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Spherical Functions with Respect to the Semisimple Symmetric Pair $(Sp(2,\mathbb{R}), SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))$

By Tomonori Moriyama

Abstract. Let π be a generalized principal series representation with respect to the Jacobi parabolic subgroup or a large discrete series representation of $G = Sp(2,\mathbb{R})$. A spherical function is the image of a K-finite vector by the intertwining operator from π to the representation induced from an irreducible unitary representation of $SL(2,\mathbb{R})^2$ in G. We obtain differential equations for the spherical functions except for a few cases. We write down the solutions of these differential equations by means of the Gaussian hypergeometric functions.

§0. Introduction

The theory of spherical functions on reductive groups over local fields frequently appears in the arithmetic and the analytic theory of automorphic forms. Corresponding to the various aspects of the theory of automorphic forms, we have to consider not only usual but also various generalized spherical functions. For example, Whittaker functions have been studied by many authors. On the other hand, Murase and Sugano[7][8][9] introduce a new kind of spherical functions on reductive groups over p-adic fields called Shintani functions and apply them to automorphic L-functions. The purpose of this paper is to investigate an archimedean analogue of Shintani functions for the reductive group $Sp(2,\mathbb{R})$.

Let us explain general frameworks of this problem. Let G be a real reductive group. Let K be a fixed maximal compact subgroup of G and R a closed subgroup of G. For an irreducible smooth representation (η, V_η) of R, we form a C^∞-induced representation $C^\infty\text{-Ind}_R^G(\eta)$ with the representation space

$$C^\infty_\eta(R\backslash G) := \{F : G \to V_\eta \mid C^\infty\text{-class},\ F(rg) = \eta(r)F(g)\ \forall(r,g) \in R \times G\},$$

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on which G acts by the right translation. Denote by π^0 the underlying $(\mathfrak{g}, K)$-module of a smooth representation π of G. For a standard representation π of G, we consider the space of algebraic intertwining operators $\mathrm{Hom}_{(\mathfrak{g},K)}(\pi^0, C_\eta^\infty(R\backslash G)^0)$. Let (τ, W_τ) be a K-type of π. For $\Phi \in \mathrm{Hom}_{(\mathfrak{g},K)}(\pi^0, C_\eta^\infty(R\backslash G)^0)$ and a specification of K-type $i \in \mathrm{Hom}_K(\tau, \pi)$, the composite $\Phi \circ i$ can be considered as a $V_\eta \otimes W_\tau^*$-valued function on G. We call this function the spherical function of type (π, η, τ). We consider the following problems:

(1) Under what assumptions on the pair (G, R), is the dimension $m(\pi, \eta)$ of the intertwining space $\mathrm{Hom}_{(\mathfrak{g},K)}(\pi^0, C_\eta^\infty(R\backslash G)^0)$ finite ?

(2) What kind of functions appear as the spherical functions of type (π, η, τ) ?

Many studies suggest that the problem (1) is closely related with the geometry of the homogeneous space $R\backslash G$. Among them when R is reductive, a general results due to Bien, Kobayashi, and Oshima [1,Theorem (5.1)] asserts that the existence of an open P'-orbit on G/P is a sufficient condition for the finiteness of $m(\pi, \eta)$ for an irreducible admissible representations π (resp. η) of G (resp. of R). Here we denote by P (resp. by P') a minimal parabolic subgroup of G (resp. of R). Further, some examples suggest that $m(\pi, \eta)$ is likely to have a small upper bound with respect to π and η when the compact forms (G^c, R^c) have multiplicity-free property. On the problem (2), Hirano[3](resp. Tsuzuki[11][12]) obtained a general result for the pair $(GL(2,\mathbb{R}), GL(1,\mathbb{R}) \times GL(1,\mathbb{R}))$ (resp. $(U(n,1), U(1) \times U(n-1,1))$ and there appear Gaussian hypergeometric functions in both cases.

In this paper, we discuss these problems for the semisimple symmetric pair $(G, R) = (Sp(2,\mathbb{R}), SL(2,\mathbb{R})^2)$. Here $Sp(2,\mathbb{R})$ stands for the real symplectic group of rank 2 (matrix size 4). Our main results are the following:

(1) Suppose that π is either a generalized principal series representation of G induced from the parabolic subgroup corresponding to the long root(the *Jacobi parabolic subgroup*) or a large discrete series representation of G and η is a (limit of) discrete series representation of R. Then we have an inequality $m(\pi, \eta) \leqslant 1$ (cf. Corollary(6.7)).

(2) Suppose π and η belong to the same classes as in (1). Then the radial parts of spherical functions attached to (π, η) can be written in terms of Gaussian hypergeometric functions, or rational functions of exponential functions (cf. Theorem (6.1),(6.2),(6.3),(6.4)).

The organizations of this paper is as follows. In §1 we introduce and fix some notation about Lie groups and Lie algebras. In §2 we recall the representation theory of G, R, and $K \cong U(2)$ needed later. In §3 we define the spherical functions of type (π, η, τ), where π is a smooth representation of G, η is a smooth representation of R, τ is a K-type of π. We discuss the restriction of the spherical functions to a split torus A, which contains a complete representative of the double coset space $R\backslash G/K$. In §4 we construct systems of differential equations satisfied by the spherical functions using two kinds of differential operators. One of them is shift operators, which are defined by means of the Schmid operator, and the other is the Casimir operator. In §5 we reduce the above differential equations to more suitable ones for our purpose. In §6, by investigating these differential equations, we prove our main results mentioned above.

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§1. Basic Notation

1.1 Lie groups, Lie algebras and a root system

Put

$$J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \in M(4, \mathbb{R}),$$

where I_2 is the identity matrix of size 2. The symplectic group $G := Sp(2, \mathbb{R})$ is given by

$$Sp(2, \mathbb{R}) := \{g \in M(4, \mathbb{R}) |\, {}^t g\, Jg = J\}.$$

Let us consider two commutative involutions θ, σ of G:

$$\theta : G \ni g \mapsto {}^t g^{-1} \in G,$$

$$\sigma : G \ni g \mapsto \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} g \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \in G.$$

Put

$$K := G^\theta, \qquad R := G^\sigma,$$

$$\mathfrak{g} := \mathrm{Lie}(G), \quad \mathfrak{k} := \mathrm{Lie}(K), \quad \mathfrak{r} := \mathrm{Lie}(R).$$

The fixed point set by the Cartan involution θ determines a maximal compact subgroup K of G, which is given by

$$K = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in Sp(2,\mathbb{R}) | A, B \in M(2,\mathbb{R}) \right\}.$$

It is isomorphic to the unitary group

$$U(2) := \{ g \in GL(2,\mathbb{C}) | \, {}^t\bar{g}\, g = I_2 \},$$

via a homomorphism

$$u : K \ni \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto A + \sqrt{-1}B \in U(2).$$

It is easily checked that

$$\mathfrak{k} = \left\{ X = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} | A, B \in M(2,\mathbb{R}), {}^tA = -A, {}^tB = B \right\}.$$

The (-1)-eigenspace $\mathfrak{p}$ of θ is

$$\begin{aligned} \mathfrak{p} &= \{ X \in \mathfrak{g} | \theta(X) = -X \} \\ &= \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} | A, B \in M(2,\mathbb{R}), {}^tA = A, {}^tB = B \right\}, \end{aligned}$$

which gives a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. The derivative of the isomorphism u, which is also denoted by u, is given by

$$u : \mathfrak{k} \ni \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto A + \sqrt{-1}B \in \mathfrak{u}(2).$$

The subgroup R is isomorphic to $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. We take a basis of $\mathfrak{r}_{\mathbb{C}}$ as follows:

$$k_1 := \left(\begin{array}{cc|cc} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ \hline i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \qquad k_2 := \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ \hline 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{array}\right),$$
$$n_1^+ := \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ \hline i & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \qquad n_2^+ := \frac{1}{2}\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & i \\ \hline 0 & 0 & 0 & 0 \\ 0 & i & 0 & -1 \end{array}\right),$$
$$n_1^- := \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ \hline -i & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \qquad n_2^- := \frac{1}{2}\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -i \\ \hline 0 & 0 & 0 & 0 \\ 0 & -i & 0 & -1 \end{array}\right).$$

We denote the (-1)-eigenspace of σ by $\mathfrak{q}$.

The simple Lie algebra $\mathfrak{g}$ has a compact Cartan subalgebra $\mathfrak{h} := \mathbb{R}T_1 \oplus \mathbb{R}T_2$, where

$$T_1 := \sqrt{-1}k_1, \qquad T_2 := \sqrt{-1}k_2.$$

We note that $\mathfrak{h}$ is a compact Cartan subalgebra of $\mathfrak{r}$, too. Let $\beta_1, \beta_2 \in \mathfrak{h}_{\mathbb{C}}{}^*$ be the dual basis of k_1, k_2. We define the set of integral weights by $\mathbb{Z}\beta_1 \oplus \mathbb{Z}\beta_2$. For each $\beta \in \mathfrak{h}_{\mathbb{C}}^*$, set

$$\mathfrak{g}_\beta := \{X \in \mathfrak{g}_{\mathbb{C}} | [H, X] = \beta(H)X, \forall H \in \mathfrak{h}_{\mathbb{C}}\}.$$

Then the root system $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ given by

$$\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) = \{\pm 2\beta_1, \pm 2\beta_2, \pm(\beta_1 \pm \beta_2)\}.$$

We fix a positive system Δ^+ of Δ as

$$\Delta^+ := \{2\beta_1, \beta_1 + \beta_2, 2\beta_2, \beta_1 - \beta_2\}.$$

For each positive root β, the root space $\mathfrak{g}_{\mathbb{C}}^{\beta}$ is spanned by X_β as follows:

$$X_{(2,0)} := \left(\begin{array}{cc|cc} 1 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ \hline i & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \qquad X_{(1,1)} := \left(\begin{array}{cc|cc} 0 & 1 & 0 & i \\ 1 & 0 & i & 0 \\ \hline 0 & i & 0 & -1 \\ i & 0 & -1 & 0 \end{array}\right),$$

$$X_{(0,2)} := \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & i \\ \hline 0 & 0 & 0 & 0 \\ 0 & i & 0 & -1 \end{array}\right), \qquad X_{(1,-1)} := \left(\begin{array}{cc|cc} 0 & 1 & 0 & -i \\ -1 & 0 & -i & 0 \\ \hline 0 & i & 0 & 1 \\ i & 0 & -1 & 0 \end{array}\right).$$

For each negative root $-\beta$, the root space $\mathfrak{g}_{\mathbb{C}}^{-\beta}$ is spanned by the root vector $X_{-\beta} := \bar{X}_\beta$. Set

$$\mathfrak{p}_{\mathbb{C}}^{+} := \mathbb{C}X_{(2,0)} \oplus \mathbb{C}X_{(1,1)} \oplus \mathbb{C}X_{(0,2)},$$

and

$$\mathfrak{p}_{\mathbb{C}}^{-} := \mathbb{C}X_{(-2,0)} \oplus \mathbb{C}X_{(-1,-1)} \oplus \mathbb{C}X_{(0,-2)}.$$

Then $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_{\mathbb{C}}^{+} \oplus \mathfrak{p}_{\mathbb{C}}^{-}$. For each root $\beta = b_1\beta_1 + b_2\beta_2 = (b_1, b_2)$, we put $\|\beta\| = \sqrt{b_1^2 + b_2^2}$. Since the set of compact positive roots is $\Delta_c^+ = \{\beta_1 - \beta_2\}$, the set of dominant integral weights with respect to Δ_c^+ is given by $\{(\lambda_1, \lambda_2) | \lambda_i \in \mathbb{Z},\ \lambda_1 \geqslant \lambda_2\}$.

If we put

$$H_1 := \left(\begin{array}{cc|cc} & 1 & & \\ 1 & & & \\ \hline & & & -1 \\ & & -1 & \end{array}\right), \qquad \text{and} \qquad a_t := \exp(tH_1),$$

then $\mathfrak{a} := \mathbb{R}H_1$ is a maximal abelian subspace in $\mathfrak{p} \cap \mathfrak{q}$. Define a vector subgroup A of G by $A := \{a_t | t \in \mathbb{R}\}$. In order to state the next lemma (a generalized Cartan decomposition) we introduce some notation.

Notation. Set

$$c(t) := \frac{1}{2}(\frac{1}{\cosh 2t} + 1), \quad s(t) := \frac{1}{2}(\frac{1}{\cosh 2t} - 1),$$

and define $\mathfrak{r}_{\mathbb{C}}$-valued functions $\mathcal{L}^{\pm}, \mathcal{M}^{\pm}$, and $\mathcal{N}$ on $\mathbb{R}$ by

$$\begin{aligned}\mathcal{L}^{\pm} &:= c(t)n_1^{\pm} - s(t)n_2^{\mp},\\ \mathcal{M}^{\pm} &:= s(t)n_1^{\mp} - c(t)n_2^{\pm},\\ \mathcal{N} &:= \tanh t \cdot k_1 - \coth t \cdot k_2.\end{aligned}$$

LEMMA (1.1). *For any $t \in \mathbb{R}^*$, we have*

$$\mathfrak{g}_{\mathbb{C}} = \mathrm{Ad}(a_{-t})\mathfrak{r}_{\mathbb{C}} + \mathfrak{a}_{\mathbb{C}} + \mathfrak{k}_{\mathbb{C}}.$$

To be more precise, each root vector X_β is decomposed as below:

(1) $X_{(2,0)} = \mathrm{Ad}(a_{-t}) \cdot 2\mathcal{L}^+ - \tanh 2t \cdot X$,

(2) $X_{(1,1)} = \mathrm{Ad}(a_{-t}) \cdot (-\mathcal{N}) + H_1 - 2\coth 2t \cdot k_2$,

(3) $X_{(0,2)} = \mathrm{Ad}(a_{-t}) \cdot (-2\mathcal{M}^+) - \tanh 2t \cdot Y$,

(4) $X_{(-2,0)} = \mathrm{Ad}(a_{-t}) \cdot 2\mathcal{L}^- + \tanh 2t \cdot Y$,

(5) $X_{(-1,-1)} = \mathrm{Ad}(a_{-t}) \cdot \mathcal{N} + H_1 + 2\coth 2t \cdot k_2$,

(6) $X_{(0,-2)} = \mathrm{Ad}(a_{-t}) \cdot (-2\mathcal{M}^-) + \tanh 2t \cdot X$,

(7) $X_{(1,-1)}, X_{(-1,1)} \in \mathfrak{k}_{\mathbb{C}}$.

Here we write

$$X := \frac{1}{2}X_{(1,-1)} = u^{-1}(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}), \qquad Y := \frac{-1}{2}X_{(-1,1)} = u^{-1}(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}).$$

The proof is direct computations.

1.2 The Jacobi parabolic subgroup

We fix a maximal parabolic subgroup P_J corresponding to the long root of the C_2-type root system, the *Jacobi parabolic subgroup* of G, as follows :

$$P_J := \left\{ \left(\begin{array}{cc|cc} * & * & * & * \\ 0 & * & * & * \\ \hline 0 & 0 & * & 0 \\ 0 & * & * & * \end{array} \right) \in G \right\}.$$

The Langlands decomposition $P_J = M_J A_J N_J$ of P_J is given by

$$M_J := \left\{ \left(\begin{array}{cc|cc} \epsilon & & & \\ & a & & b \\ \hline & & \epsilon & \\ & c & & d \end{array} \right) | \epsilon \in \{\pm 1\}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) \right\},$$

$$A_J := \{\mathrm{diag}(t, 1, t^{-1}, 1) | t \in \mathbb{R}_{>0}\},$$

and

$$N_J := \left\{ \left(\begin{array}{cc|cc} 1 & * & * & * \\ 0 & 1 & * & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & * & 1 \end{array} \right) \in G \right\}.$$

Here $\mathrm{diag}(a_1, a_2, a_3, a_4)$ denotes the diagonal matrix whose (i,i)-components are given by a_i. Put $\mathfrak{a}_J :=\mathrm{Lie}(A_J)$.

§2. Representations of K, R and G

In this section we collect some basic facts about the representations of K, R and G. In §2.1 and §2.2, we describe all the irreducible unitary representations of K and R, respectively. In §2.3 we recall the generalized principal series representations and the discrete series representations of G.

2.1 Irreducible K-modules

For our later computation, we recall some of the results about the representation theory of K. Since K is isomorphic to the unitary group $U(2)$ of degree 2, the irreducible finite-dimensional representations of K are parametrized by the set of their highest weights relative to Δ_c^+:

$$\{\lambda = \lambda_1\beta_1 + \lambda_2\beta_2 = (\lambda_1, \lambda_2) \in \mathfrak{h}^* | \lambda_i \in \mathbb{Z},\ \lambda_1 \geqslant \lambda_2\}.$$

For each dominant integral weight $\lambda = (\lambda_1, \lambda_2)$, we set $d = d_\lambda = \lambda_1 - \lambda_2 (\geqslant 0)$. Then the degree of the representation $(\tau_\lambda, W_\lambda)$ associated to λ is $d+1$. We can take a basis $\{w_k | 0 \leqslant k \leqslant d\}$ in W_λ so that the representation of $\mathfrak{k}_{\mathbb{C}}$ associated to τ_λ is given by

$$\tau_\lambda(k_1)w_k = (k + \lambda_2)w_k;$$
$$\tau_\lambda(k_2)w_k = (-k + \lambda_1)w_k;$$

$$\tau_\lambda(X)w_k = (k+1)w_{k+1};$$
$$\tau_\lambda(Y)w_k = (d+1-k)w_{k-1} = \{d-(k-1)\}w_{k-1}.$$

We call this basis the *standard basis* of τ. If we want to refer explicitly to the dominant weight λ, we denote w_k by w_k^λ.

The vector space $\mathfrak{p}_\mathbb{C}$ becomes a K-module via the adjoint representation of K. It is easily checked that $\mathfrak{p}_\mathbb{C}^+ \cong W_{(2,0)}$ and the correspondence of the bases is given by

$$(X_{(0,2)}, X_{(1,1)}, X_{(2,0)}) \mapsto (w_0, w_1, w_2).$$

Similarly for $\mathfrak{p}_\mathbb{C}^-$, we have $\mathfrak{p}_\mathbb{C}^- \cong W_{(0,-2)}$ and the correspondence of the bases is given by

$$(X_{(-2,0)}, X_{(-1,-1)}, X_{(0,-2)}) \mapsto (w_0, -w_1, w_2).$$

Let us consider the tensor products $W_\lambda \otimes \mathfrak{p}_\mathbb{C}^\pm$.

LEMMA (2.1). (i) *The tensor product $W_\lambda \otimes \mathfrak{p}_\mathbb{C}^+$ has the decomposition into irreducible factors as*

$$W_\lambda \otimes \mathfrak{p}_\mathbb{C}^+ = \begin{cases} W_{(\lambda_1+2,\lambda_2)} \oplus W_{(\lambda_1+1,\lambda_2+1)} \oplus W_{(\lambda_1,\lambda_2+2)} & if \lambda_1 > \lambda_2, \\ W_{(\lambda_1+2,\lambda_2)} & if \lambda_1 = \lambda_2. \end{cases}$$

Here we understand $W_{(\lambda_1,\lambda_2)} = 0$ for $\lambda_1 < \lambda_2$.

(ii) *The tensor product $W_\lambda \otimes \mathfrak{p}_\mathbb{C}^-$ has the decomposition into irreducible factors as*

$$W_\lambda \otimes \mathfrak{p}_\mathbb{C}^- = \begin{cases} W_{(\lambda_1,\lambda_2-2)} \oplus W_{(\lambda_1-1,\lambda_2-1)} \oplus W_{(\lambda_1-2,\lambda_2)} & if \lambda_1 > \lambda_2, \\ W_{(\lambda_1,\lambda_2-2)} & if \lambda_1 = \lambda_2. \end{cases}$$

Here we understand $W_{(\lambda_1,\lambda_2)} = 0$ for $\lambda_1 < \lambda_2$.

Let P^{up}, P^{even} and P^{down} be the projectors from $W_\lambda \otimes \mathfrak{p}_\mathbb{C}^+$ to factors $W_{(\lambda_1+2,\lambda_2)}$, $W_{(\lambda_1+1,\lambda_2+1)}$ and $W_{(\lambda_1,\lambda_2+2)}$, respectively. Also denote the projectors from $W_\lambda \otimes \mathfrak{p}_\mathbb{C}^-$ to factors $W_{(\lambda_1,\lambda_2-2)}$, $W_{(\lambda_1-1,\lambda_2-1)}$ and $W_{(\lambda_1-2,\lambda_2)}$ by the same symbols P^{up}, P^{even} and P^{down}, respectively. We will write these projectors explicitly in the next lemma.

LEMMA (2.2). (i) *Set* $\mu = (\lambda_1 + 2, \lambda_2)$ *or* $(\lambda_1, \lambda_2 - 2)$. *Then, up to scalar multiple, the projector* P^{up} *is given by*

$$P^{up}(w_k^\lambda \otimes w_2) = \frac{(k+2)(k+1)}{2} w_{k+2}^\mu \qquad (0 \leqslant k \leqslant d);$$
$$P^{up}(w_k^\lambda \otimes w_1) = (k+1)(d+1-k) w_{k+1}^\mu \qquad (0 \leqslant k \leqslant d);$$
$$P^{up}(w_k^\lambda \otimes w_0) = \frac{(d+1-k)(d+2-k)}{2} w_k^\mu \qquad (0 \leqslant k \leqslant d).$$

(ii) *Set* $\nu = (\lambda_1 + 1, \lambda_2 + 1)$ *or* $(\lambda_1 - 1, \lambda_2 - 1)$. *Then, up to scalar multiple, the projector* P^{even} *is given by*

$$P^{even}(w_k^\lambda \otimes w_2) = (k+1) w_{k+1}^\nu \qquad (0 \leqslant k \leqslant d);$$
$$P^{even}(w_k^\lambda \otimes w_1) = (d-2k) w_k^\nu \qquad (0 \leqslant k \leqslant d);$$
$$P^{even}(w_k^\lambda \otimes w_0) = -(d+1-k) w_{k-1}^\nu \qquad (0 \leqslant k \leqslant d).$$

(iii) *Set* $\pi = (\lambda_1, \lambda_2 + 2)$ *or* $(\lambda_1 - 2, \lambda_2)$. *Then, up to scalar multiple, the projector* P^{down} *is given by*

$$P^{down}(w_k^\lambda \otimes w_2) = w_k^\pi \qquad (0 \leqslant k \leqslant d);$$
$$P^{down}(w_k^\lambda \otimes w_1) = -2 w_{k-1}^\pi \qquad (0 \leqslant k \leqslant d);$$
$$P^{down}(w_k^\lambda \otimes w_0) = w_{k-2}^\pi \qquad (0 \leqslant k \leqslant d).$$

Here we understand that w_k^ν, *or* w_k^π *is zero for* $k < 0$, *or* $k > d_\nu$ *or* $k > d_\pi$.

The proof of the above lemmas is easy. It is enough to find out the highest weight vectors in $W_\lambda \otimes \mathfrak{p}_{\mathbb{C}}^{\pm}$ corresponding to the direct factors W_μ, W_ν and W_π, respectively. The other vectors can be obtained by operating Y.

2.2 Unitary representations of R

Since R is identified with $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$, each irreducible unitary representation η can be written uniquely of the form $\eta = \eta_1 \boxtimes \eta_2$, where $\eta_i (i = 1, 2)$ is an irreducible unitary representation of $SL(2, \mathbb{R})$. So, we have only to recall the irreducible unitary representations of $SL(2, \mathbb{R})$. Here we describe them infinitesimally (see HOWE AND TAN [4, Ch.III, p.100]).

As a maximal compact subgroup of $SL(2,\mathbb{R})$, we take $K' := SO(2)$. Define a basis $\{k, n^+, n^-\}$ of $\mathfrak{sl}(2,\mathbb{C})$ by

$$k := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad n^+ := \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad n^- := \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}.$$

We introduce some $\mathfrak{sl}(2,\mathbb{C})$-modules:

(1) Lowest weight module $V_\mu (\mu \in \mathbb{C})$: V_μ has a basis of k-eigen vectors $\{v_i | i = 0, 1, 2, \dots\}$ such that

$$kv_i = (\mu + 2i)v_i, \quad n^+ v_i = v_{i+1}, \quad n^- v_i = -i(\mu + i - 1)v_{i-1}.$$

(2) Highest weight module $\bar{V}_\mu (\mu \in \mathbb{C})$: $\bar{V}_\mu$ has a basis of k-eigen vectors $\{v_i | i = 0, -1, -2, \dots\}$ such that

$$kv_i = (\mu + 2i)v_i, \quad n^+ v_i = -i(\mu + i + 1)v_{i+1}, \quad n^- v_i = v_{i-1}.$$

(3) $\mathcal{S}^{s,+} (s \in \mathbb{C})$: $\mathcal{S}^{s,+}$ has a basis of k-eigen vectors $\{v_i | i \in \mathbb{Z}\}$ such that

$$kv_i = 2iv_i, \quad n^+ v_i = (-\frac{s}{2} + i)v_{i+1}, \quad n^- v_i = (-\frac{s}{2} - i)v_{i-1}.$$

(4) $\mathcal{S}^{s,-} (s \in \mathbb{C})$: $\mathcal{S}^{s,-}$ has a basis of k-eigen vectors $\{v_i | i \in \mathbb{Z}\}$ such that

$$kv_i = (2i+1)v_i, \quad n^+ v_i = (\frac{-s+1}{2} + i)v_{i+1}, \quad n^- v_i = (\frac{-s-1}{2} - i)v_{i-1}.$$

We call the basis $\{v_i\}$ introduced above the *standard basis*.

Theorem (2.3 Unitary dual of $SL(2,\mathbb{R})$). *Suppose that (η, V_η) is an irreducible unitary representation of $SL(2,\mathbb{R})$, then the $\mathfrak{sl}(2,\mathbb{C})$-module structure of the space V_η^0 of K'-finite vectors in V_η is equivalent to one of the following*:

(1) *Trivial representation* 1.

(2) *(Limits of) discrete series representation V_l $(l \in \{1, 2, \dots, \})$ or $\bar{V}_l$ $(l \in \{-1, -2, \dots, \})$. In this case we write $\eta = D_l$, and call l the Blattner parameter of D_l.*

(3) *Principal series representations $\mathcal{S}^{-1+it,+}$ for $t \in \mathbb{R}_{\geqslant 0}$ or $\mathcal{S}^{-1+it,-}$ for $t \in \mathbb{R}_{>0}$. In this case we write $\eta = P^{-1+it,+}$ or $P^{-1+it,-}$.*

(4) *Complementary series representations $\mathcal{S}^{s,+}$ for $s \in (-1, 0)$. In this case we write $\eta = P^{s,+}$.*

We denote by $\hat{R}$ the set of equivalent classes of irreducible unitary representations of R.

2.3 The standard representations of G

In this subsection we review some facts on standard representations of G.

2.3.1 *The generalized principal series representations*

A discrete series representation σ of the semisimple part M_J of P_J is of the form $\sigma = \epsilon \boxtimes D_\lambda (|\lambda| \geqslant 2)$, where $\epsilon : \{\pm 1\} \to \mathbb{C}^*$ is a character, D_λ is a discrete series representation of $SL(2,\mathbb{R})$ with the Blattner parameter λ (see §2.2). For an element of $\nu_J \in \mathfrak{a}^*_{J,\mathbb{C}}$, let $\exp(\nu_J) : A_J \ni a_J \mapsto a_J^{\nu_J} \in \mathbb{C}^*$ be the corresponding character of A_J. Define a representation $\sigma \otimes \nu_J$ of P_J by

$$\sigma \otimes \nu_J(p_J) = \sigma(m_J) a_J^{\nu_J}, \qquad \text{for } p_J = m_J a_J n_J \in P_J = M_J A_J N_J.$$

Then the generalized principal series representation $\pi(\sigma, \nu_J)$ of G is defined as the induced representation $C^\infty\text{-}\mathrm{Ind}_{P_J}^G(\sigma \otimes (\nu_J + \rho_J))$ of G with the representation space

$$\begin{aligned} &C^\infty\text{-}\mathrm{Ind}_{P_J}^G(\sigma \otimes (\nu_J + \rho_J)) \\ &:= \{F : G \overset{C^\infty}{\to} V_\sigma | F(m_J a_J n_J g) = \sigma(m_J) a^{\nu_J + \rho_J} F(g), \\ &\qquad\qquad \forall (m_J, a_J, n_J, g) \in M_J \times A_J \times N_J \times G\}, \end{aligned}$$

on which G acts by right translation.

We describe the K-types of the generalized principal series representations $\pi(\sigma, \nu_J)$.

PROPOSITION (2.4). *Let $\pi(\sigma, \nu_J)$ be the generalized principal series representationof G with $\sigma = (\epsilon, D_\lambda)$ and $\nu_J \in \mathfrak{a}^*_{J,\mathbb{C}}$. Then for a dominant integral weight $q = (q_1, q_2) \in \mathbb{Z}^{\oplus 2} (q_1 \geqslant q_2)$, the irreducible representation $\tau_{(q_1,q_2)}$ of K occurs in $\pi(\sigma, \nu_J)$ with multiplicity*

$$\begin{aligned} &[\pi(\sigma, \nu_J) : \tau_q] \\ &= \sharp\{m \in \mathbb{Z} | m \equiv \lambda \pmod 2, \mathrm{sgn}(\lambda) m \geqslant |\lambda|, \\ &\qquad\qquad (-1)^{q_1 + q_2 - m} = \epsilon, q_2 \leqslant m \leqslant q_1\}. \end{aligned}$$

In particular,

(1) *when* $\epsilon = (-1)^\lambda$ *and* $\lambda \geqslant 2$, *then each of* $\tau_{(q,q)}$ $(q \in \mathbb{Z}, q \equiv \lambda \pmod 2, q \geqslant \lambda)$ *or* $\tau_{(\lambda,q)}$ $(q \in \mathbb{Z}, q \equiv \lambda \pmod 2, \lambda \geqslant q)$ *occurs in* $\pi(\sigma, \nu_J)$ *with multiplicity one.*

(2) *when* $\epsilon = (-1)^{\lambda+1}$ *and* $\lambda \geqslant 2$, *then each of* $\tau_{(q,q-1)}$ $(q \in \mathbb{Z}, q \equiv \lambda \pmod 2, q \geqslant \lambda)$ *or* $\tau_{(\lambda,q-1)}$ $(q \in \mathbb{Z}, q \equiv \lambda \pmod 2, \lambda \geqslant q-1)$ *occurs in* $\pi(\sigma, \nu_J)$ *with multiplicity one.*

PROOF. Consider the restriction of σ to $K \cap M_J$:

$$\sigma|_{K\cap M_J} = \sum_{\omega \in \widehat{K\cap M_J}} [\sigma : \omega]\omega.$$

Here $[\sigma : \omega]$ is the multiplicity of ω in $\sigma|_{K\cap M_J}$. Since $K \cap M_J \cong \{\pm 1\} \times SO(2)$, any $\omega \in \widehat{K \cap M_J}$ is specified by its value $\omega(\gamma)$ at $\gamma := \mathrm{diag}(-1, 1, -1, 1)$ and the restriction $\omega|_{SO(2)}$. We define a character $\chi_m (m \in \mathbb{Z})$ of $SO(2)$ by

$$\chi_m(r_\theta) = \exp(im\theta),$$

with $r_\theta \in SO(2)$ being the rotation with angle θ. Then the K-type theorem for D_λ (see §2.2) implies that the multiplicity of $\omega = (\omega(\gamma), \chi_m)$ is given by

$$[\sigma : \omega] = \begin{cases} 1, & \text{if } m \equiv \lambda \pmod 2, \mathrm{sgn}(\lambda)m \geqslant |\lambda|, \omega(\gamma) = \epsilon, \\ 0, & \text{otherwise.} \end{cases}$$

The Frobenius reciprocity implies that the multiplicity of $\tau = \tau_{(q_1,q_2)} \in \hat{K}$ in $\pi(\sigma, \nu_J)$ is given by

$$[\pi(\sigma, \nu_J) : \tau] = \sum_{\omega \in \widehat{K\cap M_J}} [\sigma|_{K\cap M_J} : \omega] \cdot [\tau|_{K\cap M_J} : \omega].$$

Since the irreducible decomposition of $\tau_{(q_1,q_2)}|_{K\cap M_J}$ is given by

$$\tau_{(q_1,q_2)}|_{K\cap M_J} = \oplus_{q_2 \leqslant m \leqslant q_1} ((-1)^{q_1+q_2-m}, \chi_m),$$

together with the above formula of $[\sigma : \omega]$, we have the former part of the proposition. The latter part of the proposition is a direct consequence of the former. □

2.3.2 *The large discrete series representations*

In this subsection we review the Harish-Chandra parametrization of the discrete series representations in our case $G = Sp(2,\mathbb{R})$ (cf, KNAPP, [5, Ch. VII]). Consider a compact Cartan subgroup $\exp(\mathfrak{h})$ of G corresponding to $\mathfrak{h}$. Then the characters of $\exp(\mathfrak{h})$ are given by

$$\exp(\mathfrak{h}) \ni \left(\begin{array}{cc|cc} \cos\theta_1 & & \sin\theta_1 & \\ & \cos\theta_2 & & \sin\theta_2 \\ \hline -\sin\theta_1 & & \cos\theta_1 & \\ & -\sin\theta_2 & & \cos\theta_2 \end{array}\right)$$
$$\mapsto \exp\{-\sqrt{-1}(m_1\theta_1 + m_2\theta_2)\} \in \mathbb{C}^*.$$

Here m_1, m_2 are some integers. The integral structure determined by the derivation of these characters coincides with what we introduced in §1. In order to parametrize the representations of discrete series of $Sp(2,\mathbb{R})$, we first enumerate all the positive systems containing Δ_c^+. There are four such positive systems:

$$\begin{aligned} &(1)\ \Delta_I^+ = \{(1,-1),(2,0),(1,1),(0,2)\};\\ &(2)\ \Delta_{II}^+ = \{(1,-1),(0,-2),(2,0),(1,1)\};\\ &(3)\ \Delta_{III}^+ = \{(1,-1),(-1,-1),(0,-2),(2,0)\};\\ &(4)\ \Delta_{IV}^+ = \{(1,-1),(-2,0),(-1,-1),(0,-2)\}. \end{aligned}$$

Let J be a variable running over the set of indices $\{I, II, III, IV\}$. Then we write $\Delta_{J,n}^+ := \Delta_J^+ \setminus \Delta_c^+$ for the set of non-compact positive roots. For each index J, define a subset Ξ_J of dominant weights by

$$\Xi_J := \{\Lambda = (\Lambda_1, \Lambda_2) | \langle \Lambda, \beta \rangle > 0,\ \forall \beta \in \Delta_J^+\}.$$

Then the set $\cup_{J=I}^{IV} \Xi_J$ gives the Harish-Chandra parametrization of the discrete series for $Sp(2,\mathbb{R})$. Let π_Λ be the discrete representation of G associated to $\Lambda \in \Xi_J$. The Blattner parameter λ_{min} of π_Λ is given by $\lambda_{min} := \Lambda - \rho_{c,J} + \rho_{n,J}$, where $\rho_{c,J}$ or $\rho_{n,J}$ is a half of the sum of compact positive roots or non-compact positive roots, respectively. The highest weights of the K-types of $\pi_\Lambda|_K$ are of the form

$$\lambda_{min} + \sum_{\alpha \in \Delta_{J,n}^+} m_\alpha \alpha \qquad \text{with } m_\alpha \in \mathbb{Z}_{\geqslant 0}.$$

Furthermore, $\tau_{\lambda_{min}}$ occurs in $\pi_\Lambda|_K$ with multiplicity one, and we call it the minimal K-type of π_Λ. A discrete series representation π_Λ is called *large* if its Harish-Chandra parameter Λ belongs to $\Xi_{II} \cup \Xi_{III}$.

DEFINITION (2.5). (1) We refer, in this paper, the generalized principal series representations $\pi = \pi(\sigma, \nu_J)$ with $\sigma = (\epsilon, D_\lambda)(\lambda \geqslant 2)$(§2.3.1) and the large discrete series representations $\pi = \pi_\Lambda(\Lambda \in \Xi_{II})$ (§2.3.2) of G as the *half-size standard representations* of G.
(2) We define *the corner K-type* τ of a half-size standard representation π of G by

$$\tau := \begin{cases} \tau_{(\lambda,\lambda)}, & \text{if } \pi = \pi(\sigma,\nu_J),\ \sigma = (D_\lambda, \epsilon),\ \epsilon = (-1)^\lambda; \\ \tau_{(\lambda,\lambda-1)}, & \text{if } \pi = \pi(\sigma,\nu_J),\ \sigma = (D_\lambda, \epsilon),\ \epsilon = (-1)^{\lambda+1}; \\ \tau_{\lambda_{\min}}, & \text{if } \pi = \pi_\Lambda\ (\Lambda \in \Xi_{II}). \end{cases}$$

REMARK. (1) The rather unfamiliar terminology "half size" standard representations is due to the fact that Bernstein degree of these representations are the half of the order of the Weyl group of G. We notice that the "shapes" of the K-types of these two kinds of representations are the same.
(2) The contragredient representation of $\pi((\epsilon, D_\lambda), \nu_J)$ is $\pi((-\epsilon, D_{-\lambda}), -\nu_J)$.
(3) The contragredient representation of π_Λ with $\Lambda = (\Lambda_1, \Lambda_2) \in \Xi_{II}$ is π_{Λ^*} with $\Lambda^* = (-\Lambda_2, -\Lambda_1) \in \Xi_{III}$.

§3. Spherical Functions

3.1 Definition of spherical functions

For $(\eta, V_\eta) \in \hat{R}$, we define a C^∞-induced module $C^\infty\text{-Ind}_R^G(\eta)$ with the representation space

$$C^\infty_\eta(R\backslash G) := \{F : G \to V_\eta |\ C^\infty\text{-class},\ F(rg) = \eta(r)F(g)\ \forall (r,g) \in R \times G\}$$

on which G acts by the right translation. For a smooth representation η of R and a finite-dimensional K-module (τ, W_τ), we denote by $C^\infty_{\eta,\tau}(R\backslash G/K)$ the space of C^∞-functions $F : G \to V_\eta \otimes W_\tau^*$ with the property

$$F(rgk) = (\eta(r) \otimes \tau^*(k)^{-1})F(g) \qquad (r,g,k) \in R \times G \times K,$$

where (τ^*, W_τ^*) stands for the contragredient representation of (τ, W_τ). For a stnadard representation π of G and a K-equivariant map $i : \tau \to \pi|_K$, we define a $\mathbb{C}$-linear map

$$i^* : \mathrm{Hom}_{(\mathfrak{g},K)}(\pi^0, C_\eta^\infty(R\backslash G)^0) \longrightarrow \mathrm{Hom}_K(\tau, C_\eta^\infty(R\backslash G)) \cong C_{\eta,\tau}^\infty(R\backslash G/K),$$

by the pullback via i. Here π^0 or $C_\eta^\infty(R\backslash G)^0$ stands for the underlying $(\mathfrak{g}, K)$-module of π or $C_\eta^\infty(R\backslash G)$, respectively. We call the image of i^* the *spherical functions of type* (π, η, τ). The main purpose of this paper is to compute the A-radial part of the spherical functions when π is a half-size standard representation and τ is its corner K-type (see Definition (2.5)). We should note that if π is irreducible, then the above map i^* is injective.

3.2 Some consequences from structure theory

We recall some structure theory of semisimple symmetric spaces, by which we can regard the spherical functions as C^∞-functions of one real variable.

Proposition (3.1). *Let R, A, and K be as in §1.*
(i) *The multiplication map $\Phi : R \times A \times K \ni (r, a, k) \mapsto rak \in G$ is a C^∞-surjection, and regular at (r, a, k) if and only if $a \neq 1$.*
(ii) *The fiber of ϕ above $g = rak$ is given by as follows*:

$$(1) \quad \Phi^{-1}(g) = \{(rl^{-1}, 1, lk) | l \in R \cap K\} \qquad \text{if } a = 1,$$

$$(2) \quad \Phi^{-1}(g) = \{(rl^{-1}, lal^{-1}, lk) | l \in N_{R\cap K}(\mathfrak{a})\} \qquad \text{if } a \neq 1.$$

Proof. See Theorem 9 and Theorem 10 of Rossmann [10]. □

Let $C_{\eta,\tau}^\infty(A)$ be the space of $V_\eta \otimes W_\tau^*$-valued C^∞-functions satisfying the following conditions $(a), (b)$ and (c):

(a) $\eta(m) \otimes \tau^*(m)\phi(a_t) = \phi(a_t)$ for any $m \in Z_{R\cap K}(\mathfrak{a})$.
(b) $\eta(m^0) \otimes \tau^*(m^0)\phi(a_t) = \phi(a_{-t})$
for a non trivial representative $m^0$of $W_{R\cap K}(\mathfrak{a}) = \{\pm 1\}$.
(c) $\eta(r) \otimes \tau^*(r)\phi(e) = \phi(e)$ for any $r \in R \cap K$.

Proposition (3.2). *The restriction map*

$$\mathrm{res}_A : C^\infty_{\eta,\tau}(R\backslash G/K) \to C^\infty_{\eta,\tau}(A)$$

is a linear injection.

Proof. This follows from Proposition (3.1). □

Owing to Proposition (3.2), we can regard a spherical function $F(g)$ of type (π, η, τ) as a function $\phi = \mathrm{res}_A(F)$ on A. In what follows, we frequently write $\phi(t)$ instead of $\phi(a_t)$. For any $\mathbb{C}$-linear map

$$\mathcal{A} : C^\infty_{\eta,\tau}(R\backslash G/K) \to C^\infty_{\eta,\tau'}(R\backslash G/K),$$

there exists a $\mathbb{C}$-linear map $\rho(\mathcal{A}) : C^\infty_{\eta,\tau}(A) \to C^\infty_{\eta,\tau'}(A)$ such that $\mathrm{res}\,|_A \circ \mathcal{A} = \rho(\mathcal{A}) \circ \mathrm{res}\,|_A$, and call $\rho(\mathcal{A})$ the A-radial part of $\mathcal{A}$.

If $C^\infty_{\eta,\tau}(A) = 0$, there is no non-zero spherical functions.

Assumption (3.3). *Hereafter, we (tacitly) assume that* $C^\infty_{\eta,\tau}(A) \neq \{0\}$.

§4. Differential Operators and Differential Equations

In this section, we construct systems of differential equations for spherical functions and calculate their radial parts.

4.1 Differential operators

Here we introduce two kinds of differential operators, that is, shift operators and the Casimir operator.

4.1.1 *Shift operators*

Before introducing shift operators, we recall the definition of the Schmid operators. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ in §1, and $\mathrm{Ad} = \mathrm{Ad}\,|_{\mathfrak{p}_\mathbb{C}}$ the adjoint representation of K on $\mathfrak{p}_\mathbb{C}$. Then we have a canonical covariant differential operator ∇_τ from $C^\infty_{\eta,\tau}(R\backslash G/K)$ to $C^\infty_{\eta,\tau\otimes\mathfrak{p}_\mathbb{C}{}^*}(R\backslash G/K)$:

$$\nabla_\tau F = \sum_i R_{X_i} F \otimes X_i, \quad F \in C^\infty_{\eta,\tau}(R\backslash G/K),$$

where $(X_i)_i$ is any fixed orthonormal basis of $\mathfrak{p}$ with respect to the Killing form B of $\mathfrak{g}$, and

$$R_X F(g) := \frac{d}{dt}(F(g\exp(tX)))|_{t=0} \quad (g \in G, X \in \mathfrak{g}).$$

We call this differential operator ∇_τ the Schmid operator.

Let $P_{\tau'} : W_\tau^* \otimes \mathfrak{p}_{\mathbf{C}} \to W_{\tau'}^*$ be the projection to an irreducible component $W_{\tau'}^*$ of the K-module $W_\tau^* \otimes \mathfrak{p}_{\mathbf{C}}$. We define a $V_\eta \otimes W_{\tau'}^*$-valued function $F' \in C_{\eta,\tau'}^\infty(R\backslash G/K)$ by $F' := P_{\tau'}(\nabla_\tau F)$. We prove the following key proposition.

PROPOSITION (4.1). *For a spherical function F of type (π,η,τ), the $V_\eta \otimes W_{\tau'}^*$-valued function F' is also a spherical function of type (π,η,τ').*

PROOF. By the definition of the spherical functions, F can be written as

$$F = \sum_{k=0}^{d} \Phi(w_k) \otimes w_k^*,$$

where $\Phi \in \mathrm{Hom}_{(\mathfrak{g},K)}(\pi, C_\eta^\infty(R\backslash G)), \{w_k\}$ is a basis of τ, $\{w_k^*\}$ is the dual basis of $\{w_k\}$. Then we have

$$\begin{aligned}\nabla_\tau(F) &= \sum_{k=0}^{d}\sum_i R_{X_i}\Phi(w_k) \otimes w_k^* \otimes X_i \\ &= \sum_{k=0}^{d}\sum_i \Phi(\pi(X_i)w_k) \otimes w_k^* \otimes X_i.\end{aligned}$$

On the other hand, we can identify $\mathfrak{p}_{\mathbf{C}}^*$ with $\mathfrak{p}_{\mathbf{C}}$ by means of the Killing form B:

$$\xi : \mathfrak{p}_{\mathbf{C}} \ni X \mapsto \xi_X \in \mathfrak{p}_{\mathbf{C}}^*, \quad \xi_X(Y) = B(X,Y) \quad (\text{for any } X, Y \in \mathfrak{p}_{\mathbf{C}}).$$

We notice that (i) $\{\xi_{X_i} \otimes w_k \in \mathfrak{p}_{\mathbf{C}}^* \otimes W_\tau\}$ is the dual basis of $\{X_i \otimes w_k^* \in \mathfrak{p}_{\mathbf{C}} \otimes W_\tau^*\}$, and (ii) the natural map

$$\mathfrak{p}_{\mathbf{C}}^* \otimes W_\tau \ni \xi_X \otimes w \mapsto \pi(X)w \in \pi|_K,$$

is a K-equivariant map from $\mathrm{Ad}^* \otimes \tau$ to $\pi|_K$. Therefore, considering the complete reducibility of finite-dimensional K-modules, we obtain the assertion. □

We compute the A-radial part $\rho(\nabla_\tau)$ of ∇_τ. As an orthonormal basis of $\mathfrak{p}$, we take

$$\left\{ C\|\beta\|(X_\beta + X_{-\beta}), \quad \frac{C\|\beta\|}{\sqrt{-1}}(X_\beta - X_{-\beta}) \quad | \, \beta \in \Delta_n^+ \right\},$$

with some constant $C > 0$ depending on the Killing form. Then

$$2\nabla_\tau F = C \sum_{\beta \in \Delta_n^+} \|\beta\|^2 R_{X_\beta} F \otimes X_{-\beta} + C \sum_{\beta \in \Delta_n^+} \|\beta\|^2 R_{X_{-\beta}} F \otimes X_\beta.$$

We write

$$\nabla_\tau^+ F = C \sum_{\beta \in \Delta_n^+} \|\beta\|^2 R_{X_\beta} F \otimes X_{-\beta}, \quad \nabla_\tau^- F = C \sum_{\beta \in \Delta_n^+} \|\beta\|^2 R_{X_{-\beta}} F \otimes X_\beta.$$

We put

$$\begin{aligned} D_\lambda^- &:= P^{down} \circ \nabla^-_{(\lambda,\lambda-2)} \circ \nabla^-_{(\lambda,\lambda)} : \\ &\quad C^\infty_{\eta,(\lambda,\lambda)}(R\backslash G/K) \to C^\infty_{\eta,(\lambda-2,\lambda-2)}(R\backslash G/K); \\ E_\lambda^- &:= P^{even} \circ \nabla^-_{(\lambda,\lambda-1)} : C^\infty_{\eta,(\lambda,\lambda-1)}(R\backslash G/K) \to C^\infty_{\eta,(\lambda-1,\lambda-2)}(R\backslash G/K). \end{aligned}$$

We call these differential operators D_λ^- and E_λ^- the *shift operators.*

4.1.2 *The Casimir operator*

The Casimir element L of $\mathfrak{g}_\mathbb{C}$ is up to constant given by

$$\begin{aligned} L =& 2k_1^2 + 2k_2^2 - 8k_1 - 4k_2 + 2X_{(2,0)} \cdot X_{(-2,0)} \\ &+ 2X_{(0,2)} \cdot X_{(0,-2)} + X_{(1,1)} \cdot X_{(-1,-1)} - X_{(1,-1)} \cdot X_{(-1,1)}. \end{aligned}$$

We can extend the action $R_Y (Y \in \mathfrak{g}_\mathbb{C})$ of $\mathfrak{g}_\mathbb{C}$ on $C^\infty_{\eta,\tau}(R\backslash G/K)$ to the universal enveloping algebra $U(\mathfrak{g}_\mathbb{C})$ of $\mathfrak{g}_\mathbb{C}$. In particular, the action R_L of the Casimir element L is defined, which we call the Casimir operator.

4.2 Differential equations

We are going to write abstractly the differential equations satisfied by the spherical functions in terms of the differential operators constructed above. We begin with the case of the generalized principal series representations.

THEOREM (4.2). *Let* $\pi = \pi(\sigma, \nu_J)$ *be a generalized principal series representation with* $\sigma = (D_\lambda, \epsilon)$. (i) *If* $\epsilon = (-1)^\lambda$, *then the spherical functions* F *of type* $(\pi, \eta, \tau_{(\lambda,\lambda)})$ *satisfies*

$$\begin{aligned} &\text{(a-1)}: \quad && D_\lambda^- F = 0, \\ &\text{(a-2)}: \quad && R_L F = \{2{\nu_J}^2 + 2(\lambda-1)^2 - 10\}F. \end{aligned}$$

(ii) *If* $\epsilon = (-1)^{\lambda+1}$, *then the spherical functions* F *of type* $(\pi, \eta, \tau_{(\lambda,\lambda-1)})$ *satisfies*

$$\begin{aligned} &\text{(b-1)}: \quad && E_\lambda^- F = 0, \\ &\text{(b-2)}: \quad && R_L F = \{2{\nu_J}^2 + 2(\lambda-1)^2 - 10\}F. \end{aligned}$$

Here we identify $\nu_J \in \mathfrak{a}_{J,\mathbb{C}}^*$ *with its value at* $\mathrm{diag}(1, 0, -1, 0) \in \mathfrak{a}_{J,\mathbb{C}}$.

PROOF. From the irreducible decomposition of $\pi|_K$ as a K-module (Proposition (2.4)) and Proposition (4.1), we have the equations (a-1) and (b-1). We will prove (a-2) and (b-2). We denote the infinitesimal character of π by χ_π. As in the proof of Proposition (4.1), we write F as

$$F = \sum_{k=0}^{d} \Phi(w_k) \otimes w_k^*,$$

where $\Phi \in \mathrm{Hom}_{(\mathfrak{g},K)}(\pi^0, C_\eta^\infty(R\backslash G)^0)$, $\{w_k\}$ is a basis of τ, $\{w_k^*\}$ is the dual basis of $\{w_k\}$. Then we have

$$R_L F = \sum_{k=0}^{d} \Phi(\chi_\pi(L) w_k) \otimes w_k^* = \chi_\pi(L) F.$$

On the other hand, the value $\chi_\pi(L)$ of the infinitesimal character χ_π of π at L is equal to $2\nu_J^2 + 2(\lambda-1)^2 - 10$ (see MIYAZAKI AND ODA, [6, Proposition (7.2)(7.3)]). Hence (a-2) and (b-2) are proved. □

The next theorem characterizes the spherical functions for the discrete series representations. This is a special case of a general result of YAMASHITA [13].

THEOREM (4.3). (i) *Let* $\pi = \pi_\Lambda$ *be a discrete series representation with* $\Lambda \in \Xi_J, \tau = \tau_\lambda$ *the minimal* K*-type of* π. *We define a differential operator* $\mathcal{D}_\lambda$, *which we also call the shift operator, by*

$$\mathcal{D}_\lambda : C^\infty_{\eta,\tau}(R\backslash G/K) \ni F \mapsto P_\lambda(\nabla_\tau F) \in C^\infty_{\eta,(\tau^-)^*}(R\backslash G/K),$$

where P_λ *is the projection to the second component* $\tau^- := \bigoplus_{\beta\in\Delta^+_{J,n}} W^*_{\lambda-\beta}$ *of* ${W_\tau}^* \otimes \mathfrak{p}_{\mathbb{C}} = \bigoplus_{\beta\in\Delta^+_{J,n}} W^*_{\lambda+\beta} \oplus \bigoplus_{\beta\in\Delta^+_{J,n}} W^*_{\lambda-\beta}$. *Then the restriction to the minimal* K*-type* τ_λ *induces an isomorphism*

$$i^* : \mathrm{Hom}_{(\mathfrak{g},K)}(\pi^0_\Lambda, C^\infty_\eta(R\backslash G)^0) \cong \ker(\mathcal{D}_\lambda).$$

(ii) *If the Harish-Chandra parameter* Λ *belongs to* Ξ_{II}, *the differential equation* $\mathcal{D}_\lambda F = 0$ *is equivalent to the system*:

$$\begin{aligned} &(\text{c-1}): \qquad P^{down} \circ \nabla^- F = 0;\\ &(\text{c-2}): \qquad P^{down} \circ \nabla^+ F = 0;\\ &(\text{c-3}): \qquad P^{even} \circ \nabla^- F = 0. \end{aligned}$$

PROOF. (i) By the same reason as Theorem (4.1), the image of the restriction map i^* is annihilated by $\mathcal{D}_\lambda$. The injectivity is remarked earlier in §3.1. Since the Blattner parameters of the contragredient representations π^*_Λ of π_Λ ($\Lambda \in \Xi_{II}$) are *far from the wall* (see [13, Definition (1.7)]), the image of i^* coincides with $\ker(\mathcal{D}_\lambda)$[13, Theorem (2.4)]. (ii) This follows from the definition of the differential operator $\mathcal{D}_\lambda$. □

4.3 The radial part of the Schmid operator

We begin with calculating the A-radial parts $\rho(R_Y)$ of the actions R_Y of $Y \in \mathfrak{g}_{\mathbb{C}}$ on $C^\infty_{\eta,\tau}(R\backslash G/K)$.

PROPOSITION (4.4). *For* $\phi \in C^\infty_{\eta,\tau}(A)$, *we have*

$$(\rho(R_{k_1})\phi)(t) = -\tau^*(k_1)\phi(t),$$

$$\begin{aligned}
\left(\rho(R_{k_2})\phi\right)(t) &= -\tau^*(k_2)\phi(t),\\
\left(\rho(R_{X_{(2,0)}})\phi\right)(t) &= \{2\eta(\mathcal{L}^+) + \tanh 2t\cdot\tau^*(X)\}\phi(t),\\
\left(\rho(R_{X_{(1,1)}})\phi\right)(t) &= \{\frac{d}{dt} - \eta(\mathcal{N}) + 2\coth 2t\cdot\tau^*(k_2)\}\phi(t),\\
\left(\rho(R_{X_{(0,2)}})\phi\right)(t) &= \{-2\eta(\mathcal{M}^+) + \tanh 2t\cdot\tau^*(Y)\}\phi(t),\\
\left(\rho(R_{X_{(-2,0)}})\phi\right)(t) &= \{2\eta(\mathcal{L}^-) - \tanh 2t\cdot\tau^*(Y)\}\phi(t),\\
\left(\rho(R_{X_{(-1,-1)}})\phi\right)(t) &= \{\frac{d}{dt} + \eta(\mathcal{N}) - 2\coth 2t\cdot\tau^*(k_2)\}\phi(t),\\
\left(\rho(R_{X_{(0,-2)}})\phi\right)(t) &= \{-2\eta(\mathcal{M}^-) - \tanh 2t\cdot\tau^*(X)\}\phi(t),\\
\left(\rho(R_{X_{(1,-1)}})\phi\right)(t) &= -2\tau^*(X)\phi(t),\\
\left(\rho(R_{X_{(-1,1)}})\phi\right)(t) &= 2\tau^*(Y)\phi(t).
\end{aligned}$$

We can deduce these formulae from the generalized Cartan decomposition (Lemma (1.1)) and the following lemma.

Lemma (4.5). *Let* $U = \mathrm{Ad}(a_{-t})(X_1\cdot X_2\cdots X_l)\cdot H_1^m\cdot Y_1\cdot Y_2\cdots Y_n$ *be an element of* $U(\mathfrak{g}_{\mathbb{C}})$, *where* $X_i\in\mathfrak{r}_{\mathbb{C}}, m\in\mathbb{Z}_{\geqslant 0}$ *and* $Y_i\in\mathfrak{k}_{\mathbb{C}}$. *Then for* $F\in C^\infty_{\eta,\tau}(R\backslash G/K)$ *we have*

$$\begin{aligned}
&[R_U F](a_t) =\\
&\eta(X_1)\circ\eta(X_2)\circ\cdots\circ\eta(X_l)\circ(\frac{d}{dt})^m\circ(-\tau^*(Y_n))\circ\cdots\\
&\qquad\circ(-\tau^*(Y_2))\circ(-\tau^*(Y_1))F(a_t).
\end{aligned}$$

This lemma can be proved by direct computation.

Proposition (4.6). *We have the following formulae for the A-radial parts of* ∇^+ *and* ∇^- :

(i) $\rho(\nabla^+)\phi$
$= 4\{2\eta(\mathcal{L}^+) + \tanh 2t(\tau^*\otimes\mathrm{Ad})(X)\}\phi\otimes X_{(-2,0)}$

$$+2\{\frac{d}{dt}-\eta(\mathcal{N})+2\coth 2t((\tau^*\otimes\mathrm{Ad})(k_2)+1)+4\tanh 2t\}\phi\otimes X_{(-1,-1)}$$
$$+4\{-2\eta(\mathcal{M}^+)+\tanh 2t(\tau^*\otimes\mathrm{Ad})(Y)\}\phi\otimes X_{(0,-2)};$$

(ii) $\rho(\nabla^-)\phi$
$$=4\{2\eta(\mathcal{L}^-)-\tanh 2t(\tau^*\otimes\mathrm{Ad})(Y)\}\phi\otimes X_{(2,0)}$$
$$+2\{(\frac{d}{dt}+\eta(\mathcal{N})-2\coth 2t((\tau^*\otimes\mathrm{Ad})(k_2)-1)+4\tanh 2t)\}\phi\otimes X_{(1,1)}$$
$$+4\{-2\eta(\mathcal{M}^-)-\tanh 2t(\tau^*\otimes\mathrm{Ad})(X)\}\phi\otimes X_{(0,2)}.$$

PROOF. (i) It is easy to see that

$$\begin{aligned}\tau^*(X)\phi\otimes X_{(-2,0)}&=(\tau^*\otimes\mathrm{Ad})(X)\phi\otimes X_{(-2,0)}-\phi\otimes[X,X_{(-2,0)}]\\&=(\tau^*\otimes\mathrm{Ad})(X)\phi\otimes X_{(-2,0)}+\phi\otimes X_{(-1,-1)}.\end{aligned}$$

Similarly we have

$$\begin{aligned}\tau^*(k_2)\phi\otimes X_{(-1,-1)}&=(\tau^*\otimes\mathrm{Ad})(k_2)\phi\otimes X_{(-1,-1)}+\phi\otimes X_{(-1,-1)},\\ \tau^*(Y)\phi\otimes X_{(0,-2)}&=(\tau^*\otimes\mathrm{Ad})(Y)\phi\otimes X_{(0,-2)}+\phi\otimes X_{(-1,-1)}.\end{aligned}$$

From these formulae and Proposition (4.4), we can compute $\rho(\nabla^+)$. The computation of $\rho(\nabla^-)$ is quite similar. □

4.4 The radial parts of the shift operators

In order to write the radial parts of the shift operators, it is better to introduce a "macro symbol" C_λ, which is given by

$$C_\lambda:=\eta(\mathcal{N})+2\lambda\coth 2t.$$

We represent the spherical function $\phi^{(0)}\in C^\infty_{\eta,\tau}(A)$ as

$$\phi^{(0)}(a)=\sum_{k=0}^{d}\phi_0^{(0)}(a)w_k^{(0)},$$

with respect to the standard basis $\{w_k^{(0)}\}$ of the representation τ^* of K. Then we can write down explicitly the differential equations (a-1), (b-1), (c-1), (c-2) and (c-3) arising from the shift operators in terms of the coefficient functions $\phi_k^{(0)}$. The computation is divided according to the dimension d of the corner K-type τ.

PROPOSITION (4.7 one-dimensional case). *Set* $\tau = \tau_{(\lambda,\lambda)}$. *The equation* (a-1) *is equivalent to*

$$\text{(A-1)}: \quad \{(\frac{d}{dt}+C_{\lambda-1}+4\tanh 2t)(\frac{d}{dt}+C_{\lambda})+8\eta(\mathcal{L}^-\cdot\mathcal{M}^-+\mathcal{M}^-\cdot\mathcal{L}^-)\}\phi_0^{(0)}=0.$$

PROOF. We put $\phi^{(1)} := \rho(\nabla^-)\phi^{(0)}, \phi^{(2)} := P^{down}(\rho(\nabla^-)\phi^{(1)}) = \rho(D_\lambda^-)\phi^{(0)}$. We define V_η-valued C^∞-functions $\phi_k^{(1)} (k=0,1,2)$, and $\phi_0^{(2)}$ on A by

$$\phi^{(1)}(a) = \sum_{0\leqslant k\leqslant 2} \phi_k^{(1)}(a) w_k^{(1)}, \quad \phi^{(2)}(a) = \phi_0^{(2)}(a) w_0^{(2)},$$

where $\{w_k^{(1)}, 0\leqslant k\leqslant 2\}$ or $\{w_0^{(2)}\}$ is the standard basis of $W_{(2-\lambda,-\lambda)}$ or $W_{(1-\lambda,1-\lambda)}$. Firstly we compute $\phi_0^{(1)}, \phi_1^{(1)}$ and $\phi_2^{(1)}$. By Lemma (2.2)(iii), we have

$$\phi^{(0)}\otimes X_{(2,0)} = \phi_0^{(0)} w_2^{(1)}; \quad \phi^{(0)}\otimes X_{(1,1)} = \phi_0^{(0)} w_1^{(1)}; \quad \phi^{(0)}\otimes X_{(2,0)} = \phi_0^{(0)} w_0^{(1)}.$$

From these formulae we obtain

$$\phi^{(1)} = 8\eta(\mathcal{L}^-)\phi_0^{(0)} w_2^{(1)} + 2(\frac{d}{dt}+\eta(\mathcal{N})+2\lambda\coth 2t)\phi_1^{(0)} w_1^{(1)} - 8\eta(\mathcal{M}^-)\phi_0^{(0)} w_0^{(1)}.$$

In other words, we have

$$\phi_2^{(1)} = 8\eta(\mathcal{L}^-)\phi_0^{(0)}; \quad \phi_1^{(1)} = 2(\frac{d}{dt}+\eta(\mathcal{N})+2\lambda\coth 2t)\phi_0^{(0)};$$
$$\phi_0^{(1)} = -8\eta(\mathcal{M}^-)\phi_0^{(0)}.$$

Using these formulae and Lemma (2.2)(i) we conclude that

$$\phi_0^{(2)} = -8\{8\eta(\mathcal{L}^-\cdot\mathcal{M}^-)+(\frac{d}{dt}+C_{\lambda-1}+4\tanh 2t)(\frac{d}{dt}+C_\lambda)+8\eta(\mathcal{M}^-\cdot\mathcal{L}^-)\}\phi_0^{(0)}.$$

Thus the proposition follows. □

PROPOSITION (4.8 two-dimensional case). *Set* $\tau = \tau_{(\lambda,\lambda-1)}$. *The equation* (b-1) *is equivalent to the system*:

$$\text{(B-1)}: \qquad (\frac{d}{dt}+C_{\lambda-1}+2\tanh 2t)\phi_0^{(0)} + 4\eta(\mathcal{M}^-)\phi_1^{(0)} = 0;$$

$$\text{(B-2)}: \qquad 4\eta(\mathcal{L}^-)\phi_0^{(0)} - (\frac{d}{dt}+C_\lambda+2\tanh 2t)\phi_1^{(0)} = 0.$$

PROOF. Put $\phi^{(1)} := \rho(E_\lambda)\phi^{(0)} = P^{even}(\rho(\nabla^-)\phi^{(0)})$. By Lemma (2.2)(ii), we have

$$\begin{aligned} P^{even}(\phi^{(0)}\otimes X_{(2,0)}) &= \phi_0^{(0)}w_1^{(1)};\\ P^{even}(\phi^{(0)}\otimes X_{(1,1)}) &= \phi_0^{(0)}w_0^{(1)} - \phi_1^{(0)}w_1^{(1)};\\ P^{even}(\phi^{(0)}\otimes X_{(0,2)}) &= -\phi_1^{(0)}w_0^{(1)}, \end{aligned}$$

where $\{w_k^{(1)}|k=0,1\}$ is the standard basis of $W_{(-\lambda+2,-\lambda+1)}$. By virtue of these formulae and Proposition (4.6)(ii) , we get

$$\begin{aligned} \phi^{(1)} &= 4\{2\eta(\mathcal{L}^-) - \tanh 2t(\tau^*\otimes \mathrm{Ad})(Y)\}\phi_0^{(0)}w_1^{(1)}\\ &\quad + 2\{(\frac{d}{dt}+\eta(\mathcal{N}) - 2\coth 2t((\tau^*\otimes\mathrm{Ad})(k_2)-1)+4\tanh 2t)\}\\ &\quad \cdot(\phi_0^{(0)}w_0^{(1)} - \phi_1^{(0)}w_1^{(1)})\\ &\quad + 4\{-2\eta(\mathcal{M}^-) - \tanh 2t(\tau^*\otimes\mathrm{Ad})(X)\}(-\phi_1^{(0)}w_0^{(1)})\\ &= 8\eta(\mathcal{L}^-)\phi_0^{(0)}w_1^{(1)} - 4\tanh 2t\,\phi_0^{(0)}w_0^{(1)}\\ &\quad + 2\{\frac{d}{dt}+\eta(\mathcal{N}) - 2\coth 2t(1-\lambda)+4\tanh 2t\}\phi_0^{(0)}w_0^{(1)}\\ &\quad - 2\{\frac{d}{dt}+\eta(\mathcal{N}) - 2\coth 2t(-\lambda)+4\tanh 2t\}\phi_1^{(0)}w_1^{(1)} \end{aligned}$$

$$+\,8\eta(\mathcal{M}^-)\phi_1^{(0)}w_0^{(1)}+4\tanh 2t\,\phi_1^{(0)}w_1^{(1)}.$$

This proves our assertion. □

PROPOSITION (4.9 $(d+1)$-dimensional case $(d\geqslant 4)$). *Set* $\tau=\tau_{(\lambda_1,\lambda_2)}$ *such that* $d=\lambda_1-\lambda_2\geqslant 4$. *Then we have the following*:
(i) *The equation* (c-1) *is equivalent to the system*:

$$\begin{aligned}(\text{C-1})_k:\quad & 2\eta(\mathcal{M}^-)\phi_{k+2}^{(0)}+(\frac{d}{dt}+C_{k+\lambda_2+1}+(d+2)\tanh 2t)\phi_{k+1}^{(0)}\\ &-2\eta(\mathcal{L}^-)\phi_k^{(0)}=0\qquad(0\leqslant k\leqslant d-2).\end{aligned}$$

(ii) *The equation* (c-2) *is equivalent to the system*:

$$\begin{aligned}(\text{C-2})_k:\quad & 2\eta(\mathcal{L}^+)\phi_{k+2}^{(0)}+(\frac{d}{dt}-C_{k+\lambda_2+1}+(d+2)\tanh 2t)\phi_{k+1}^{(0)}\\ &-2\eta(\mathcal{M}^+)\phi_k^{(0)}=0\qquad(0\leqslant k\leqslant d-2).\end{aligned}$$

(iii) *The equation* (c-3) *is equivalent to the system*:

$$\begin{aligned}(\text{C-3})_k:\quad & 4(d-k)\eta(\mathcal{M}^-)\phi_{k+1}^{(0)}+(d-2k)(\frac{d}{dt}+C_{k+\lambda_2}+2\tanh 2t)\phi_k^{(0)}\\ &+4k\,\eta(\mathcal{L}^-)\phi_{k-1}^{(0)}=0\qquad(0\leqslant k\leqslant d).\end{aligned}$$

PROOF. We prove (ii) only. The proofs of (i) and (iii) are similar. Set $\phi^{(1)}:=P^{down}(\rho(\nabla^+))\phi^{(0)}$. From Lemma (2.2)(ii), we have

$$P^{down}(\phi^{(0)}\otimes X_{(-2,0)})=\sum_{k=0}^{d}\phi_k^{(0)}w_{k-2}^{(1)};$$
$$P^{down}(\phi^{(0)}\otimes X_{(-1,-1)})=\sum_{k=0}^{d}2\phi_k^{(0)}w_{k-1}^{(1)};$$
$$P^{down}(\phi^{(0)}\otimes X_{(0,-2)})=\sum_{k=0}^{d}\phi_k^{(0)}w_k^{(1)},$$

where $\{w_k^{(1)} | 0 \leqslant k \leqslant d-2\}$ is the standard basis of $W_{(-\lambda_2-2,-\lambda_1)}$. Combining these formulae with Proposition (4.6)(i) we have

$$\begin{aligned}
\phi^{(1)} &= 4\{2\eta(\mathcal{L}^+) + \tanh 2t(\tau^* \otimes \mathrm{Ad})(X)\}(\sum_{k=0}^{d} \phi_k^{(0)} w_{k-2}^{(1)}) \\
&+ 2\{\frac{d}{dt} - \eta(\mathcal{N}) + 2\coth 2t((\tau^* \otimes \mathrm{Ad})(k_2) + 1) + 4\tanh 2t\} \\
&\cdot (\sum_{k=0}^{d} 2\phi_k^{(0)} w_{k-1}^{(1)}) \\
&+ 4\{-2\eta(\mathcal{M}^+) + \tanh 2t(\tau^* \otimes \mathrm{Ad})(Y)\}(\sum_{k=0}^{d} \phi_k^{(0)} w_k^{(1)}) \\
&= \sum_{k=0}^{d} [\, 8\eta(\mathcal{L}^+)\phi_k^{(0)} w_{k-2}^{(1)} + 4(k-1)\tanh 2t\, \phi_k^{(0)} w_{k-1}^{(1)} \\
&+ 4\{\frac{d}{dt} - \eta(\mathcal{N}) + 2\coth 2t(-k-\lambda_2) + 4\tanh 2t\}\phi_k^{(0)} w_{k-1}^{(1)} \\
&- 8\eta(\mathcal{M}^+)\phi_k^{(0)} w_k^{(1)} + 4(d-k-1)\tanh 2t \phi_k^{(0)} w_{k-1}^{(1)} \,] \\
&= \sum_{k=0}^{d-2} \{8\eta(\mathcal{L}^+)\phi_{k+2}^{(0)} + 4(\frac{d}{dt} - C_{k+\lambda_2+1} + (d+2)\tanh 2t)\phi_{k+1}^{(0)} \\
&\qquad - 8\eta(\mathcal{M}^+)\phi_k^{(0)}\} w_k^{(1)}.
\end{aligned}$$

This proves (ii). □

4.5 The radial part of the Casimir operator

We are going to write down the differential equations arising from the Casimir operator in terms of the coefficient functions $\phi_k^{(0)}$.

PROPOSITION (4.10). (i) *If* $\tau = \tau_{(\lambda,\lambda)}$, *then the equation* (a-2) *is equivalent to* (A-2) *below*:

$$\begin{aligned}
\{(&\frac{d}{dt} - C_{\lambda-1} + 4\tanh 2t)(\frac{d}{dt} + C_\lambda) \\
&+ 8\eta(\mathcal{L}^+ \cdot \mathcal{L}^- + \mathcal{M}^+ \cdot \mathcal{M}^-) + 4\lambda^2 - 12\lambda\}\phi_0^{(0)} = \chi_\pi(L)\phi_0^{(0)}.
\end{aligned}$$

(ii) *If* $\tau = \tau_{(\lambda,\lambda-1)}$, *then the equation* (b-2) *is equivalent to the system*:

$$\begin{aligned}
\text{(B-3)}: \quad & \{(\frac{d}{dt} - C_{\lambda-2} + 2\tanh 2t)(\frac{d}{dt} + C_{\lambda-1} + 2\tanh 2t) \\
& + 8\eta(\mathcal{L}^+ \cdot \mathcal{L}^- + \mathcal{M}^- \cdot \mathcal{M}^+) \\
& + 4\lambda^2 - 8\lambda - 6\}\phi_0^{(0)} - 4\tanh 2t \cdot \eta(\mathcal{L}^+ + \mathcal{M}^-)\phi_1^{(0)} = \chi_\pi(L)\phi_0^{(0)}; \\
\text{(B-4)}: \quad & \{(\frac{d}{dt} - C_{\lambda-1} + 2\tanh 2t)(\frac{d}{dt} + C_{\lambda} + 2\tanh 2t) \\
& + 8\eta(\mathcal{L}^- \cdot \mathcal{L}^+ + \mathcal{M}^+ \cdot \mathcal{M}^-) \\
& + 4\lambda^2 - 8\lambda - 6\}\phi_1^{(0)} + 4\tanh 2t \cdot \eta(\mathcal{L}^- + \mathcal{M}^+)\phi_0^{(0)} = \chi_\pi(L)\phi_1^{(0)}.
\end{aligned}$$

PROOF. Our first task is to express the Casimir element as a linear combination of such elements as $\mathrm{Ad}(a_{-t})(U_1) \cdot H_1^m \cdot U_2$ $(U_1 \in U(\mathfrak{r}_{\mathbb{C}}), m \in \mathbb{Z}_{\geqslant 0}, U_2 \in U(\mathfrak{k}_{\mathbb{C}}))$. Using Proposition (1.1)(the generalized Cartan decomposition), we have

$$\begin{aligned}
X_{(2,0)}X_{(-2,0)} &= (2\,\mathrm{Ad}(a_{-t})\mathcal{L}^+ - \tanh 2t \cdot X)(2\,\mathrm{Ad}(a_{-t})\mathcal{L}^- + \tanh 2t \cdot Y) \\
&= 4\,\mathrm{Ad}(a_{-t})(\mathcal{L}^+ \cdot \mathcal{L}^-) - 2\tanh 2t\, X \cdot (\mathrm{Ad}(a_{-t})\mathcal{L}^-) \\
&+ 2\tanh 2t(\mathrm{Ad}(a_{-t})\mathcal{L}^+) \cdot Y - \tanh^2 2t\, X \cdot Y \\
&= 4\,\mathrm{Ad}(a_{-t})(\mathcal{L}^+ \cdot \mathcal{L}^-) + 2\tanh 2t(\mathrm{Ad}(a_{-t})\mathcal{L}^+) \cdot Y \\
&- \tanh^2 2t\, X \cdot Y \\
&- \tanh 2t\{2(\mathrm{Ad}(a_{-t})\mathcal{L}^-) \cdot X + [X, X_{(-2,0)} - \tanh 2t \cdot Y]\} \\
&= 4\,\mathrm{Ad}(a_{-t})(\mathcal{L}^+ \cdot \mathcal{L}^-) + 2\tanh 2t(\mathrm{Ad}(a_{-t})\mathcal{L}^+) \cdot Y \\
&- \tanh^2 2t\, X \cdot Y \\
&- \tanh 2t\{2(\mathrm{Ad}(a_{-t})\mathcal{L}^-) \cdot X - X_{(-1,-1)} - \tanh 2t(k_1 - k_2)\}.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
X_{(0,2)}X_{(0,-2)} &= 4\,\mathrm{Ad}(a_{-t})(\mathcal{M}^+ \cdot \mathcal{M}^-) - 2\tanh 2t(\mathrm{Ad}(a_{-t})\mathcal{M}^+) \cdot X \\
&- \tanh^2 2t\, Y \cdot X \\
&+ \tanh 2t\{2\,\mathrm{Ad}(a_{-t})\mathcal{M}^- \cdot Y + X_{(-1,-1)} - \tanh 2t(k_1 - k_2)\}; \\
X_{(1,1)}X_{(-1,-1)} &= -\mathrm{Ad}(a_{-t})(\mathcal{N}^2) + H_1^2 - 4\coth^2 2t \cdot k_2^2
\end{aligned}$$

$$- 4\coth 2t(\mathrm{Ad}(a_{-t})\mathcal{N}) \cdot k_2 + 2k_1 + 2k_2 + 2\coth 2t \cdot H_1.$$

Getting these formulae together, we obtain

$$\begin{aligned} L = {} & 2k_1^2 + 2k_2^2 - 6k_1 + 6k_2 - 4\coth^2 2t \cdot k_2^2 - 2\tanh^2 2t(X \cdot Y + Y \cdot X) \\ & + 4X \cdot Y + \mathrm{Ad}(a_{-t})(8\mathcal{L}^+ \cdot \mathcal{L}^- + 8\mathcal{M}^+ \cdot \mathcal{M}^- - \mathcal{N}^2 + 4\tanh 2t \cdot \mathcal{N}) \\ & + 4\frac{\tanh 2t}{\cosh 2t}\{\mathrm{Ad}(a_{-t})(n_1^+ - n_2^-) \cdot Y - \mathrm{Ad}(a_{-t})(n_1^- - n_2^+) \cdot X)\} \\ & - 4\coth 2t\, \mathrm{Ad}(a_{-t})\mathcal{N} \cdot k_2 + H_1^2 + (4\tanh 2t + 2\coth 2t)H_1. \end{aligned}$$

Now we can compute the A-radial parts of the Casimir operator by using Lemma (4.5). □

§5. Reduction of Differential Equations

In this section we reduce the systems of differential equations constructed in the previous section to suitable ones for our later computations in §6.

5.1 The one-dimensional case

First we treat the case of one-dimensional corner K-types.

Proposition (5.1). *The system of differential equations* (A-1) (A-2) *is equivalent to the system*:

$$\begin{aligned} \text{(A-3)}: \quad & \{(\frac{d}{dt} + 2\tanh 2t - \frac{C_{\lambda-1}}{\cosh 2t})(\frac{d}{dt} + 2\tanh 2t + C_\lambda) \\ & + c(t)(\frac{8}{\cosh 2t}\eta(n_1^+ n_1^- + n_2^- n_2^+) + 2(\lambda^2 - \nu_J^2)) \\ & - \frac{16 s(t)}{\cosh 2t}\eta(n_1^+ n_2^+)\}\phi_0^{(0)} = 0, \\ \text{(A-4)}: \quad & \{(\frac{d}{dt} + 2\tanh 2t + \frac{C_{\lambda-1}}{\cosh 2t})(\frac{d}{dt} + 2\tanh 2t + C_\lambda) \\ & + s(t)(\frac{8}{\cosh 2t}\eta(n_1^- n_1^+ + n_2^+ n_2^-) - 2(\lambda^2 - \nu_J^2)) \\ & - \frac{16 c(t)}{\cosh 2t}\eta(n_1^- n_2^-)\}\phi_0^{(0)} = 0. \end{aligned}$$

Proof. To get (A-3) or (A-4) from (A-1) and (A-2), it is enough to compute $s(t)$ (A-1) $+c(t)$ (A-2) or $c(t)$ (A-1) $+s(t)$ (A-2), respectively. The converse implication is now trivial. □

5.2 The two-dimensional case

Next we proceed to the case of two-dimensional corner K-types.

Proposition (5.2). *Suppose that* $\phi^{(0)} \in C^{\infty}_{\eta,\tau}(A)$ *satisfies the system of the differential equation* (B-1), (B-2), (B-3), *and* (B-4). *Then we have the following formulae from* (B-5) *to* (B-8).

$$\begin{aligned}
\text{(B-5)}:\quad & \{(\frac{d}{dt} + 3\tanh 2t - \frac{C_{\lambda-2}}{\cosh 2t})(\frac{d}{dt} + 2\tanh 2t + C_{\lambda-1}) \\
& + c(t)(\frac{-8}{\cosh 2t}\eta(n_1^+ n_1^- - n_2^- n_2^+) + 2(\lambda-1)^2 - 2\nu_J^2)\}\phi_0^{(0)} \\
& + \frac{4}{\cosh 2t}(\frac{d}{dt} + C_\lambda + \tanh 2t - 2\tanh t)\eta(n_1^+)\phi_1^{(0)} = 0, \\
\text{(B-6)}:\quad & \{(\frac{d}{dt} + 3\tanh 2t + \frac{C_{\lambda-1}}{\cosh 2t})(\frac{d}{dt} + 2\tanh 2t + C_{\lambda}) \\
& + s(t)(\frac{-8}{\cosh 2t}\eta(n_1^- n_1^+ - n_2^+ n_2^-) - 2(\lambda-1)^2 + 2\nu_J^2)\}\phi_1^{(0)} \\
& - \frac{4}{\cosh 2t}(\frac{d}{dt} + C_{\lambda-1} + \tanh 2t + 2\tanh t)\eta(n_1^-)\phi_0^{(0)} = 0, \\
\text{(B-7)}:\quad & \{(\frac{d}{dt} + 3\tanh 2t - \frac{C_{\lambda-1}}{\cosh 2t})(\frac{d}{dt} + 2\tanh 2t + C_{\lambda}) \\
& + c(t)(\frac{8}{\cosh 2t}\eta(n_1^- n_1^+ - n_2^+ n_2^-) + 2(\lambda-1)^2 - 2\nu_J^2)\}\phi_1^{(0)} \\
& + \frac{4}{\cosh 2t}(\frac{d}{dt} + C_{\lambda-1} + \tanh 2t + 2\coth t)\eta(n_2^+)\phi_0^{(0)} = 0, \\
\text{(B-8)}:\quad & \{(\frac{d}{dt} + 3\tanh 2t + \frac{C_{\lambda-2}}{\cosh 2t})(\frac{d}{dt} + 2\tanh 2t + C_{\lambda-1}) \\
& + s(t)(\frac{8}{\cosh 2t}\eta(n_1^+ n_1^- - n_2^- n_2^+) - 2(\lambda-1)^2 + 2\nu_J^2)\}\phi_0^{(0)} \\
& - \frac{4}{\cosh 2t}(\frac{d}{dt} + C_\lambda + \tanh 2t - 2\coth t)\eta(n_2^-)\phi_1^{(0)} = 0.
\end{aligned}$$

Proof. Computing $4\eta(\mathcal{L}^+)\cdot$(B-2) $-$ (B-3) (resp. $4\eta(\mathcal{M}^+)\cdot$(B-1) $-$ (B-4)), we obtain the following equation (B-9) (resp. (B-10)):

$$\begin{aligned}
\text{(B-9)}:\ & \{(\frac{d}{dt} - C_{\lambda-2} + 2\tanh 2t)(\frac{d}{dt} + C_{\lambda-1} + 2\tanh 2t) \\
& + \eta(-8\mathcal{L}^+\cdot\mathcal{L}^+ + 8\mathcal{M}^+\cdot\mathcal{M}^-) + 4\lambda^2 - 8\lambda - 6 - \chi_\pi(L)\}\phi_0^{(0)}
\end{aligned}$$

$$+ 4\{\eta(\mathcal{L}^+)(\frac{d}{dt} + C_\lambda + 2\tanh 2t) - \tanh 2t\, \eta(\mathcal{L}^+ + \mathcal{M}^-)\}\phi_1^{(0)} = 0;$$

$$\text{(B-10)}: \ \{(\frac{d}{dt} - C_{\lambda-1} + 2\tanh 2t)(\frac{d}{dt} + C_\lambda + 2\tanh 2t)$$
$$+ \eta(8\mathcal{L}^- \cdot \mathcal{L}^+ - 8\mathcal{M}^+ \cdot \mathcal{M}^-) + 4\lambda^2 - 8\lambda - 6 - \chi_\pi(L)\}\phi_0^{(0)}$$
$$- 4\{\eta(\mathcal{M}^+)(\frac{d}{dt} + C_{\lambda-1} + 2\tanh 2t) - \tanh 2t\, \eta(\mathcal{L}^- + \mathcal{M}^-)\}\phi_0^{(0)}.$$

Now it is easy to check that

$$\text{(B-5)} = s(t)(\frac{d}{dt} + C_\lambda + 2\tanh 2t - 2\tanh t)\,\text{(B-1)} - c(t)\,\text{(B-9)};$$
$$\text{(B-6)} = c(t)(\frac{d}{dt} + C_{\lambda-1} + 2\tanh 2t + 2\tanh t)\,\text{(B-2)} + s(t)\,\text{(B-10)};$$
$$\text{(B-7)} = s(t)(\frac{d}{dt} + C_{\lambda-1} + 2\tanh 2t + 2\coth t)\,\text{(B-2)} + c(t)\,\text{(B-10)};$$
$$\text{(B-8)} = c(t)(\frac{d}{dt} + C_\lambda + 2\tanh 2t - 2\coth t)\,\text{(B-1)} - s(t)\,\text{(B-9)}\,. \ \square$$

5.3 The $(d+1)$-dimensional case $(d \geqslant 4)$

Finally we treat the case of the large discrete series representations.

Proposition (5.3). *The system of the differential equations* $\text{(C-1)}_{0\leqslant k\leqslant d-2}, \text{(C-3)}_{0\leqslant k\leqslant d}$ *is equivalent to the system of differential equations* $\text{(C-4)}_{0\leqslant k\leqslant d-1}, \text{(C-5)}_{0\leqslant k\leqslant d-1}$ *below*: $\text{(C-4)}_{k+1}\,(-1 \leqslant k \leqslant d-2)$:

$$4\eta(\mathcal{M}^-)\phi_{k+2}^{(0)} + (\frac{d}{dt} + C_{k+\lambda_2+1} + 2(k+2)\tanh 2t)\phi_{k+1}^{(0)} = 0,$$

$\text{(C-5)}_{k-1}\,(1 \leqslant k \leqslant d)$:

$$(\frac{d}{dt} + C_{k+\lambda_2} + 2(d-k+1)\tanh 2t)\phi_k^{(0)} - 4\eta(\mathcal{L}^-)\phi_{k-1}^{(0)} = 0.$$

Proof. We have only to check

$$\text{(C-4)}_{k+1} = 2(k+1)\,\text{(C-1)}_k + \text{(C-3)}_{k+1} \qquad (0 \leqslant k \leqslant d-2);$$
$$\text{(C-5)}_{k-1} = 2(d-k)\,\text{(C-1)}_{k-1} - \text{(C-3)}_k \qquad (1 \leqslant k \leqslant d-1);$$

$$\text{and}\quad (\text{C-4})_0 = (\text{C-3})_0, \qquad (\text{C-5})_{d-1} = (\text{C-3})_d \,. \ \square$$

PROPOSITION (5.4). *Suppose that $\phi^{(0)} \in C^\infty_{\eta,\tau}(A)$ satisfies the systems of differential equations $(\text{C-1})_{0\leqslant k\leqslant d-2}, (\text{C-2})_{0\leqslant k\leqslant d-2}$, and $(\text{C-3})_{0\leqslant k\leqslant d}$. Then the following formulae from $(\text{C-6})_k$ to (C-11) hold.*

$$\begin{aligned}(\text{C-6})_{2\leqslant k\leqslant d}: \ & \frac{2}{\cosh 2t}\eta(n_1^+)\phi_k^{(0)} + (\frac{d}{dt} + (d+2)\tanh 2t - \frac{C_{k+\lambda_2-1}}{\cosh 2t})\phi_{k-1}^{(0)} \\ & + \frac{2}{\cosh 2t}\eta(n_2^+)\phi_{k-2}^{(0)} = 0, \\ (\text{C-7})_{0\leqslant k\leqslant d-2}: \ & \frac{-2}{\cosh 2t}\eta(n_2^-)\phi_{k+2}^{(0)} + (\frac{d}{dt} + (d+2)\tanh 2t + \frac{C_{k+\lambda_2+1}}{\cosh 2t})\phi_{k+1}^{(0)} \\ & + \frac{-2}{\cosh 2t}\eta(n_1^-)\phi_k^{(0)} = 0,\end{aligned}$$

$$\begin{aligned}(\text{C-8}) : & \{(\frac{d}{dt} + (d+2)\tanh 2t - \frac{C_{\lambda_2-1}}{\coth 2t})(\frac{d}{dt} + 2\tanh 2t + C_{\lambda_2}) \\ & \qquad + \frac{8c(t)}{\cosh 2t}\eta(n_2^- n_2^+)\}\phi_0^{(0)} \\ & + 4\{s(t)(\frac{d}{dt} + (d+2)\tanh 2t - \frac{C_{\lambda_2-1}}{\cosh 2t}) - \frac{\tanh 2t}{\cosh 2t}\}\eta(n_1^+)\phi_1^{(0)} \\ & + \frac{8c(t)}{\cosh 2t}\eta(n_1^+ n_2^-)\phi_2^{(0)} = 0, \\ (\text{C-9}) : & \{(\frac{d}{dt} + (d+2)\tanh 2t + \frac{C_{\lambda_1-1}}{\cosh 2t})(\frac{d}{dt} + 2\tanh 2t + C_{\lambda_1}) \\ & \qquad + \frac{8s(t)}{\cosh 2t}\eta(n_2^+ n_2^-)\}\phi_d^{(0)} \\ & - 4\{c(t)(\frac{d}{dt} + (d+2)\tanh 2t + \frac{C_{\lambda_1-1}}{\cosh 2t}) - \frac{\tanh 2t}{\cosh 2t}\}\eta(n_1^-)\phi_{d-1}^{(0)} \\ & + \frac{8s(t)}{\cosh 2t}\eta(n_1^- n_2^+)\phi_{d-2}^{(0)} = 0, \\ (\text{C-10}) : & \{(\frac{d}{dt} + (d+2)\tanh 2t - \frac{C_{\lambda_1-1}}{\cosh 2t})(\frac{d}{dt} + 2\tanh 2t + C_{\lambda_1}) \\ & \qquad + \frac{8c(t)}{\cosh 2t}\eta(n_1^- n_1^+)\}\phi_d^{(0)} \\ & + 4\{s(t)(\frac{d}{dt} + (d+2)\tanh 2t - \frac{C_{\lambda_1-1}}{\cosh 2t}) - \frac{\tanh 2t}{\cosh 2t}\}\eta(n_2^+)\phi_{d-1}^{(0)}\end{aligned}$$

$$+\frac{8c(t)}{\cosh 2t}\eta(n_1^- n_2^+)\phi_{d-2}^{(0)}=0,$$

$$(\text{C-11}):\{(\frac{d}{dt}+(d+2)\tanh 2t+\frac{C_{\lambda_2-1}}{\coth 2t})(\frac{d}{dt}+2\tanh 2t+C_{\lambda_2}) + \frac{8s(t)}{\cosh 2t}\eta(n_1^+ n_1^-)\}\phi_0^{(0)}$$
$$-4\{c(t)(\frac{d}{dt}+(d+2)\tanh 2t+\frac{C_{\lambda_2-1}}{\cosh 2t})-\frac{\tanh 2t}{\cosh 2t}\}\eta(n_2^-)\phi_1^{(0)}$$
$$+\frac{8s(t)}{\cosh 2t}\eta(n_1^+ n_2^-)\phi_2^{(0)}=0.$$

Proof. Direct computations show that

$$(\text{C-6})_k=-s(t)((\text{C-4})_{k-1}+(\text{C-5})_{k-2})+2c(t)\,(\text{C-2})_{k-2};$$
$$(\text{C-7})_k=c(t)((\text{C-4})_{k+1}+(\text{C-5})_k)-2s(t)\,(\text{C-2})_k;$$

$$(\text{C-8})=(\frac{d}{dt}+(d+2)\tanh 2t-\frac{C_{\lambda_2-1}}{\cosh 2t})\,(\text{C-4})_0+4c(t)\eta(n_2^-)\cdot(\text{C-6})_2;$$
$$(\text{C-9})=(\frac{d}{dt}+(d+2)\tanh 2t+\frac{C_{\lambda_1-1}}{\cosh 2t})\,(\text{C-5})_{d-1}-4s(t)\eta(n_2^+)\cdot(\text{C-7})_{d-2};$$
$$(\text{C-10})=(\frac{d}{dt}+(d+2)\tanh 2t-\frac{C_{\lambda_1-1}}{\cosh 2t})\,(\text{C-5})_{d-1}+4c(t)\eta(n_1^-)\cdot(\text{C-6})_d;$$
$$(\text{C-11})=(\frac{d}{dt}+(d+2)\tanh 2t+\frac{C_{\lambda_2-1}}{\cosh 2t})\,(\text{C-4})_0-4s(t)\eta(n_1^+)\cdot(\text{C-7})_0\,.\;\square$$

§6. Explicit Formulae and the Main Theorems

Throughout this section, we retain the assumption that π is a half-size standard representation of G and $\tau=\tau_{(\lambda_1,\lambda_2)}$ is its corner K-type (see Definition (2.5)). In what follows, we denote $\dim_{\mathbb{C}} W_\tau=\lambda_1-\lambda_2$ by d_π or simply by d. When $d=0$ or 1, we also write $\tau=\tau_{(\lambda,\lambda)}$ or $\tau=\tau_{(\lambda,\lambda-1)}$. We use the standard bases $\{v_i\}$ and $\{w_k\}$ introduced in §2 in order to represent the spherical functions $\phi\in C^\infty_{\eta,\tau}(A)$.

In this section, we write down concretely the differential equations satisfied by the spherical functions of type (π,η,τ), where η is a tensor product of (limits of) discrete series representation or the trivial representation of $SL(2,\mathbb{R})$. By investigating these differential equations, we obtain the upper bounds of the dimensions of the algebraic intertwining spaces $\mathrm{Hom}_{(\mathfrak{g},K)}(\pi^0, C_\eta^\infty(R\backslash G)^0)$. In some cases, we write down the solutions of these differential equations. Let ${}_2F_1(a,b,c;z)$ be the Gaussian hypergeometric function which is given by the series expansion around the origin

$$ {}_2F_1(a,b,c;z) := \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}, $$

for $a, b \in \mathbb{C}$ and $c \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$.

Roughly speaking, types of the differential equations depend on the representation η of R and not on the representation π of G. Thus we separate our problem according to the representation η of R. Here is the first part of the main results:

THEOREM (6.1). *Define subsets $D_>^{(+,+)}$ and $D_<^{(+,+)}$ of $\hat{R}$ by*

$$ D_>^{(+,+)} := \{D_{l_1} \boxtimes D_{l_2}; l_1 - d \geqslant l_2 > 0\}, $$
$$ D_<^{(+,+)} := \{D_{l_1} \boxtimes D_{l_2}; l_2 - d \geqslant l_1 > 0\}. $$

Suppose that η belongs to $D_>^{(+,+)}$ or $D_<^{(+,+)}$ and that $\phi \in C_{\eta,\tau}^\infty(A)$ is a spherical function of type (π, η, τ). Then we have the following assertions:
(1) (i) *Suppose that $\eta \in D_>^{(+,+)}$. Then $\phi(t) \in C_{\eta,\tau}^\infty(A)$ can be written as*

$$ (*1) \quad \phi(t) = \sum_{k=0}^{d} \sum_{i=0}^{\infty} \phi_{i,k}(t) v_i \otimes v_{j(i)+k} \otimes w_k, $$

where $j(i) := (l_1 - l_2 - d)/2 + i$.
(ii) *Suppose that $\eta \in D_<^{(+,+)}$. Then $\phi(t) \in C_{\eta,\tau}^\infty(A)$ can be written as*

$$ (*2) \quad \phi(t) = \sum_{k=0}^{d} \sum_{j=0}^{\infty} \phi_{j,k}(t) v_{i(j)-k} \otimes v_j \otimes w_k, $$

where $i(j) := (-l_1 + l_2 + d)/2 + j$.
(2) *Define a C^∞-function $u(t)$ on $\mathbb{R}$ by $u(t) := \phi_{0,0}(t)$ or $\phi_{0,d}(t)$ according as $\eta \in D_>^{(+,+)}$ or $D_<^{(+,+)}$. Then $\phi(t)$ is uniquely determined by $u(t)$.*

(3) *$u(t)$ satisfies the following differential equation*:

$$
\text{(D-1)}: \quad
\begin{aligned}
&\{(\frac{d}{dt} + (d+2)\tanh 2t \\
&\qquad - \frac{1}{\cosh 2t}(c_1 \tanh t - c_2 \coth t + 2(c_3 - 1)\coth 2t)) \\
&(\frac{d}{dt} + 2\tanh 2t + c_1 \tanh t - c_2 \coth t + 2c_3 \coth 2t) \\
&+ (\frac{1}{\cosh 2t} + 1)(\frac{A}{\cosh 2t} + B)\}u = 0,
\end{aligned}
$$

with the constants:

$$(c_1, c_2, c_3) := (l_1, l_1 - d, \lambda_2) \ (\textit{resp.}\ (l_2 - d, l_2, \lambda_1)) \ \textit{if}\ \eta \in D_{>}^{(+,+)} (\textit{resp.}\ D_{<}^{(+,+)}).$$

Here A is given by

$$A := \begin{cases} (-l_1 + l_2 + d - 2)(l_1 + l_2 - d) & \textit{if}\ \eta \in D_{>}^{(+,+)}, \\ (l_1 - l_2 + d - 2)(l_1 + l_2 - d) & \textit{if}\ \eta \in D_{<}^{(+,+)}, \end{cases}$$

and $B := {\lambda_2}^2 - \nu_J^2$ (resp. 0) if $d = 0, 1$ (resp. $d \geqslant 4$).
(4) *Put $\epsilon(\pi, \eta) := l_1 + \lambda_1$ (resp. $l_2 + \lambda_1$) if $\eta \in D_{>}^{(+,+)}$ (resp. $D_{<}^{(+,+)}$). Then $u(t)$ is an even or odd function on $\mathbb{R}$ according as $\epsilon(\pi, \eta) \equiv 0$ or $1 \pmod 2$.*
(5) *The differential equation* (D-1) *has a unique, up to constant multiple, C^∞-solution $u(t)$ satisfying the parity condition in* (4)*. Further the unique C^∞-solution $u(t)$ is given by*

$$
\begin{aligned}
u(t) &= (\tanh t)^{|\kappa|}(\tanh^2 t + 1)^{-\mu_0}(\tanh t \cdot \tanh 2t - 1)^{\mu_1} \\
&\quad \times {}_2F_1(a, b, 1 + |\kappa|; \tanh t \cdot \tanh 2t),
\end{aligned}
$$

with the constants

$$(\kappa, \mu_0) := \begin{cases} (l_1 - \lambda_1, \max\{(-\lambda_1 - \lambda_2)/2, (-2l_1 + d)/2\}) & \textit{if}\ \eta \in D_{>}^{(+,+)}; \\ (l_2 - \lambda_1, \max\{(-\lambda_1 - \lambda_2)/2, (-2l_2 + d)/2\}) & \textit{if}\ \eta \in D_{<}^{(+,+)}; \end{cases}$$

and $\mu_1 := (\lambda_1 + 2 - \nu_J)/2$ (*resp.* $(d+2)/2$) *if* $d = 0, 1$ (*resp.* $d \geqslant 4$). *Here* a, b *are given by*

$$(a,b) := \begin{cases} ((l_1 - l_2 - d)/2 + \mu_0 + \mu_1, (l_1 + l_2 - d - 2)/2 + \mu_0 + \mu_1) \\ \hfill \text{if } \eta \in D_{>}^{(+,+)}; \\ ((-l_1 + l_2 - d)/2 + \mu_0 + \mu_1, (l_1 + l_2 - d - 2)/2 + \mu_0 + \mu_1) \\ \hfill \text{if } \eta \in D_{<}^{(+,+)}. \end{cases}$$

Proof. (1) We prove (i). It is easily checked that

$$Z_{R\cap K}(\mathfrak{a}) = \left\{ m_\theta := \left(\begin{array}{cc|cc} \cos\theta & & -\sin\theta & \\ & \cos\theta & & \sin\theta \\ \hline \sin\theta & & \cos\theta & \\ & -\sin\theta & & \cos\theta \end{array} \right) \;\middle|\; \theta \in \mathbb{R} \right\}.$$

We write ϕ as

$$\phi(a_t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{d} \phi_{i,j,k}(a_t) v_i \otimes v_j \otimes w_k,$$

with the standard basis of V_η and W_τ^*. Then the condition (a) in the definition of $C_{\eta,\tau}^\infty(A)$ implies that the relation

$$\exp(\sqrt{-1}(-l_1 - 2i + l_2 + 2j + d - 2k)\theta)\phi_{i,j,k}(a_t) = \phi_{i,j,k}(a_t),$$

hold for any θ, i, j, k. Hence $\phi_{i,j,k}(a_t)$ is identically zero unless $-l_1 - 2i + l_2 + 2j + d - 2k = 0$. This proves (i). The case of (ii) can be proved in the same way.

(2) Suppose at first $\eta \in D_{>}^{(+,+)}$. (i) Suppose $d = 0$. If we apply (A-4) to ϕ, we have

$$\begin{aligned} &\{(\frac{d}{dt} + 2\tanh 2t + \frac{1}{\cosh 2t}((c_1 + 2i)\tanh t - (c_2 + 2j(i))\coth t \\ &\qquad + 2(\lambda - 1)\coth 2t)) \\ &(\frac{d}{dt} + 2\tanh 2t + (c_1 + 2i)\tanh t - (c_2 + 2j(i))\coth t + 2\lambda\coth 2t) \end{aligned}$$

$$+ s(t)(\frac{8}{\cosh 2t}((-i-1)(l_1+i)+(-j(i))(l_2+j(i)-1))$$
$$- 2(\lambda^2-\nu_J^2))\}\phi_{i,0}(t)$$
$$=\frac{16c(t)}{\cosh 2t}(-i-1)(l_1+i)(-j(i)-1)(l_2+j(i))\phi_{i+1,0}(t).$$

Thus $\phi_{i+1,0}(t)$ is determined recursively by $u(t) = \phi_{0,0}(t)$. (ii) Suppose $d = 1$. Then we can notice that $\phi_{i+1,1}$ is determined by $\phi_{i,1}$ and $\phi_{i+1,0}$ by means of (B-1) and that $\phi_{i+1,0}$ is determined by $\phi_{i,0}$ and $\phi_{i,1}$ by means of (B-2). Thus ϕ is determined by $u = \phi_{0,0}$. (iii) Suppose $d \geqslant 4$. Then we can notice that $\phi_{i+1,k+1}$ is determined by $\phi_{i,k+1}$ and $\phi_{i+1,k}$ by means of $(\text{C-4})_k(i \geqslant 0,\ 0 \leqslant k \leqslant d-1)$ and that $\phi_{i+1,k-1}$ is determined by $\phi_{i,k-1}$ and $\phi_{i,k}$ by means of $(\text{C-5})_k(i \geqslant 0, 0 \leqslant k \leqslant d-2)$. Hence ϕ is determined by $u = \phi_{0,0}$.

Next we suppose that $\eta \in D_<^{(+,+)}$. Then we use (A-4), (B-1) and (B-2), or $(\text{C-4})_k$ and $(\text{C-5})_k$, according as $d = 0, d = 1$, or $d \geqslant 4$ to prove our assertion.

(3) Suppose that $\eta \in D_>^{(+,+)}$. In order to get the differential equation (D-1), it is enough to apply (A-3), (B-5) or (C-8) to ϕ according as $d = 0$, $d = 1$ or $d \geqslant 4$.

Next we suppose that $\eta \in D_<^{(+,+)}$. Then we use (A-3), (B-7) or (C-10) according as $d = 0$, $d = 1$ or $d \geqslant 4$.

(4) As a non-trivial representative of $W_{R\cap K}(\mathfrak{a}) := N_{R\cap K}(\mathfrak{a})/Z_{R\cap K}(\mathfrak{a}) \cong \{\pm 1\}$, we can take $m^0 := \mathrm{diag}(-1,1,-1,1)$. If $\eta \in D_>^{(+,+)}$, from the condition (b) in the definition of $C_{\eta,\tau}^\infty(A)$, we conclude that

$$(-1)^{l_1+\lambda_1-k}\phi_{i,k}(a_t) = \phi_{i,k}(a_{-t}).$$

This proves our assertion in this case. The case of $\eta \in D_<^{(+,+)}$ is the same.

(5) (i) The case of $\eta \in D_>^{(+,+)}$. Suppose that $d = 0$ or 1 and $\epsilon(\pi,\eta) \equiv 0 \pmod 2$. We make changes of variables from t to $z = (\tanh t)^2$. Then $u(z)$ satisfies the following differential equation of the Fuchsian type with (at most) four regular singularities:

$$(\natural)\quad \{(\frac{d}{dz})^2 + \sum_{a=-1,0,1}\frac{\alpha_a}{z-a}\frac{d}{dz} + \sum_{a=-1,0,1}(\frac{\beta_a}{(z-a)^2}+\frac{\gamma_a}{(z-a)})\}u = 0,$$

with

$$\alpha_{-1} = 2 - c_2, \qquad \alpha_0 = 1, \qquad \alpha_1 = -1 - \lambda_1,$$
$$\beta_{-1} = (-A - 4c_2)/4, \qquad \beta_0 = -(c_2 - \lambda_2)^2/4,$$
$$\beta_1 = (4 + B + 4d + d^2 + 4\lambda_2 + 2d\lambda_2)/4,$$

$$\begin{aligned}\gamma_{-1} &= (-4 - 3A - B - 4c_2 - 4c_2^2 + 4d - 2c_2 d + 4\lambda_2)/8,\\ \gamma_0 &= (4 + A + B + c_2 + c_2^2 + d + \lambda_2 - \lambda_2^2)/2,\\ \gamma_1 &= (-12 - A - 3B - 8d + 2c_2 d - 8\lambda_2 + 4\lambda_2^2)/8.\end{aligned}$$

Here we use the relation $c_1 = c_2 + d$. Put $\tilde{u}(z) := z^{-(c_2+\lambda_1)/2}u(z)$. Then Proposition (A.3) in Appendix 2 tells us that $\tilde{u}(z)$ satisfies a differential equation of the Fuchsian type which has (at most) three regular singularities at $z = 0, 1$ and -1. Computing the characteristic exponents at these singularities, we have

$$\tilde{u}(z) \in P\left\{\begin{matrix} 0 & 1 & -1 & \\ \mu_0 & (\lambda_1 + 2 + \nu_J)/2 & (l_1 - l_2 - d)/2 & ; z \\ \mu_0 - |\kappa| & (\lambda_1 + 2 - \nu_J)/2 & (l_1 + l_2 - d - 2)/2 & \end{matrix}\right\},$$

where P stands for the P-function of Riemann. Keeping the relation $\mu_0 + (c_2 + \lambda_1)/2 = |\kappa|/2\,(\in \mathbb{Z})$ in mind, we obtain our assertion in this case. The other cases can be proved in the same manner.
(ii) The case of $\eta \in D_<^{(+,+)}$. If we interchange l_1 with l_2 in the differential equation (D-1) for $\eta \in D_>^{(+,+)}$, we get the differential equation (D-1) for $\eta \in D_<^{(+,+)}$. Here we use the identity $2\coth 2t = \tanh t + \coth t$. Thus, our assertion for $\eta \in D_<^{(+,+)}$ follows from that for $\eta \in D_>^{(+,+)}$ immediately. □

The second part of the main result is:

THEOREM (6.2). *Define subsets* $D_<^{(-,-)}$ *and* $D_>^{(-,-)}$ *of* $\hat{R}$ *by*

$$D_<^{(-,-)} := \{D_{l_1} \boxtimes D_{l_2}; 0 > l_2 \geqslant l_1 + d\},$$
$$D_>^{(-,-)} := \{D_{l_1} \boxtimes D_{l_2}; 0 > l_1 \geqslant l_2 + d\}.$$

If η belongs to $D_<^{(-,-)}$ or $D_>^{(-,-)}$, then there are no non-zero spherical functions of type (π, η, τ).

PROOF. Our first task is to establish the following lemma.

LEMMA. (1) (i) *Suppose that $\eta \in D_<^{(-,-)}$. Then $\phi(t) \in C^\infty_{\eta,\tau}(A)$ can be written as*

$$(*3) \quad \phi(t) = \sum_{k=0}^{d} \sum_{i=0}^{-\infty} \phi_{i,k}(t) v_i \otimes v_{j(i)+k} \otimes w_k,$$

where $j(i) := (l_1 - l_2 - d)/2 + i$.
(ii) *Suppose that $\eta \in D_>^{(-,-)}$. Then $\phi(t) \in C^\infty_{\eta,\tau}(A)$ can be written as*

$$(*4) \quad \phi(t) = \sum_{k=0}^{d} \sum_{j=0}^{-\infty} \phi_{j,k}(t) v_{i(j)-k} \otimes v_j \otimes w_k,$$

where $i(j) := (-l_1 + l_2 + d)/2 + j$.
(2) *Define a C^∞-function $u(t)$ on $\mathbb{R}$ by $u(t) := \phi_{0,d}(t)$ or $\phi_{0,0}(t)$ according as $\eta \in D_<^{(-,-)}$ or $D_>^{(-,-)}$. Then $\phi(t)$ is uniquely determined by $u(t)$.*
(3) *$u(t)$ satisfies the following differential equation*:

$$\begin{aligned}
&\{(\frac{d}{dt} + (d+2)\tanh 2t + \frac{1}{\cosh 2t}(c_1 \tanh t - c_2 \coth t \\
&\quad + 2(c_3 - 1)\coth 2t)) \\
\text{(D-2)}: \quad &(\frac{d}{dt} + 2\tanh 2t + c_1 \tanh t - c_2 \coth t + 2c_3 \coth 2t) \\
&+ (\frac{1}{\cosh 2t} - 1)(\frac{A}{\cosh 2t} - B)\}u = 0,
\end{aligned}$$

with the constants:

$$(c_1, c_2, c_3) := (l_1, l_1 + d, \lambda_1) \ (\textit{resp. } (l_2 + d, l_2, \lambda_2)) \quad \textit{if } \eta \in D_<^{(-,-)} \ (\textit{resp. } D_>^{(-,-)}).$$

Here A is given by

$$A := \begin{cases} (-l_1 + l_2 - d + 2)(l_1 + l_2 + d) & \textit{if } \eta \in D_<^{(-,-)} \\ (l_1 - l_2 - d + 2)(l_1 + l_2 + d) & \textit{if } \eta \in D_>^{(-,-)}. \end{cases}$$

and B *is the same value as in* Theorem (6.1)*.*
(4) *Put* $\epsilon(\pi,\eta) := l_1 + \lambda_2$ *(resp.* $l_2 + \lambda_2$*) if* $\eta \in D_<^{(-,-)}$ *(resp.* $D_>^{(-,-)}$*). Then* $u(t)$ *is an even or odd function on* $\mathbb{R}$ *according as* $\epsilon(\pi,\eta) \equiv 0$ *or* $1 \pmod 2$*.*

PROOF OF LEMMA. (1) We can prove this in the same way as Theorem (6.1)(1).
(2) This can be proved in the same manner as Theorem (6.1)(2) by using (A-3), (B-1) (B-2) or $(\text{C-4})_k$ $(\text{C-5})_k$, according as $d = 0$, $d = 1$ or $d \geqslant 4$ to obtain our assertion.
(3) Suppose that $\eta \in D_<^{(-,-)}$. In order to get the differential equation (D-2), it is enough to apply (A-4), (B-6) or (C-9) to ϕ according as $d = 0, d = 1$ or $d \geqslant 4$. Next we suppose that $\eta \in D_>^{(-,-)}$. Then we use (A-4), (B-8) or (C-11) according as $d = 0, d = 1$ or $d \geqslant 4$.
(4) This can be proved in the same manner as Theorem (6.1)(4). □

Return to the proof of Theorem (6.2).
(i) The case of $\eta \in D_<^{(-,-)}$. Suppose that $\epsilon(\pi,\eta) \equiv 1 \pmod 2$. Then $u_1(t) := (\tanh t)^{-1} u(t)$ is an even function on $\mathbb{R}$. We make changes of variables from t to $z = (\tanh t)^2$. Then $u_1(z)$ satisfies the differential equation (♮) in the proof of Theorem (6.1) with the constants:

$$\alpha_{-1} = 2 + c_2, \quad \alpha_0 = 1 - c_2 + \lambda_1, \quad \alpha_1 = -1 - \lambda_1, \quad \beta_{-1} = (-A + 4c_2)/4,$$

$$\beta_0 = (c_2 - \lambda_1 - 1)(c_2 - \lambda_1 + 1)/4, \quad \beta_1 = (4 + B - d^2 + 4\lambda_1 + 2d\lambda_1)/4,$$

$$\begin{aligned} \gamma_{-1} &= (-4 - A + B + 8c_2 + 4c_2^2 - 2c_2 d - 4\lambda_1)/8, \\ \gamma_0 &= (c_2 - \lambda_1 - 1)(-c_2 - \lambda_1 - 3)/2, \\ \gamma_1 &= (-8 + A - B + 2c_2 d - 12\lambda_1 - 4\lambda_1^2)/8. \end{aligned}$$

Here we use the relation $c_1 = c_2 - d$. The characteristic exponents at $z = 0$ are $(c_2 - \lambda_1 - 1)/2$ and $(c_2 - \lambda_1 + 1)/2$. Keeping in mind $0 > l_2 \geqslant l_1 + d$ and $\lambda_1 \geqslant 2$ (see §2), we know $(c_2 - \lambda_1 + 1)/2 \leqslant (l_2 - \lambda_1 + 1)/2 \leqslant -1$. Hence we have our assertion in this case. The case of $\epsilon(\pi,\eta) \equiv 0 \pmod 2$ is similar.
(ii) The case of $\eta \in D_>^{(-,-)}$. If we interchange l_1 with l_2 in the differential equation (D-2) for $\eta \in D_<^{(-,-)}$, we get the differential equation (D-2) for

$\eta \in D_{>}^{(-,-)}$. Here we use the identity $2\coth 2t = \tanh t + \coth t$. Thus, our assertion for $\eta \in D_{>}^{(-,-)}$ follows from that for $\eta \in D_{<}^{(-,-)}$ immediately. $\square$

REMARK. We can prove by virtue of Proposition (A3) in Appendix 2 that the singularity of ($\natural$) at $z = 0$ is apparent. That is, if we put $\tilde{u}(z) := z^{(c_2-\lambda_1)/2}u(z)$, then $\tilde{u}(z)$ satisfies the differential equation of the same form as ($\natural$) with $\alpha_0 = \beta_0 = \gamma_0 = 0$.

The third part of the main results is:

THEOREM (6.3). *Define subsets $D^{(+,-)}$ and $D^{(-,+)}$ of $\hat{R}$ by*

$$D^{(+,-)} := \{D_{l_1} \boxtimes D_{l_2}; l_1 > 0 > l_2\}, \quad D^{(-,+)} := \{D_{l_1} \boxtimes D_{l_2}; l_1 < 0 < l_2\}.$$

Suppose that η belongs to one of these subsets and that $\phi \in C^\infty_{\eta,\tau}(A)$ is a spherical function of type (π, η, τ). Then we have the following assertions:
(1) *If η belongs to $D^{(+,-)}$ (resp.$D^{(-,+)}$), then $\phi \in C^\infty_{\eta,\tau}(A)$ can be written in the same form as* (∗1) *(resp.* (∗3)*) with $j(i) := (l_1 - l_2 - d)/2 + i$. In particular, we may assume $|l_1 - l_2| \leqslant d$ and $d \geqslant 4$.*
(2) *Define a C^∞-function $u(t)$ on $\mathbb{R}$ by $u(t) := \phi_{0,-j(0)}(t)$. Then $\phi(t)$ is uniquely determined by $u(t)$.*
(3) *$u(t)$ satisfies the following differential equation*

$$\text{(D-3)}: \quad \{\frac{d}{dt} + c_1 \tanh t - c_2 \coth t + c_3 \tanh 2t + c_4 \coth 2t\}u(t) = 0,$$

with the constants:

$$(c_1, c_2, c_3, c_4) := \begin{cases} (l_1, l_2, 2 + d - l_1 + l_2, \lambda_1 + \lambda_2 - l_1 + l_2) \\ \qquad\qquad\qquad \text{if } \eta \in D^{(+,-)}; \\ (l_1, l_2, 2 + d + l_1 - l_2, \lambda_1 + \lambda_2 - l_1 + l_2) \\ \qquad\qquad\qquad \text{if } \eta \in D^{(-,+)}. \end{cases}$$

(4) *If we put $\epsilon(\pi, \eta) := (-2c_2 + c_4)/2$, then $u(t)$ is an even or odd function on $\mathbb{R}$ according as $\epsilon(\pi, \eta) \equiv 0$ or $1 \pmod 2$.*
(5) *Put $\alpha_{-1} := c_3/2$, $\alpha_0 := (-2c_2 + c_4)/4$, $\alpha_1 := (-c_1 + c_2 - c_3 - c_4)/2$.*

(i) *If* $\alpha_0 \leqslant 0$ (*that is*, $l_1+l_2 \geqslant \lambda_1+\lambda_2$), *then the differential equation* (D-3) *has a unique, up to constant multiple,* C^∞*-solution* $u(t)$ *on* $\mathbb{R}$ *satisfying the parity condition in* (4), *which is given by*

$$u(t) = (\tanh^2 t+1)^{-\alpha_{-1}}(\tanh t)^{-2\alpha_0}(\tanh^2 t-1)^{-\alpha_1}.$$

(ii) *If* $\alpha_0 > 0$, *then there are no non-zero spherical functions of type* (π, η, τ).

PROOF. (1) We can prove this in the same way as Theorem (6.1)(1).
(2) This can be prove by using the formulae from $(\text{C-4})_k$ to $(\text{C-7})_k$.
(3) Suppose $\eta \in D^{(+,-)}$(resp. $D^{(-,+)}$). Applying $(\text{C-4})_{-j(0)}$(resp. $(\text{C-5})_{-j(0)-1}$), we have the differential equation (D-3).
(4) This can be proved in the same manner as Theorem (6.1)(4).
(5) Suppose $\epsilon(\pi,\eta) \equiv 0 \pmod 2$. By changing variables from t to $z = (\tanh t)^2$, we obtain

$$\{\frac{d}{dz} + \sum_{a=-1,0,1} \frac{\alpha_a}{(z-a)}\}u = 0.$$

Here the constants are as in the theorem. The unique, up to constant multiple, solution of this differential equation is given by

$$u(z) = (z+1)^{-\alpha_{-1}} z^{-\alpha_0}(z-1)^{-\alpha_1}.$$

Hence we have the desired assertion in this case. The case of $\epsilon(\pi,\eta) \equiv 1 \pmod 2$ is similar. □

The fourth part of the main results is:

THEOREM (6.4). *Define subsets* $D_\sim^{(+,+)}$, $D_\sim^{(-,-)}$ *of* $\hat{R}$ *by*

$$D_\sim^{(+,+)} := \{D_{l_1} \boxtimes D_{l_2}; l_1, l_2 > 0, |l_1 - l_2| < d\},$$
$$D_\sim^{(-,-)} := \{D_{l_1} \boxtimes D_{l_2}; l_1, l_2 < 0, |l_1 - l_2| < d\}.$$

Suppose that η *belongs to* $D_\sim^{(+,+)} \cup D_\sim^{(-,-)}$ *and that* $\phi \in C^\infty_{\eta,\tau}(A)$ *is a spherical function of type* (π, η, τ). *Then we have the following assertions*:

(1) *Suppose that* η *belongs to* $D_{\sim}^{(+,+)}$ (*resp.* $D_{\sim}^{(-,-)}$), *then* $\phi(t) \in C_{\eta,\tau}^{\infty}(A)$ *can be written in the same form as* $(*2)$ (*resp.* $(*4)$) *with* $i(j) := (-l_1 + l_2 + d)/2 + j$.
(2) *If we define a* C^{∞}*-function* $u(t)$ *on* $\mathbb{R}$ *by* $u(t) := \phi_{0,i(0)}(t)$, *then* $\phi(t)$ *is uniquely determined by* $u(t)$.
(3) $u(t)$ *satisfies the following differential equation*:

$$(\text{D-4}) : \{\frac{d}{dt} + (d+2)\tanh 2t - \frac{1}{\cosh 2t}(c_1 \tanh t - c_2 \coth t + c_3 \coth 2t)\}u(t) = 0,$$

with the constants

$$(c_1, c_2, c_3) := \begin{cases} (l_1, l_2, -l_1 + l_2 + \lambda_1 + \lambda_2) & \text{if } \eta \in D_{\sim}^{(+,+)}, \\ (-l_1, -l_2, l_1 - l_2 - \lambda_1 - \lambda_2) & \text{if } \eta \in D_{\sim}^{(-,-)}. \end{cases}$$

(4) *If we put* $\epsilon(\pi, \eta) := (-l_1 - l_2 + \lambda_1 + \lambda_2)/2$, *then* $u(t)$ *is an even or odd function on* $\mathbb{R}$ *according as* $\epsilon(\pi, \eta) \equiv 0$ *or* $1 \pmod 2$.
(5) *We put* $\alpha_{-1} := (2 + d - c_1 - c_2)/2$, $\alpha_0 := (2c_2 - c_3)/4$ *and* $\alpha_1 := (-2 - d)/2$.
(i) *If* $\alpha_0 \leqslant 0$, *then the differential equation* (D-4) *has a unique, up to constant multiple,* C^{∞}*-solution on* $\mathbb{R}$ *satisfying the parity condition in* (4), *which is given by*

$$u(t) = (\tanh^2 t + 1)^{-\alpha_{-1}} (\tanh t)^{-2\alpha_0} (\tanh^2 t - 1)^{-\alpha_1}.$$

(ii) *If* $\alpha_0 > 0$ (*in particular, if* $\eta \in D_{\sim}^{(-,-)}$), *then there are no non-zero spherical functions of type* (π, η, τ).

Proof. (1) This can be proved in the same way as Theorem (6.1)(1). Remark that we may assume $d \geqslant 4$ by Assumption (3.3)
(2) The coefficient functions $\phi_{i,j}$ are determined from $u(t)$ by means of the formulae $(\text{C-4})_k$ and $(\text{C-7})_k$.
(3) If $\eta \in D_{\sim}^{(+,+)}$ (resp. $D_{\sim}^{(-,-)}$), then in order to get the differential equation (D-4) we have only to apply $(\text{C-6})_{i(0)+1}$ (resp. $(\text{C-7})_{i(0)-1}$) to ϕ .
(4) We can prove this in the same way as Theorem (6.1) (4).

(5) Suppose $\epsilon(\pi,\eta)\equiv 0 \pmod 2$. If we make changes of variables from t to $z=(\tanh t)^2$, then $u(z)$ satisfies the differential equation

$$\{\frac{d}{dz}+\sum_{a=-1,0,1}\frac{\alpha_a}{(z-a)}\}u=0.$$

Here the constants α_{-1},α_0 and α_1 are as in the theorem. From this, our assertion follows easily. The case of $\epsilon(\pi,\eta)\equiv 1 \pmod 2$ is similar. □

The fifth part of the main results is:

THEOREM (6.5). *Suppose that η is the trivial representation of R. Then there are no non-zero spherical functions of type (π,η,τ).*

PROOF. Firstly we should notice that d must be an even integer by Assumption (3.3). Thus, a spherical function $\phi\in C^\infty_{\eta,\tau}(A)$ of type (π,η,τ) can be written as

$$\phi(t)=u(t)w_{d/2}.$$

(i) The case of $d=0$. Put $\tau=\tau_{(\lambda,\lambda)}$ $(\lambda\geqslant 2)$. From the differential equations (A-1) and (A-2), we know $u(t)$ satisfies the system of differential equations consisting of $(*)$ and $(**)$ below:

$$(*):\quad \{(\frac{d}{dt}+4\tanh 2t+2(\lambda-1)\coth 2t)(\frac{d}{dt}+2\lambda\coth 2t)\}u(t)=0,$$
$$(**):\quad \{(\frac{d}{dt}+4\tanh 2t-2(\lambda-1)\coth 2t)(\frac{d}{dt}+2\lambda\coth 2t)-A\}u(t)=0,$$

where $A:=-2(\lambda-2)^2+2\nu_J^2$. Subtracting $(**)$ from $(*)$, we have

$$(***):\qquad \{\frac{d}{dt}+\frac{A}{4(\lambda-1)}\tanh 2t+2\lambda\coth 2t\}u=0.$$

The unique, up to constant multiple, solution of $(***)$ is given by

$$u(t)=(\tanh^2 t+1)^{-A/(8\lambda-8)}(\tanh t)^{-\lambda}(\tanh^2 t-1)^{\lambda+A/(8\lambda-8)}.$$

However, since $\lambda\geqslant 2$, this is not smooth at $t=0$. Thus our assertion follows for the case of $d=0$.

(ii) The case of $d \geqslant 4$. From $(\text{C-2})_{d/2-1}$ and $(\text{C-4})_{d/2}$, we have the following system of differential equations satisfied by $u(t)$ consisting of ($\flat$) and ($\flat\flat$) below:

$$(\flat) \quad \{\frac{d}{dt} + (d+2)\tanh 2t - (\lambda_1 + \lambda_2)\coth 2t\}u = 0;$$
$$(\flat\flat) \quad \{\frac{d}{dt} + (d+2)\tanh 2t + (\lambda_1 + \lambda_2)\coth 2t\}u = 0.$$

Subtracting ($\flat$) from ($\flat\flat$), we obtain $\{(\lambda_1 + \lambda_2)\coth 2t\}u = 0$. On the other hand, $\lambda_1 + \lambda_2 \geqslant 2$ (see §2.3). Hence our assertion follows for the case of $d \geqslant 4$, too. □

The last part of the main results is:

THEOREM (6.6). *Define subsets TD^+, TD^-, DT^+ and DT^- of $\hat{R}$ by*

$$TD^+ := \{1 \boxtimes D_{l_2}; l_2 > 0\}, \quad TD^- := \{1 \boxtimes D_{l_2}; l_2 < 0\},$$
$$DT^+ := \{D_{l_1} \boxtimes 1; l_1 > 0\}, \quad DT^- := \{D_{l_1} \boxtimes 1; l_1 < 0\}.$$

Suppose that η belongs to one of these subsets and that $\phi \in C^\infty_{\eta,\tau}(A)$ is a spherical function of type (π, η, τ). Then we have the following assertions:
(1) (i) *Suppose that η belongs to TD^+ or TD^-. Then $\phi \in C^\infty_{\eta,\tau}(A)$ can be written as*

$$(*5) \quad \phi(t) = \sum_{k=0}^{d} \phi_k(t) v_{j(k)} \otimes w_k,$$

where $j(k) := (-l_2 - d)/2 + k$.
(ii) *Suppose that η belongs to DT^+ or DT^-. Then $\phi \in C^\infty_{\eta,\tau}(A)$ can be written as*

$$(*6) \quad \phi(t) = \sum_{k=0}^{d} \phi_k(t) v_{i(k)} \otimes w_k,$$

where $i(k) := (-l_1 + d)/2 - k$.
(2) *Define a C^∞-function $u(t)$ on $\mathbb{R}$ by $u(t) := \phi_{-j(0)}(t)$ or $\phi_{i(0)}(t)$ according as $\eta \in TD^+ \cup TD^-$ or $\eta \in DT^+ \cup DT^-$. Then $\phi(t)$ is uniquely determined by $u(t)$.*

(3) (i) *Suppose that* $\eta \in TD^+$ (*resp.* DT^+), $d \geqslant 4$ *and* $\lambda_1 + \lambda_2 = l_2$ (*resp.* $\lambda_1 + \lambda_2 = l_1$). *Then* $u(t)$ *satisfies the differential equation* (D-3) *in Theorem* (6.3) *with the constants*:

$$(c_1, c_2, c_3, c_4) := \begin{cases} (0, \lambda_1 + \lambda_2, -2\lambda_2 + 2, 2(\lambda_1 + \lambda_2)) & \textit{if } \eta \in TD^+; \\ (\lambda_1 + \lambda_2, 0, -2\lambda_2 + 2, 0) & \textit{if } \eta \in DT^+. \end{cases}$$

This differential equation has a unique, up to constant multiple, C^∞-*solution on* $\mathbb{R}$, *which is given by*

$$u(t) = (\tanh^2 t + 1)^{\lambda_2 - 1} (\tanh^2 t - 1)^{(\lambda_1 - \lambda_2 - 2)/2}.$$

(ii) *In the other cases,* $u(t)$ *is identically zero. Therefore, there are no non-trivial spherical functions of type* (π, η, τ).

PROOF. (1)(2) We can prove these in the same way as Theorem (6.1)(1)(2). Remark that we may assume $d \geqslant 1$ by Assumption (3.3).
(3) (i) Suppose $\eta \in TD^+$ and $d \geqslant 4$. Applying $(\text{C-5})_{-j(0)-1}$ and $(\text{C-6})_{-j(0)+1}$, we have

$$(\natural): \quad \{\frac{d}{dt} + (-l_2 + d + 2)\tanh 2t - l_2 \coth t \\ + (l_2 + \lambda_1 + \lambda_2)\coth 2t\} u(t) = 0,$$

$$(\natural\natural): \quad \{(\frac{d}{dt} + (d+2)\tanh 2t \\ - \frac{1}{\cosh 2t}(-l_2 \coth t + (l_2 + \lambda_1 + \lambda_2)\coth 2t)\} u(t) = 0$$

Subtracting $(\natural)$ from $(\natural\natural)$, we get

$$(-l_2 + \lambda_1 + \lambda_2) \coth t \, u(t) = 0.$$

Thus we conclude that $u(t)$ must be identically zero unless $l_2 = \lambda_1 + \lambda_2$. Inserting this relation to $(\natural)$, we obtain the desired differential equation and its solution. The case of DT^+ can be proved in the same manner by using $(\text{C-4})_{i(0)}$ and $(\text{C-6})_{i(0)+1}$.
(ii) (ii-a) The case of $d = 1$. If η belongs to TD^+, then it follows from Assumption (3.3) $l_2 = 1$. Applying (B-2) to ϕ, we know that $u(t)$ satisfies

the differential equation (D-3) with $(c_1, c_2, c_3, c_4) := (0, 1, 2, 2\lambda)$. By the proof of Theorem (6.3), $u(t)$ is smooth only when $\alpha_0 := (-2c_2 + c_4)/4 = (\lambda - 1)/2 \leqslant 0$. But this contradicts $\lambda \geqslant 2$ (see §2.3). The proofs for the case of $\eta \in TD^- \cup DT^+ \cup DT^-$ are quite analogous.
(ii-b) The case of $d \geqslant 4$. Suppose that $\eta \in TD^-$. Applying $(\text{C-4})_{-j(0)}$ to $\phi(t)$, we conclude that $u(t)$ satisfies the differential equation (D-3) in Theorem (6.3) with the constants:

$$(c_1, c_2, c_3, c_4) := (0, l_2, d + l_2 + 2, l_2 + \lambda_1 + \lambda_2).$$

Again, by the proof of Theorem (6.3), $u(t)$ is smooth only when $\alpha_0 := (-2c_2 + c_4)/4 = (-l_2 + \lambda_1 + \lambda_2) \leqslant 0$. But this is impossible (see §2.3). The proof for the case of $\eta \in DT^-$ is quite analogous. □

From the main theorems proved above, we can know the upper bounds of the dimensions of the intertwining spaces. For an irreducible half-size standard representation π of G and an irreducible unitary representation η of R, we put

$$m(\pi, \eta) := \dim_{\mathbb{C}} \mathrm{Hom}_{(\mathfrak{g},K)}(\pi^0, C^\infty_\eta(R\backslash G)^0).$$

Corollary (6.7). *Let $\eta = D_{l_1} \boxtimes D_{l_2}$ be a (limit of) discrete series representation of R.*
(1) *If η satisfies one of the following conditions, then we have $m(\pi, \eta) \leqslant 1$.*
(i) $\eta \in D^{(+,+)}_{>} \cup D^{(+,+)}_{<}$ *and* $l_1 + l_2 \equiv d \pmod 2$;
(ii) $\eta \in D^{(+,-)} \cup D^{(-,+)}$, $l_1 + l_2 \equiv d \pmod 2$, $|l_1 - l_2| \leqslant d$ *and* $l_1 + l_2 \geqslant \lambda_1 + \lambda_2$;
(iii) $\eta \in D^{(+,+)}_{\sim} \cup D^{(-,-)}_{\sim}$, $l_1 + l_2 \equiv d \pmod 2$ *and* $l_1 + l_2 \leqslant \lambda_1 + \lambda_2$.
(2) *In the other cases, we have $m(\pi, \eta) = 0$.*

Proof. This follows from the theorems from (6.1) to (6.4). □

Remark. (1) By "Frobenius reciprocity", our result should be related to the problem of the spectral decomposition of the restriction $\pi|_R$ of π to R. There is a result of Harris and Kudla[2, Theorem (2.4.1)] on the multiplicities of the discrete part of $\pi|_R$ when π is a large discrete series representation. Our result in Corollary (6.7) is compatible with theirs to a certain extent.
(2) Hayata [14] investigates the same problem as this paper for the semisimple symmetric pair $(SU(2,2), Sp(2,\mathbb{R}))$.

Appendix 1

In this appendix, we shall give parallel results on the spherical functions of type (π, η, τ) for other irreducible unitary representations η, which we do not treat in §6.

Theorem (A1). *Define subsets DP^+, TP, PD^+ and PT of $\hat{R}$ by*

$$DP^+ := \{D_{l_1} \boxtimes P^{-1+it,\pm}; l_1 > 0\}, \quad TP := \{1 \boxtimes P^{-1+it,\pm}\},$$
$$PD^+ := \{P^{-1+it,\pm} \boxtimes D_{l_2}; l_2 > 0\}, \quad PT := \{P^{-1+it,\pm} \boxtimes 1\}.$$

Suppose that η belongs to one of these subsets and that $\phi \in C^\infty_{\eta,\tau}(A)$ is a spherical function of type (π, η, τ). Then we have the following assertions:
(1) (i) *If $\eta \in DP^+$, then $\phi \in C^\infty_{\eta,\tau}(A)$ can be written in the same form as* $(*1)$ *with*

$$j(i) := (l_1 - d)/2 + i \,(\textit{resp. } (l_1 - d - 1)/2 + i), \ \textit{if } \eta_2 = P^{s,+} \,(\textit{resp. } P^{s,-}).$$

(ii) *If $\eta \in TP$, then $\phi \in C^\infty_{\eta,\tau}(A)$ can be written in the same form as* $(*5)$ *with*

$$j(k) := (-d)/2 + k \,(\textit{resp. } (-d - 1)/2 + k), \ \textit{if } \eta_2 = P^{s,+} \,(\textit{resp. } P^{s,-}).$$

(iii) *If $\eta \in PD^+$, then $\phi \in C^\infty_{\eta,\tau}(A)$ can be written in the same form as* $(*2)$ *with*

$$i(j) := (l_2 + d)/2 + j \,(\textit{resp. } (l_2 + d - 1)/2 + j), \ \textit{if } \eta_1 = P^{s,+} \,(\textit{resp. } P^{s,-}).$$

(iv) *If $\eta \in PT$, then $\phi \in C^\infty_{\eta,\tau}(A)$ can be written in the same form as* $(*6)$ *with*

$$i(k) := d/2 - k \,(\textit{resp. } (d - 1)/2 - k), \ \textit{if } \eta_1 = P^{s,+} \,(\textit{resp. } P^{s,-}).$$

(2) *Define a C^∞-function $u(t)$ on $\mathbb{R}$ by $u(t) := \phi_{0,0}(t), \phi_0(t), \phi_{0,d}(t)$ or $\phi_d(t)$ according as $\eta \in DP^+$, TP, PD^+ or PT. Then $\phi(t)$ is uniquely determined by $u(t)$.*

(3) $u(t)$ satisfies the differential equation (D-1) *in Theorem* (6.1) *with the constants*:

$$(c_1, c_2, c_3) := \begin{cases} (l_1, l_1 - d, \lambda_2) & \text{if } \eta \in DP^+; \\ (0, -d, \lambda_2) & \text{if } \eta \in TP; \\ (l_2 - d, l_2, \lambda_1) & \text{if } \eta \in PD^+; \\ (-d, 0, \lambda_1) & \text{if } \eta \in PT. \end{cases}$$

The constant A in (D-1) *is defined by the equations*

$$\begin{cases} 4n_2^- n_2^+ v_{j(0)} = A v_{j(0)} & \text{if } \eta \in DP^+ \cup TP; \\ 4n_1^- n_1^+ v_{i(0)-d} = A v_{i(0)-d} & \text{if } \eta \in PD^+ \cup PT, \end{cases}$$

and B is as in Theorem (6.1).
(4) If $u(t)$ does not vanish identically, then it must be a constant multiple of

$$u(t) := (\tanh t)^{|\kappa|} (\tanh^2 t + 1)^{-\mu_0} (\tanh t \cdot \tanh 2t - 1)^{\mu_1} \times {}_2F_1(a, b, 1 + |\kappa|; \tanh t \cdot \tanh 2t),$$

where the constant μ_1 are the same as in Theorem (6.1)(5) *and*

$$(\kappa, \mu_0) := \begin{cases} (c_2 - \lambda_2, \max\{(-\lambda_1 - \lambda_2)/2, (-2c_2 - d)/2\}) \\ \qquad \text{if } \eta \in DP^+ \cup TP; \\ (c_2 - \lambda_1, \max\{(-\lambda_1 - \lambda_2)/2, (-2c_2 + d)/2\}) \\ \qquad \text{if } \eta \in PD^+ \cup PT; \end{cases}$$

Here a, b are given by

$$(a, b) := \begin{cases} ((l_1 - s - d - 2)/2 + \mu_0 + \mu_1, (l_1 + s - d)/2 + \mu_0 + \mu_1) \\ \qquad \text{if } \eta \in DP^+; \\ ((-s - d - 2)/2 + \mu_0 + \mu_1, (s - d)/2 + \mu_0 + \mu_1) \\ \qquad \text{if } \eta \in TP; \\ ((-s + l_2 - d - 2)/2 + \mu_0 + \mu_1, (s + l_2 - d)/2 + \mu_0 + \mu_1) \\ \qquad \text{if } \eta \in PD^+; \\ ((-s - d - 2)/2 + \mu_0 + \mu_1, (s - d)/2 + \mu_0 + \mu_1) \\ \qquad \text{if } \eta \in PT. \end{cases}$$

Proof. The proof is quite analogous to Theorem (6.1). □

Theorem (A2). *Define subsets* DP^- *and* PD^- *by*

$$DP^- := \{D_{l_1} \boxtimes P^{-1+it,\pm}; l_1 < 0\}, \quad PD^- := \{P^{-1+it,\pm} \boxtimes D_{l_2}; l_2 < 0\}.$$

Suppose that η *belongs to* DP^- *or* PD^- *and that* $\phi \in C^\infty_{\eta,\tau}(A)$ *is a spherical function of type* (π, η, τ). *Then we have the following assertions*:
(1) (i) *If* $\eta \in DP^-$, *then* $\phi \in C^\infty_{\eta,\tau}(A)$ *can be written in the same form as* $(*3)$ *with*

$$j(i) := (l_1 - d)/2 + i \,(\textit{resp. } (l_1 - d - 1)/2 + i), \ \textit{if } \eta_2 = P^{s,+} \,(\textit{resp. } P^{s,-}).$$

(ii) *If* $\eta \in PD^-$, *then* $\phi \in C^\infty_{\eta,\tau}(A)$ *can be written in the same form as* $(*4)$ *with*

$$i(j) := (l_2 + d)/2 + j \,(\textit{resp. } (l_2 + d - 1)/2 + j), \ \textit{if } \eta_1 = P^{s,+} \,(\textit{resp. } P^{s,-}).$$

(2) *Define a* C^∞*-function* $u(t)$ *on* $\mathbb{R}$ *by* $u(t) := \phi_{0,d}(t) \,(\textit{resp. } \phi_{0,0}(t))$ *if* $\eta \in DP^-$ (*resp.* PD^-). *Then* ϕ *is uniquely determined by* $u(t)$.
(3) $u(t)$ *satisfies the differential equation* (D-2) *in the proof of Theorem* (6.2) *with the constants*:

$$(c_1, c_2, c_3) := (l_1, l_1 + d, \lambda_1) \,(\textit{resp. } (l_2 + d, l_2, \lambda_2)) \ \textit{if } \eta \in DP^- \,(\textit{resp. } DP^-).$$

The constant A *in* (D-2) *is defined by the equations*

$$\begin{cases} 4n_2^+ n_2^- v_{j(0)+d} = A v_{j(0)+d} & \textit{if } \eta \in DP^-; \\ 4n_1^+ n_1^- v_{i(0)} = A v_{i(0)} & \textit{if } \eta \in PD^-. \end{cases}$$

and B *is the same value as in Theorem* (6.1).
(4) *If we put* $\epsilon(\pi, \eta) := l_1 + \lambda_2 \,(\textit{resp. } l_2 + \lambda_2)$ *if* $\eta \in DP^-$ (*resp.* PD^-), *then* $u(t)$ *is an even or odd function on* $\mathbb{R}$ *according as* $\epsilon(\pi, \eta) \equiv 0$ *or* $1 \pmod 2$.
(5) *Let* $s(\pi, \eta)$ *be the dimension of the space of* C^∞*-functions satisfying the differential equation* (D-2) *with the parity condition in* (4). *Then we have*
(i) *if* η *belongs to* DP^-, *then* $s(\pi, \eta) = 2, 1$ *or* 0 *according as* $l_1 - \lambda_2 \geqslant 0, l_1 - \lambda_2 = -1, -2$ *or* $l_1 - \lambda_2 \leqslant -3$;

(ii) *if* η *belongs to* PD^-, *then* $s(\pi,\eta) = 2, 1$ *or* 0 *according as* $l_2 - \lambda_2 \geqslant 0, l_2 - \lambda_2 = -1, -2$ *or* $l_2 - \lambda_2 \leqslant -3$.

PROOF. The proof is quite analogous to Theorem (6.2). □

REMARK. From Theorem (A2), we know $m(\pi,\eta) \leqslant 2$ for an irreducible generalized principal series representation π of G and $\eta \in DP^- \cup DT^-$. It seems beyond the scope of our method to answer the question whether this estimate is best possible or not.

Appendix 2

In this appendix we collect some basic facts about the second-order differential equations of the Fuchsian type, which we use in the proof of Theorem (6.1).

PROPOSITION (A3). *Let* $S := \{a_1, a_2, \cdots, a_n\}$ *be a finite subset of* $\mathbb{C}$. *Consider the following second-order ordinary differential equation on* $\mathbb{P}^1(\mathbb{C})$:

$$(\sharp) \quad \{(\frac{d}{dz})^2 + \sum_{a\in S} \frac{\alpha_a}{z-a}\frac{d}{dz} + \sum_{a\in S}(\frac{\beta_a}{(z-a)^2} + \frac{\gamma_a}{(z-a)})\}u = 0,$$

with $\alpha_a, \beta_a, \gamma_a \in \mathbb{C}$.
(1) *The differential equation* ($\sharp$) *is of the Fuchsian type if and only if the relation* $\sum_{a\in S}\gamma_a = 0$ *holds.*
From now on, we assume $\sum_{a\in S}\gamma_a = 0$.
(2) *Set*

$$\alpha_\infty := 2 - \sum_{a\in S}\alpha_a; \quad \beta_\infty := \sum_{a\in S}(\beta_a + a\gamma_a); \quad \gamma_\infty := \sum_{a\in S}(2a\beta_a + a^2\gamma_a).$$

Then the characteristic exponents at $a \in S \cup \{\infty\}$ *are given by the solutions of the indicial equation*

$$\rho^2 + (\alpha_a - 1)\rho + \beta_a = 0.$$

(3) *If we put* $\tilde{u}(z) := (z-a_1)^{-\kappa}u(z)$, *then* $\tilde{u}(z)$ *satisfies the following differential equation*

$$(\sharp\sharp) \quad \{(\frac{d}{dz})^2 + \sum_{a\in S} \frac{\alpha'_a}{z-a}\frac{d}{dz} + \sum_{a\in S}(\frac{\beta'_a}{(z-a)^2} + \frac{\gamma'_a}{(z-a)})\}\tilde{u} = 0,$$

with

$$\begin{aligned}
\alpha'_{a_1} &:= \alpha_{a_1} + 2\kappa; & \alpha'_{a_i} &:= \alpha_{a_i} \quad (2 \leqslant i \leqslant n);\\
\beta'_{a_1} &:= \beta_{a_1} + \kappa(\alpha_{a_1} + \kappa - 1); & \beta'_{a_i} &:= \beta_{a_i} \quad (2 \leqslant i \leqslant n);\\
\gamma'_{a_1} &:= \gamma_{a_1} + \sum_{i=2}^{n} \frac{\alpha_{a_i}\kappa}{a_1 - a_i}; & \gamma'_{a_i} &:= \gamma_{a_i} - \frac{\alpha_{a_i}\kappa}{a_1 - a_i} \quad (2 \leqslant i \leqslant n).
\end{aligned}$$

(4) *Put*

$$\alpha'_\infty := 2 - \sum_{a\in S} \alpha'_a; \quad \beta'_\infty := \sum_{a\in S}(\beta'_a + a\gamma'_a); \quad \gamma_\infty := \sum_{a\in S}(2a\beta'_a + a^2\gamma'_a).$$

Then we have

$$\begin{aligned}
&\alpha'_\infty = \alpha_\infty - 2\kappa; \qquad \beta'_\infty := \beta_\infty + \kappa(1 - \alpha_\infty + \kappa);\\
&\gamma'_\infty = \gamma_\infty + 2a_1\kappa(\kappa - 1) + \sum_{i=1}^{n}(a_1 + a_i)\alpha_{a_i}\kappa.
\end{aligned}$$

(5) *If there exists a constant* $\kappa \in \mathbb{C}$ *such that* $\alpha'_\infty = \beta'_\infty = \gamma'_\infty = 0$ (*resp.* $\alpha'_{a_1} = \beta'_{a_1} = \gamma'_{a_1} = 0$), *then* $z = \infty$ (*resp.* $z = a_1$) *is an apparent singularity of the differential equation* ($\sharp$).

PROOF. (1) This is well-known. (2) Rewrite the differential equation ($\sharp$) in the coordinate $w = 1/z$. (3) (4) These can be proved by direct computation. (5) This is clear. □

REMARK. We call $z_0 \in S \cup \{\infty\}$ an apparent singularity of ($\sharp$) if the fundamental system of solutions around z_0 is spanned by meromorphic functions.

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Graduate School of Mathematical Sciences
The University of Tokyo
3-8-1 Komaba, Meguro-Ku
Tokyo 153-8914, Japan
E-mail: moriyama@ms318sun.ms.u-tokyo.ac.jp