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Upper Bounds for the Eigenvalues of the Laplacian on Forms on Certain Riemannian Manifolds

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Abstract. We have two kinds of upper bounds of the eigenvalues of the Laplacian on forms on compact Riemannian manifolds. One is implicit in terms of the Ricci curvature and the injective radius. The other is explicit for a class of Riemannian manifolds.

1. Introduction

We shall study the eigenvalues of the Laplacian on forms on a connected compact oriented Riemannian manifold (M, g). We denote by $\lambda_k^{(p)}(M, g)$ the k-th *non-negative* eigenvalue of the Laplacian on p-forms with the multiplicity counted. In general, in spite of rich results for the eigenvalues of the Laplacian on functions, little is known for those on forms. So we are interested in estimating the eigenvalues on forms. J. Dodziuk found interesting estimates [D-82] by using S. Y. Cheng's method in [C-75] for functions. His estimates are implicit and require the information of the covariant dirivative of the curvature tensor. In this paper, we shall improve his estimates from the two different points of view:

(1) weakening assumptions at the sacrifice of showing bounds explicitly,

(2) showing bounds explicitly by requiring more assumptions.

For point of view (1), we obtain the following estimates.

Theorem 1.1 (Implicit Bounds). *Let (M^m, g) be an m-dimensional connected compact oriented Riemannian manifold without boundary. If $|\,\mathrm{Ric}\,| \leq \Lambda$ and $\mathrm{inj}(M, g) \geq i_0 > 0$, then for $p = 1, 2, \cdots, m-1$ and $k = 1, 2, \cdots$, we have*

$$\lambda_k^{(p)}(M, g) \leq C(m, p, k, \Lambda, i_0),$$

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where $\operatorname{inj}(M,g)$ *is the injective radius of* (M,g).

To obtain these estimates, we essentially use the *a priori* estimates of the Christoffel symbols in terms of harmonic coordinates in Main Lemma 2.2 of [Ad-90].

For point of view (2), we obtain the following estimates.

THEOREM 1.2 (Explicit Bounds). *Let* (M^m,g) *be an* m*-dimensional connected compact oriented Riemannian manifold without boundary. Suppose that the metric* g *is written as* $g=dr^2+f(r)^2h$ *for the geodesic polar coordinates* $(r,u^1,\cdots,u^{m-1})$ *on a certain geodesic ball* $B(x_0,\rho)$*, where* h *is the canonical metric of* S^{m-1}*. If* $(m-1)\alpha\le \operatorname{Ric}_{B(x_0,\rho)}$ *and* $K_{B(x_0,\rho)}\le \beta$ $(\alpha<0<\beta)$*, then we obtain for* $p=1,2,\cdots,m-1$ *and* $k=1,2,\cdots$,

(1) *when* $p\le \dfrac{m+1}{2}$,

$$\lambda_k^{(p)}(M,g)\le\left(\frac{\pi}{R}\right)^2+\frac{(m-2p+1)(m-2p+3)}{4}(-\alpha)\left(\frac{\cosh(\sqrt{-\alpha}R)}{\sinh(\sqrt{-\alpha}R)}\right)^2 +\frac{m-2p+1}{2}\beta+(p-1)(m-p+1)\frac{\beta}{\sin^2(\sqrt{\beta}R)},$$

(2) *when* $\dfrac{m-1}{2}\le p$,

$$\lambda_k^{(p)}(M,g)\le\left(\frac{\pi}{R}\right)^2+\frac{(m-2p-1)(m-2p-3)}{4}(-\alpha)\left(\frac{\cosh(\sqrt{-\alpha}R)}{\sinh(\sqrt{-\alpha}R)}\right)^2 -\frac{m-2p-1}{2}\beta+(p+1)(m-p-1)\frac{\beta}{\sin^2(\sqrt{\beta}R)},$$

where $R:=\dfrac{1}{k+1}\min\{\rho,\dfrac{\pi}{2\sqrt{\beta}}\}$.

REMARK 1.3. When $p=\dfrac{m}{2}$, both estimates (1) and (2) coincide with each other. For compact Riemannian manifolds of constant curvature, sharper estimates are given by J. Eichhorn ([E-84] Satz 4.2).

For the proof, using J. Eichhorn's method in [E-84] which yields upper bounds for compact Riemannian manifolds of constant curvature, we reduce our problem to that of ordinary differential operators. We estimate their eigenvalues by means of the min-max principle. For this purpose, we need to control $f(r)$ and its derivatives, which is given by conditions on curvature bounds.

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2. Proof of Theorem 1.1

The next lemma plays the fundamental role in this paper.

LEMMA 2.1 (Fundamental Lemma). *Let (M, g) be a connected compact oriented Riemannian manifold without boundary. If $\{\Omega_i\}_{i=1,\cdots,k}$ are the k disjoint domains with C^∞ boundaries, then we have*

$$\lambda_k^{(p)}(M, g) \leq \max\{\lambda_1^{(p)}(\Omega_1), \lambda_1^{(p)}(\Omega_2), \cdots, \lambda_1^{(p)}(\Omega_k)\},$$

where $\lambda_1^{(p)}(\Omega_i)$ is the first eigenvalue of Ω_i with respect to the induced metric under the Dirichlet condition for p-forms, i.e. vanishing at boundary.

We can prove this lemma by the same method as Lemma 3.3 in [T-98]. Note that C. Anné's theorem [Ae-89] always implies $\lambda_1^{(p)}(\Omega) > 0$. The following lemma follows from Main Lemma 2.2 in [Ad-90] (cf. E. Hebey [H-96], p.5).

LEMMA 2.2 (Anderson). *Let (M^m, g) be an m-dimensional connected compact oriented Riemannian manifold without boundary. Suppose that $|\operatorname{Ric}| \leq \Lambda$, $\operatorname{inj}(M, g) \geq i_0 > 0$. Then there exist harmonic coordinates $(x^1, \cdots, x^m)$ on a ball $B(x_0, r_H)$, where the r_H depends only on m, Λ and i_0, such that for any $y \in B(x_0, r_H)$*

(1) $g_{ij}(x_0) = \delta_{ij}$,

(2) $2^{-1}(\delta_{ij}) \leq (g_{ij}(y)) \leq 2\,(\delta_{ij})$ (*as bilinear forms*),

(3) $|\Gamma_{ij}^k(y)| \leq C(m, \Lambda, i_0)$.

LEMMA 2.3. *Under the same assumption in Lemma 2.2, we obtain for* $p = 1, \cdots, m-1$

$$\lambda_1^{(p)}(B(x_0, r_H)) \leq C(m, p, \Lambda, i_0).$$

PROOF. By applying bounds in Lemma 2.2 to Proposition 2.3 in [D-82], we immediately obtain this lemma. □

Thus by the same argument as Theorem 4.2 in [D-82] using Lemma 2.1 and Berger's isoembolic inequality $\mathrm{vol}(M, g) \geq \left(\frac{\mathrm{inj}(M, g)}{\pi}\right)^m \mathrm{vol}(S^m(1))$ ([Sa-96], p.252), we can prove Theorem 1.1. □

3. Eichhorn's Method

For the sake of the readers, we shall recall Eichhorn's method from [E-84] S.289–S.293, [E-81] S. 16–S.20, S.27–S.29, and [E-83] S.25–S.35.

Let (M^m, g) be an m-dimensional connected compact oriented Riemannian manifold without boundary. Suppose that the metric g is written as

$$g = dr^2 + f(r)^2 h$$

for the geodesic polar coordinates $(r, u^1, \cdots, u^{m-1})$, where the metric h is of $S^{m-1}(1)$ and the function $f(r)$ is C^∞ on $(0, \rho)$ and satisfies $f(0) = 0, f'(0) = 1$ and $f(r) > 0$ for $r > 0$ ([D-79], p.395). For example, the metric of a compact space of conatant δ-curvature is given as $f(r) = s_\delta(r)$, where

$$s_\delta(r) := \begin{cases} \dfrac{\sin(\sqrt{\delta} r)}{\sqrt{\delta}} & (\delta > 0), \\ r & (\delta = 0), \\ \dfrac{\sinh(\sqrt{-\delta} r)}{\sqrt{-\delta}} & (\delta < 0). \end{cases}$$

We denote by $\omega = a \wedge dr + b$ a p-form on the anulus $\Omega = (r_0, r_1) \times S^{m-1}$ in $B(x_0, \rho)$, where a, b are respectively $p-1$, p-forms on Ω which do not involve dr. We obtain

$$\begin{aligned}\Delta(a\wedge dr+b) = &\left\{\frac{1}{f^2}\Delta_0 a-(m-2p+1)\left(\frac{ff''-(f')^2}{f^2}a+\frac{f'}{f}\frac{\partial a}{\partial r}\right)\right.\\ &\left.-\frac{\partial^2 a}{\partial r^2}+(-1)^p\frac{2f'}{f^3}\delta_0 b\right\}\wedge dr\\ &+\left\{\frac{1}{f^2}\Delta_0 b-(m-2p-1)\frac{f'}{f}\frac{\partial b}{\partial r}-\frac{\partial^2 b}{\partial r^2}+(-1)^p\frac{2f'}{f}d_0 a\right\}\end{aligned}$$

(see [E-81] S.16 –S.17), where we denote by d_0, δ_0, Δ_0 the corresponding operators on (S^{m-1}, h). If we denote $\omega = a\wedge dr + b$ by $\begin{pmatrix} a\wedge dr \\ b\end{pmatrix}$, then we may write

$$\Delta\begin{pmatrix} a\wedge dr \\ b\end{pmatrix} = \begin{pmatrix}\Delta_{11} & \Delta_{12}\\ \Delta_{21} & \Delta_{22}\end{pmatrix}\begin{pmatrix} a\wedge dr \\ b\end{pmatrix},$$

where

$$\begin{cases} \Delta_{11}(a\wedge dr) &= \left\{-\dfrac{\partial^2}{\partial r^2}a-(m-2p+1)\right.\\ &\qquad \left.\cdot\left(\dfrac{ff''-(f')^2}{f^2}+\dfrac{f'}{f}\dfrac{\partial}{\partial r}\right)a+\dfrac{1}{f^2}\Delta_0 a\right\}\wedge dr,\\ \Delta_{12}(b) &= (-1)^p\dfrac{2f'}{f^3}\delta_0 b\wedge dr,\\ \Delta_{21}(a\wedge dr) &= (-1)^p\dfrac{2f'}{f}d_0 a,\\ \Delta_{22}(b) &= -\dfrac{\partial^2}{\partial r^2}b-(m-2p-1)\dfrac{f'}{f}\dfrac{\partial}{\partial r}b+\dfrac{1}{f^2}\Delta_0 b.\end{cases}$$

We set the two linear subspaces

$$\begin{aligned}\tilde{A} &:= \{a\wedge dr\in A^p(\Omega)|\ a \text{ is an } L^2(p-1)\text{-form not including } dr\},\\ \tilde{B} &:= \{b\in A^p(\Omega)|\ b \text{ is an } L^2p\text{-form not including } dr\}.\end{aligned}$$

We shall define two inner products on $\tilde{A}$ and $\tilde{B}$ respectively. On $\tilde{A}$, for $a_1\wedge dr,\ a_2\wedge dr\in\tilde{A}$,

$$\begin{aligned}(a_1\wedge dr,\, a_2\wedge dr)_A &:= \int_\Omega \langle a_1,\, a_2\rangle_g v_g,\\ (a_1\wedge dr,\, a_2\wedge dr)_{A'} &:= \int_{r_0}^{r_1}(a_1,\, a_2)_0 dr,\end{aligned}$$

and on $\tilde{B}$, for $b_1,\ b_2 \in \tilde{B}$,

$$(b_1,\, b_2)_B := \int_\Omega \langle b_1,\, b_2\rangle_g v_g,$$
$$(b_1,\, b_2)_{B'} := \int_{r_0}^{r_1} (b_1,\, b_2)_0 dr.$$

Here $(\ ,\ )_0$ means the L^2 inner product on the canonical sphere (S^{m-1}, h). Then we define the two Hilbert spaces A and A' by the completion of $\tilde{A}$ with respect to $(\ ,\ )_A$ and $(\ ,\ )_{A'}$. Similarly, we define the two Hilbert spaces B and B' by the completion of $\tilde{B}$ with respect to $(\ ,\ )_B$ and $(\ ,\ )_{B'}$. If we define two linear operators $\Phi_{A'}$ from A' to A and $\Phi_{B'}$ from B' to B respectively by

$$\begin{aligned} \Phi_{A'}(a \wedge dr) &= f^{-\frac{m-2p+1}{2}} a \wedge dr, \quad \text{for } a \wedge dr \in A', \\ \Phi_{B'}(b) &= f^{-\frac{m-2p-1}{2}} b, \quad \text{for } b \in B', \end{aligned}$$

then we see that $\Phi_{A'}$ and $\Phi_{B'}$ are isometric. We define $\Delta' : A' \oplus B' \longrightarrow A' \oplus B'$ by

$$\begin{aligned} \Delta' &:= \begin{pmatrix} \Phi_{A'}^{-1} & 0 \\ 0 & \Phi_{B'}^{-1} \end{pmatrix} \circ \Delta \circ \begin{pmatrix} \Phi_{A'} & 0 \\ 0 & \Phi_{B'} \end{pmatrix} \\ &= \begin{pmatrix} \Delta'_{11} & \Delta'_{12} \\ \Delta'_{21} & \Delta'_{22} \end{pmatrix}, \end{aligned}$$

where

$$\begin{cases} \Delta'_{11}(a \wedge dr) &= \Bigg\{ -\dfrac{\partial^2}{\partial r^2} a + \Bigg[\dfrac{(m-2p+1)(m-2p+3)}{4} \left(\dfrac{f'}{f}\right)^2 \\ & \qquad -\dfrac{m-2p+1}{2} \dfrac{f''}{f} \Bigg] a + \dfrac{1}{f^2} \Delta_0 a \Bigg\} \wedge dr, \\ \Delta'_{12}(b) &= (-1)^p \dfrac{2f'}{f^2} \delta_0 b \wedge dr, \\ \Delta'_{21}(a \wedge dr) &= (-1)^p \dfrac{2f'}{f^2} d_0 a, \\ \Delta'_{22}(b) &= -\dfrac{\partial^2}{\partial r^2} b + \Bigg[\dfrac{(m-2p-1)(m-2p-3)}{4} \left(\dfrac{f'}{f}\right)^2 \\ & \qquad + \dfrac{m-2p-1}{2} \dfrac{f''}{f} \Bigg] b + \dfrac{1}{f^2} \Delta_0 b. \end{cases}$$

Diagram 3.1.

$$\begin{array}{ccc} A \oplus B & \xrightarrow{\ \Delta\ } & A \oplus B \\ {\scriptstyle \Phi_{A'}}\Big\uparrow{\scriptstyle \Phi_{B'}} & & {\scriptstyle \Phi_{A'}^{-1}}\Big\downarrow{\scriptstyle \Phi_{B'}^{-1}} \\ A' \oplus B' & \xrightarrow{\ \Delta'\ } & A' \oplus B' \end{array}$$

Since Δ is equivalent to Δ', $\mathrm{Spec}^{(p)}(\Delta_F;\Omega) = \mathrm{Spec}^{(p)}(\Delta'_F;\Omega)$, where the subscript "$F$" means the Friedrichs extension. So we shall consider the spectrum of Δ'. We denote by $\mathrm{Spec}_d(\Delta_0^{(p)})$, $\mathrm{Spec}_\delta(\Delta_0^{(p)})$ the eigenvalues of $\Delta_0^{(p)} = \Delta_{(S^{m-1},h)}^{(p)}$ on closed, co-closed p-forms for (S^{m-1},h) respectively, which were computed by S. Gallot et D. Meyer [GM-75].

Lemma 3.2 (Eichhorn). *The spectrum* $\mathrm{Spec}^{(p)}(\Delta_F;\Omega)$ *consists of the pure point spectrum and we obtain*

$$\mathrm{Spec}^{(p)}(\Delta_F;\Omega) = \bigcup_{\nu\in\mathrm{Spec}_d(\Delta_0^{(p-1)})} \mathrm{Spec}(D_{11,\nu,F}) \cup \bigcup_{\mu\in\mathrm{Spec}_\delta(\Delta_0^{(p)})} \mathrm{Spec}(D_{22,\mu,F}) \cup \bigcup_{\mu\in\mathrm{Spec}_\delta(\Delta_0^{(p-1)})\cap\mathrm{Spec}_d(\Delta_0^{(p)})} \mathrm{Spec}(\Delta'_{2,\mu,F}).$$

Here, we set

$$\begin{cases} D_{11,\nu}(\varphi) := -\dfrac{d^2}{dr^2}\varphi + \left[\dfrac{(m-2p+1)(m-2p+3)}{4}\left(\dfrac{f'}{f}\right)^2 - \dfrac{m-2p+1}{2}\dfrac{f''}{f}\right]\varphi + \dfrac{\nu}{f^2}\varphi, \\ D_{22,\mu}(\varphi) := -\dfrac{d^2}{dr^2}\varphi + \left[\dfrac{(m-2p-1)(m-2p-3)}{4}\left(\dfrac{f'}{f}\right)^2 + \dfrac{m-2p-1}{2}\dfrac{f''}{f}\right]\varphi + \dfrac{\mu}{f^2}\varphi, \\ \Delta'_{2,\mu}\begin{pmatrix} a\wedge dr \\ b \end{pmatrix} := \begin{pmatrix} \Delta'_{11}(a\wedge dr) + (-1)^p\dfrac{2f'}{f^2}\delta_0 b\wedge dr \\ \Delta'_{22}(b) + (-1)^p\dfrac{2f'}{f^2}d_0 a \end{pmatrix}, \end{cases}$$

where $\varphi \in C_0^\infty(r_0, r_1)$ *and* a, b *are the* $p-1$, p*-forms respectively satisfying that*

$$\Delta_0^{(p-1)} a = \mu a, \quad \delta_0 a = 0, \quad \Delta_0^{(p)} b = \mu b, \quad d_0 b = 0.$$

We immediately see the following proposition from Lemma 3.2.

Proposition 3.4. *For* $\nu = (p-1)(m-p+1)$, $\mu = (p+1)(m-p-1)$, *we have*

$$\lambda_1^{(p)}(\Omega) \leq \min\{\lambda_1(D_{11,\nu,F}),\ \lambda_1(D_{22,\mu,F})\}.$$

Using the min-max principle, we have the following.

Proposition 3.4. *Let the notation be the above. We have*

$$\lambda_1(D_{11,\nu,F}) = \inf_{\varphi \in C_0^\infty(r_0,r_1)} \frac{\displaystyle\int_{r_0}^{r_1} \left\{ (\varphi')^2 + \frac{(m-2p+1)(m-2p+3)}{4} \left(\frac{f'}{f}\right)^2 \varphi^2 - \frac{m-2p+1}{2} \frac{f''}{f} \varphi^2 + (p-1)(m-p+1) \frac{1}{f^2} \varphi^2 \right\} dr}{\displaystyle\int_{r_0}^{r_1} \varphi^2 dr},$$

$$\lambda_1(D_{22,\mu,F}) = \inf_{\varphi \in C_0^\infty(r_0,r_1)} \frac{\displaystyle\int_{r_0}^{r_1} \left\{ (\varphi')^2 + \frac{(m-2p-1)(m-2p-3)}{4} \left(\frac{f'}{f}\right)^2 \varphi^2 + \frac{m-2p-1}{2} \frac{f''}{f} \varphi^2 + (p+1)(m-p-1) \frac{1}{f^2} \varphi^2 \right\} dr}{\displaystyle\int_{r_0}^{r_1} \varphi^2 dr},$$

where $\nu = (p-1)(m-p+1)$, $\mu = (p+1)(m-p-1)$.

Thus, we have only to estimate the eigenvalue $\lambda_1(D_{22,\nu,F})$ or $\lambda_1(D_{22,\mu,F})$ from above.

4. Estimates of $f(r)$ and $\dfrac{f'(r)}{f(r)}$

Let a Riemannian manifold (M, g) be as in Section 3. First we begin with estimates of $f(r)$.

PROPOSITION 4.1 (Estimates of f(r)). *If* $(m-1)\alpha \le \mathrm{Ric}_{B(x_0,\rho)}$ *and* $K_{B(x_0,\rho)} \le \beta$ $(\alpha < 0 < \beta)$*, then for* $0 < r < \min\{\rho, \frac{\pi}{\sqrt{\beta}}\}$, $s_\beta(r) \le f(r) \le s_\alpha(r)$.

PROOF. We use the notation in [Sa-96]. Since $\theta(r, \partial_r) = f(r)^{m-1}$ for $0 < r < \rho$, we see the inequality $f(r) \le s_\alpha(r)$ directly from Bishop's comparison theorem ([Sa-96], p.154).

On the other hand, if we take the Jacobi field $Y(r) = f(r)e(r)$, where $e(r)$ is a unit vector field perpendicular to a geodesic γ, we see that $y_\beta(r) = s_\beta(r)$ and $|Y(r)| = f(r)$ for $0 < r < \min\{\rho, \frac{\pi}{\sqrt{\beta}}\}$. Hence we obtain the inequality $s_\beta(r) \le f(r)$ from Rauch's comparison theorem (Theorem 2.7 in [Sa-96], p.152). □

LEMMA 4.2. *If* $(m-1)\alpha \le \mathrm{Ric}_{B(x_0,\rho)} \le (m-1)\beta$ $(\alpha < 0 < \beta)$*, for* $0 < r < \min\{\rho, \frac{\pi}{\sqrt{\beta}}\}$, $\alpha \le -\frac{f''}{f}(r) \le \beta$.

PROOF. If we set $(u^0, u^1, \cdots, u^{m-1}) = (r, u^1, \cdots, u^{m-1})$, then the metric g is written as $g_{a0} = \delta_{a0}, g_{ij} = f^2 h_{ij}$ for all $a = 0, 1, \cdots, m-1,\ i, j = 1, 2, \cdots, m-1$. Since $\Gamma^a_{00} = 0, \Gamma^i_{0j} = \frac{f'}{f}\delta^i_j$, we obtain $R^i{}_{0i0} = \partial_i \Gamma^i_{00} - \partial_0 \Gamma^i_{i0} + \sum_{a=0}^{m-1} \Gamma^i_{ia}\Gamma^a_{00} - \sum_{a=0}^{m-1} \Gamma^i_{0a}\Gamma^a_{i0} = -\frac{f''}{f}$. Thus we see that $\mathrm{Ric}(\partial_r, \partial_r) = \sum_{i=1}^{m-1} R^i{}_{0i0} = -(m-1)\frac{f''}{f}$ and $g(\partial_r, \partial_r) \equiv 1$, hence Lemma 4.2 follows. □

Next we shall estimate $\dfrac{f'(r)}{f(r)}$.

LEMMA 4.3 (Riccati Comparison Theorem). *Let* $\psi_1,\ \psi_2$ *be* C^∞ *functions on* $(0, \rho)$ *such that* $\psi_1' + {\psi_1}^2 \le -\delta$ *and* $\psi_2' + {\psi_2}^2 \ge -\delta$.

(1) *If* $\psi_1(r_0) \ge \psi_2(r_0)$ *for some* $r_0 \in (0, \rho)$*, then* $\psi_1(r) \ge \psi_2(r)$ *for any* $r \le r_0$.

(2) *If* $\psi_1(r_0) \le \psi_2(r_0)$ *for some* $r_0 \in (0, \rho)$*, then* $\psi_1(r) \le \psi_2(r)$ *for any* $r \ge r_0$.

PROOF. (See [K-89], (1.6.2).) For the sake of the readers, we shall write the proof. If we set $\Psi(r) := (\psi_1(r) - \psi_2(r)) \exp\{\int_c^r \{\psi_1(t) + \psi_2(t)\}dt\}$ for $c > 0$, then we have $\Psi' = \{({\psi_1}' + {\psi_1}^2) - ({\psi_2}' + {\psi_2}^2)\} \exp\{\int_c^r \{\psi_1 + \psi_2\}dt\} \leq 0$ by the assumption of ψ_1, ψ_2. If $\psi_1(r_0) \geq \psi_2(r_0)$ i.e. $\Psi(r_0) \geq 0$ for some $r_0 \in (0, \rho)$, $\Psi(r) \geq 0$ i.e. $\psi_1(r) \geq \psi_2(r)$ for any $0 < r \leq r_0$. (2) is similar, too. □

Now we set the two functions $\psi(r) := \frac{f'(r)}{f(r)}$, $\psi_\delta(r) := \frac{s'_\delta(r)}{s_\delta(r)}$ on $(0, \rho)$.

PROPOSITION 4.4. *If $(m-1)\alpha \leq \mathrm{Ric}_{B(x_0,\rho)} \leq (m-1)\beta$ ($\alpha < 0 < \beta$), then for $0 < r < \rho$, $\psi_\beta(r) \leq \psi(r) \leq \psi_\alpha(r)$, i.e. $\frac{s'_\beta}{s_\beta}(r) \leq \frac{f'}{f}(r) \leq \frac{s'_\alpha}{s_\alpha}(r)$.*

PROOF. (See [K-89], (1.6.3).) For the sake of the readers, we shall write the proof. We prove $\psi(r) \leq \psi_\alpha(r)$ on $(0, \rho)$ by a contradiction. Suppose that there is an r_0, $0 < r_0 < \rho$, such that $\psi(r_0) > \psi_\alpha(r_0)$. By the continuity, we have $\psi(r_0) \geq \psi_\alpha(r_0 - \varepsilon)$ for some $\varepsilon > 0$. Since Lemma 4.2 and $\psi' = \frac{f''f - (f')^2}{f^2} = \frac{f''}{f} - \psi^2$, we have $\psi' + \psi^2 \leq -\alpha$. Of course, ${\psi_\alpha}' + {\psi_\alpha}^2 = -\alpha$. If we apply Lemma 4.3 for $\psi_1(r) := \psi(r)$ and $\psi_2(r) := \psi_\alpha(r - \varepsilon)$, we obtain $\psi(r) \geq \psi_\alpha(r - \varepsilon)$ for any $r \leq r_0$. But this is a contradiction because of $\lim_{r \downarrow \varepsilon} \psi_\alpha(r - \varepsilon) = \infty$. It is also similar that $\psi(r) \geq \psi_\beta(r)$ on $(0, \rho)$. □

Because for $0 < r < \min\{\rho, \frac{\pi}{2\sqrt{\beta}}\}$, $\frac{s'_\beta}{s_\beta}(r) = \sqrt{\beta}\frac{\cos(\sqrt{\beta}r)}{\sin(\sqrt{\beta}r)} \geq 0$, we obtain the following.

COROLLARY 4.5. $0 \leq \left(\frac{s'_\beta}{s_\beta}(r)\right)^2 \leq \left(\frac{f'}{f}(r)\right)^2 \leq \left(\frac{s'_\alpha}{s_\alpha}(r)\right)^2.$

5. Proof of Theorem 1.2

We set

$$R_i := \frac{1}{k+1} \min\{\rho, \frac{\pi}{2\sqrt{\beta}}\} \times i, \quad (i = 0, 1, 2, \cdots, k+1).$$

Note that $R_1 = R_{i+1} - R_i$. We take the disjoint annuli $\Omega_i := (R_i, R_{i+1}) \times S^{m-1}$ in $B(x_0, \rho)$ and estimate $\lambda_1^{(p)}(\Omega_i)$ from above for $i = 1, 2, \cdots, k$. We

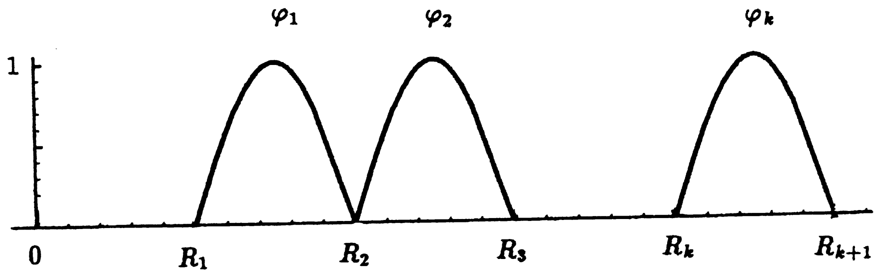

Figure 1: test functions

use the min-max principle for the operators $D_{11,\nu}, D_{22,\mu}$ on Ω_i showed in Proposition 3.4. Then we may take the test function on (R_i, R_{i+1}) as

$$\varphi_i(r) := \sin\left(\frac{\pi}{R_1}(r - R_i)\right)$$

(see Figure 1).

First we have $\displaystyle\int_{R_i}^{R_{i+1}} \varphi_i(r)^2 dr = \frac{R_1}{2}$, and $\displaystyle\int_{R_i}^{R_{i+1}} \varphi_i'(r)^2 dr = \frac{\pi^2}{2R_1}$.

When $p \le \dfrac{m+1}{2}$, from Corollary 4.5 and the monotone decreasing of $\left(\frac{s_\alpha'}{s_\alpha}(r)\right)^2$, we see that

$$\begin{aligned}
&\frac{(m-2p+1)(m-2p+3)}{4}\int_{R_i}^{R_{i+1}}\left(\frac{f'}{f}(r)\right)^2\varphi_i(r)^2 dr\\
&\le \frac{(m-2p+1)(m-2p+3)}{4}\left(\frac{s_\alpha'}{s_\alpha}(R_1)\right)^2\int_{R_i}^{R_{i+1}}\varphi_i(r)^2 dr\\
&= \frac{(m-2p+1)(m-2p+3)}{4}\left(\frac{s_\alpha'}{s_\alpha}(R_1)\right)^2\frac{R_1}{2},
\end{aligned}$$

and from Lemma 4.2 we see that

$$\begin{aligned}
-\frac{m-2p+1}{2}\int_{R_i}^{R_{i+1}}\frac{f''}{f}(r)\varphi_i(r)^2 dr &\le \frac{m-2p+1}{2}\beta\int_{R_i}^{R_{i+1}}\varphi_i(r)^2 dr\\
&= \frac{m-2p+1}{2}\beta\frac{R_1}{2}.
\end{aligned}$$

When $\dfrac{m-1}{2} \leq p$, we similarly see that

$$\frac{(m-2p-1)(m-2p-3)}{4}\int_{R_i}^{R_{i+1}}\left(\frac{f'}{f}\right)^2 {\varphi_i}^2 dr \leq \frac{(m-2p-1)(m-2p-3)}{4}\left(\frac{s'_\alpha}{s_\alpha}(R_1)\right)^2 \frac{R_1}{2}$$

and that

$$\frac{m-2p-1}{2}\int_{R_i}^{R_{i+1}}\frac{f''}{f}{\varphi_i}^2 dr \leq \frac{-(m-2p-1)}{2}\,\beta\,\frac{R_1}{2}.$$

Finally from Proposition 4.1, we have

$$\int_{R_i}^{R_{i+1}}\left(\frac{\varphi_i(r)}{f(r)}\right)^2 dr \leq \frac{1}{s_\beta^2(R_1)}\int_{R_i}^{R_{i+1}}\varphi_i(r)^2 dr = \frac{1}{s_\beta^2(R_1)}\frac{R_1}{2}.$$

Thus we obtain the estimates of $\lambda_k^{(p)}(\Omega_i)$ uniformly for i. Hence from Lemma 2.1 we obtain Theorem 1.2. □

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