J. Math. Sci. Univ. Tokyo
6 (1999), 49–86.

The Mellin Transformation of Strongly Increasing Functions

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Abstract. The definition of the Mellin transformation is modified in a way suitable for the study of some classes of functions with exponential growth at zero.

Introduction

The (local) Mellin transform of a continuous function f on $I = (0, t], t > 0$ can be defined by

$$\mathcal{M}_t f(z) = \int_0^t f(x) x^{-z-1} dx.$$

To make the integral convergent it is required usually that f has polynomial growth at zero i.e. $|f(x)| \leq Cx^v$ with some $v \in \mathbb{R}$. Then $\mathcal{M}_t f$ is a holomorphic function on $\{\operatorname{Re} z < v\}$. If we further assume that f is a generalized analytic function, i.e. a function representable in the form $f(x) = S[x^{\cdot}]$, where S is a Laplace distribution supported by $Z \subset v + \overline{\mathbb{R}}_+$, then $\mathcal{M}f$ extends holomorphically to a function on $\mathbb{C} \setminus Z$ and $\mathcal{M}f$ determines uniquelly the S ([SZ]). For a generalized analytic function f the set $\operatorname{supp} S$ can be interpreted as the set of those exponents $\alpha \in \mathbb{R}$ which enter into decomposition of f into powers x^α and $(\ln x)^k x^\alpha$, $0 \leq k \leq m$ with some $m \in \mathbb{N}_0$. It appears that generalized analytic functions are defined on the universal covering space $\widetilde{B}(\rho)$ of the punctured disc $B(\rho) \setminus \{0\}, \rho > 0$ and the information about S is also carried by a pair of functions

$$\mathcal{M}_t^{\pm} f(z) = \int_{\gamma^{\pm}(t)} f(x) x^{-z-1} dx,$$

1991 *Mathematics Subject Classification*. 30D15, 44A10, 46F12.
Key words: Laplace, Taylor and Mellin transformations, Laplace ultradistributions.
Supported by KBN grant no. 2 P03A 006 08.

where $\gamma^{\pm}(t) = \{x \in \widetilde{B}(\rho) : \mp \arg x > 0, |x| = t\}, t < \rho$. Now $\mathcal{M}_t^{\pm} f \in \mathcal{O}(\{\pm \operatorname{Im} z > 0\})$ and the definition of $\mathcal{M}_t^{\pm} f$ requires only some estimation of f on $\gamma^{\pm}(t)$ (e.g. polynomial growth in $\arg x$) and not on the interval $(0,t]$. Using this observation we extend the definition of the Mellin transformation to some classes of holomorphic functions on $\widetilde{B}(\rho; r) = \widetilde{B}(\rho) \setminus \widetilde{B}(r)$ with $r < \rho$. In fact we study the Mellin transforms of functions $f \in \mathcal{O}^{(N)}\big(\widetilde{B}(\rho; r)\big)$, where $\mathcal{O}^{(N)}\big(\widetilde{B}(\rho; r)\big)$ is the image under the Taylor transformation of the space $L^{(N_p)\prime}_{(\ln r, \ln \rho)}(\mathbb{R})$ of Laplace ultradistributions on $\mathbb{R}$, (see Section 2) i. e. $f \in \mathcal{O}^{(N)}\big(\widetilde{B}(\rho; r)\big)$ if and only if f is representable in the form

$$f(x) = S[x^{\cdot}] \text{ for } r < |x| < \rho$$

with some $S \in L^{(N_p)\prime}_{(\ln r, \ln \rho)}(\mathbb{R})$. Then $\mathcal{M}_t^{\pm} f \in \mathcal{O}(\{\pm \operatorname{Im} z > 0\})$, has the growth of type $t^{-\operatorname{Re} z} \exp\{N^*\big(L/|\operatorname{Im} z|\big)\}$ near the real axis (N^* is the growth function of the sequence (N_p)) and the difference of the boundary values of $\mathcal{M}_t^{+} f$ and $\mathcal{M}_t^{-} f$ is equal to $2\pi i \cdot S$.

A special attention is paid on the study of the space $\mathcal{O}^{(N)}_{(M^*)}\big(\widetilde{B}(\rho)\big)$ of holomorphic functions on $\widetilde{B}(\rho)$ bounded by

$$C \exp\{M^*(K/|x|) + N(L|\ln x|)\}$$

for $|x|$ close to zero. To justify our interest in this space let us recall that solutions of differential equations of type $P(x^{1+k}D)f = 0$, where P is a polynomial and $k > 0$, have the above growth with $M^*(\rho) = \rho^k$. More generally - as will be proved in the subsequent paper ([Ł3]) - the same remains true for solutions of linear differential equations with analytic coefficients at zero. On the other hand, if $f(x) = S[x^{\cdot}]$ with $S \in L^{(N_p)\prime}_{(\emptyset, \ln \rho)}(\overline{\mathbb{R}}_+)$ and $J(D) = \sum_{k=0}^{\infty} a_k D^k$ is an ultradifferential operator of class (M_p) then $u = J(D)f \in \mathcal{O}^{(N)}_{(M^*)}\big(\widetilde{B}(\rho)\big)$. Since $\mathcal{M}_t f \in \mathcal{O}(\mathbb{C} \setminus \overline{\mathbb{R}}_+)$ and $\mathcal{M}_t(Df)(z) = (z+1)\mathcal{M}_t f(z+1) + f(t)t^{-z-1}$ we can also define the Mellin transform of u by

$$\mathcal{M}_t u(z) = \sum_{k=0}^{\infty} a_k \frac{\Gamma(z+k+1)}{\Gamma(z+k)} \mathcal{M}_t f(z+k) \mod \mathcal{O}^{\exp}(\mathbb{C}).$$

It appears that the series on the right hand side of the above formula is locally uniformly convergent on $\mathbb{C} \setminus \mathbb{R}$ and the definition of $\mathcal{M}_t u$ is consistent with the previous one.

In the final section we compute some explicit examples of Taylor and Mellin tranforms. In particular, we compute the Mellin transform of $f(x) = \exp\{1/x^k\}, k > 0$. Since $\mathcal{M}_1 f$ is the Laplace transform of $g(y) = f(\exp\{-y\})$ this is equivalent to the computation of the Laplace transform of $g(y) = \exp\{\exp\{y^k\}\}$. In the case $k = 1$ similar result was obtained by Deakin in [D] by means of asymptotic expansions. Let us also point out here that Komatsu defined in [K2] the Laplace transformation for functions with arbitrary growth but he has not given any explicitly evaluated transforms of functions with superexponential growth.

Finally let us remark that throughout the paper one can omit the symbols (N) and (N_p) working with the spaces of Laplace distributions and replacing $\exp\{N\big(K \cdot \rho(z)\big)\}$ and $\exp\{N^*\big(K \cdot \rho(z)\big)\}$ by $\big(\rho(z)\big)^K$.

1. Preliminaries

Throughout the paper $(M_p)_{p\in\mathbb{N}_0}$, $(N_p)_{p\in\mathbb{N}_0}$ are sequences of positive numbers with $M_0 = N_0 = 0$. We assume that both sequences satisfy the conditions

(M.1) $\quad M_p^2 \le M_{p-1}M_{p+1}$ for $p \in \mathbb{N}$;

(M.2) $\quad M_p \le H^p M_q M_{p-q}$ for $p \in \mathbb{N},\ 0 < q \le p$ with some $H < \infty$;

(M.3) $\quad \displaystyle\sum_{p=q}^{\infty} \frac{M_{p-1}}{M_p} \le Aq\frac{M_q}{M_{q+1}}$ for $q \in \mathbb{N}$ with some $A < \infty$.

We refer to [K1] or [M] for the significance of these conditions. We associate with the sequence (M_p) the *weight function*

$$M(\rho) = \sup_{p\in\mathbb{N}_0} \ln\frac{\rho^p}{M_p},\ \rho > 0 \tag{1}$$

and the *growth function*

$$M^*(\rho) = \sup_{p\in\mathbb{N}_0} \ln\frac{\rho^p p!}{M_p},\ \rho > 0. \tag{2}$$

The weight function M is an increasing, convex function in $\ln\rho$, which vanishes for $\rho \in (0, 1]$ and satisfies ([K1], [BMT], Remark 8.9)

$$M(2\rho) \le C\big(M(\rho) + 1\big) \text{ for } \rho > 0; \tag{3}$$

$$\lim_{\rho\to\infty}\frac{\ln\rho}{M(\rho)}=0; \tag{4}$$

$$\int_0^\infty \frac{M(t\rho)}{t^2}dt \le C\big(M(\rho)+1\big) \text{ for } \rho>0. \tag{5}$$

By (3) it follows that ([BMT], Lemma 1.2)

$$M(\rho_1+\rho_2)\le C\big(M(\rho_1)+M(\rho_2)+1\big) \text{ for } \rho_1,\rho_2>0. \tag{6}$$

The growth function M^* is also increasing and vanishing near zero. Furthermore by (2) and (M.2) we get

$$\rho\exp\{M^*(\rho)\}\le C\exp\{M^*(H\rho)\} \text{ for } \rho>0 \tag{7}$$

where H is the constant in (M.2). We also have

$$\sum_{n=0}^\infty \frac{n!\rho^n}{M_n}\le 2\exp\{M^*(2\rho)\} \text{ for } \rho>0 \tag{8}$$

and

$$\sum_{n=0}^\infty \frac{\rho^n}{M_n}\le 2\exp\{M(2\rho)\} \text{ for } \rho>0. \tag{8'}$$

Put $m_p=M_p/M_{p-1}$ for $p\in\mathbb{N}$ and

$$m(\rho)=|\{p\in\mathbb{N}: m_p\le\rho\}|,\ m^*(\rho)=|\{p\in\mathbb{N}: m_p/p\le\rho\}| \text{ for } \rho>0.$$

Then ([R], p.65)

$$M(\rho)=\int_0^\rho \frac{m(\lambda)}{\lambda}d\lambda,\ M^*(\rho)=\int_0^\rho \frac{m^*(\lambda)}{\lambda}d\lambda \text{ for } \rho>0. \tag{9}$$

Lemma 1. *For every $c > 0$ one can find $H(c) < \infty$ such that*

$$M^*(\rho) \le cM^*(r\rho) \text{ for } r \ge H(c),\ \rho > 0. \tag{10}$$

Proof. Put $N_p = \min_{0\le q\le p} M_q M_{p-q}$ for $p \in \mathbb{N}_0$. Then ([R], p. 55) $n_{2p-1} = n_{2p} = m_p$, $p \in \mathbb{N}$. So $2m^*(\lambda) \le n^*(\lambda)$ for $\lambda \ge 0$. By (9) and (M.2) we get

$$2M^*(\rho) \le N^*(\rho) \le \sup_{p\in\mathbb{N}_0} \ln \frac{(H\rho)^p p!}{M_p} = M^*(H\rho) \text{ for } \rho > 0.$$

Fix $c > 0$ and take $N \in \mathbb{N}_0$ such that $2^N \ge c^{-1}$. Iterating the above inequality N times we get (10) with $H(c) = H^N$. □

We say that functions f and g defined in a neighbourhood of ∞ are equivalent, and we write $f(\rho) \sim g(\rho)$ as $\rho \to \infty$, if there are positive constants C and L such that

$$C^{-1} f(L^{-1}\rho) \le g(\rho) \le Cf(L\rho) \text{ for } \rho \text{ big enough.} \tag{11}$$

In a similar way we define $f(\rho) \sim g(\rho)$ as $\rho \to 0$.

The *Young conjugate* of the weight function M is defined by

$$\omega^*(t) = \sup_{\rho>0} \big(M(\rho) - \rho t\big) \text{ for } t > 0. \tag{12}$$

It follows by the Stirling formula that ([PV], Lemma 5.6)

$$\omega^*(1/\rho) \sim M^*(\rho) \text{ as } \rho \to \infty$$

and by (5) we get

Lemma 2 ([F], Lemma 2.3). *For every $L < \infty$ there exists $C_L < \infty$ such that*

$$M^*(L\rho) \le C_L\big(M^*(\rho) + 1\big) \text{ for } \rho > 0. \tag{13}$$

The most important example of a sequence (M_p) satisfying (M.1) – (M.3) is the Gevrey sequence $M_p = (p!)^s$ for $p \in \mathbb{N}_0$ where $s > 1$. Then $M(\rho) \sim \rho^{1/s}$ and $M^*(\rho) \sim \rho^{1/(s-1)}$ as $\rho \to \infty$.

An entire function $P(z) = \sum_{k=0}^{\infty} a_k z^k$ is called a *symbol of class* (M_p) if it satisfies one of the following equivalent conditions ([K1], Propositions 4.5 and 4.6)

(i) There are constants $L < \infty$ and $C < \infty$ such that

$$|a_k| \le CL^k/M_k \text{ for } k \in \mathbb{N}_0;$$

(ii) There are constants $L < \infty$ and $C < \infty$ such that

$$|P(z)| \le C\exp\{M(L|z|)\} \text{ for } z \in \mathbb{C};$$

(iii) P has Hadamard's factorization

$$P(z) = az^{\nu_0} \prod_{k=\nu_0+1}^{\infty} \Big(1 - \frac{z}{c_k}\Big)$$

and there are constants $L < \infty$ and $C < \infty$ such that

$$\int_0^{\rho} \frac{\nu(\lambda) - \nu_0}{\lambda} d\lambda \le M(L\rho) + C \text{ for } \rho > 0,$$

where $\nu(\lambda) = |\{k \in \mathbb{N}_0 : |c_k| \le \lambda\}|$.

Definition ([Ł2]). Let $(N_p)_{p\in\mathbb{N}_0}$ be a sequence of positive numbers satisfying the conditions (M.1) — (M.3) and $N_0 = 1$. Let $\nu \in \mathbb{R} \cup \{-\infty\}$ and $\omega \in \mathbb{R} \cup \{\infty\}$. The space $L^{(N_p)\prime}_{(\nu,\omega)}(\mathbb{R})$ of *Laplace ultradistributions on* $\mathbb{R}$ is defined as the dual space of

$$L^{(N_p)}_{(\nu,\omega)}(\mathbb{R}) = \varinjlim_{a>\nu, b<\omega} L^{(N_p)}_{a,b}(\mathbb{R})$$

where for any $a, b \in \mathbb{R}$

$$L_{a,b}^{(N_p)}(\mathbb{R}) = \varprojlim_{h>0} L_{a,b,h}^{(N_p)}(\mathbb{R})$$

with

$$(14)\ \ L_{a,b,h}^{(N_p)}(\mathbb{R}) = \{\varphi \in C^\infty(\mathbb{R}) : \|\varphi\|_{a,b,h}^{(N_p)} = \sup_{x\in\mathbb{R}} \sup_{\alpha\in\mathbb{N}_0} \frac{|D^\alpha \varphi(x)|\kappa_{a,b}(x)}{h^\alpha N_\alpha} < \infty \ \}$$

and

$$\kappa_{a,b}(x) = \begin{cases} e^{-ax} & \text{for } x < 0, \\ e^{-bx} & \text{for } x \geq 0. \end{cases} \tag{15}$$

The space $L_{(\emptyset,\omega)}^{(N_p)\prime}(\overline{\mathbb{R}}_+)$ of *Laplace ultradistributions on* $\overline{\mathbb{R}}_+$ is defined in an analogous way replacing $L_{a,b,h}^{(N_p)}(\mathbb{R})$ by

$$\begin{aligned}(16)\quad & L_{\emptyset,b,h}^{(N_p)}(\overline{\mathbb{R}}_+) \\ & = \{\varphi \in C^\infty(\overline{\mathbb{R}}_+) : \|\varphi\|_{\emptyset,b,h}^{(N_p)} = \sup_{x\in\overline{\mathbb{R}}_+} \sup_{\alpha\in\mathbb{N}_0} \frac{|D^\alpha \varphi(x)|e^{-bx}}{h^\alpha N_\alpha} < \infty \ \}.\end{aligned}$$

We have topological inclusion

$$L_{(\emptyset,\omega)}^{(N_p)\prime}(\overline{\mathbb{R}}_+) \hookrightarrow L_{(\nu,\omega)}^{(N_p)\prime}(\mathbb{R}) \text{ for any } \nu \in \mathbb{R} \cup \{-\infty\}.$$

Immediately by the definition we get

Lemma 3. *A linear functional S on $L_{(\nu,\omega)}^{(N_p)}(\mathbb{R})$ (resp. $L_{(\emptyset,\omega)}^{(N_p)}(\overline{\mathbb{R}}_+)$) belongs to $L_{(\nu,\omega)}^{(N_p)\prime}(\mathbb{R})$ (resp. $L_{(\emptyset,\omega)}^{(N_p)\prime}(\overline{\mathbb{R}}_+)$) iff for every $a > \nu, b < \omega$ one can find $h > 0$ such that*

$$|S[\varphi]| \leq C\|\varphi\|_{a,b,h}^{(N_p)} \text{ for } \varphi \in L_{a,b}^{(N_p)}(\mathbb{R}) \tag{17}$$

(*resp.* $\qquad |S[\varphi]| \leq C\|\varphi\|_{\emptyset,b,h}^{(N_p)}$ for $\varphi \in L_{\emptyset,b}^{(N_p)}(\overline{\mathbb{R}}_+)$).

Let $P(z) = \sum_{k=0}^{\infty} a_k z^k$ be a symbol of class (N_p). Then an *ultradifferential operator* $P(D)$ *of class* (N_p) defines linear continuous mappings

$$P(D) : L^{(N_p)}_{(\nu,\omega)}(\mathbb{R}) \to L^{(N_p)}_{(\nu,\omega)}(\mathbb{R}),\ L^{(N_p)\prime}_{(\nu,\omega)}(\mathbb{R}) \to L^{(N_p)\prime}_{(\nu,\omega)}(\mathbb{R})$$

where for $S \in L^{(N_p)\prime}_{(\nu,\omega)}(\mathbb{R}), \varphi \in L^{(N_p)}_{(\nu,\omega)}(\mathbb{R})$

$$P(D)S[\varphi] \stackrel{\text{def}}{=} S[P^*(D)\varphi] \text{ with } P^*(D) = \sum_{k=0}^{\infty}(-1)^k a_k D^k.$$

The space of Laplace ultradistributions can be characterized as follows.

Theorem 1 (Structure theorem, [Ł2], Theorem 7). *In order that an ultradistribution* $S \in D^{(N_p)\prime}(\mathbb{R})$ *belong to* $L^{(N_p)\prime}_{(\nu,\omega)}(\mathbb{R})$ *it is necessary and sufficient that for any* $a > \nu, b < \omega$ *there are ultradifferential operators* $J_a(D), J_b(D)$ *of class* (N_p) *and functions* $S_a, S_b \in C^0(\mathbb{R})$ *such that* $\operatorname{supp} S_a \subset \overline{\mathbb{R}}_-, |S_a(x)| \le Ce^{-ax}$ *for* $x \le 0$, $\operatorname{supp} S_b \subset \overline{\mathbb{R}}_+, |S_b(x)| \le Ce^{-bx}$ *for* $x \ge 0$ *and*

$$S = J_a(D)S_a + J_b(D)S_b \textit{ in } L^{(N_p)\prime}_{(a,b)}(\mathbb{R}).$$

Definition ([Ł2]). Let W be a tubular neighbourhood of $\mathbb{R}$ i.e. $W = \mathbb{R} + i\Omega$ where Ω is a neighbourhood of zero. Put $W^\pm = W \cap \{\pm \operatorname{Im} z > 0\}$. Let $H \in \mathcal{O}(W^\pm)$ and let $\nu \in \mathbb{R} \cup \{-\infty\}, \omega \in \mathbb{R} \cup \{\infty\}$. Assume that for any $a > \nu, b < \omega$, $H(\cdot \pm iy) \in L^{(N_p)\prime}_{a,b}(\mathbb{R})$ for $0 < \pm y$ small enough. If for any $\varphi \in L^{(N_p)}_{(\nu,\omega)}(\mathbb{R})$ there exists the limit $\lim_{\pm y \to 0} H(\cdot \pm iy)[\varphi] \stackrel{\text{def}}{=} b^\pm H[\varphi]$ then, by the Banach-Steinhaus theorem, $b^\pm H \in L^{(N_p)\prime}_{(\nu,\omega)}(\mathbb{R})$ and we say that H has the *boundary value* $b^\pm H$ from above (below) in $L^{(N_p)\prime}_{(\nu,\omega)}(\mathbb{R})$.

In the sequel we shall assume that $\nu < \omega$. Let $z \in \mathbb{C}$. Denote by $\exp_z$ the function

$$\mathbb{R} \ni x \to \exp_z(x) \stackrel{\text{def}}{=} e^{xz}.$$

Then $\exp_z$ is an $L^{(N_p)}_{(\nu,\omega)}(\mathbb{R})$-valued holomorphic function on $\{\nu < \operatorname{Re} z < \omega\}$ and for any $\nu < a < b < \omega, h > 0$ we have

$$\|\exp_z\|^{(N_p)}_{a,b,h} = \exp\{N(|z|/h)\} \text{ for } a \le \operatorname{Re} z \le b.$$

Thus, the *Laplace transform* of $S \in L^{(N_p)\prime}_{(\nu,\omega)}(\mathbb{R})$ defined by

$$\mathcal{L}S(z) = S[\exp_z] \text{ for } \nu < \operatorname{Re} z < \omega$$

is a holomorphic function on $\nu < \operatorname{Re} z < \omega$ and by Lemma 3 it satisfies for any $\nu < a < b < \omega$

$$(18)\qquad |\mathcal{L}S(z)| \le C \exp\{N(K_{a,b}|z|)\} \text{ for } a \le \operatorname{Re} z \le b$$

with some $K_{a,b} < \infty$ and $C < \infty$.

Conversely assume that $F \in \mathcal{O}(\{\nu < \operatorname{Re} z < \omega\})$ satisfies (18) with F in place of $\mathcal{L}S$. Then for any $\nu < a < b < \omega$ one can find a symbol $P_{a,b}$ of class (N_p) not vanishing on $\{\operatorname{Re} z < b + 1\}$ such that

$$\frac{\exp\{N(K_{a,b}|z|)\}}{|P_{a,b}(z)|} \le \frac{1}{(1+|z|)^2} \text{ for } a \le \operatorname{Re} z \le b$$

([Ł1], Lemma 3). Thus, modifying the proof of Theorem 3.6.1 of [Z] we find that there exists an $S \in L^{(N_p)\prime}_{(\nu,\omega)}(\mathbb{R})$ such that $F(z) = \mathcal{L}S(z)$ for $\nu < \operatorname{Re} z < \omega$.

Let $S \in L^{(N_p)\prime}_{(\nu,\omega)}(\mathbb{R})$. We define the *Taylor transform* of S by

$$\mathcal{T}S(x) = S[x^{\cdot}] \text{ for } x \in \widetilde{B}(e^{\omega}; e^{\nu}).$$

To describe the image of $L^{(N_p)\prime}_{(\nu,\omega)}(\mathbb{R})$ under the Taylor transformation we introduce the space

$$(19)\qquad \begin{aligned} &\mathcal{O}^{(N)}\big(\widetilde{B}(e^{\omega}; e^{\nu})\big) = \Big\{u \in \mathcal{O}\big(\widetilde{B}(e^{\omega}; e^{\nu})\big) : \\ &\textit{for every } e^{\nu} < r < t < e^{\omega} \textit{ there exist } K \textit{ and } C < \infty \textit{ such that} \\ &|u(x)| \le C \exp\{N\big(K|\ln x|\big)\} \textit{ for } r \le |x| \le t\Big\}. \end{aligned}$$

Since

$$\mathcal{T}S(x) = \mathcal{L}S(\ln x) \text{ for } x \in \widetilde{B}(e^{\omega}; e^{\nu})$$

we have

Theorem 2. *The Taylor transformation is an isomorphism of* $L^{(N_p)\prime}_{(\nu,\omega)}(\mathbb{R})$ *onto the space* $\mathcal{O}^{(N)}\big(\widetilde{B}(e^{\omega}; e^{\nu})\big)$.

Analogously we have

Theorem 3 ([Ł1], Theorem 6). *The Taylor transformation is an isomorphism of* $L^{(N_p)\prime}_{(\emptyset,\omega)}(\overline{\mathbb{R}}_+)$ *onto the space* $\mathcal{O}^{(N)}\big(\widetilde{B}(e^{\omega}, \emptyset)\big)$, *where*

$$\mathcal{O}^{(N)}\big(\widetilde{B}(e^{\omega}, \emptyset)\big) = \Big\{u \in \mathcal{O}\big(\widetilde{B}(e^{\omega})\big) : \text{ for every } t < e^{\omega}$$
$$\text{there exist } K \text{ and } C < \infty \text{ such that (19) holds for } |x| \le t\Big\}.$$

The elements of $\mathcal{O}^{(N)}\big(\widetilde{B}(e^{\omega}, \emptyset)\big)$ are called *generalized analytic functions*.

2. The Mellin Transformation

Let $u \in \mathcal{O}^{(N)}\big(\widetilde{B}(e^{\omega}, \emptyset)\big)$ and $t < e^{\omega}$. Then the *(local) Mellin transform* of u is defined by

$$\mathcal{M}_t u(z) = \int_0^t u(x) x^{-z-1} dx. \tag{20}$$

Since $|u(x)| \le C\exp\{N(K|\ln x|)\} \le C_{\varepsilon} x^{-\varepsilon}$ for $0 < x \le t$ with any $\varepsilon > 0$ the integral converges for $\operatorname{Re} z < 0$. Let $\operatorname{Re} z < 0$ with $\pm \operatorname{Im} z > 0$. Then by the Cauchy formula we have

$$\int_0^t u(x) x^{-z-1} dx = \int_{\gamma^{\pm}(t)} u(x) x^{-z-1} dx$$

where $\gamma^\pm(t) = \{x \in \widetilde{B}(e^\omega; e^\nu) : \mp \arg x \geq 0, |x| = t\}$ and the orientation of $\pm\gamma^\pm(t)$ is positive. Since the right hand side of above equality converges locally uniformly on $\{\pm \operatorname{Im} z > 0\}$ we get $\mathcal{M}_t u \in \mathcal{O}(\mathbb{C} \setminus \overline{\mathbb{R}}_+)$.

Now let $u \in \mathcal{O}^{(N)}\big(\widetilde{B}(e^\omega; e^\nu)\big)$ with $\nu < \omega$. Inspired by the above considerations we define the *Mellin transforms* $\mathcal{M}_t^\pm u$ by

$$\mathcal{M}_t^\pm u(z) = \int\limits_{\gamma^\pm(t)} u(x) x^{-z-1} dx \text{ for } \pm \operatorname{Im} z > 0 \tag{21}$$

with $\gamma^\pm(t)$ as above.

To study properties of $\mathcal{M}_t^\pm u$ observe that

$$\mathcal{M}_t^\pm u(z) = \pm i t^{-z} \int\limits_0^\infty u(te^{\mp i\varphi}) e^{\pm i\varphi z} d\varphi \tag{22}$$

and since $|u(te^{\mp i\varphi})| \leq C\exp\{N\big(K(|\ln t| + \varphi)\big)\} \leq C_\varepsilon e^{\varepsilon\varphi}$ for any $\varepsilon > 0$, the integral converges locally uniformly in $\{z \in \mathbb{C} : \pm \operatorname{Im} z > 0\}$. So $\mathcal{M}_t^\pm u \in \mathcal{O}(\{\pm \operatorname{Im} z > 0\})$. To estimate the integral in (22) write $z = \alpha + i\beta$. Then by the definition of $N(\rho)$ we derive for $0 < \beta \leq 1$ with $\widetilde{K} = K\max(1, |\ln t|)$

$$|\int\limits_0^\infty u(te^{-i\varphi}) e^{i\varphi z} d\varphi| \leq C \int\limits_0^\infty \exp\{N\big(\widetilde{K}(1+\varphi)\big) - \varphi\beta\} d\varphi$$

$$= C \int\limits_0^\infty \sup_{p \in \mathbb{N}_0} \frac{\widetilde{K}^p (1+\varphi)^p}{N_p} e^{-\varphi\beta/2} \cdot e^{-\varphi\beta/2} d\varphi \leq C \sup_{p \in \mathbb{N}_0} \frac{\widetilde{K}^p}{N_p} \Big(\frac{2p}{\beta}\Big)^p e^{-p+\beta/2} \cdot \frac{2}{\beta}$$

since $\sup\limits_{\varphi \geq 0} \{(1+\varphi)^p e^{-\varphi\beta/2}\} = (2p)^p \beta^{-p} e^{-p+\beta/2}$.

Similary if $\beta > 1$ writing $e^{-\varphi\beta} = e^{-\varphi} \cdot e^{-\varphi(\beta-1)}$ we obtain

$$|\int\limits_0^\infty u(te^{-i\varphi}) e^{i\varphi z} d\varphi| \leq C \sup_{p \in \mathbb{N}_0} \frac{\widetilde{K}^p}{N_p} p^p e^{-p+1} \cdot \frac{1}{\beta - 1}.$$

Thus, by the Stirling formula, the definition of $N^*(\rho)$ and (7) we get with some $L < \infty$

$$(23)\qquad |\mathcal{M}_t^{\pm}u(z)| \le \begin{cases} Ct^{-\operatorname{Re} z}\exp\{N^*(L/|\operatorname{Im} z|)\} & \text{for } 0 < \pm\operatorname{Im} z \le 1,\\ Ct^{-\operatorname{Re} z}/|\operatorname{Im} z| & \text{for } \pm\operatorname{Im} z \ge 1.\end{cases}$$

Our next aim is to compute the boundary value of $\mathcal{M}_t^{\pm}u$. To this end fix $e^{\nu} < r < t < e^{\omega}$ and take K such that (19) is true. By Lemma 3 of [Ł1] there exists a symbol P of class (N_p) not vanishing on $\{z : \operatorname{Re} z < \ln t + 1\}$ such that

$$(24)\qquad |\frac{1}{P(\ln x)}| \le \frac{\exp\{-N(K|\ln x|)\}}{(1+|\ln x|)^2} \text{ for } |x| \le t.$$

Put

$$(25)\qquad v(x) = u(x)/P(\ln x) \text{ for } |x| \le t.$$

Then $|v(x)| \le C(1+|\ln x|)^{-2}$ for $r \le |x| \le t$. So for $r \le \tau \le t$, $\mathcal{M}_\tau^{\pm}v \in \mathcal{O}(\{\pm\operatorname{Im} z > 0\}) \cap C^0(\{\operatorname{Im} z \ge 0\})$ and the function

$$(26)\qquad \mathbb{R} \ni \alpha \to g(\alpha) = \frac{1}{2\pi i}\big(\mathcal{M}_\tau^{+}v(\alpha) - \mathcal{M}_\tau^{-}v(\alpha)\big)$$

does not depend on the choise of τ. Observe that

$$g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(\tau e^{i\varphi})(\tau e^{i\varphi})^{-\alpha-i\beta}d\varphi_{|\beta=0}$$
$$= \frac{1}{2\pi i}\int_{\ln\tau+i\mathbb{R}} u(e^{\zeta})e^{-\zeta(\alpha+i\beta)}d\zeta_{|\beta=0}.$$

Thus, by Lemma 3.6.1 of [Z], $g \in L^{(N_p)\prime}_{(\ln r,\ln t)}(\mathbb{R})$ and

$$\mathcal{T}g(x) = \mathcal{L}g(\ln x) = v(e^{\ln x}) = v(x) \text{ for } r < |x| < t.$$

Finally put $S = P^*(D)g \in L^{(N_p)\prime}_{(\ln r,\ln t)}(\mathbb{R})$. Then

$$\mathcal{T}S(x) = P(\ln x)\mathcal{T}g(x) = P(\ln x)v(x) = u(x) \text{ for } r < |x| < t.$$

Since the above holds for any $e^\nu < r < t < e^\omega$, by the uniqueness theorem ([Z], Theorem 5.3.2) we get $S \in L^{(N_p)\prime}_{(\nu,\omega)}(\mathbb{R})$ and $\mathcal{T}S(x) = u(x)$ for $e^\nu < |x| < e^\omega$. Thus, we have proved

THEOREM 4. *Let $u \in \mathcal{O}^{(N)}\big(\widetilde{B}(e^\omega; e^\nu)\big)$ with $\nu < \omega$. Then for any $e^\nu < t < e^\omega$, $\mathcal{M}_t^\pm u \in \mathcal{O}(\{\pm \operatorname{Im} z > 0\})$ satisfies (23) with some $L < \infty$ and $C < \infty$. Furthermore the difference of boundary values $S \overset{\text{def}}{=} \frac{1}{2\pi i}\big(b^+(\mathcal{M}_t^+ u) - b^-(\mathcal{M}_t^- u)\big) \in L^{(N_p)\prime}_{(\nu,\omega)}(\mathbb{R})$ is independent of t and $\mathcal{T}S = u$.*

REMARK 1. For $e^\nu < r < t < e^\omega$ the difference $\mathcal{M}_t^\pm u - \mathcal{M}_r^\pm u$ extends holomorphically to an entire function $G_{r,t}$ which satisfies

$$|G_{r,t}(z)| \le C \frac{\kappa_{\ln t, \ln r}(\operatorname{Re} z)}{1 + |\operatorname{Im} z|}. \tag{27}$$

with $\kappa_{\ln t, \ln t}$ given by (15).

PROOF. Indeed

$$\mathcal{M}_t^\pm u(z) - \mathcal{M}_r^\pm u(z) = \int_r^t u(x) x^{-z-1} dx \text{ for } \pm \operatorname{Im} z > 0$$

and the integral converges for $z \in \mathbb{C}$. Now since

$$z \int_r^t u(x) x^{-z-1} dx = u(r) r^{-z} - u(t) t^{-z} + \int_r^t Du(x) x^{-z} dx, \ z \in \mathbb{C}$$

and Du is bounded on $[r,t]$ we get (27). □

THEOREM 5. *Let $\nu < \omega$ and let W be a bounded tubular neighbourhood of $\mathbb{R}$. Let $\{H_t\}_{t \in (e^\nu, e^\omega)}$ be a family of functions $H_t \in \mathcal{O}(W \setminus \mathbb{R})$ satisfying: for every $\varepsilon > 0$ there exists $L < \infty$ such that for any closed tubular set $\widetilde{W} \subset\subset W$*

$$|H_t(z)| \le C \exp\big\{N^*\big(L/|\operatorname{Im} z|\big)\big\} \cdot \kappa_{\ln t + \varepsilon, \ln t - \varepsilon}(\operatorname{Re} z) \text{ for } z \in \widetilde{W} \setminus \mathbb{R} \tag{28}$$

Assume that for any $e^\nu < r < t < e^\omega$, $H_t^\pm - H_r^\pm$ *extends holomorphically to a function* $H_{r,t} \in \mathcal{O}(V)$ *with some tubular neighbourhood* V *of* $\mathbb{R}$ *such that for any* $\varepsilon > 0$ *and* $\widetilde{V} \subset\subset V$

$$|H_{r,t}(z)| \le C\kappa_{\ln t+\varepsilon,\ln r-\varepsilon}(\operatorname{Re} z) \text{ for } z \in \widetilde{V}.$$

Then one can find a unique $u \in \mathcal{O}^{(N)}\big(\widetilde{B}(e^\omega; e^\nu)\big)$ *such that for any* $e^\nu < t < e^\omega$, $\mathcal{M}_t^\pm u - H_t^\pm$ *extends to a function* F_t *holomorphic on a tubular neighbourhood* V *of* $\mathbb{R}$ *satisfying with any* $\varepsilon > 0$ *and* $\widetilde{V} \subset\subset V$

$$|F_t(z)| \le C\kappa_{\ln t+\varepsilon,\ln r-\varepsilon}(\operatorname{Re} z) \text{ for } z \in \widetilde{V}. \tag{29}$$

Proof. It follows by Proposition 2 of [Ł2] that H_t admits boundary values $S_t^\pm \overset{\text{def}}{=} b^\pm(H_t^\pm) \in L^{(N_p)\prime}_{(\ln t,\ln t)}(\mathbb{R})$. Since for any $e^\nu < r < t < e^\omega$

$$S_t^+ - S_t^- = b^+(H_t^+) - b^-(H_t^-) = b^+(H_r^+ + H_{r,t}) - b^-(H_r^- + H_{r,t}) = S_r^+ - S_r^-,$$

the difference $S = S_t^+ - S_t^-$ does not depend on t and thus defines an element of $L^{(N_p)\prime}_{(\nu,\omega)}(\mathbb{R})$. Put $u = 2\pi i\mathcal{T}S$. Then by Theorem 2, $u \in \mathcal{O}^{(N)}\big(\widetilde{B}(e^\omega; e^\nu)\big)$. Let $\widetilde{S} = b^+(\mathcal{M}_t^+ u) - b^-(\mathcal{M}_t^- u)$. It follows by Theorem 4 that $\widetilde{S} \in L^{(N_p)\prime}_{(\nu,\omega)}(\mathbb{R})$ and $2\pi i\mathcal{T}\widetilde{S} = u$. So by Theorem 2, $\widetilde{S} = S$ and $b^+(\mathcal{M}_t^+ u - H_t^+) = b^-(\mathcal{M}_t^- u - H_t^-)$. Thus, it follows by the proof of Theorem 9 of [Ł2] that $\mathcal{M}_t^\pm u - H_t^\pm$ extends to a function F_t holomorphic on a tubular neighbourhood V of $\mathbb{R}$ satisfying (29) with any $\varepsilon > 0$ and $\widetilde{V} \subset\subset V$. To show the uniqueness of u assume that $\widetilde{u} \in \mathcal{O}^{(N)}\big(\widetilde{B}(e^\omega; e^\nu)\big)$ is another function such that $\mathcal{M}_t^\pm \widetilde{u} - H_t^\pm$ extends to $\widetilde{F}_t \in \mathcal{O}(V)$ and satisfying (29). Since $\mathcal{M}_t^\pm(u - \widetilde{u}) = F_t - \widetilde{F}_t$, $\mathcal{M}_t^\pm(u - \widetilde{u})$ is an entire function which satisfies by Theorem 4 and the Phragmén-Lindelöf theorem ([B])

$$|\mathcal{M}_t^\pm(u - \widetilde{u})(z)| \le C\frac{t^{-\operatorname{Re} z}}{1 + |\operatorname{Im} z|} \text{ for } z \in \mathbb{C}.$$

Hence by the Liouville theorem $\mathcal{M}_t^\pm(u - \widetilde{u}) = 0$ and so $u = \widetilde{u}$. □

3. The Space $\mathcal{O}^{(N)}_{(M^*)}\big(\widetilde{B}(\rho)\big)$

Definition. Let $\rho > 0$.

$$\begin{aligned}&\mathcal{O}^{(N)}_{(M^*)}\big(\widetilde{B}(\rho)\big) = \Big\{u \in \mathcal{O}\big(\widetilde{B}(\rho)\big) : \\ &\text{for every } r < \rho \text{ one can find } H < \infty \text{ and } K < \infty \text{ such that} \\ &|u(x)| \le C\exp\big\{M^*(H/|x|) + N(K|\ln x|)\big\} \text{ for } |x| \le r\Big\}.\end{aligned} \tag{30}$$

Let $u \in \mathcal{O}^{(N)}_{(M^*)}\big(\widetilde{B}(e^\omega)\big)$. Since $\mathcal{O}^{(N)}_{(M^*)}\big(\widetilde{B}(e^\omega)\big) \subset \mathcal{O}^{(N)}\big(\widetilde{B}(e^\omega;0)\big)$ it follows by Theorem 4 that for any $t < e^\omega$, $\mathcal{M}^{\pm}_t u \in \mathcal{O}(\{\pm \operatorname{Im} z > 0\})$ satisfies (23) and with $S = \frac{1}{2\pi i}\big(b^+(\mathcal{M}^+_t u) - b^-(\mathcal{M}^-_t u)\big) \in L^{(N_p)\prime}_{(-\infty,\omega)}(\mathbb{R})$ we have $\mathcal{T}S = u$. This time however we have a little more information. Namely repeating the computations leading to (23) we find that the constant C in (23) can be choosen as $C = \widetilde{C}\exp\{M^*(H/t)\}$ where $\widetilde{C}$ does not depend on $0 < t \le r$ with any $r < e^\omega$.

To compute the difference $b^+(\mathcal{M}^+_t u) - b^-(\mathcal{M}^-_t u)$ let us define

$$M_*(r) = \sup_{\rho \ge 0}\big(r\rho - M^*(e^\rho)\big) \text{ for } r > 0. \tag{31}$$

Proposition 1. *Let $t > 0$. Then for every $c > 0$ one can find $H = H(c)t < \infty$ such that*

$$t^{-\alpha}\exp\big\{-(1/c)M_*(-c\alpha)\big\} \le \inf_{\tau \le t}\exp\big\{M^*\big(H/\tau\big)\big\}\tau^{-\alpha} \text{ for } \alpha < 0, \tag{32}$$

and for every $H < \infty$ one can find $c = c(H/t) > 0$ such that

$$\inf_{\tau \le t}\exp\big\{M^*\big(H/\tau\big)\big\}\tau^{-\alpha} \le Ct^{-\alpha}\exp\big\{-(1/c)M_*(-c\alpha)\big\} \text{ for } \alpha < 0. \tag{33}$$

Proof. Fix $c > 0$ and note that

$$(1/c)M_*(-c\alpha) = \sup_{\rho \ge 0}\big\{-\alpha\rho - (1/c)M^*(e^\rho)\big\} \text{ for } \alpha < 0.$$

Thus applying Lemma 1 we derive for $\alpha < 0$

$$
\begin{aligned}
t^{-\alpha}\exp\big\{-(1/c)M_*(-c\alpha)\big\} &\le \exp\Big\{-\big[\alpha\ln t + \sup_{\rho\ge 0}\big(-\alpha\rho - M^*(H(c)e^{\rho})\big)\big]\Big\} \\
&\overset{\tau = te^{-\rho}}{=} \exp\Big\{-\sup_{\tau\le t}\big(\alpha\ln\tau - M^*\big(H/\tau\big)\big)\Big\} \\
&= \inf_{\tau\le t}\exp\big\{M^*\big(H/\tau\big)\big\}\tau^{-\alpha} \text{ where } H = H(c)t.
\end{aligned}
$$

To prove the second part fix $H < \infty$ and observe that by Lemma 2 we can find $C_H = C(H/t) < \infty$ such that

$$M^*\big(He^{\rho}/t\big) \le C_H\big(M^*(e^{\rho}) + 1\big) \text{ for } \rho \ge 0.$$

So we estimate for $\alpha < 0$

$$
\begin{aligned}
\inf_{\tau\le t}\exp\big\{M^*\big(H/\tau\big)\big\}\tau^{-\alpha} &= \exp\Big\{-\sup_{\tau\le t}\big(\alpha\ln\tau - M^*(H/\tau)\big)\Big\} \\
&\le \exp\Big\{-\alpha\ln t + C_H - \sup_{\rho\ge 0}\big(-\alpha\rho - C_H M^*(e^{\rho})\big)\Big\} \\
&= Ct^{-\alpha}\exp\Big\{-C_H\sup_{\rho\ge 0}\big(-\alpha\rho - M^*\big(e^{C_H\rho}\big)\big)\Big\} \\
&= Ct^{-\alpha}\exp\big\{-C_H M_*\big(-\alpha/C_H\big)\big\} \text{ where } C = e^{C_H}.
\end{aligned}
$$

Thus we get (33) with $c = (C_H)^{-1}$. $\square$

Now fix $t < e^{\omega}$ and define g by (26) with v given by (25) and P by (24). Since g does not depend on the choise of $0 < \tau \le t$ we get by (33)

$$
\begin{aligned}
|g(\alpha)| &\le C\inf_{\tau\le t}\exp\{M^*(H/\tau)\}\tau^{-\alpha} \\
&\le Ct^{-\alpha}\exp\big\{-(1/c)M_*(-c\alpha)\big\} \textit{ for } \alpha < 0
\end{aligned}
$$

with some $c = c(H/t) > 0$. So we have

Theorem 6. *Let $u \in \mathcal{O}^{(N)}_{(M^*)}\big(\widetilde{B}(e^{\omega})\big)$ and fix $t < e^{\omega}$. Then one can find $H < \infty, L < \infty$ and $C < \infty$ such that for $0 < \tau \le t$*

$$
(34)\qquad |\mathcal{M}^{\pm}_{\tau}u(z)| \le C\exp\{M^*(H/\tau)\}\tau^{-\operatorname{Re} z} \times \begin{cases} \exp\{N^*(L/|\operatorname{Im} z|)\} & \textit{for } 0 < \pm\operatorname{Im} z \le 1, \\ 1/|\operatorname{Im} z| & \textit{for } \pm\operatorname{Im} z \ge 1. \end{cases}
$$

Furthermore there exists a symbol P of class (N_p) and a function $g \in C^0(\mathbb{R})$ satisfying

$$|g(\alpha)| \leq C \cdot \begin{cases} t^{-\alpha}\exp\big\{-(1/c)M_*(-c\alpha)\big\} & \text{for } \alpha < 0, \\ t^{-\alpha} & \text{for } \alpha \geq 0 \end{cases} \tag{35}$$

with some $c > 0$ and M_* given by (31) such that $u(x) = \mathcal{T}S(x)$ for $|x| < t$ with $S = P^*(D)g$.

REMARK 2. If $M_p = (p!)^s$ with $s > 1$ then $\exp M_*(\rho) \sim \rho^{(s-1)\rho}$ as $\rho \to \infty$.

Now we shall give converse statements to those in Theorem 6.

THEOREM 7. *Let $S \in L^{(N_p)\prime}_{(-\infty,\omega)}(\mathbb{R})$. Assume that for every $t < e^\omega$ there exist a symbol P_t of class (N_p) and $g_t \in C^0(\mathbb{R})$ such that for any $\varepsilon > 0$*

$$|g_t(\alpha)| \leq C_\varepsilon \cdot \begin{cases} \big(te^\varepsilon\big)^{-\alpha}\exp\big\{-(1/c)M_*(-c\alpha)\big\} & \text{for } \alpha < 0, \\ \big(te^{-\varepsilon}\big)^{-\alpha} & \text{for } \alpha \geq 0 \end{cases} \tag{36}$$

with some $c > 0$ independent of $\varepsilon > 0$ and

$$S = P_t^*(D)g_t \text{ in } L^{(N_p)\prime}_{(-\infty,\ln t)}(\mathbb{R}).$$

Then $u = \mathcal{T}S \in \mathcal{O}^{(N)}_{(M^*)}\big(\widetilde{B}(e^\omega)\big)$

PROOF. Fix $r < e^\omega$ and choose $r < t < e^\omega$. Put $v^-(x) = \int\limits_{-\infty}^{0} g_t(\alpha)x^\alpha d\alpha$, $v^+(x) = \int\limits_{0}^{\infty} g_t(\alpha)x^\alpha d\alpha$ and $v = v^- + v^+$. Then choosing $\varepsilon = 1/2\min(\ln 2, \ln t/r)$ in (36) we get for $|x| \leq r$

$$|v^+(x)| \leq C_\varepsilon \int\limits_0^\infty \Big(\frac{e^\varepsilon|x|}{t}\Big)^\alpha d\alpha \leq C_r.$$

Now, by (32) and the Fatou lemma we derive for $|x| \le r$

$$|v^-(x)| \le C_\varepsilon \int_{-\infty}^{0} \inf_{\tau \le t} \exp\big\{M^*(H/\tau)\big\}\big(e^{-\varepsilon}|x|/\tau\big)^\alpha d\alpha$$
$$\le C_\varepsilon \inf_{\tau \le t} \exp\big\{M^*(H/\tau)\big\}(\ln|x| - \ln\tau - \varepsilon)^{-1}$$
$$\overset{\tau=|x|/2}{\le} C_r \exp\big\{M^*(2H/|x|)\big\}.$$

So $|v(x)| \le C\exp\big\{M^*(2H/|x|)\big\}$ for $|x| \le r$. Since for $|x| \le r$

$$u(x) = \mathcal{T}\big(P_t^*(D)g_t\big)(x) = P_t(\ln x)\mathcal{T}g_t(x) = P_t(\ln x)v(x)$$

and

$$|P_t(\ln x)| \le C\exp\{N(K|\ln x|)\} \text{ with some } K < \infty$$

we get the conclusion. □

Theorem 8. *Let $\{H_\tau\}_{\tau<e^\omega}$ be a family of functions $H_\tau \in \mathcal{O}(\mathbb{C} \setminus \mathbb{R})$ satisfying for any $t < e^\omega$ and $0 < \tau \le t$*

$$(34') \qquad |H_\tau(z)| \le C\exp\{M^*(H/\tau)\}\tau^{-\operatorname{Re} z} \times \begin{cases} \exp\{N^*(L/|\operatorname{Im} z|)\} & \text{for } 0 < \pm\operatorname{Im} z \le 1, \\ 1/|\operatorname{Im} z| & \text{for } \pm\operatorname{Im} z \ge 1 \end{cases}$$

with some C, H and L independent of $0 < \tau \le t$. If for any $0 < r < t < e^\omega$, $H_t^\pm - H_r^\pm$ extends holomorphically to an entire function $G_{r,t}$ satisfying (27) then one can find a unique $u \in \mathcal{O}_{(M^*)}^{(N)}\big(\widetilde{B}(e^\omega)\big)$ such that $H_\tau = \mathcal{M}_\tau u$ for $\tau < e^\omega$.

Proof. As in the proof of Theorem 5, H_τ admits boundary values $b^\pm(H_\tau) \in L_{(\ln\tau,\ln\tau)}^{(N_p)\prime}(\mathbb{R})$ and $S = b^+(H_\tau) - b^-(H_\tau)$ does not depend on $0 < \tau < e^\omega$. Hence $S \in L_{(-\infty,\omega)}^{(N_p)\prime}(\mathbb{R})$. Now, since C, H and L does not depend on $0 < \tau \le t$ following the proof of Proposition 2 of [Ł2] we find by (33) that for any $t < e^\omega$ there exist a symbol P_t of class (N_p) and $g_t \in C^0(\mathbb{R})$ satisfying (36) such that $S = P_t^*(D)g_t$ in $L_{(-\infty,\ln t)}^{(N_p)\prime}(\mathbb{R})$. Hence by Theorem

7, $u \overset{\text{def}}{=} 2\pi i \mathcal{T} S \in \mathcal{O}^{(N)}_{(M^*)}\big(\widetilde{B}(e^{\omega})\big)$. Put $F_\tau = \mathcal{M}_\tau u - H_\tau$. Then F_τ is an entire function satisfying

$$|F_\tau(z)| \leq \frac{C}{1 + |\operatorname{Im} z|} \cdot \exp\{M^*(H/\tau)\}\tau^{-\operatorname{Re} z} \text{ for } z \in \mathbb{C}$$

and the Liouville theorem implies $F_\tau \equiv 0$. The uniqueness of u follows by Theorem 4. □

4. Relations between $\mathcal{O}^{(N)}\big(\widetilde{B}(\rho, \emptyset)\big)$ and $\mathcal{O}^{(N)}_{(M^*)}\big(\widetilde{B}(\rho)\big)$

Obviously $\mathcal{O}^{(N)}\big(\widetilde{B}(\rho, \emptyset)\big) \subset \mathcal{O}^{(N)}_{(M^*)}\big(\widetilde{B}(\rho)\big)$ for any growth function M^*. The theorem below implies that applying an ultradifferential operator P of class (M_p) to $f \in \mathcal{O}^{(N)}\big(\widetilde{B}(\rho, \emptyset)\big)$ we get an element of $\mathcal{O}^{(N)}_{(M^*)}\big(\widetilde{B}(\rho)\big)$.

Theorem 9. *Let $u \in \mathcal{O}\big(\widetilde{B}(\rho)\big)$. Assume that for every $t < \rho$ one can find an ultradifferential operator P_t of class (M_p) and $f_t \in \mathcal{O}^{(N)}_{(M^*)}\big(\widetilde{B}(t)\big)$ such that*

$$u(x) = P_t(D) f_t(x) \text{ for } |x| < t. \tag{37}$$

Then $u \in \mathcal{O}^{(N)}_{(M^*)}\big(\widetilde{B}(\rho)\big)$.

Proof. Fix $r < t$ and choose $r < r' < t < \rho$. Take $x \in \widetilde{B}(t)$ with $|x| \leq r$ and write for $n \in \mathbb{N}_0$

$$D^n f_t(x) = \frac{n!}{2\pi i} \int\limits_{\gamma_x} \frac{f(\zeta)}{(\zeta - x)^{n+1}} d\zeta,$$

where $\gamma_x = \{\zeta \in \widetilde{B}(t) : |\zeta - x| = |x|/R\}$ with $R = \max\big(2, r/(r' - r)\big)$. Since for $\zeta \in \gamma_x$

$$\begin{aligned} |f_t(\zeta)| &\leq C \exp\{M^*(H/|\zeta|) + N(K|\ln \zeta|)\} \\ &\leq C \exp\{M^*(2H/|x|) + N\big(K(|\ln x| + \ln 2 + \pi/6)\big)\} \end{aligned}$$

we get by (6)

$$|D^n f_t(x)| \le n!(R/|x|)^n \cdot \sup_{\zeta\in\gamma_x} |f_t(\zeta)|$$
$$\le C_1 n!(R/|x|)^n \exp\{M^*(2H/|x|) + N(K|\ln x|)\}.$$

Now let $P_t(D) = \sum\limits_{n=0}^{\infty} a_n D^n$ with $|a_n| \le CL^n/M_n$. Then for $|x| \le r$ we get by (8)

$$|u(x)| \le C \sum_{n=0}^{\infty} \frac{n!}{M_n}\Big(\frac{LR}{|x|}\Big)^n \exp\{M^*(2H/|x|) + N(K|\ln x|)\}$$
$$\le 2C\exp\{M^*(\widetilde{H}/|x|) + N(K|\ln x|)\}$$

where $\widetilde{H} = \max(2H, 2LR)$. Since $r < \rho$ was arbitrary this implies that $u \in \mathcal{O}^{(N)}_{(M^*)}\big(\widetilde{B}(\rho)\big)$. $\square$

Corollary 1. *Let $u \in \mathcal{O}\big(\widetilde{B}(\rho)\big)$. If for every $t < \rho$ one can find an ultradifferential operator P_t of class (M_p) and $f_t \in \mathcal{O}^{(N)}\big(\widetilde{B}(t,\emptyset)\big)$ such that (37) holds then $u \in \mathcal{O}^{(N)}_{(M^*)}\big(\widetilde{B}(\rho)\big)$.*

We would like to show a converse result to that of Corollary 1. Namely that for every $u \in \mathcal{O}^{(N)}_{(M^*)}\big(\widetilde{B}(\rho)\big)$ and $t < \rho$ there exist an ultradifferential operator P_t of class (M_p) and $f_t \in \mathcal{O}^{(N)}\big(\widetilde{B}(t)\big)$ such that (37) holds. However we are not able to prove this conjecture in general, we have only a weaker result.

Theorem 10. *Let $u \in \mathcal{O}^{(N)}_{(M^*)}\big(\widetilde{B}(\rho)\big)$ and assume that $M^*(r) \sim r^{1/(s-1)}$ as $r \to \infty$ with $s > 1$. Then for every $t < \rho$ one can find an ultradifferential operator $J(D)$ of class $(p!)^s$ and a sequence of functions*

$$v_\nu \in \mathcal{O}\big(\{x \in \widetilde{B}(t) : |\arg x - 2\nu\pi| < 3\pi/2\}\big),\ \nu \in \mathbb{Z}$$

satisfying with some $\delta > 0$ and $C,\ \widetilde{K} < \infty$ not depending on $\nu \in \mathbb{Z}$

(38) $|v_\nu(x)| \le C\exp\{N\big(\widetilde{K}|\nu|\big)\}$ *for* $x \in \widetilde{B}(t)$ *with* $|\arg x - 2\nu\pi| \le \pi + \delta$

and

$$J(D)v_\nu = u \text{ on } \widetilde{B}(t) \cap \{|\arg x - 2\nu\pi| < 3\pi/2\}.$$

Furthermore for any $\beta > 0$ *one can find* $\widetilde{H}$, $\widetilde{K}$ *and* $C < \infty$ *such that*

$$|v_\nu(x)| \le C\exp\{M^*\big(\widetilde{H}/|x|\big) + N\big(\widetilde{K}|\nu|\big)\} \tag{39}$$

for $x \in \widetilde{B}(t)$ *with* $|\arg x - (2\nu+1)\pi| \le \pi/2 - \beta$.

We procede the proof of Theorem 10 by two lemmas

Lemma 4. *Let* $m_p \sim p^s$ *with* $s > 1$. *Put*

$$P(\zeta) = \prod_{p=1}^{\infty} \big(1 + \frac{\zeta}{m_p}\big) \text{ for } \zeta \in \mathbb{C}.$$

Then there exist $\psi > 0$ *and* $c > 0$ *such that*

$$\ln|P(\zeta)| \ge c|\zeta|^{1/s} \text{ for } \zeta \in A_\psi \stackrel{\text{def}}{=} \{\zeta \in \mathbb{C} : |\arg\zeta| \le \psi + \pi/2\}.$$

Hence

$$|1/P(\zeta)| \le C\exp\{-M(k|\zeta|)\} \tag{40}$$

$$\text{for } \zeta \in A_\psi \text{ with some } C < \infty \text{ and } k > 0.$$

Furthermore for any $\delta > 0$ *we can find* $K_\delta < \infty$ *such that*

$$\Big|\frac{1}{P(\zeta)}\Big| \le C\exp\big\{M\big(K_\delta|\zeta|\big)\big\} \text{ for } |\arg\zeta| \le \pi - \delta. \tag{41}$$

Proof. Take $0 < \psi < \pi/2\min(1, s-1)$. Since P does not vanish on A_ψ it is sufficient to show that $\operatorname{Re}\ln P(\zeta) \ge c|\zeta|^{1/s}$ for $\zeta \in A_\psi$ with $|\zeta|$ big enough. To this end recall that ([R])

$$\ln\widetilde{P}(\zeta) = \zeta \int_{m_0}^{\infty} \frac{m(\rho)}{\rho(\rho+\zeta)} d\rho,$$

where $m(\rho) \overset{\text{def}}{=} |\{p \in \mathbb{N} : m_p \le \rho\}|$ and write

$$\zeta \int_{m_0}^{\infty} \frac{m(\rho)}{\rho(\rho+\zeta)} d\rho = \int_{m_0}^{|\zeta|} + \int_{|\zeta|}^{|\zeta|/\sin\psi} + \int_{|\zeta|/\sin\psi}^{\infty} \frac{m(\rho)}{\rho} \frac{\zeta}{\rho+\zeta} d\rho$$
$$\overset{\text{def}}{=} I_1(\zeta) + I_2(\zeta) + I_3(\zeta).$$

Since $m_p \sim p^s$ we can find $C < \infty$ such that $1/C\rho^{1/s} \le m(\rho) \le C\rho^{1/s}$ for $\rho \ge m_0$. If $0 < \rho \le |\zeta|$ then $|\arg\big(\frac{\zeta}{\rho+\zeta}\big)| \le \pi/4 + \psi/2$ and $|\frac{\zeta}{\rho+\zeta}| \ge \sqrt{2}/2$. So $\mathrm{Re}\,\big(\frac{\zeta}{\zeta+\rho}\big) \ge \frac{\sqrt{2}}{2}\cos(\pi/2+\psi/2) \overset{\text{def}}{=} C_\psi > 0$ and

$$\mathrm{Re}\, I_1(\zeta) \ge \int_{m_0}^{|\zeta|} \frac{C_\psi}{C} \rho^{1/s-1} d\rho = \frac{C_\psi s}{C}\big(|\zeta|^{1/s} - m_0^{1/s}\big).$$

Now observe that $\mathrm{Re}\, I_2(\zeta) \ge 0$ and $\mathrm{Re}\, I_3(\zeta) \ge 0$ if $\mathrm{Re}\,\zeta \ge 0$. Finally, if $\zeta \in A_\psi$ with $\mathrm{Re}\,\zeta < 0$ and $\rho \ge |\zeta|/\sin\psi$ then

$$-\mathrm{Re}\,\Big(\frac{\zeta}{\rho+\zeta}\Big) \le \Big|\frac{\zeta}{\rho+\zeta}\Big| \sin\psi \le \frac{|\zeta|}{\rho}\sin\psi.$$

So

$$-\mathrm{Re}\, I_3(\zeta) \le \int_{|\zeta|/\sin\psi}^{\infty} C\rho^{1/s-2}|\zeta|\sin\psi d\rho = \frac{C(s-1)}{s}(\sin\psi)^{2-1/s}|\zeta|^{1/s}.$$

Hence chosing $\psi > 0$ and $c > 0$ small enough we get the desired estimation of $\mathrm{Re}\ln P$.

To show (41) we repeat the estimations of $-\mathrm{Re}\, I_3$ with ψ replaced by $\pi/2 - \delta$. □

The next lemma improves Lemma 11.4 of [K1].

Lemma 5. *Let $m_p \sim p^s$ with $s > 1$ and $L < \infty$. Put*

$$J(\zeta) = (1+\zeta)^2 \prod_{p=1}^{\infty} \big(1 + \frac{L\zeta}{m_p}\big) \text{ for } \zeta \in \mathbb{C} \tag{42}$$

and define the Green kernel for J

$$G(z)=\frac{1}{2\pi i}\int_0^\infty \frac{e^{z\zeta}}{J(\zeta)}d\zeta \text{ for } \operatorname{Re} z<0. \tag{43}$$

Then $G\in\mathcal{O}(\{\operatorname{Re} z<0\})$ *can be holomorphically continued to the Riemann domain* $\{z\in\widetilde{\mathbb{C}}:-\pi/2<\arg z<5\pi/2\}$ *on which we have*

$$J(D)G(z)=-\frac{1}{2\pi i}\frac{1}{z}.$$

For any $0<\varphi<\pi/2$ *there exist* $K_\varphi<\infty$ *and* $C_\varphi<\infty$ *such that*

$$|G(z)|\le C_\varphi \exp\Big\{M^*\Big(\frac{K_\varphi}{|z|}\Big)\Big\} \tag{44}$$

$$\text{for } z\in B_\varphi \overset{\text{def}}{=} \{z\in\widetilde{\mathbb{C}}:-\varphi\le\arg z\le 2\pi+\varphi\}.$$

Furthermore, one can find $0<\psi<\pi/2$ *such that G is bounded on* B_ψ *and*

$$|g(z)|\le C\sqrt{|z|}\exp\Big\{-M^*\big(\frac{kL}{|z|}\big)\Big\} \text{ for } |\arg z|\le\psi \tag{45}$$

with some $k>0$*, where*

$$g(z)=G_+(z)-G_-(e^{2\pi i}z) \text{ for } \operatorname{Re} z>0,$$

with G_+ *being the branch of G on* $\{-\pi/2<\arg z<\pi/2\}$ *and* G_- *that on* $\{3\pi/2<\arg z<5\pi/2\}$*.*

Proof. We have only to prove the estimations of G and g. To this end observe that

$$G(z)=\frac{1}{2\pi i}\int_{l_\alpha}\frac{e^{z\zeta}}{J(\zeta)}d\zeta \text{ for } \{-\alpha+\pi/2<\arg z<3\pi/2-\alpha\} \tag{46}$$

where $l_\alpha=\{\zeta\in\mathbb{C}:\arg\zeta=\alpha\}$ with $-\pi<\alpha<\pi$. Since by Lemma 4 there exists $\widetilde{\psi}>0$ such that $1/J$ is bounded on $\{|\arg\zeta|\le\widetilde{\psi}+\pi/2\}$ we get the boundedness of G on B_ψ with any $\psi < \widetilde{\psi}$. Now to show (44) it is sufficient to estimate G on $\{-\varphi \le \arg z \le 0\}$ and on $\{2\pi \le \arg z \le 2\pi + \varphi\}$. On the first set G is defined by (46) with $\pi/2 + \varphi < \alpha < \pi$. So by (41)

$$|G(z)| \le C_\alpha \int_0^\infty \exp\{r|z|\cos(\arg z + \alpha) + M(K_\alpha r)\}dr$$
$$\le C_\alpha \int_0^\infty \exp\{-\delta_{\alpha,\varphi} r|z| + M(K_\alpha r)\}dr$$

with some $\delta_{\alpha,\varphi} > 0$. Now using the definitions of M and M^* we conclude that G satisfies (44). The estimation on the second set is derived in an analogous way.

Now we shall prove (45). To this way fix $0 < \psi < \pi/2$ and observe that

$$g(z) = \frac{1}{\pi i} \int_{\widetilde{l}_\psi} \frac{e^{z\zeta}}{J(\zeta)} d\zeta \text{ for } |\arg z| \le \psi,$$

where $\widetilde{l}_\psi \overset{\text{def}}{=} \{\zeta \in \mathbb{C} : |\arg \zeta| = \psi + \pi/2\}$. Now by Lemma 4 if ψ is small enough we can replace $\widetilde{l}_\psi$ under the integral sign by $r + \widetilde{l}_\psi$ with any $r > 0$. Note that if $\zeta \in r + \widetilde{l}_\psi$ and $|\arg z| \le \psi$ than $|\zeta| \ge r\cos\psi$ and $\operatorname{Re}(z\zeta) \le |z|r$. We also have

$$\int_{r+\widetilde{l}_\psi} \frac{|d\zeta|}{|1+\zeta|^2} = \frac{\pi + 2\psi}{(1+r)\cos\psi} \le \frac{2\pi}{(1+r)\cos\psi}.$$

Thus, emploing estimate (40) we derive

$$|g(z)| \le \inf_{r>0} \frac{1}{2\pi} \int_{r+\widetilde{l}_\psi} \frac{|e^{z\zeta}|}{|J(\zeta)|} |d\zeta|$$
$$\le \inf_{r>0} C \int_{r+\widetilde{l}_\psi} \frac{|d\zeta|}{|1+\zeta|^2} \exp\{-M(kLr\cos\psi) + |z|r\}$$

$$\leq \frac{C}{\cos\psi} \inf_{r>0, p\in\mathbb{N}_0} \frac{1}{\sqrt{r}} \frac{M_p}{(kLr\cos\psi)^p} e^{|z|r}.$$

Finally, putting $r = p/|z|$, by the Stirling formula we arrive at (45). □

Remark 3. If $m_p = p^s$, $s > 1$ then (40) holds for $\zeta \in A_\psi$ with any $0 < \psi < \pi/2 \min(1, s-1)$ ([B], Theorem 4.1.1). Hence also in (45) one can choose any $0 < \psi < \pi/2 \min(1, s-1)$.

Proof of Theorem 10. Fix $t < \rho$ and choose $t < x_0 < \rho$. Let H and K be such that (30) holds for $|x| \leq x_0$. Let G be the Green kernel for

$$J(\zeta) = (1+\zeta)^2 \prod_{p=1}^{\infty} \Big(1 + L\zeta/p^s\Big)$$

where L will be chosen later. For $\nu \in \mathbb{Z}$ put $x_\nu = x_0 e^{2\nu\pi i}$ Then for $x \in \widetilde{B}(x_0)$ with $|\arg x - 2\nu\pi| < 3\pi/2$ one can find a closed curve $\gamma_\nu \subset \widetilde{B}(x_0)$ with endpoints at x_ν, encircling x once in the positive direction and such that $-\pi/2 < \arg(y-x) < 5\pi/2$ for $y \in \gamma_\nu$. Define

$$v_\nu(x) = \int_{\gamma_\nu} G(y-x)u(y)dy \text{ for } x \in \widetilde{B}(x_0) \text{ with } |\arg x - 2\nu\pi| < 3\pi/2.$$

Then $v_\nu \in \mathcal{O}\big(\{x \in \widetilde{B}(x_0) : |\arg x - 2\nu\pi| < 3\pi/2\}\big)$, v_ν does not depend on the choise of γ_ν satisfying the above properties and $J(D)v_\nu = u$ on $\widetilde{B}(t) \cap \{|\arg x - 2\nu\pi| < 3\pi/2\}$. Hence we only need to show (38) and (39). First we shall show (38) for $x \in S_\nu^{\delta,+} \stackrel{\text{def}}{=} \{x \in \widetilde{B}(t) : 0 \leq \arg x - 2\nu\pi \leq \pi - \delta\}$ where $0 < \delta < \pi/2$. To this end take a convenient curve $\gamma_\nu = \gamma_\nu^1 \cup \gamma_\nu^2 \cup \gamma_\nu^3$. To define γ_ν^j, $j = 1, 2, , 3$ denote by $\overline{x}$ a point of $\widetilde{\mathbb{C}}$ with $|\overline{x}| = x_0$ such that $\overline{x} = x + \tau_0$ with some $\tau_0 \in \mathbb{R}_+$. Set $\gamma_\nu^1 = \{y \in \widetilde{B}(\rho) : |y| = x_0, 2\nu\pi \leq \arg y \leq \arg\overline{x}\}$; $\gamma_\nu^3 = -\gamma_\nu^1$; γ_ν^2 - a curve from $\overline{x}$ to $\overline{x}$ encircling once the interval $[x, \overline{x}]$ and close to that interval. Define

$$v_\nu^j(x) = \int_{\gamma_\nu^j} G(y-x)u(y)dy, \ j = 1, 2, 3.$$

Note that for $x \in S_\nu^{\delta,+}$ and $y \in \gamma_\nu^1 \cup \gamma_\nu^3$ we have $-\varphi \le \arg(y-x) \le 2\pi + \varphi$ with some $\varphi < \pi/2$, $|y-x| \ge x_0 - t$, $|y| = x_0$ and $|\arg y - 2\nu\pi| < \pi/2$. So by estimations of the Green kernel G and of u we conclude that on $S_\nu^{\delta,+}$, v_ν^1 and v_ν^3 are bounded by $C\exp\{N(\widetilde{K}|\nu|)\}$ with C and $\widetilde{K}$ independent of $\nu \in \mathbb{Z}$. Now

$$v_\nu^2(x) = \int_0^{\tau_0} g(\tau)u(x+\tau)d\tau$$

Observe that for $x \in S_\nu^{\delta,+}$ and $0 \le \tau \le \tau_0$ we have $0 \le \arg(x+\tau) - 2\nu\pi \le \pi - \delta$ and $|x+\tau| \ge \tau \sin\delta$. Since for any $\varepsilon > 0$

$$N\big(K|\ln\rho|\big) + M^*\big(H/\rho\big) \le CM^*\big((H+\varepsilon)/\rho\big) \text{ for } 0<\rho<1 \tag{47}$$

and by (45)

$$|g(\tau)| \le C\sqrt{\tau}\exp\{-M^*(kL/\tau)\} \text{ for } \tau > 0$$

with some $k > 0$, choosing $L > H/(k\sin\delta)$ in the definition of J we obtain by (6)

$$\begin{aligned} |v_\nu^2(x)| &\le C\int_0^{\tau_0} \sqrt{\tau}\exp\{-M^*\Big(\frac{kL}{\tau}\Big) + M^*\Big(\frac{H}{|x+\tau|}\Big) + N\big(K|\ln(x+\tau)|\big)\}d\tau \\ &\le C\exp\{N\big(\widetilde{K}|\nu|\big)\} \end{aligned}$$

with C and $\widetilde{K} = (2\pi+1)K$ independent of $\nu \in \mathbb{Z}$. Analogously (38) holds for $x \in S_\nu^{\delta,-} \overset{\text{def}}{=} \{x \in \widetilde{B}(t) : -\pi + \delta \le \arg x - 2\nu\pi \le 0\}$ with $0 < \delta < \pi/2$.

Next we estimate v_ν on $\{|\arg x - (2\nu+1)\pi| \le \delta\}$ where $0 < \delta < \psi$ with $\psi > 0$ as in Lemma 5. To this end find a curve γ such that $-\psi \le \arg(y-x) \le 2\pi + \psi$, $|y| \ge |x|\sin(\psi - \delta)$ for $y \in \gamma$. Then

$$v_\nu(x) = \int_{\widetilde{\gamma}} g(\tau)u(x+\tau)d\tau$$

where $\widetilde{\gamma}$ is a curve connecting 0 to $x_\nu - x$ such that $|\arg\tau| \le \psi$ and $|x+\tau| \ge |\tau|\sin(\psi-\delta)$ for $\tau \in \widetilde{\gamma}$. So taking $L > H/(k\sin(\psi - \delta))$ in the definition of J, where $k > 0$ is a constant in (45) we get by (6)

$$|v_\nu(x)| = |\int_{\widetilde{\gamma}} g(\tau)u(x+\tau)d\tau|$$

$$\leq C \sup_{\tau\in\widetilde{\gamma}} \sqrt{|\tau|} \exp\{-M^*\Big(\frac{kL}{|\tau|}\Big) + M^*\Big(\frac{L}{|x+\tau|}\Big) + N\big(K|\ln(x+\tau)|\big)\}$$
$$\leq C \exp\{N\big(\widetilde{K}|\nu|\big)\}$$

for $|\arg x - (2\nu+1)\pi| \leq \delta$ which ends the proof of (38).

To show (39) fix $\beta > 0$ and observe that if $x \in \widetilde{B}(t)$ with $|\arg x - (2\nu+1)\pi| \leq \pi/2 - \beta$ than we can choose a curve γ_ν in such a way that $4|y| \geq |x|/\sin\beta, 4|y-x| \geq |x|/\sin\beta, |\arg y| \leq 1 + |\arg x|$ and $-\pi/2 + \beta/4 \leq \arg(y-x) \leq 2\pi + \pi/2 - \beta/4$ for $y \in \gamma_\nu$. Thus by (44) and (47) we estimate

$$|v_\nu(x)| \leq C_\beta \sup_{y\in\gamma_\nu} \exp\{M^*\Big(\frac{K_\beta}{|y-x|}\Big) + M^*\Big(\frac{H}{|y|}\Big) + N\big(K|\ln y|\big)\}$$
$$\leq C_\beta \exp\{M^*\Big(\frac{4K_\beta}{|x\sin\beta|}\Big) + M^*\Big(\frac{4K+1}{|x|\sin\beta}\Big) + N\big(K(1+|\arg x|\big)\}$$

and we get (39) with $\widetilde{H} = C(s)\max(4K_\beta, 4H+1)/\sin\beta$, $\widetilde{K} = (2\pi+1)K$. □

5. Relation between $\mathcal{M}g$ and $\mathcal{M}(J(D)g)$

Let $J(D) = \sum_{k=0}^{\infty} a_k D^k$ be an ultradifferential operator of class (M_p). Let $S \in L^{(N_p)\prime}_{(\emptyset,\omega)}(\overline{\mathbb{R}}_+)$ and $g = \mathcal{T}S$. Put

$$u(x) = J(D)g(x) \text{ for } 0 < |x| < e^\omega.$$

Then by Corollary 1, $u \in \mathcal{O}^{(N)}_{(M^*)}\big(\widetilde{B}(e^\omega)\big)$. So $\mathcal{M}_t u$, $t < e^\omega$ defined by (21) satisfies the estimation in Theorem 6. In this section we define the Mellin transform of χu, χ – a (smooth) cut-off function, by means of the Mellin transform of χg. First of all we shall study properties of $\mathcal{M}\chi$ and $\mathcal{M}(\chi g)$.

Lemma 6. *Let χ be a cut-off function of class (M_p) equal to one on $(0,r]$ and zero on $[t,\infty)$ where $0 < r < t$. Then $\Psi = \mathcal{M}\chi \in \mathcal{O}(\mathbb{C}\setminus\{0\})$ has simple pole at zero with residuum -1 and satisfies for any $L < \infty$*

$$|\Psi(z)| \leq C_L/|z| \exp\{-M(L|z|)\}\kappa_{\ln t,\ln r}(\operatorname{Re} z) \quad \textit{for } z \neq 0. \tag{48}$$

Furthermore for a fixed $z \in \mathbb{C} \setminus \overline{\mathbb{R}}_+$ *the function*

$$\Psi_z(x) \overset{\text{def}}{=} \Psi(z-x) \text{ for } x \in \overline{\mathbb{R}}_+ \tag{49}$$

belongs to $L_b^{(N_p)}(\overline{\mathbb{R}}_+)$ *with* $b > t$ *and for any* $h > 0$, $L < \infty$

$$\|\Psi_z\|_{\emptyset,b,h}^{(N_p)} \le C(L)\exp\{N^*\Big(\frac{2H}{hd(z)}\Big) - M(Ld(z))\}\kappa_{\ln t,\ln r}(\operatorname{Re} z) \quad \text{for } z \in \mathbb{C} \setminus \overline{\mathbb{R}}_+ \tag{50}$$

where $d(z) \overset{\text{def}}{=} \operatorname{dist}(z, \overline{\mathbb{R}}_+)$ *and* H *is the constant in* (M.2).

Proof. Since

$$z\Psi(z) = \int_r^t \frac{d\chi}{dx} x^{-z} dx \in \mathcal{O}(\mathbb{C})$$

and $z\Psi(z)_{|z=0} = -1$ the first assertion is clear. To show the second one put

$$\mu(y) = e^{-y}\frac{d\chi}{dx}(e^{-y}) \text{ for } y \in \mathbb{R}.$$

Since the function $\mathbb{R}_+ \ni x \to -\ln x$ is analytic on $\mathbb{R}_+$ it follows by the Roumieu theorem ([R], Théorème 13) that $\mu \in D^{(M_p)}(\mathbb{R})$ with $\operatorname{supp}\mu \subset [-\ln t, -\ln r]$. We also have

$$z\Psi(z) = \int_r^t \frac{d\chi}{dx} x^{-z} dx \overset{x=e^{-y}}{=} \int_{-\ln t}^{-\ln r} \mu(y) e^{yz} dy = \mathcal{L}\mu(z) \text{ for } z \in \mathbb{C}$$

and for any $k \in N_0$

$$z^k \mathcal{L}\mu(z) = \mathcal{L}(D^k\mu)(z) \text{ for } z \in \mathbb{C}.$$

So for any $L < \infty$ we can find $C_L < \infty$ such that

$$|z\Psi(z)| \le C_L \frac{M_k}{L^k |z|^k} \kappa_{\ln t,\ln r}(\operatorname{Re} z) \text{ for } z \ne 0.$$

Taking the infimum over $k \in \mathbb{N}_0$ we arrive at (48).

To show the last statement fix $z \in \mathbb{C}\backslash\overline{\mathbb{R}}_+$, and take $\rho = d(z)/2$ if $d(z) \le 2$ and $\rho = 1$ otherwise. Then

$$D^\alpha \Psi_z(x) = \frac{\alpha!}{2\pi} \int_0^{2\pi} \Psi_z(x + \rho e^{i\varphi})\rho^{-\alpha} e^{-i\varphi\alpha} d\varphi \text{ for } x \in \overline{\mathbb{R}}_+.$$

So for $d(z) \ge 2$

$$\begin{aligned} |D^\alpha \Psi_z(x)| &\le \alpha! \sup_{0\le\varphi\le 2\pi} |\Psi(z - x - e^{i\varphi})| \\ &\le C_L \alpha! \exp\{-M(Ld(z))\}\kappa_{\ln t, \ln r}(\operatorname{Re} z - x) \end{aligned}$$

and since $r < t < e^b$

$$\|\Psi_z\|^{(N_p)}_{\emptyset,b,h} \le C_L \exp\{N^*\big(1/h\big) - M(Ld(z))\}\kappa_{\ln t, \ln r}(\operatorname{Re} z).$$

Now in the case $0 < d(z) \le 2$

$$|D^\alpha \Psi_z(x)| \le \widetilde{C}_L \alpha! \frac{2^{\alpha+1}}{\big(d(z)\big)^{\alpha+1}} \kappa_{\ln t, \ln r}(\operatorname{Re} z - x).$$

So

$$\|\Psi_z\|^{(N_p)}_{\emptyset,b,h} \le C(L) \frac{2}{d(z)} \exp\{N^*\Big(\frac{2}{hd(z)}\Big) - M(Ld(z))\}\kappa_{\ln t, \ln r}(\operatorname{Re} z)$$

and by (7) we get (50). □

PROPOSITION 2. *Let $S \in L^{(N_p)\prime}_{(\emptyset,\omega)}(\overline{\mathbb{R}}_+)$ and $g = \mathcal{T}S$. Let χ be a cut-off function as in Lemma 6 with $r < t < e^\omega$. Then*

$$G \overset{\text{def}}{=} \mathcal{M}(\chi g) \in \mathcal{O}(\mathbb{C} \setminus \overline{\mathbb{R}}_+)$$

and for every $L < \infty$ one can find $C_L < \infty$ such that with $K < \infty$ independent of L

$$(51) \quad |G(z)| \le C_L \exp\{N^*\Big(\frac{K}{d(z)}\Big) - M(Ld(z))\}\kappa_{\ln t, \ln r}(\operatorname{Re} z) \text{ for } z \in \mathbb{C} \setminus \overline{\mathbb{R}}_+.$$

Proof. The holomorphicity of G follows from the continuity of S by the standard arguments. To show (51) observe that

$$G(z) = \mathcal{M}(\chi\mathcal{T}S)(z) = \int_{\mathbb{R}_+} \chi(x)S[x^{\cdot}]x^{-z-1}dx$$
$$= S[\int_{\mathbb{R}_+} \chi(x)x^{\cdot-z-1}dx] = S[\Psi_z]$$

with Ψ_z given by (49). The exchange of the integral with the action of S is legitimate by the continuity if S. Now applying Lemma 3 with $\ln t < b < \omega$ we get the existence of $h > 0$ such that

$$|S[\Psi_z]| \leq C\|\Psi_z\|^{(N_p)}_{\emptyset,b,\omega}.$$

Hence (51) follows by (50). □

Returning to the considerations from the begining of this section observe that $\chi u - J(D)(\chi g) \stackrel{\text{def}}{=} \varphi$ belongs to $D^{(M_p)}([r,t])$. As in the proof of Lemma 6 we find that $\mathcal{M}\varphi$ is an entire function satisfying with any $L < \infty$

$$(52) \qquad |\mathcal{M}\varphi(z)| \leq C_L \exp\{-M(L|z|)\}\kappa_{\ln t,\ln r}(\operatorname{Re} z) \text{ for } z \in \mathbb{C}.$$

Since for any $k \in \mathbb{N}_0$

$$\mathcal{M}(D^k\chi g)(z) = \frac{\Gamma(z+k+1)}{\Gamma(z+1)}\mathcal{M}(\chi g)(z+k)$$

we define the Mellin transform of χu by

$$(53) \quad \mathcal{M}(\chi u)(z) = \sum_{k=0}^{\infty} a_k \frac{\Gamma(z+k+1)}{\Gamma(z+1)}\mathcal{M}(\chi g)(z+k) + \mathcal{M}\varphi(z) \text{ for } z \in \mathbb{C}\setminus\mathbb{R}.$$

To justify this definition we shall show that the series on the right hand side of (53) converges locally uniformly on $\mathbb{C}\setminus\mathbb{R}$. To this end put

$$G(z) = \mathcal{M}(\chi g)(z) \text{ for } z \in \mathbb{C}\setminus\overline{\mathbb{R}}_+.$$

Observe that for $k \in \mathbb{N}_0$

$$\frac{\Gamma(z+k+1)}{\Gamma(z+1)} = \sum_{j=0}^{k} \left\{ \begin{matrix} k \\ k-j \end{matrix} \right\} z^j$$

where

$$\left\{ \begin{matrix} k \\ 0 \end{matrix} \right\} \stackrel{\text{def}}{=} 1 \text{ and } \left\{ \begin{matrix} k \\ l \end{matrix} \right\} \stackrel{\text{def}}{=} \sum_{1 \le j_1 < \ldots < j_l \le k} j_1 \cdot \ldots \cdot j_l \text{ for } 1 \le l \le k, k \in \mathbb{N}_0.$$

So

$$\sum_{k=0}^{\infty} a_k \frac{\Gamma(z+k+1)}{\Gamma(z+1)} \mathcal{M}(\chi g)(z+k) = \sum_{k=0}^{\infty} a_k \Big(\sum_{j=0}^{k} \left\{ \begin{matrix} k \\ k-j \end{matrix} \right\} z^j \Big) G(z+k)$$
$$= \sum_{n=0}^{\infty} z^n \Big(\sum_{k=n}^{\infty} a_k \left\{ \begin{matrix} k \\ k-n \end{matrix} \right\} G(z+k) \Big)$$
$$= \sum_{n=0}^{\infty} z^n \Big(\sum_{l=0}^{\infty} a_{n+l} \left\{ \begin{matrix} n+l \\ l \end{matrix} \right\} G(z+n+l) \Big).$$

Now

$$\left\{ \begin{matrix} n+l \\ l \end{matrix} \right\} = \frac{(n+l)!}{n!} \cdot \sum_{1 \le j_1 < \cdots < j_l \le n+l} \frac{j_1 \cdot \ldots \cdot j_l}{(n+1) \cdot \ldots \cdot (n+l)}$$
$$\le \frac{(n+l)!}{n!} \binom{n+l}{l} \le 4^{n+l} l!.$$

Thus, if $|a_k| \le C L_1^k M_k$ for $k \in \mathbb{N}_0$ with some $L_1 < \infty$, using (51), (M.1) and (8) we derive for $\operatorname{Re} z \ge 0$, $z \notin \overline{\mathbb{R}}_+$

$$|\sum_{l=0}^{\infty} a_{n+l} \left\{ \begin{matrix} n+l \\ l \end{matrix} \right\} G(z+n+l)|$$
$$\le C \sum_{l=0}^{\infty} \frac{L_1^{n+l}}{M_{n+l}} 4^{n+l} l!$$

$$
\begin{aligned}
&\times \exp\{N^*\Big(\frac{K}{d(z+n+l)}\Big) - M\big(Ld(z+n+l)\big)\}r^{-\operatorname{Re} z-n-l} \\
&\le C\frac{(4L_1/r)^n}{M_n}\exp\{N^*\Big(\frac{K}{d(z)}\Big) - M\big(Ld(z)\big)\}\sum_{l=0}^{\infty}\frac{l!(4L_1/r)^l}{M_l}r^{-\operatorname{Re} z} \\
&\le 2C\frac{(4L_1/r)^n}{M_n}\exp\{M^*\Big(\frac{8L_1}{r}\Big) + N^*\Big(\frac{K}{d(z)}\Big) - M\big(Ld(z)\big)\}r^{-\operatorname{Re} z}.
\end{aligned}
$$

Now let $\operatorname{Re} z < 0$ with $\operatorname{Im} z \neq 0$ and let $k \in \mathbb{N}_0$ be such that $-k-1 \le \operatorname{Re} z < -k$. Put $k_n = k - n$ if $n \le k$ and $k_n = -1$ if $n > k$. Then we estimate (omiting the sum $\sum_{l=0}^{k_n}$ if $k_n = -1$)

$$
\begin{aligned}
&|\sum_{l=0}^{\infty} a_{n+l}\begin{Bmatrix} n+l \\ l \end{Bmatrix} G(z+n+l)| \\
&\le C\Big[\sum_{l=0}^{k_n}\frac{(4L_1)^n}{M_n}\frac{(4L_1)^l l!}{M_l} \\
&\qquad \times \exp\{N^*\Big(\frac{K}{d(z+n+l)}\Big) - M\big(Ld(z+n+l)\big)\}t^{-\operatorname{Re} z-n-l} \\
&+ \sum_{l=k_n+1}^{\infty}\frac{(4L_1)^n}{M_n}\frac{(4L_1)^l l!}{M_l} \\
&\qquad \times \exp\{N^*\Big(\frac{K}{d(z+n+l)}\Big) - M\big(Ld(z+n+l)\big)\}r^{-\operatorname{Re} z-n-l}\Big] \\
&\le Ct^{-\operatorname{Re} z}\Big[\frac{(4L_1/t)^n}{M_n}\exp\{N^*\Big(\frac{K}{|\operatorname{Im} z|}\Big) - M\big(L|\operatorname{Im} z|\big)\}\sum_{l=0}^{k_n}\frac{(4L_1/t)^l l!}{M_l} \\
&+ \Big(\frac{r}{t}\Big)^{-\operatorname{Re} z}\frac{(4L_1/r)^n}{M_n}\exp\{N^*\Big(\frac{K}{|\operatorname{Im} z|}\Big) - M\big(L|\operatorname{Im} z|\big)\}\sum_{l=k_n+1}^{\infty}\frac{(4L_1/r)^l l!}{M_l}\Big] \\
&\le 2Ct^{-\operatorname{Re} z}\Big[\frac{(4L_1/t)^n}{M_n}\exp\{M^*\Big(\frac{8L_1}{t}\Big) + N^*\Big(\frac{K}{|\operatorname{Im} z|}\Big) - M\big(L|\operatorname{Im} z|\big)\} \\
&+ \frac{(4L_1/r)^n}{M_n}\exp\{M^*\Big(\frac{8L_1}{r}\Big) + N^*\Big(\frac{K}{|\operatorname{Im} z|}\Big) - M\big(L|\operatorname{Im} z|\big)\}\Big] \\
&\le 4C\frac{(4L_1/r)^n}{M_n}\exp\{M^*\Big(\frac{8L_1}{r}\Big) + N^*\Big(\frac{K}{|\operatorname{Im} z|}\Big) - M\big(L|\operatorname{Im} z|\big)\}t^{-\operatorname{Re} z}.
\end{aligned}
$$

Thus, by the above estimations we conclude that the series

$$\sum_{n=0}^{\infty} z^n \Big(\sum_{l=0}^{\infty} a_{n+l} \begin{Bmatrix} n+l \\ l \end{Bmatrix} G(z+n+l) \Big)$$

converges locally uniformly on $\mathbb{C} \setminus \mathbb{R}$ to a function $F \in \mathcal{O}(\mathbb{C} \setminus \mathbb{R})$ and having in mind (52), by (8′) we obtain

THEOREM 11. *Let $J(D)$ be an ultradifferential operator of class (M_p), $S \in L^{(N_p)\prime}_{(\emptyset,\omega)}(\overline{\mathbb{R}}_+)$, $g = \mathcal{T}S$ and $u = J(D)g$. Let χ be a cut-off function of class (M_p) equal to one on $(0, r]$ and zero on $[t, \infty)$ where $0 < r < t < e^{\omega}$. Then $\mathcal{M}(\chi u)$ defined by (53) is a holomorphic function on $\mathbb{C} \setminus \mathbb{R}$. Furthermore, one can find $H < \infty$ and $K < \infty$ such that for any $L < \infty$ there exists $C_L < \infty$ (independent of r) such that*

$$\begin{aligned} &|\mathcal{M}(\chi u)(z)| \\ &\le C_L \exp\{M^*\Big(\frac{H}{r}\Big) + N^*\Big(\frac{K}{|\operatorname{Im} z|}\Big) - M\big(L|\operatorname{Im} z|\big) + M\Big(\frac{H|z|}{r}\Big)\} \\ &\times \kappa_{\ln t, \ln r}(\operatorname{Re} z) \text{ for } z \in \mathbb{C} \setminus \mathbb{R}. \end{aligned} \tag{54}$$

We also have $u = \mathcal{T}T$ where

$$\begin{aligned} T &= \frac{1}{2\pi i}\Big(b^+\big(\mathcal{M}^+(\chi u)\big) - b^-\big(\mathcal{M}^-(\chi u)\big)\Big) \\ &= \sum_{k=0}^{\infty} a_k \frac{\Gamma(\alpha+k+1)}{\Gamma(\alpha+1)} S(\cdot + k) \text{ in } L^{(N_p)\prime}_{(-\infty,\omega)}(\mathbb{R}). \end{aligned} \tag{55}$$

REMARK 4. If in Theorem 11 we take as a cut-off function χ the characteristic function χ_t of the interval $(0, t]$ then since for $k \in \mathbb{N}$

$$\mathcal{M}_t^{\pm} D^k g(z) = \frac{\Gamma(z+k+1)}{\Gamma(z+1)} \mathcal{M}_t^{\pm} g(z+k) + \sum_{l=0}^{k-1} \frac{\Gamma(z+k-l)}{\Gamma(z+1)} g^{(l)}(t) t^{-z-k+l} \quad \text{for } \pm \operatorname{Im} z > 0$$

we obtain

$$\mathcal{M}_t^{\pm} u(z) = \sum_{k=0}^{\infty} a_k \frac{\Gamma(z+k+1)}{\Gamma(z+1)} \mathcal{M}_t^{\pm} g(z+k) + F(z) \text{ for } \pm \operatorname{Im} z > 0$$

where

$$F(z)=\sum_{k=0}^{\infty}\frac{\Gamma(z+k+1)}{\Gamma(z+1)}\Big(\sum_{l=0}^{\infty}a_{k+l+1}g^{(l)}(t)\Big)t^{-z-k-1}\ \text{for } z\in\mathbb{C}$$

is an entire function satisfying with some $C<\infty$ and $L<\infty$

$$|F(z)|\le C\exp\{M\big(L|z|\big)\}t^{-\operatorname{Re} z}\ \text{for } z\in\mathbb{C}.$$

(The estimation of F can be derived following the lines of the proof of Theorem 11 with $G(z)=t^{-z-1}$.)

Remark 5. If we assume in Theorem 11 that S is a Laplace distribution $S\in L'_{(\emptyset,\omega)}(\overline{\mathbb{R}}_+)$ then the factor $\exp\{N^*(K/|\operatorname{Im} z|)\}$ in (54) can be replaced by $|\operatorname{Im} z|^{-K}$. Also in previous theorems we can omit the symbol (N) (or (N_p)) replacing $\exp\{N\big(K\cdot\rho(z)\big)\}$ and $\exp\{N^*\big(K\cdot\rho(z)\big)\}$ by $\big(\rho(z)\big)^K$.

6. Examples

Example 1. Let $S=\delta'_{(0)}\in L'_{(\emptyset,\infty)}(\overline{\mathbb{R}}_+)$ and $f(x)=\mathcal{T}S(x)=-\ln x$ for $x\in\widetilde{\mathbb{C}}$. For $k\in\mathbb{N}$ put

$$J_k(z)=\sum_{j=1}^{\infty}a_jz^{kj}\ \text{with } a_j=\frac{(-1)^{kj}}{(kj-1)!j!}.$$

Then $J_k(D)$ is an ultradifferential operator of class $(p!)^s$ with $s=1+1/k$. Put $u=1+J_k(D)f$. Then

$$u(x)=1+\sum_{j=1}^{\infty}a_jD^{kj}(-\ln x)=\sum_{j=0}^{\infty}\frac{1}{j!}\frac{1}{x^{kj}}=\exp\{\frac{1}{x^k}\}\ \text{for } x\neq 0.$$

So $u\in\mathcal{O}_{(M^*)}(\widetilde{\mathbb{C}})$ with $M^*(\rho)\sim\rho^k$ as $\rho\to\infty$. For $0<t<\infty$ we compute

$$\mathcal{M}_t^{\pm}u(z)\overset{(21)}{=}\int_{\gamma^{\pm}(t)}e^{1/x^k}x^{-z-1}dx=\int_{\gamma^{\pm}(t)}\Big(\sum_{j=0}^{\infty}\frac{1}{j!}\frac{1}{x^{kj}}\Big)x^{-z-1}dx$$

$$= \sum_{j=0}^{\infty} \frac{1}{j!} \int\limits_{\gamma^{\pm}(t)} x^{-kj-z-1} dx \tag{56}$$

$$= \sum_{j=0}^{\infty} \frac{1}{j!} \frac{t^{-kj-z}}{-kj-z} \text{ for } \pm \operatorname{Im} z > 0.$$

So by Theorem 4

$$u = \mathcal{T}T \tag{57}$$

where

$$T = \frac{1}{2\pi i}\Big(b^{+}\big(\mathcal{M}_t^{+}u\big) - b^{-}\big(\mathcal{M}_t^{-}u\big)\Big) = \sum_{j=0}^{\infty} \frac{1}{j!}\delta_{(-kj)} \in L'_{(-\infty,\infty)}(\mathbb{R}).$$

We also have

$$T = \delta_{(0)} + \sum_{j=1}^{\infty} a_j \frac{\Gamma(\alpha+kj+1)}{\Gamma(\alpha+1)} S(\cdot + kj) \text{ in } L'_{(-\infty,\infty)}(\mathbb{R})$$

Note that if $u(x) = \exp\{1/x^k\}$ that the formulae (56) and (57) remain true under the assumption $k > 0$.

Observe also that (since $\mathcal{M}_1\big(u(x)\big) = \mathcal{L}\big(u(e^{-y})\big)$, $\mathcal{L}$ - the Laplace transformation) we can write

$$\mathcal{L}\big(\exp\{e^{y^k}\}\big)(z) = \mathcal{M}_1\big(\exp\{1/x^k\}(z) = \sum_{j=0}^{\infty} \frac{1}{j!} \frac{1}{-kj-z} \text{ for } z \in \mathbb{C} \setminus \mathbb{R},$$

which in the case $k = 1$ agrees with the formula (3.2) of [D].

Example 2. Under the notation of Example 1

$$\text{if } a_j = \frac{(-1)^{(k+1)j}}{(kj-1)!j!} \text{ then } u(x) = \exp\{\frac{-1}{x^k}\}$$

$$\text{for } x \neq 0 \text{ and } S = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \delta_{(-kj)}.$$

Note here that

$$\mathcal{M}_t^{\pm}u(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{t^{-kj-z}}{-kj-z} \text{ for } \pm \operatorname{Im} z > 0,$$

but the integral

$$\int_0^t u(x)x^{-z-1}dz$$

defines an entire function. The inequality between $\mathcal{M}_t^{\pm}u$ and the last integral is caused by the fact that u is not a generalized analytic function.

Example 3. Let $S(\alpha) = \exp\{-\alpha^2\}$ for $\alpha \in \mathbb{R}$ and $u = \mathcal{T}S$. Then

$$\begin{aligned} u(x) &= \int_{\mathbb{R}} \exp\{-\alpha^2\}x^{\alpha}d\alpha = \exp\{\frac{\ln^2 x}{4}\} \int_{\mathbb{R}} \exp\{-\Big(\alpha + \frac{\ln x}{2}\Big)^2\}d\alpha \\ &= \sqrt{\pi}x^{(1/4)\ln x} \text{ for } x \in \widetilde{\mathbb{C}}. \end{aligned}$$

Since $|u(x)| = \sqrt{\pi}\exp\{1/4\big(\ln^2|x| - \arg^2 x\big)\}$, $u \in \mathcal{O}_{(M^*)}(\widetilde{\mathbb{C}})$ for $M^*(\rho) \sim \rho^k$ as $\rho \to \infty$ with any $k > 0$. Now for $0 < t < \infty$ and $\pm \operatorname{Im} z > 0$ we compute

$$\begin{aligned} \mathcal{M}_t^{\pm}u(z) &= \int_{\gamma^{\pm}(t)} \int_{\mathbb{R}} \exp\{-\alpha^2\}x^{\alpha}d\alpha x^{-z-1}dx \\ &= \int_{\mathbb{R}} \exp\{-\alpha^2\} \int_{\gamma^{\pm}(t)} x^{\alpha-z-1}dxd\alpha = \int_{\mathbb{R}} \exp\{-\alpha^2\}\frac{t^{\alpha-z}}{\alpha - z}d\alpha. \end{aligned}$$

Note that $\mathcal{M}_t^{\pm}u$ can not be holomorphically extended to any larger domain than $\{\pm \operatorname{Im} z > 0\}$.

Example 4. Let $s = 1 + 1/k$ with $k > 0$ and

$$u(x) = \sum_{j=0}^{\infty} \frac{1}{(j!)^s} D^j\Big(\frac{1}{-\ln x}\Big) \text{ for } x \in \widetilde{B}(1).$$

Then by Corollary 1, $u \in \mathcal{O}_{(M^*)}\big(\widetilde{B}(1)\big)$ with $M^*(\rho) \sim \rho^k$ as $\rho \to \infty$. Since $1/(-\ln x) = \int_0^\infty x^\alpha d\alpha$ for $|x| < 1$ we compute for $0 < t < 1$ and $\pm \operatorname{Im} z > 0$

$$\mathcal{M}_t^{\pm} u(z) = \sum_{j=0}^{\infty} \frac{1}{(j!)^s} \mathcal{M}_t^{\pm}\Big(D^j\Big(\frac{1}{-\ln x}\Big)\Big)(z) \mod \mathcal{O}^{\exp}(\mathbb{C})$$
$$= \sum_{j=0}^{\infty} \frac{1}{(j!)^s} \frac{\Gamma(z+j+1)}{\Gamma(z+1)} G(z+j) \mod \mathcal{O}^{\exp}(\mathbb{C})$$

where

$$G(\zeta) = \mathcal{M}_t\Big(\frac{1}{-\ln x}\Big)(\zeta) = \int_0^\infty \frac{t^{\alpha-\zeta}}{\alpha-\zeta} d\zeta \text{ for } \zeta \in \mathbb{C} \setminus \overline{\mathbb{R}}_+.$$

Since $1/(2\pi i) b(G) = Y$ (Y-the Heaviside function) we obtain by Theorem 4, $u = \mathcal{T}S$ with

$$S(\alpha) = \sum_{j=0}^{\infty} \frac{1}{(j!)^s} \frac{\Gamma(\alpha+j+1)}{\Gamma(\alpha+1)} Y(\alpha+j) \text{ for } \alpha \in \mathbb{R}.$$

Acknowledgment. The author is greatful to the referee for hinting a mistake in the previous version of the proof of (23).

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(Received February 12, 1997)
(Revised September 21, 1998)

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