J. Math. Sci. Univ. Tokyo
6 (1999), 41–47.

Maximal Quasiprojective Subsets and the Kleiman-Chevalley Quasiprojectivity Criterion

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Abstract. We prove that any complete Q-factorial variety contains only finitely many maximal open quasiprojective subsets.

Let X be a normal variety defined over an algebraically closed field of any characteristic. By $Div(X)$, respectively $Car(X)$ denote the group of Weil (resp. Cartier) divisors on X. We prove the following theorem

Theorem A. *Let X be a complete normal variety such that $(Div(X)/Car(X)) \otimes \mathbf{Q}$ is finite dimensional (in particular X can be Q-factorial or rational). Then X contains only finitely many maximal (in the sense of inclusion) open quasiprojective subsets.*

Remark. The conclusion of the above theorem holds for any open subset $X' \subset X$. However it is not clear that any normal variety X' such that $(Div(X')/Car(X')) \otimes \mathbf{Q}$ is finite dimensional admits an open embedding into a complete normal variety with the above mentioned property. This is clearly true for smooth varieties defined over a field of characteristic 0. In this case we can complete our variety by the Nagata theorem (see [6]) and then apply the Hironaka resolution theorem (see [4]).

As a simple corollary of Theorem A we get

Theorem B. *Let X' be a normal variety for which there exists an open embedding $X' \subset X$ into a complete normal variety X such that $(Div(X)/Car(X)) \otimes \mathbf{Q}$ is finite dimensional. Then X' is quasiprojective iff any finite subset of X' is contained in some open affine subset of X'.*

1991 *Mathematics Subject Classification.* Primary 14C20; Secondary 14C22, 14C10.
*The author is in part supported by Polish KBN Grant.

This generalizes the Kleiman-Chevalley criterion stated for smooth and complete varieties. (See [5] Chapter IV, Section 2, Theorem 3 or [3] Chapter I, Section 9, Theorem 9.1.)

Proof of Theorem B. Suppose X' is not quasiprojective. Then it contains finitely many maximal open quasiprojective sets $U_1, ..., U_k$. Let $x_i \in X' \backslash U_i$. Then $\{x_1, ..., x_k\}$ is contained in some open affine subset and hence in some maximal open quasiprojective set U_i, a contradiction to the choice of $x_1, ..., x_k$. The converse is evident. □

Theorem B can also be stated in a relative form:

Theorem C. *Let X' be as above and $Z \subset X'$ be any subset of X'. Then Z is contained in some open quasiprojective subset $U \subset X'$ iff any finite subset of Z is contained in some open affine subset of X'.*

Proof of Theorem C. Choose $x_i \in Z \backslash X_i$ and follow the proof of Theorem B. □

A consequence of Theorem B is the following

Theorem D. *Let X' be as in Theorem B and let G be a connected algebraic group acting on X'. Let $U \subset X'$ be any open quasiprojective subset. Then $G \cdot U$ is also quasiprojective.*

Remark. This is analogous to the Theorem of Sumihiro which says that on a normal variety with an action of a linear group G, each point has a G-invariant open quasiprojective neighbourhood (see [9]).

Proof of Theorem D. Let $\{x_1, ..., x_k\}$ be any finite subset of $G \cdot V = \bigcup_{g \in G} g \cdot V$. For any x_i the set $G_i := \{g \in G : x_i \in g \cdot V\}$ is non-empty and open. Since G is connected we have $\bigcap_{i=1}^k G_i \neq \emptyset$. Then for any $g \in \bigcap_{i=1}^k G_i$ we have $\{x_1, ..., x_k\} \subset g \cdot V$. The set $g \cdot V$ is open quasiprojective, hence it contains an open affine set $U \subset g \cdot V$ containing all x_i. We are done by Theorem B. □

Proof of the Main Theorem

Proof of Theorem A. Let D be a Weil divisor on a normal variety X. We say that D is *very ample* on an open subset $U \subset X$ iff there exists an open embedding of U into a projective variety Y and a very ample divisor D_0 on Y such that $D_{0|U} = D_{|U}$. We say that D is *ample* on U iff a positive multiple of D is very ample on U.

For any two Weil divisors D_1 and D_2 on a complete normal variety X we write $D_1 \equiv D_2$ iff $D_1 - D_2$ is a Cartier divisor numerically equivalent to 0.

Lemma 1. *Let X be a complete normal variety, D_1 and D_2 be Weil divisors such that $D_1 \equiv D_2$. Then, for any open $U \subset X$ the divisor D_1 is ample on U iff D_2 is ample on U.*

Proof. By assumption there is an ample divisor D_0 on a projective variety $X_0 \supset U$ such that the restrictions of D_0 and D_1 to U are equal. By a theorem of Nagata ([7] Theorem 3.2) we can find X^0 containing U, dominating X and obtained from X_0 by a join of finitely many blow-ups X_i with centers C_i disjoint from U. Let $p_i : X_i \to X$ be the blow-up with center C_i. Then $X^0 = X_1 * ... * X_k$. Let $s_i : X^0 \to X_i$ and $p : X^0 \to X$ be the standard projections. For any p_i, let $E_i := p_i^{-1}(C_i)$ be the exceptional divisor. Then $-E_i$ is relatively very ample, and by [2], II, 4.6.13 (ii) $D_i := n_i \cdot p_i^*(D_0) - E_i$ is very ample for $n_i >> 0$. Finally, $D := \sum_{i=1}^k s_i^*(D_i)$ is ample on X^0. Note that $D_{|U} = mD_{1|U}$ for $m = n_1 + ... + n_k$. The Cartier divisor $D' := D + m \cdot p^*(D_2 - D_1)$ is numerically equivalent to D, hence it is ample by the Seshadri criterion ([8]). But $D'_{|U} = mD_{2|U}$. □

Lemma 2. *Let X be a normal variety. Assume that D is ample on both U_1 and U_2. If $(U_1 \backslash U_2) \cup (U_2 \backslash U_1)$ is of codimension at least 2 in $U_1 \cup U_2$, then D is ample on $U_1 \cup U_2$.*

Proof. We can choose $n >> 0$ such that nD has no base points on $U_1 \cup U_2$, and sections of nD intersect properly each curve meeting $U_1 \cup U_2$. Thus nD defines a quasifinite morphism $p : U_1 \cup U_2 \to \mathbf{P}(nD)$. By the Zariski Theorem we can factor p as $U_1 \cup U_2 \xrightarrow{i} Z \xrightarrow{\pi} \mathbf{P}(nD)$ where i is an open immersion and π is finite. Then $nD = \pi^*(O(1))_{|U_1 \cup U_2}$ is ample because π preserves ampleness (see [3], Chapter 1, Proposition 4.4). □

Lemma 3 (Z.Jelonek). *Let X be any normal variety and $U \subseteq X$ be maximal open quasiprojective. Then $X \backslash U$ is of codimension at least 2.*

Proof. We may assume X to be complete. By the Nagata theorem there is a projective X^0 containing U and dominating X. Let $p : X^0 \to X$ be the standard projection. Then p^{-1} defines an open embedding into the projective variety X^0 outside the exceptional locus S, which by normality and the Zariski theorem, is of codimension ≥ 2 in X ([10]). Hence $X \backslash S$ is quasiprojective and contains U. By maximality, $U = X \backslash S$. □

From now on, let X be a complete normal variety with $\dim_{\mathbf{Q}}((Div(X)/Car(X)) \otimes \mathbf{Q}) < \infty$. For a complete variety X let $r = r(X) := \dim((Div(X)/\equiv) \otimes \mathbf{Q})$ where $\equiv$ has been defined before. This is a finite number by finiteness of $\dim_{\mathbf{Q}}((Div(X)/Car(X)) \otimes \mathbf{Q})$ and $\dim_{\mathbf{Q}}((Car(X)/\equiv) \otimes \mathbf{Q})$ (see [5]).

Lemma 4. *Let $X' \subset X$ be an open subset and let $U_1, ..., U_s$ with $s > r$ be open quasiprojective subsets of X'. Assume $X' \backslash U_i$ is of codimension at least 2 in X'. Then for some pairwise distinct indices $i_1, ..., i_k, i_{k+1}, ..., i_{s'} \in \{1, ..., s\}$ where $1 \leq k < s'$ the set $U := (U_{i_1} \cap ... \cap U_{i_k}) \cup (U_{i_{k+1}} \cap ... \cap U_{i_s})$ is quasiprojective.*

Proof. Let D_i be a divisor on X such that D_i is ample on U_i. Then we can find $i_1, ..., i_k, i_{k+1}, ..., i_{s'}$ such that $\sum_{j=1}^{k} n_{i_j} D_{i_j} \equiv \sum_{j=k+1}^{s} n_{i_j} D_{i_j}$ with all n_{i_j} positive. Note that by Lemmas 1 and 2, $\sum_{j=1}^{k} n_{i_j} D_{i_j}$ is ample on U. □

Lemma 5. *Let $U_1, ..., U_s$ be open quasiprojective sets of X' as above. Assume $X' \backslash U_i$ are of dimension $\leq l \leq dim(X) - 2$ and have no common components. Let U be as in the statement of Lemma 4. Then $dim(X' \backslash U) \leq l - 1$.*

Proof. Follows directly from the definition of U. □

Now we prove Theorem A along the following lines. Given a variety X (not necessarily complete) satisfying the condition

$(*)$ X contains infinitely many maximal open quasiprojective subsets.

Let $l(X)$ be the maximal dimension of their complements. We will construct an open subset $X' \subset X$ such that X' satisfies $(*)$ and that $l(X') < l(X)$.

Set $n = \dim(X)$. We say that an open subset U of X has property P(k) for $-1 \le k \le n-1$ iff

1. $\dim X \backslash U \le k$

2. Each component of dimension k of $X \backslash U$ is contained in the complements of only finitely many maximal open quasiprojective sets.

(We mean here that $\dim(\emptyset) = -1$.)

Let U_1^{n-2} be a maximal open quasiprojective subset of X. By Lemma 3 we see that $\dim X \backslash U_1^{n-2} \le n-2$. Remove from X, one by one, all components of $X \backslash U_1^{n-2}$ which are of dimension $n-2$ and contained in the complement of infinitely many maximal open quasiprojective sets. As a result we get X' such that $U_1^{n-2} \subset X' \subset X$ and X' satisfies $(*)$. Observe that U_1^{n-2} has property P$(n-2)$ on X'. By abuse of notation write X for X'. Because of $(*)$ and by the fact that U_1^{n-2} has property P$(n-2)$ on the new X, we can find open qusiprojective U_2^{n-2} in the new X for which $X \backslash U_1^{n-2}$ and $X \backslash U_2^{n-2}$ have no common component of dimension $n-2$. By an analogous procedure of removing components we can assume that U_2^{n-2} has property P$(n-2)$ on some new X' satisfying $(*)$. Again we rename X' as X.

Continuing this process we find $U_1^{n-2}, U_2^{n-2}, ..., U_{r+1}^{n-2}$ such that all sets satisfy condition P$(n-2)$ on the varying X, and $X \backslash U_i^{n-2}$ for $i = 1, ..., r+1$ have no common component.

Note that by shrinking X we are also shrinking its open subsets U_1^{n-2}, $U_2^{n-2}, ..., U_{r+1}^{n-2}$. However all these subsets are still quasiprojective and have property P$(n-2)$ on shrinked X, and X still satisfies condition $(*)$.

Apply Lemma 4 to the sets $U_1^{n-2}, ..., U_{r+1}^{n-2}$ and call the resulting set U_1^{n-3}. By Lemma 5 $\dim X \backslash U_1^{n-3} \le n-3$. As before by removing "bad" components of dimension $n-3$ we can assume that U_1^{n-3} has property P$(n-3)$. Now we construct $U_{r+1}^{n-2}, ..., U_{2r+2}^{n-2}$ which satisfy condition P$(n-2)$ on X and such that the $X \backslash U_i^{n-2}$ have no common components and do not contain any components of dimension $n-3$ of $X \backslash U_1^{n-3}$. The last condition can be maintained since U_1^{n-3} has property P$(n-3)$. Then we find $U_2^{n-3} = U$ for $U_{r+1}^{n-2}, ..., U_{2r+2}^{n-2}$ as in Lemma 4. Again by continuing this process we construct $U_1^{n-3}, U_2^{n-3}, ..., U_{r+1}^{n-3}$ and then find U_1^{n-4} and so on. Finally we get quasiprojective $X = U_1^{-1}$ containing infinitely many maximal quasiprojective sets. □

Remark. As was noted by Z.Jelonek one can easily prove Theorem A for smooth normal surfaces.

Proof. Let X be a normal surface. Then X contains finitely many singular points $\{x_1, ..., x_n\}$. Let $U \subset X$ be a maximal open quasiprojective subset. Resolve all singular points which are not in U. We get a variety $\tilde{X}$ which is projective by a Zariski theorem ([11]). Let $V := X \setminus \{x_1, ..., x_n\} \setminus U$. Then $U \subseteq V \subseteq X$ and by the above $V \subseteq \tilde{X}$ is quasiprojective. Finally $U = V$ by the maximality of U. □

Acknowledgements. Theorem A was conjectured by Professor Andrzej Białynicki-Birula (in the case when X is smooth or normal) (see [1]). I thank Zbigniew Jelonek from Uniwersytet Jagielloński (Kraków) for numerous discussions which were very stimulating for me. He has independently obtained Theorem A under the condition that the vector space $Cl(X) \otimes \mathbf{Q}$ is finite dimensional and in the above mentioned case of normal surface. On the other hand by slight modification of the proof one can prove Theorem A in the case when X is normal and contains only isolated singularities. I would also like to thank Michel Brion, Gottfried Barthel, Michał Szurek and Jarosław Wiśniewski for advice and help.

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(Received September 4, 1998)

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