\widehat{G})^q = \widehat{G}$, i.e.,

$$\sigma_l(g^i, g^j) = y^{lq_{ij}},$$

for $0 \leq i, j \leq q-1$, where q_{ij} is the quotient of the division of $i + j$ by q. Then $H^2(G, \widehat{G}) = \{[\sigma_l] : 0 \leq l \leq q-1\}$, and $[\sigma_l] \neq [\sigma_h]$ if $l \neq h$. Observe also that the corresponding central extension, $\widehat{G} \times_{\sigma_l} G$ coincides with $\mathbb{Z}/(q^2)$ if $l \neq 0$, and with $\mathbb{Z}/(q) \times \mathbb{Z}/(q)$ if $l = 0$. For each pair of units modulo p,

m and t, such that $m, t \neq 1 (\operatorname{mod} p)$, and $m^q = t^q = 1 (\operatorname{mod} p)$, denote the associated Hopf algebra as in (2.3.1).

PROPOSITION (2.4.1). i). *The Hopf algebras* $\mathbf{k}N \rtimes^t_m \mathbf{k}^\Gamma \#_{\sigma_l} \mathbf{k}G$ *fall into* q *isomorphism classes. Moreover, for fixed units modulo* p *of order* q, m *and* t, *the family*

$$A_l := \mathbf{k}N \rtimes^t_m \mathbf{k}^\Gamma \#_{\sigma_l} \mathbf{k}G, \quad 0 \leq l \leq q-1,$$

is a complete system of representatives of the isomorphism classes.

ii). *We have* $G(A_0) \backsimeq \mathbb{Z}/(q) \times \mathbb{Z}/(q)$, *and* $G(A_l) \backsimeq \mathbb{Z}/(q^2)$, *for all* $1 \leq l \leq q-1$.

We will see in §4 that these Hopf algebras are all self dual (see (4.2.4)).

PROOF. Assertion ii) follows from Proposition (2.2.4) and the remarks above.

To prove i), let m, m', t and t', be units modulo p of order q, and let $\sigma = \sigma_l$ be a 2-cocycle. We first claim that $\mathbf{k}N \rtimes^t_m \mathbf{k}^\Gamma \#_{\sigma_l} \mathbf{k}G \backsimeq \mathbf{k}N \rtimes^{t'}_{m'} \mathbf{k}^\Gamma \#_{\sigma_h} \mathbf{k}G$, for some $0 \leq h \leq q-1$.

Let, as in (2.3.2), α and β be units modulo q, such that $m = {m'}^\alpha$ and $t = {t'}^\beta$. Let $f_\alpha : \mathbf{k}G \to \mathbf{k}G$ and $f_\beta : \mathbf{k}^G \to \mathbf{k}^G$, be the Hopf algebra automorphisms given respectively by $f_\alpha(g^i) = g^{i\alpha}$, and $\langle f_\beta(k), u^j \rangle = \langle k, u^{j\overline{\beta}} \rangle$, for all $0 \leq i, j \leq q-1$, $k \in \mathbf{k}^G$, where $\overline{\beta}$ denotes the inverse of β modulo q. Then, again using Lemma (1.3.10), we find that

$$\operatorname{id} \otimes f_\beta \otimes f_\alpha : \mathbf{k}N \rtimes^t_m \mathbf{k}^\Gamma \#_{\sigma_l} \mathbf{k}G \to \mathbf{k}N \rtimes^{t'}_{m'} \mathbf{k}^\Gamma \#_{\sigma_h} \mathbf{k}G,$$

is a Hopf algebra isomorphism, where σ_h, $0 \leq h \leq q-1$, is the cocycle arising from σ_l as in (1.3.10). This proves the first claim.

It turns out also from the above argument, that $[\sigma_h] = 1$ iff $[\sigma_l] = 1$, iff $l = 0$. On the other hand, it is clear that if $l \neq 0$, then $\mathbf{k}N \rtimes^t_m \mathbf{k}^\Gamma \#_{\sigma_l} \mathbf{k}G$ is not isomorphic to $\mathbf{k}N \rtimes^t_m \mathbf{k}^\Gamma \# \mathbf{k}G$, because by ii), in the former case the group-likes form a cyclic group and in the first not. Hence, it follows that the Hopf algebras $\mathbf{k}N \rtimes^t_m \mathbf{k}^\Gamma \# \mathbf{k}G$ fall into one isomorphism class.

The Proposition will be proved if we show that for $1 \leq l \neq h \leq q-1$, and for fixed units modulo p of order q, m and t, the Hopf algebras A_l and

A_h are not isomorphic. For this we will adapt some of the arguments in [G, Lemma 3.7] Assume for simplicity, that $l = 1$.

Suppose, on the contrary, that there exists a Hopf algebra isomorphism $F : A_1 \to A_l$. As in [G, 3.5], it is not difficult to see that A_1, A_l both have a unique Hopf subalgebra B of dimension pq (the same for both cases). Namely, $B = \mathbf{k}N \rtimes^t \mathbf{k}^G \backsimeq \mathbf{k}^{N \rtimes G}$, where $N \rtimes G$ as in the proof of (2.3.2) is the only (up to isomorphism) non-abelian group of order pq. Hence, F induces by restriction a Hopf algebra automorphism $F : B \to B$. Then F satisfies

$$F(y) = y, \quad F(x) = x^r \# k_F(x),$$

for some $1 \le r \le p-1$, and $k_F(x) \in \mathbf{k}^G$. Now, in A_1, we have the relation $g^q = y$, while in A_l, $g^q = y^l$. So that

$$y = F(y) = F(g^q) = F(g)^q. \tag{*}$$

But F induces by restriction an isomorphism between $G(A_1) = \widehat{G} \times_{\sigma_1} G$ and $G(A_l) = \widehat{G} \times_{\sigma_l} G$, hence

$$F(g) = y^{ls} \# g^v, \tag{**}$$

for some $1 \le v \le q-1$, and $0 \le s \le q-1$. Comparing (*) and (**), we find that

$$F(g) = y^{ls} \# g^{\bar{l}},$$

for some $0 \le s \le q-1$, where $1 \le \bar{l} \le q-1$ is the inverse of l modulo q. Now we compute

$$\begin{aligned} F((1\#1\#g)(x\#1\#1)) &= F(g.x\#1\#g) \\ &= F(x^m\#1\#g) \\ &= F((x^m\#1\#1)(1\#1\#g)) \\ &= F(x)^m F(g) \\ &= (x^{mr}\#(k_F(x))^m\#1)(1\#y^{ls}\#g^{\bar{l}}) \\ &= x^{mr}\#(k_F(x))^m y^{ls}\#g^{\bar{l}}, \end{aligned} \tag{2.4.2}$$

and on the other hand,

$$\begin{aligned} F(1\#1\#g)F(x\#1\#1) &= (1\#y^{ls}\#g^{\bar{l}})(x^r\#k_F(x)\#1) \\ &= x^{m^{\bar{l}}r}\#k_F(x)y^{ls}\#g^{\bar{l}}. \end{aligned} \tag{2.4.3}$$

Putting together (2.4.2) and (2.4.3), we get that

$$\begin{aligned} mr = m^{\bar{l}} r (\operatorname{mod} p) \quad &\Longleftrightarrow \quad m = m^{\bar{l}} (\operatorname{mod} p) \quad \Longleftrightarrow \quad \bar{l} = 1 (\operatorname{mod} p) \\ &\Longleftrightarrow \quad l = 1, \end{aligned}$$

as claimed. $\square$

Fix m, t in (2.4.1) such that $mt = 1(\operatorname{mod} p)$. Then A_1 is isomorphic to the Hopf algebra A_{qp} in [G, §3]. Also, A_0 is isomorphic to $\mathcal{A}_{qp}$ [G, Prop. 3.12].

REMARK (2.4.5). Observe that, as coalgebras (and also as algebras, see §4)

$$A_l \; \backsimeq \; \underbrace{\mathbf{k} \oplus \cdots \oplus \mathbf{k}}_{q^2 \text{ times}} \oplus \underbrace{M_q(\mathbf{k}) \oplus \cdots \oplus M_q(\mathbf{k})}_{p-1 \text{ times}},$$

for all $0 \leq l \leq q-1$.

2.5 Hopf algebras arising from dual pairs over finite fields

In this section we fix a prime number p, an integer $s \geq 1$ and call $q := p^s$. Let $\mathbb{F} = \mathbb{F}_q$ denote the field with q elements. For a finite dimensional vector space N over $\mathbb{F}$, we shall denote the underlying abelian subgroup also by N. As a corollary of Proposition (2.1.5) we obtain the following.

PROPOSITION (2.5.1). *Let N be a finite dimensional vector space over $\mathbb{F}$. Let G and Γ be non-trivial subgroups of $GL(N)$ commuting with each other and let $[\sigma] \in H^2(G, \widehat{\Gamma})$. Then there is an associated Hopf algebra $A = \mathbf{k}N \rtimes \mathbf{k}^\Gamma \#_\sigma \mathbf{k}G$ of dimension $q^{\dim N}|G||\Gamma|$.*

We list below some explicit examples to be further discussed in §4.

(2.5.2). As an application of Proposition (2.5.1), we present now an alternative construction of a family of semisimple Hopf algebras built in [W, Ch. IV].

Let $N = \mathbb{F}_q$ be the finite field with q elements and let G the cyclic group of non-zero elements in $\mathbb{F}_q$ acting on N by left multiplication. Set $a = q-1$. Take $G = \Gamma$ and denote as above $G = \{1, g, \dots, g^{a-1}\}$, for some non-zero $g \in \mathbb{F}_{p^r}$.

Let $\lambda \in \mathbf{k}^\times$ be a primitive a-th. root of unity; then $\widehat{G} = \{1, y, \dots, y^{a-1}\}$, where $\langle y, g\rangle = \lambda$. Pick $[\sigma] \in H^2(G, \widehat{G})$ the 2-cocycle given by (2.2.1), corresponding to y. So that we obtain a Hopf algebra $\mathcal{H}$ (which is semisimple if the characteristic of $\mathbf{k}$ does not divide $a(a+1)$) of dimension $a^2(a+1)$. Also, by (1.5.3) $\mathcal{H}$ is non-commutative and non-cocommutative.

Let F be the semidirect product group $F = N \rtimes G$, with respect to the above action. In [W, Ch. IV] a Hopf algebra structure is given on the semisimple $\mathbf{k}$-algebra $\mathbf{k}^{(a^2)} \times M_a(\mathbf{k})^{(a)}$. It is also shown there that this Hopf algebra, denoted by H, is self dual and contains a Hopf subalgebra isomorphic to $\mathbf{k}^F$. To prove this, it is shown in [W, (4.1.7)] that H has a basis $\{x^i, d^t_{jl} :\ 0 \le i \le a^2-1,\ 0 \le t, j, l \le a-1\}$, where the x^i's are group-likes and the d^t_{jl}'s are matrix counits. However, the ultimate reason of the self-duality of H appeared not clear to the present authors.

Define $F : \mathcal{H} \to H$, in the form

$$F(y^l \# g^i) = x^{al+i},$$
$$F(g^h \# y^l \# g^i) = \sum_{j=0}^{a-1} \lambda^{l(h-j)} d^i_{hj},$$

for all $0 \le i, h, l \le a-1$. Then F is an isomorphism of Hopf algebras.

(2.5.3). Let now $N = M(d, \mathbb{F})$, $G = \Gamma = GL(d, \mathbb{F})$ acting on N by $\mu(g)X = gX$, $\theta(g)X = Xg^{-1}$. Clearly, $\widehat{\Gamma} = \widehat{G/[G,G]} \simeq \widehat{GL(1, \mathbb{F})} \simeq \widehat{\mathbb{F}^\times}$. Hence, for any cocycle $[\sigma] \in H^2(GL(d, \mathbb{F}), \mathbb{F}^\times)$ we obtain a Hopf algebra of dimension $q^{d^2}(q^d-1)^2 \dots (q^d - q^{d-1})^2$.

Observe that non-trivial 2-cocycles $[\sigma] \in H^2(GL(d, \mathbb{F}), \mathbb{F}^\times)$ may be obtained by composing with the non-trivial characters (i.e., powers of the determinant) of $GL(d, \mathbb{F})$, from cocycles $\mathbb{F}^\times \times \mathbb{F}^\times \to \widehat{\mathbb{F}^\times}$, cf. (2.2.1).

When $d = 1$, G and Γ are cyclic. If $q = p$ is prime, then also N is cyclic and we are in the situation of (2.2). For q not prime, the smallest example has dimension 36. For $d > 1$, the smallest example is obtained when $q = 2 = d$. In this case the resulting Hopf algebra has dimension $2^6.3^2 = 576$.

§3. Duality

3.1 Generalities

Suppose that H and K are bialgebras over $\mathbf{k}$. A bilinear form $\langle\ ,\ \rangle$:

$H \times K \to \mathbf{k}$ is called a *left duality,* if it satisfies:

(3.1.1). $\langle 1, k\rangle = \epsilon(k)$, for all $k \in K$.

(3.1.2). $\langle h\tilde{h}, k\rangle = \langle h, k_{(1)}\rangle\langle \tilde{h}, k_{(2)}\rangle$, for all $h, \tilde{h} \in H$, $k \in K$.

The form $\langle\ ,\ \rangle : H \times K \to \mathbf{k}$ is called a *right duality,* if it satisfies:

(3.1.3). $\langle h, 1\rangle = \epsilon(h)$, for all $h \in H$.

(3.1.4). $\langle h, k\tilde{k}\rangle = \langle h_{(1)}, k\rangle\langle h_{(2)}, \tilde{k}\rangle$, for all $h \in H$, $k, \tilde{k} \in K$.

We say that $\langle\ ,\ \rangle : H \times K \to \mathbf{k}$ is a *duality,* if it is both a left and a right duality.

REMARKS. Suppose K is finite dimensional over $\mathbf{k}$. So that $K^* = \mathrm{Hom}(K, \mathbf{k})$ is a bialgebra over $\mathbf{k}$. This amounts to say that the evaluation map $\langle\ ,\ \rangle : K^* \times K \to \mathbf{k}$ is a non-degenerate duality.

(3.1.5). A left (respectively right) duality $\langle\ ,\ \rangle : H \times K \to \mathbf{k}$ induces an algebra map (respectively a coalgebra map) $\Psi : H \to K^*$, in the form $\Psi(h)(k) := \langle h, k\rangle$, $\forall h \in H$, $k \in K$. The assignment $\langle\ ,\ \rangle \mapsto \Psi$, gives a bijective correspondance between the left (respectively right) dualities $\langle\ ,\ \rangle : H \times K \to \mathbf{k}$ and the algebra maps (respectively coalgebra maps) $H \to K^*$. The form $\langle\ ,\ \rangle$ is non-degenerate iff Ψ is an isomorphism. Under this equivalence, dualities $\langle\ ,\ \rangle : H \times K \to \mathbf{k}$ become identified with bialgebra maps $H \to K^*$.

(3.1.6). If both H and K are Hopf algebras, then the map induced by a duality $\langle\ ,\ \rangle : H \times K \to \mathbf{k}$ is a Hopf algebra map $\Psi : H \to K^*$, because any bialgebra map $H \to K$ must preserve the antipode. In particular, a Hopf algebra H is self-dual iff there exists a non-degenerate duality $\langle\ ,\ \rangle : H \times H \to \mathbf{k}$.

(3.1.7). A left duality $\langle\ ,\ \rangle : K \times H \to \mathbf{k}$, induces left and right K-module structures on H, denoted by $\rightharpoonup$ and $\leftharpoonup$, respectively, in the form

$$k \rightharpoonup h = \langle k, h_2\rangle h_1, \quad h \leftharpoonup k = \langle k, h_1\rangle h_2,$$

for $h \in H$, $k \in K$. If $\langle\ ,\ \rangle$ is a duality then $\rightharpoonup$ and $\leftharpoonup$ are K-module algebra structures.

3.2 Twisted duality

Throughout $(K, H, R, ., \rho, \sigma)$ and $(K', H', R', .', \rho', \sigma')$ will denote two basic datum. We introduce in this subsection (see (3.2.5)), a "twisted" bilinear form $R \rtimes K\#_\sigma H \times R' \rtimes K'\#_{\sigma'} H' \to \mathbf{k}$. We also give sufficient conditions for it to be a duality.

Let A be a coalgebra and B an algebra. Let $\star$ denote the convolution product in $\mathrm{Hom}(A, B)$. The group of convolution invertible maps in $\mathrm{Hom}(A, B)$ will be denoted by $\mathrm{Reg}(A, B)$. Similarly, the subgroups of $\mathrm{Reg}(A, B)$ consisting, respectively, of the maps f such that $f(1) = 1$, and $\epsilon \circ f = \epsilon$, will be denoted by $\mathrm{Reg}_1(A, B)$ and $\mathrm{Reg}_\epsilon(A, B)$; also $\mathrm{Reg}_{1,\epsilon}(A, B)$ denotes the subgroup $\mathrm{Reg}_\epsilon(A, B) \cap \mathrm{Reg}_1(A, B)$.

For $f \in \mathrm{Reg}(A, B)$, we denote by $\delta f \in \mathrm{Reg}(A \otimes A, B)$, the map given by

$$\delta f(a \otimes b) = f(a_1)f(b_1)f^{-1}(a_2b_2),$$

for all $a, b \in A$. See [AD, 3.1].

Let

(3.2.1). $\langle\ ,\ \rangle : K \times H' \to \mathbf{k}$,

(3.2.2). $\langle\ ,\ \rangle : H \times K' \to \mathbf{k}$,

be non-degenerate dualities. Let also

(3.2.3). $\phi : H \times H' \to \mathbf{k}$,

(3.2.4). $(\ |\ ) : R \times R' \to \mathbf{k}$,

be respectively a convolution invertible bilinear map, and a bilinear map.

By means of the dualities (3.2.1) and (3.2.2), the left coactions $\rho : R \to K \otimes R$ and $\rho' : R' \to K' \otimes R'$ give rise respectively to right actions $R \otimes H' \to R$ and $R' \otimes H \to R'$ in the form

$$r.h' := \langle r_{-1}, h' \rangle r_0, \quad r'.h := \langle h, r'_{-1} \rangle r'_0,$$

for all $r \in R$, $r' \in R'$, $h \in H$, $h' \in H'$.

The map ϕ in (3.2.3) induces invertible linear maps $\widetilde{\phi} : H \to (H')^*$ and $\overline{\phi} : H' \to H^*$, determined respectively by

$$\widetilde{\phi}(h)(h') = \phi(h, h') = \overline{\phi}(h')(h),$$

$\forall\ h \in H$, $h' \in H'$.

Define a bilinear form $\langle\ |\ \rangle_\phi : R \rtimes K\#_\sigma H \times R' \rtimes K'\#_{\sigma'} H'$, by

$$(3.2.5) \qquad \langle r\#k\#h, r'\#k'\#h' \rangle_\phi := (r|r')\ \langle k, h'_1 \rangle\ \langle h_1, k' \rangle\ \phi(h_2, h'_2),$$

for $r \in R$, $r' \in R'$, $k \in K$, $k' \in K'$, $h \in H$, $h' \in H'$.

Theorem (3.2.6). *Assume that the following conditions hold:*

(3.2.7). *$\phi(1,h') = \epsilon(h')$, $\phi(h,1) = \epsilon(h)$, for all $h \in H$, $h' \in H'$.*

(3.2.8). *The bilinear form (3.2.4) is (H,H')-invariant, that is,*

$$(r.h'|r') = (r|h'.r') \quad \textit{and} \quad (h.r|r') = (r|r'.h),$$

for all $r \in R$, $r' \in R'$, $h \in H$, $h' \in H'$.

Then

(3.2.9). *The bilinear form $\langle\ ,\ \rangle_\phi$ is a left duality iff (3.2.4) is and $\sigma = \delta\widetilde{\phi}$.*

(3.2.10). *The bilinear form $\langle\ ,\ \rangle_\phi$ is a right duality iff (3.2.4) is and $\sigma' = \delta\overline{\phi}$.*

(3.2.11). *The bilinear form $\langle\ ,\ \rangle_\phi$ is a duality iff (3.2.4) is a duality, $\sigma = \delta\widetilde{\phi}$ and $\sigma' = \delta\overline{\phi}$. In such case, $\langle\ ,\ \rangle_\phi$ is non-degenerate iff (3.2.4) is.*

PROOF. (3.2.9). For $r \in R$, $r' \in R'$, $k \in K$, $k' \in K'$, $h \in H$, $h' \in H'$, we have

$$\langle 1\#1\#1, r'\#k'\#h'\rangle_\phi = (1|r')\ \langle 1, h'_1\rangle\ \langle 1, k'\rangle\ \phi(1, h'_2) = (1|r')\epsilon(k')\epsilon(h'),$$

by (3.2.7). So $\langle\ ,\ \rangle_\phi$ satisfies (3.1.1) iff (3.2.4) does.

Condition (3.1.2) is

$$(*)\qquad \langle (r\#k\#h)(s\#g\#l), r'\#k'\#h'\rangle_\phi$$
$$= \langle r\#k\#h, (r'\#k'\#h')_1\rangle_\phi \langle s\#g\#l, (r'\#k'\#h')_2\rangle_\phi,$$

for all $r, s \in R$, $r' \in R'$, $k, g \in K$, $k' \in K'$, $h, l \in H$, $h' \in H'$. Hence, if $\langle\ ,\ \rangle_\phi$ satisfies (3.1.2) so does (3.2.4): just take $k = g = 1$, $h = l = 1$, $k' = 1$, $h' = 1$.

Conversely, assume that the bilinear form (3.2.4) satisfies (3.1.2). The left hand side of (*) equals

$$\begin{aligned}&\langle (r\#k\#h)(s\#g\#l), r'\#k'\#h'\rangle_\phi \\ &\quad = \langle r(h_1.s)\#kg\sigma(h_2,l_1)\#h_3l_2, r'\#k'\#h'\rangle_\phi \\ &\quad = (r(h_1.s)|r')\ \langle kg\sigma(h_2,l_1), h'_1\rangle\ \langle h_3l_2, k'\rangle\ \phi(h_4l_3, h'_2).\end{aligned}$$

On the other hand, the right hand side of (*) is

$$\begin{aligned}&\langle r\#k\#h, (r'\#k'\#h')_1\rangle_\phi \langle s\#g\#l, (r'\#k'\#h')_2\rangle_\phi \\ &\quad = \langle r\#k\#h, {r'}^1\#{r'}^2_{-1}k'_1\#h'_1\rangle_\phi \langle s\#g\#l, {r'}^2_0\#k'_2\#h'_2\rangle_\phi\end{aligned}$$

$$= (r|{r'}^1)(s|{r'}^2_0)\ \langle k, h'_1\rangle\langle g, h'_3\rangle\ \langle h_1, {r'}^2_{-1}k'_1\rangle\langle l_1, k'_2\rangle\ \phi(h_2, h'_2)\phi(l_2, h'_4)$$
$$= (r|{r'}^1)(s|{r'}^2.h_1)\ \langle k, h'_1\rangle\langle g, h'_3\rangle\ \langle h_2, k'_1\rangle\langle l_1, k'_2\rangle\ \phi(h_3, h'_2)\phi(l_2, h'_4)$$
$$= (r|{r'}^1)(h_1.s|{r'}^2)\ \langle k, h'_1\rangle\langle g, h'_3\rangle\ \langle h_2, k'_1\rangle\langle l_1, k'_2\rangle\ \phi(h_3, h'_2)\phi(l_2, h'_4) \qquad \text{by (3.2.8)}$$
$$= (r(h_1.s)|r')\ \langle k, h'_1\rangle\langle g, h'_3\rangle\ \langle h_2, k'_1\rangle\langle l_1, k'_2\rangle\ \phi(h_3, h'_2)\phi(l_2, h'_4).$$

Since (3.2.4) satisfies (3.1.2), we see that (*) is equivalent to

$$(**) \quad \langle kg\sigma(h_1, l_1), h'_1\rangle\ \langle h_2l_2, k'\rangle\ \phi(h_3l_3, h'_2) = \langle k, h'_1\rangle\langle g, h'_3\rangle\ \langle h_1, k'_1\rangle\langle l_1, k'_2\rangle\ \phi(h_2, h'_2)\phi(l_2, h'_4).$$

Now, we have by (1.3.1) and (1.2.8),

$$\begin{aligned}\langle kg\sigma(h_1, l_1), h'_1\rangle\ \langle h_2l_2, k'\rangle\ \phi(h_3l_3, h'_2)&\\ &= \langle k, h'_1\rangle\langle g, h'_3\rangle\ \langle \sigma(h_1, l_1), h'_2\rangle\ \langle h_2l_2, k'\rangle\ \phi(h_3l_3, h'_4)\\ &= \langle k, h'_1\rangle\langle g, h'_3\rangle\ \langle \sigma(h_2, l_2), h'_2\rangle\ \langle h_1l_1, k'\rangle\ \phi(h_3l_3, h'_4),\end{aligned}$$

and hence (**) is equivalent to

$$(***) \qquad \langle \sigma(h_1, l_1), h'_1\rangle\ \phi(h_2l_2, h'_2) = \phi(h, h'_1)\phi(l, h'_2).$$

This is in turn equivalent to

$$\sigma(h, l) = \widetilde{\phi}(h_1)\widetilde{\phi}(l_1)\ \widetilde{\phi}^{-1}(h_2l_2) = (\delta\widetilde{\phi})(h, l),$$

for all $h, l \in H$, as claimed.

As to (3.2.10), it is similarly proved using (3.2.7) that $\langle\ ,\ \rangle_\phi$ satisfies (3.1.3) iff (3.2.4) does. Condition (3.1.4), reads in this case

$$(*) \quad \langle r\#k\#h, r'(h'_1.s')\#k'g'\sigma'(h'_2, l'_1)\#h'_3l'_2\rangle_\phi = \langle r^1\#r^2_{-1}k_1\#h_1, r'\#k'\#h'\rangle_\phi\ \langle r^2_0\#k_2\#h_2, s'\#g'\#l'\rangle_\phi,$$

for all $r \in R$, $s', r' \in R'$, $k \in K$, $k', g' \in K'$, $h \in H$, $h', l' \in H'$. So that if $\langle\ ,\ \rangle_\phi$ satisfies (3.1.4), so does (3.2.4).

Using (3.2.8), this reduces to

$$(**)\quad \langle k, h'_2 l'_2\rangle \; \langle h_1, k'\rangle \; \langle h_3, g'\rangle \; \langle h_2 | \sigma'(h'_2, l'_1)\rangle \; \phi(h_4, h'_3 l'_3) \\ = \langle k_1, h'_1\rangle\langle k_2, l'_1\rangle \; \langle h_1, k'\rangle \; \langle h_3, g'\rangle \; \phi(h_2, h'_2) \; \phi(h_4, l'_2)$$

Now, again using (1.2.8) and (1.3.1), this reduces to

$$\langle h_1, \sigma'(h'_1, l'_1)\rangle \; \phi(h_2, h'_2 l'_2) = \phi(h_1, h') \; \phi(h_2, l'),$$

or equivalently to

$$\sigma'(h', l') = \overline{\phi}(h'_1)\overline{\phi}(l'_1)\overline{\phi}^{-1}(h'_2 l'_2) = (\delta\overline{\phi})(h', l'),$$

as claimed.

It remains to discuss the non-degeneracy of $\langle\ ,\ \rangle_\phi$. It is clear that if (3.2.5) is non-degenerate, then (3.2.4) also is. To prove the converse, it is enough to take R trivial, i.e., to show that the bilinear form

$$\langle\ |\ \rangle_\phi : K\#_\sigma H \times K'\#_{\sigma'} H' \to \mathbf{k}, \quad \langle k\#h, k'\#h'\rangle_\phi := \langle k, h'_1\rangle \; \langle h_1, k'\rangle \; \phi(h_2, h'_2),$$

is non-degenerate. Now, define $\zeta : K\#_\sigma H \times K'\#_{\sigma'} H' \to \mathbf{k}$ in the form

$$\zeta(k\#h, k'\#h') = \epsilon(k)\epsilon(k')\phi^{-1}(h, h'),$$

for $k \in K$, $k' \in K'$, $h \in H$, $h' \in H'$. Denote also by $(\ ,\ ) : K\#_\sigma H \times K'\#_{\sigma'} H' \to \mathbf{k}$, the non-degenerate bilinear form $(\ ,\ ) = \langle\ |\ \rangle_{\epsilon\otimes\epsilon}$.

Then, it is easily verified that the identity

$$\langle\ |\ \rangle_\phi \star \zeta = (\ ,\ )$$

holds in $\operatorname{Hom}(K\#_\sigma H \otimes K'\#_{\sigma'} H', \mathbf{k})$.

Denote by $A := K\#_\sigma H$, and $A' := K'\#_{\sigma'} H'$. Let $S := \{x \in A : \langle x, y\rangle_\phi = 0, \forall y \in A'\}$. As $\langle\ |\ \rangle_\phi$ is a right duality, $\Delta(S) \subseteq S \otimes A + A \otimes S$. On the other hand, let $W := \{x \in A : \zeta(x, y) = 0, \forall y \in A'\}$. We will show that $S \subseteq W$ and hence, that $\Delta(S) \subseteq S \otimes A + A \otimes W$. For this, let $x = \sum_i k_i\#h_i \in S$, then for all $k' \in K'$, and for all $h' \in H'$,

$$0 = \langle x, k'\#h'\rangle_\phi$$

$$= \sum_i \langle k_i, h'_1\rangle\langle h_{i,1}, k'\rangle\phi(h_2, h'_2)$$
$$= \langle \sum_i \langle k_i, h'_1\rangle h_{i,1}\phi(h_2, h'_2), k'\rangle.$$

Hence, by the non-degeneracy of the form $H \times K' \to \mathbf{k}$ in (3.2.2), we have

$$\sum_i \langle k_i, h'_1\rangle\phi(h_2, h'_2)h_{i,1} = 0,$$

for all $h' \in H'$. Putting $h' = 1$, we obtain,

$$0 = \sum_i \langle k_i, 1\rangle\phi(h_2, 1)h_1 = \sum_i \epsilon(k_i)h_i.$$

Thus, for all $k' \in K'$, and for all $h' \in H'$,

$$\begin{aligned}\zeta(x, k'\#h') &= \epsilon(k')\sum_i \epsilon(k_i)\phi^{-1}(h_i, h') \\ &= \epsilon(k')\phi^{-1}(\sum_i \epsilon(k_i)h_i, h') = 0,\end{aligned}$$

and the claimed inclusion follows. Now, if $y \in A'$, then

$$\begin{aligned}0 &= \langle x_1, y_1\rangle_\phi\zeta(x_2, y_2) \\ &= (x \mid y),\end{aligned}$$

hence $x = 0$. Similarly, using that $\langle \mid \rangle_\phi$ is a left duality, one can prove that $S' = \{y \in A' : \langle x, y\rangle_\phi = 0, \forall x \in A\} = 0$, and hence (3.2.11) follows. □

REMARK (3.2.12). Assume that (3.2.4) is non-degenerate. The argument of the proof of (3.2.11) shows that under the assumption that $\langle \mid \rangle_\phi$ is a right duality, the radical $S := \{x \in A : \langle x, y\rangle_\phi = 0, \forall y \in A'\}$ is zero. In the finite dimensional case, this suffices for the non-degeneracy of $\langle \mid \rangle_\phi$, since $\dim A = \dim A'$.

3.3 The finite dimensional case

We apply now Theorem (3.2.6) to show that certain basic datum, satisfying notably a symmetry condition on the cocycle, give rise to self-dual bialgebras. Compare with Proposition (1.6.3).

Let $(H, K, R, ., \rho, \sigma)$ be a basic data, such that H, K, R are *finite dimensional.* Call $A := R \rtimes K\#_{\sigma}H$ the associated bialgebra. Let also $\langle\ ,\ \rangle : K \times H \to \mathbf{k}$ be a non-degenerate duality. So that ρ gives rise to a right H-action $R \otimes H \to R$, $r.h := \langle r_{-1}, h\rangle r_0$.

For $f \in \mathrm{Reg}_{1,\epsilon}(H, K)$, let ${}^t f \in \mathrm{Reg}_{1,\epsilon}(H, K)$ be defined by $\langle {}^t f(h), l\rangle = \langle f(l), h\rangle$, for $h, l \in H$.

Theorem (3.3.1). *Suppose* $(\ |\ ) : R \times R \to \mathbf{k}$ *is a non-degenerate* (H, H)*- invariant bilinear map, that is*

$$(h.r|s) = (r|s.h), \quad \textit{and} \quad (r.h|s) = (r|h.s), \tag{3.3.2}$$

for all $r, s \in R$, $h \in H$. *Assume that* $\sigma = \delta f$, $f \in \mathrm{Reg}_{1,\epsilon}(H, K)$. *Then*

(3.3.3). *If* $(\ |\ )$ *is a left duality,* $A \backsimeq A^*$ *as algebras, and hence as coalgebras.*

(3.3.4). *If* $(\ |\ )$ *is a duality and* $\sigma = \delta({}^t f)$*, then* $A \backsimeq A^*$ *as bialgebras.*

Proof. Claim (3.3.3) follows from (3.2.9) and Remark (3.2.12), taking $\langle\ ,\ \rangle$ for (3.2.1), the non-degenerate duality $\langle\ ,\ \rangle^{\mathrm{op}} : H \times K \to \mathbf{k}$ for (3.2.2), and $\phi : H \times H \to \mathbf{k}$, $\phi(h, l) = \langle f(h), l\rangle$, $\forall h, l \in H$, for (3.2.3). Observe that $\widetilde{\phi} : H \to H^*$ coincides with f when identifying $K \backsimeq H^*$ by means of $\langle\ ,\ \rangle$.

Claim (3.3.4) follows from (3.2.11), observing that when identifying $K \backsimeq H^*$ by means of $\langle\ ,\ \rangle$, $\overline{\phi}$ coincides with ${}^t f$. □

§4. Examples (Continuation)

In this section, we investigate conditions under which the examples shown in §2 are self dual. We assume in what follows that the base field $\mathbf{k}$ is algebraically closed.

4.1

Keep the notation and hypothesis of (2.1). Then

(4.1.1). A non-degenerate duality $\langle\ ,\ \rangle : H \times K \to \mathbf{k}$ as in (3.3), amounts to a group isomorphism $\Gamma \to G$.

(4.1.2). Identify $\Gamma = G$ via a fixed isomorphism. A non-degenerate bilinear map $(\ |\ ) : R \times R \to \mathbf{k}$ is (H, H)- invariant, iff it satisfies

$$(\theta(g)(r) \mid \ \mu(g)(s)) = (r \mid s) = (\mu(g)(r) \mid \theta(g)(s)),$$

for all $g \in G$, $r, s \in R$.

(4.1.3). Let Π be the multiplicative subgroup of $(\mathbf{k}^{\Gamma})^{\times}$ defined by

$$\Pi = \{f \in (\mathbf{k}^{\Gamma})^{\times} : f(1) = 1\}.$$

Then $\widehat{G}$ is a subgroup of Π. Consider the trivial G-action on Π. Condition $\sigma = \delta f$, $f \in \mathrm{Reg}_{1,\epsilon}(\mathbf{k}G, \mathbf{k}^{\Gamma})$, amounts in this case to $[\sigma] = 1$ in $H^2(G, \Pi)$.

(4.1.4). Condition $\sigma = \delta f = \delta(^t f)$, amounts to the aditional condition $f^{-1}(^t f) \in \mathrm{Hom}_{\mathrm{groups}}(G, \Pi)$.

Putting together (2.1) and §3, we get the following proposition.

Proposition (4.1.5). *Let R be a bialgebra, Γ, G finite groups, and $[\sigma] \in H^2(G, \widehat{\Gamma})$, satisfying the hypothesis of Proposition (2.1.5). Assume that:*

i). $G \backsimeq \Gamma$.

ii). *There is a non-degenerate duality $(\ |\ ) : R \times R \to \mathbf{k}$, satisfying (4.1.2).*

iii). *$\sigma = \delta f$, for some $f : G \to (\mathbf{k}^G)^{\times}$, such that $f(g)(1) = 1 = f(1)(g)$, $\forall g \in G$, and $f^{-1}(^t f) \in \mathrm{Hom}_{\mathrm{groups}}(G, (\mathbf{k}^G)^{\times})$.*

Then the bialgebra $R \rtimes \mathbf{k}^G \#_{\sigma} \mathbf{k}G$ in (2.1.5) is self dual.

Proof. Follows from Theorem (3.3.1) and the remarks above. □

4.2

Recall the construction in (2.2), and assume that $\Gamma = G$. We assume in this subsection that the field $\mathbf{k}$ is algebraically closed.

(4.2.1). Let Π be as in (4.1.3). In view of the correspondence $H^2(G, \Pi) \backsimeq \Pi/\Pi^a$ mencioned in (2.2.1), and as Π is a divisible abelian group, we have $H^2(G, \Pi) = 1$.

Let $[\sigma] \in H^2(G, \widehat{G})$ be the cocycle in (2.2.1) corresponding to an element $\chi \in \widehat{G}$, where $\langle \chi, g \rangle = \nu \in \mathbf{k}$. Let $\omega \in \mathbf{k}$ such that $\omega^a = \nu$. Define a function $f_\omega : G \to (\mathbf{k}^G)^{\times}$, in the form

$$f_\omega(g^i)(g^j) := \omega^{ij},$$

for $0 \leq i, j \leq a-1$. Then $f_\omega \in \Pi$, $f_\omega = {}^t f_\omega$, and $\sigma = \delta f_\omega$. In fact, for $0 \leq i, j, l \leq a-1$,

$$\begin{aligned} \delta f(g^i, g^j)(g^l) &= f(g^i)(g^l)\ f(g^j)(g^l)\ f(g^i g^j)(g^l)^{-1} \\ &= \omega^{(i+j)l} \omega^{-r_{ij}l} = \omega^{(i+j-r_{ij})l} \\ &= \omega^{(aq_{ij})l} = \nu^{q_{ij}l} = \chi^{q_{ij}}(g^l). \end{aligned}$$

So that conditions (4.1.3) and (4.1.4) are always satisfied when G is a cyclic group.

(4.2.2). Let $R = \mathbf{k}N$, $\theta, \mu : \Gamma = G \to \mathrm{Aut}(N)$, and $[\sigma] \in H^2(G, \widehat{G})$. *Assume in adition that* $mt = 1 (\mathrm{mod}\, n)$. This amounts to requiring that $\theta(g) = \mu(g^{-1})$.

Let $\zeta \in \mathbf{k}$ be a primitive n-th root of unity. Hence we have a duality $(\ \mid\ ) : N \times N \to \mathbf{k}$, defined by

$$(x^i \mid x^j) = \zeta^{ij}, \tag{4.2.3}$$

for $0 \leq i, j \leq n$. In particular,

$$(\theta(g)(x^i) \mid\ \mu(g)(x^j)) = (x^{im} \mid x^{jt}) = \zeta^{mtij} = \zeta^{ij} = (x^i \mid x^j).$$

So that (4.2.2) satisfies (4.1.2).

We summarize this remarks in the following corollary.

COROLLARY (4.2.4). *Let A be the Hopf algebra of dimension nab constructed in (2.2.4). Assume that $\Gamma = G$, and $mt = 1 (\mathrm{mod}\, n)$. Then A is self dual.*

In particular, we obtain the self-duality of the Hopf algebras of dimension pq^2 in (2.4.1). See [G].

4.3 Self dual Hopf algebras arising from dual pairs

We switch now to the setting in (2.5). So we suppose that N is a finite dimensional vector space over a finite field $\mathbb{F} = \mathbb{F}_q$; $G = \Gamma$ is a non-trivial finite group, $\mu, \theta : G \to GL(N)$ are group homomorphisms such that $[\mu(G), \theta(G)] = 1$ and $[\sigma] \in H^2(G, \widehat{G})$. Assume $\mathbf{k}$ is algebraically closed.

Let $[\ ,\ ] : N \times N \to \mathbb{F}$ be a non-degenerate bilinear form over $\mathbb{F}$. Let $\omega \in \mathbf{k}$ be a primitive q-th root of unity. Define $(\ |\ ) : \mathbf{k}N \times \mathbf{k}N \to \mathbf{k}$ by

$$(4.3.1) \qquad (x|y) = \omega^{[x,y]}, \qquad x, y \in N.$$

Then $(\ |\ )$ is non-degenerate and satisfies (4.1.2) whenever $[\ ,\ ]$ does.

(4.3.2). Consider in (2.5.1), the *trace map* on $\mathbb{F}_{q^r}$, $\mathrm{Tr} : \mathbb{F}_{q^r} \to \mathbb{Z}_q$, $\mathrm{Tr}(s) = \sum_f f(s)$, for $s \in \mathbb{F}_{q^r}$; where f ranges over the Galois group of the extension $\mathbb{F}_{q^r}/\mathbb{F}_q$. It is well known that the bilinear form $[x, y] \mapsto \mathrm{Tr}(xy)$ is non-degenerate and it clearly satisfies (4.1.2). Then using Proposition (4.1.5) and (4.2.1), we find that the associated Hopf algebra, $\mathcal{H}$, is self dual.

(4.3.3). In (2.5.2), consider the usual trace map on $M(d, \mathbb{F})$. It is well known that the bilinear form $(x, y) \mapsto \mathrm{Tr}(xy)$ is non-degenerate and satisfies (4.1.2). Then, applying (4.2.1), we find that the associated Hopf algebra is self dual whenever σ satisfies iii) in Proposition (4.1.5).

§5. Self Dual Hopf Data

5.1

In this section, we broach the following question: let H be a finite dimensional Hopf algebra, $\rightharpoonup: H^* \otimes H \to H$ a left weak action and $\sigma : H^* \otimes H^* \to H$ a 2-cocycle. Let $\rho : H^* \to H^* \otimes H$ be the weak coaction and $\tau : H^* \to H \otimes H$ be the dual cocycle obtained respectively from $\rightharpoonup$ and σ by transposition. When is $(\rightharpoonup, \sigma, \rho, \tau)$ a Hopf data?

We follow again [A, §3]. We assume that the map σ satisfies (1.2.2), (1.2.3) as it should. A straightforward checking shows that ρ is a weak coaction and τ a dual cocycle. We need then to investigate the compatibility conditions [A, (3.1.7 - 11)]. For convenience, we introduce maps $\leftharpoonup: H^* \otimes H^* \to H^*$, $\leftharpoondown: H \otimes H^* \to H$ and $\rightharpoondown: H^* \otimes H^* \to H^*$ by

$$\langle \beta, \alpha \rightharpoonup h \rangle = \langle \beta \leftharpoonup \alpha, h \rangle = \langle \alpha, h \leftharpoondown \beta \rangle = \langle \beta \rightharpoondown \alpha, h \rangle;$$

all these expressions are also equal to $\langle \rho(\beta), a \otimes \alpha \rangle$. Here $h \in H$, $\alpha, \beta \in H^*$.

The answer to our initial question is summarized in the following result.

Proposition (5.1.1). *The collection $(\rightharpoonup, \sigma, \rho, \tau)$ is a Hopf data if and only if they satisfy (1.2.5),*

$$(5.1.2) \qquad \epsilon(\alpha \rightharpoonup a) = \langle \alpha, 1 \rangle \epsilon(a),$$

$$
\begin{aligned}
(5.1.3)\qquad & (h \leftharpoondown (\alpha_1\beta_1))\,\sigma(\alpha_2 \otimes \beta_2) \\
& \qquad = \sigma(\alpha_1 \otimes \beta_1)\,(\beta_2 \rightharpoonup (h_1 \leftharpoondown \alpha_2))\,(h_2 \leftharpoondown \beta_3),
\end{aligned}
$$

$$
\begin{aligned}
(5.1.4)\qquad & \sigma(\alpha_1 \otimes \beta_1)\,(h \leftharpoondown (\alpha_2\beta_2)) \\
& \qquad = (h_1 \leftharpoondown \alpha_1)\,(\alpha_2 \rightharpoonup (h_2 \leftharpoondown \beta_1))\,\sigma(\alpha_3 \otimes \beta_2),
\end{aligned}
$$

$$
(5.1.5)\quad \langle \alpha_1 \otimes (\alpha_2 \rightharpoonup h)\,,\, \big(\beta_1 \rightharpoonup h'\big) \otimes \beta_2 \rangle = \langle (\alpha_1 \rightharpoonup h) \otimes \alpha_2,\, \beta_1 \otimes \big(\beta_2 \rightharpoonup h'\big) \rangle,
$$

$$
\begin{aligned}
(5.1.6)\quad & \langle \sigma(\alpha_1 \otimes \beta_1),\, \gamma_1\mu_1 \rangle\, \langle \alpha_2\beta_2,\, \sigma(\gamma_2 \otimes \mu_2) \rangle \\
& = \langle \alpha_1,\, \sigma(\gamma_1 \otimes \mu_1) \rangle\, \langle \beta_1,\, \sigma\left((\gamma_2 \rightharpoondown (\alpha_2 \leftharpoonup \mu_2) \otimes (\mu_3 \leftharpoonup \alpha_3)\right)\rangle \\
& \qquad \langle \sigma\left((\alpha_4 \leftharpoonup \mu_4) \otimes (\beta_2 \leftharpoonup (\mu_5 \leftharpoonup \alpha_5))\right),\, \gamma_3 \rangle\, \langle \sigma(\alpha_6 \otimes \beta_3),\, \mu_6 \rangle.
\end{aligned}
$$

PROOF. The conditions (1.2.5) and (5.1.2) are the half of [A, (3.1.7)]; by transposition we get the other half. The conditions (5.1.3), (5.1.4), (5.1.5) and (5.1.6) correspond respectively to [A, (3.1.8), (3.1.9), (3.1.10), (3.1.11)]. We leave to the reader these lenghty but straightforward computations. □

It follows immediately from (5.1.2) that

$$
(5.1.7)\qquad \langle \beta \leftharpoonup \alpha, 1 \rangle = \langle \beta \rightharpoondown \alpha, 1 \rangle = \langle \alpha, 1 \rangle \langle \beta, 1 \rangle.
$$

We discuss in the rest of this section some examples of self dual Hopf algebras obtained from Proposition 5.1.1.

5.2

Let us seek for weak actions $\rightharpoonup: H^* \otimes H \to H$ producing a self dual Hopf data with trivial cocycle: $\sigma(\alpha \otimes \beta) = \langle \alpha, 1 \rangle \langle \beta, 1 \rangle$. The conditions stated above are simplified in the following way. First, (1.2.3) says that $\rightharpoonup$ is an algebra action. Using (5.1.7) we see that (5.1.6) is void in this case. So, we have only to impose (5.1.5) and the following translation of (5.1.3, 4):

$$
\begin{aligned}
(5.2.1)\qquad h \leftharpoondown (\alpha\beta) &= (\beta_1 \rightharpoonup (h_1 \leftharpoondown \alpha_1))\,(h_2 \leftharpoondown \beta_2) \\
&= (h_1 \leftharpoondown \alpha_1)\,(\alpha_2 \rightharpoonup (h_2 \leftharpoondown \beta_1))\,.
\end{aligned}
$$

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Facultad de Matemática, Astronomía y Física
Universidad Nacional de Córdoba
(5000) Ciudad Universitaria
Córdoba, Argentina
E-mail: andrus@mate.uncor.edu
natale@mate.uncor.edu