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On the Singularities of Non-Analytic Szegö Kernels

By Joe Kamimoto

Abstract. The *CR* manifold $M_m = \{(z_1, z_2) \in \mathbb{C}^2;\ \Im z_2 = [\Re z_1]^{2m}\}(m = 2, 3, \ldots)$ is a counterexample, which was given by Christ and Geller, to analytic hypoellipticity of $\bar\partial_b$ and real analyticity of the Szegö kernel. In order to give a direct interpretation for the breakdown of real analyticity of the Szegö kernel, we give a Borel summation type representation of the Szegö kernel in terms of simple singular solutions of the equation $\bar\partial_b u = 0$.

0. Introduction

In [10], Christ and Geller gave the following remarkable counterexample to analytic hypoellipticity of $\bar\partial_b$ for real analytic *CR* manifolds of finite type (in the sense of Kohn or D'Angelo):

Theorem 0.1. *On the three-dimensional CR manifold* $M_m = \{(z_1, z_2) \in \mathbb{C}^2; \Im z_2 = [\Re z_1]^{2m}\}\,(m = 2, 3, \ldots)$, $\bar\partial_b$ *fails to be analytically hypoelliptic* (*in the modified sense of* [10]).

Moreover analytic hypoellipticity of $\bar\partial_b$ is closely connected with real analyticity of the Szegö kernel off the diagonal. By considering the Szegö kernel as a singular solution of the equation $\bar\partial_b u = 0$, Christ and Geller obtained Theorem 0.1 as a corollary to the following theorem.

Theorem 0.2. *The Szegö kernel of* $M_m\,(m = 2, 3, \ldots)$ *fails to be real analytic off the diagonal.*

The proof of Theorem 0.2 by Christ and Geller [10] is based on certain formula of Nagel [23]. Although their proof is logically clear, it seems difficult to understand the singularity of the Szegö kernel of M_m directly,

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since their proof is established by contradiction. On the other hand, Christ ([5],[6]) directly constructed singular solutions of $\bar{\partial}_b u = 0$ and proved Theorem 0.1. In this paper, we give an integral representation of the Szegö kernel of M_m in terms of the singular solutions of Christ. Since the singular solutions of Christ are substantially simpler, our integral representation makes it easy to understand the singularity of the Szegö kernel of M_m. Moreover we give the direct proof of Theorem 0.2. We also give a similar representation of the Bergman kernel of the domain $\{(z_1, z_2) \in \mathbb{C}^2; \Im z_2 > [\Re z_1]^{2m}\}$ $(m = 2, 3, \ldots)$ on the boundary.

We remark that our study is based on the argument due to Christ ([6],§7), and that our result can be considered as an improvement of Proposition 7.2 in [6].

In general we consider the hypersurface

$$M_P := \{(z_1, z_2) \in \mathbb{C}^2; \Im z_2 = P(z_1)\},$$

where $P : \mathbb{C} \to \mathbb{R}$ is a real analytic function. We assume that $\triangle P$ is non-negative and does not vanish identically on any open set in $\mathbb{C}$. Such a surface is pseudoconvex and of finite type. A nonvanishing, antiholomorphic, tangent vector field is $\partial/\partial \bar{z}_1 - 2i(\partial P/\partial \bar{z}_1)\partial/\partial \bar{z}_2$. As coordinates for the surface we use $\mathbb{C} \times \mathbb{R} \ni (z = x + iy, t) \mapsto (z, t + iP(z))$; the vector field pulls back to $\bar{\partial}_b = \partial/\partial \bar{z} - i(\partial P/\partial \bar{z})\partial/\partial t$. Let $\bar{\partial}_b^*$ denote the formal adjoint of $\bar{\partial}_b$ with respect to the Lebesgue measure on $\mathbb{C} \times \mathbb{R}$. We say that $\bar{\partial}_b$ is *analytically hypoelliptic on M_P (in the modified sense of* [10]) if whenever $\bar{\partial}_b u$ is real analytic in an open set U and $u = \bar{\partial}_b^* v$ for some $v \in L^2$ in U, u is real analytic in U. In the usual sense, $\bar{\partial}_b$ is not even C^∞ hypoelliptic, but it is known that if $\bar{\partial}_b \bar{\partial}_b^* u \in C^\infty$, then $\bar{\partial}_b^* u \in C^\infty$([20]).

Let $S((z,t);(w,s))$ be the Szegö kernel of M_P; that is, the distribution kernel associated to the operator defined by the orthogonal projection of $L^2(\mathbb{C} \times \mathbb{R})$, with respect to the Lebesgue measure, onto the kernel of $\bar{\partial}_b$. It is known that the Szegö kernel is C^∞ off the diagonal ([24]).

There are many interesting studies on real analyticity for $\bar{\partial}_b$ and Szegö kernels. For certain strictly pseudoconvex *CR* manifolds M_P (i.e. $\triangle P > 0$), it is known in [12] that $\bar{\partial}_b$ is analytically hypoelliptic. In the weakly pseudoconvex case, there are many important results (see the references in [10]), but the precise condition for them to be real analyticity is still unknown.

In investigating the properties of Szegö kernels, it is an important problem to find a *good* expression for them. Although the Szegö kernels for some classes of *CR* manifolds (or the Bergman kernels for some domains) are explicitly computed in closed form (see the reference in [2]), almost all Szegö kernels seem impossible to be written in closed form. Therefore the following two types of the integral representation for the Szegö kernels are useful to analyse its singularities (e.g. [3],[10],[14]).

The representations of first type are obtained in [13],[22],[21],[23], [15], [26]. The Szegö kernel of M_P (P satisfies certain conditions) can be represented as follows:

$$S((z,t);(w,s)) = c\int_0^\infty\int_{-\infty}^\infty \frac{\tau\exp(\tau(\eta(z+\bar{w})-P(z)-P(w)-i(s-t)))}{\displaystyle\int_{-\infty}^\infty \exp(2\tau(r\eta-P(r)))dr}d\eta d\tau. \tag{0.1}$$

The representations of this type can be obtained by using the generalized Paley-Wiener theorem. Those of second type are represented as the Borel (or Mittag-Leffler) summation of some countably many functions. For example, Bonami and Lohoué [3] gave a representation for the Szegö kernel of the *CR* manifold $\{z\in\mathbb{C}^n; \sum_{j=1}^n |z_j|^{2m_j}=1\}$ ($m_j\in\mathbb{N}$):

$$S(z,w) = c\int_0^\infty e^{-p}\left[\prod_{j=1}^n\sum_{\nu_j=0}^\infty \frac{(z_j\bar{w}_jp^{1/m_j})^{\nu_j}}{\Gamma(\nu_j/m_j+1/m_j)}\right]p^{\sum_{j=1}^n 1/m_j-1}dp.$$

The purpose of this paper is to give a representation of the Borel summation type for the Szegö kernel of M_m by using the representation of first type, which is obtained by Nagel ([23]). Since the Szegö kernel is expressed by the superposition of certain simple singular solutions of $\bar{\partial}_b u=0$ due to Christ, the structure of the singularity can be understood directly. The singularity of the Szegö kernel is almost equal to that of Christ's singular solution involving the first eigenfunction of the certain ordinary differential operator (see §6).

In this paper, we analyse the counterexample of Christ and Geller directly by using the classical asymptotic analysis for ordinary differential equations with irregular singular points. In particular, some techniques to obtain the asymptotic expansion of the functions admitting an integral representation or a Taylor series expansion are very useful for our computation

([11],[28]). Explicitly the properties of the entire function:

$$\varphi(x) = \int_{-\infty}^{\infty} e^{-2(w^{2m}-xw)} dw \tag{0.2}$$

play an important role in the breakdown of real analyticity. The function φ appears in many mathematical subjects and its properties have been studied in detail (ref. the Introduction in [18]).

The plan of this paper is as follows. We state our results and outline of the proofs in Section 1. In Section 2, we recall the direct construction of the singular solutions of $\bar{\partial}_b u = 0$ in [5],[6], which are used in our representations. In Section 3, we establish our theorems. In Sections 4,5, we give the proofs of propositions and lemmas respectively, which are necessary for the proofs of our theorems. In Section 6, we directly show the failure of real analyticity of the Szegö kernel by using our representation.

In this paper, we use c or C $(C(X_1, X_2, \cdots))$ for various constants (depending on $X_1, X_2, \cdots$) without further comment.

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1. Statement of Main Results

For $M_m = \{(z_1, z_2) \in \mathbb{C}^2; \Im z_2 = [\Re z_1]^{2m}\}$ $(m = 2, 3, \ldots)$, Christ in [5],[6] constructed the singular solutions of the equation $\bar{\partial}_b u = 0$ $(u = \bar{\partial}_b^* v, v \in L^2)$ by applying the partial Fourier transformation and by solving a certain simple ordinary differential equation (see §2). The following functions $S_j^v(z,t)$ $(j \in \mathbb{N})$ are slight generalization of the singular solutions of Christ. Let $\varphi(x)$ be the function defined as in (0.2) in the Introduction. It is known that φ has infinitely many zeros and that all of them are simple and exist on the imaginary axis ([25],[19]). We denote them by $\pm i a_j$ $(j \in \mathbb{N})$, where the a_j's are positive and arranged in the increasing order. (More detailed information of φ is given in Subsection 3.1.) Let $S_j^v(z,t)$ be defined by

$$S_j^v(z,t) = \frac{2\pi i}{\varphi'(ia_j)} \int_0^{\infty} e^{it\tau} e^{-x^{2m}\tau} e^{\sigma(y) i a_j z \tau^{1/(2m)}} \tau^{v+1/m} d\tau, \tag{1.1}$$

for $y \neq 0$, $j \in \mathbb{N}$, $v \geq 0$ where $\sigma(y)$ is the sign of y $(z = x+iy)$. It is easy to check that the S_j^v's are not real analytic on $\{(0+iy,0); y \in \mathbb{R}\}$. Besides this, the S_j^v's belong to sth order Gevrey class G^s for all $s \geq 2m$, but no better on $\{(0+iy,0); y \neq 0\}$, where $G^s := \{f; \exists C > 0 \,\text{s.t.}\, |\partial^\alpha f| \leq C^{|\alpha|}\Gamma(s|\alpha|)\ \forall \alpha\}$.

For M_m $(m = 2,3,\ldots)$, define the distribution $K(z,t) = S((z,t);(0,0))$. Then K is a C^∞ function away from $(0,0)$ ([24]). We give a representation of K in terms of the singular solutions $\{S_j^0\}_{j\in\mathbb{N}}$.

THEOREM 1.1. *Suppose* $|\arg z \pm \pi/2| < \pi/(4m-2)$. *Then we have*

$$K(z,t) = c^S \int_0^\infty e^{-p} H(z,t;p)dp, \tag{1.2}$$

where

$$H(z,t;p) = \sum_{j=1}^\infty \frac{S_j^0(z,t)}{\Gamma(f_j+1)} p^{f_j}, \tag{1.3}$$

for some sequence $f_j = j + O(j^{-1}) (> 0)$ *as* $j \to \infty$ *and* c^S *is a constant. Here the series* (1.3) *absolutely converges with respect to* $p \geq 0$ *for fixed* (z,t) *; moreover there exist a positive constant* $C^S(z)$ *depending on* z *such that*

$$|H(z,t;p)| \leq C^S(z) p^{m/(4m-2)} \quad \text{for } p \geq 1. \tag{1.4}$$

Let $B((z_1,z_2);(w_1,w_2))$ be the Bergman kernel of the domain $D \subset \mathbb{C}^n$; that is, the distribution kernel for the orthogonal projection of $L^2(D)$ onto the subspace of holomorphic functions. Then B extends to a C^∞ function on $\overline{D} \times \overline{D}$ minus the diagonal ([24]). When $D = D_m := \{(z_1,z_2) \in \mathbb{C}^2; \Im z_2 > [\Re z_1]^{2m}\}$ $(m = 2,3,\ldots)$, we obtain a similar representation of $K^B(z,t) := B((z,t+ix^{2m});(0,0))$.

THEOREM 1.2. *Suppose* $|\arg z \pm \pi/2| < \pi/(4m-2)$. *Then we have*

$$K^B(z,t) = c^B \int_0^\infty e^{-p} H^B(z,t;p)dp, \tag{1.5}$$

with

$$H^B(z,t;p) = \sum_{j=1}^\infty \frac{S_j^1(z,t)}{\Gamma(f_j+1)} p^{f_j}, \tag{1.6}$$

where f_j's are as in Theorem 1.1 and c^B is a constant. Here the series (1.6) *absolutely converges with respect to $p \geq 0$ for fixed (z, t) ; moreover there exist a positive constant $C^B(z)$ depending on z such that,*

$$\left|H^B(z, t; p)\right| \leq C^B(z) p^{m/(4m-2)} \quad \text{for } p \geq 1. \tag{1.7}$$

REMARKS. 1) If we change the order of the sum and the integral in (1.2),(1.5) formally, we obtain the formal sum of the form $c \sum_{j=1}^{\infty} S_j^v(z, t)$. However the formal sum is not convergent in the usual sense. We shall show this fact in Subsection 5.6.

2) We conjecture that $O(j^{-1})$ can be omitted in the above theorems.

3) We do not know whether the estimates (1.4),(1.7) are optimal.

Outline of the proofs. We explain the idea of the proof of Theorem 1.1 roughly. The proof of Theorem 1.2 is given in the same fashion.

First let us recall the argument due to Christ [6] and consider his result. His computation starts from the formula (0.1) due to Nagel [23]. Normalizing (0.1), we have

$$K(z, t) = c^S \int_0^{\infty} e^{it\tau} e^{-x^{2m}\tau} \left[\int_{-\infty}^{\infty} e^{z\tau^{1/(2m)}v} \frac{1}{\varphi(v)} dv\right] \tau^{1/m} d\tau. \tag{1.8}$$

As mentioned above, the function φ has countably many zeros on the imaginary axis. Therefore by shifting the integral contour with respect to v, applying the residue formula and changing the order of the sum and the integral, the following formula is obtained:

$$K(z, t) = c^S \sum_{j=1}^{n-1} S_j^0(z, t) + E_n(z, t), \tag{1.9}$$

S_j^0's are as in (1.1). But $|E_n|$ does not become small as $n \to \infty$ for fixed (z, t), so (1.9) can not be interpreted as a *standard* asymptotic expansion. (Of course the sum $\sum_{j=1}^{\infty} S_j^0(z, t)$ does not converge in the usual sense (see §5.6)). Our purpose of this paper is to regularize the above expansion of Christ. Our method to deal with the divergent sum is based on the idea of the exact WKB method (ref. [27],[1], e.g.). We remark that the alternation of the series $\{S_j^0(z, t)\}_j$ strongly effects our computation.

Introducing the positive large parameter q, we consider the integral

$$K(z,t;q) = c^S \int_0^\infty e^{it\tau} e^{-x^{2m}\tau} \left[\int_\Gamma e^{z\tau^{1/(2m)}v} \frac{q^{-F(v)-1}}{\varphi(v)} dv \right] \tau^{1/m} d\tau, \tag{1.10}$$

where F and Γ are an appropriate function and contour, respectively. Note that $K(z,t;1) = K(z,t)$. By the same procedure as above, the convergent sum $\sum_{j=1}^\infty S_j^0(z,t)q^{-f_j-1}$ can be obtained, where f_j is as in the theorem. Next we apply the Borel and Laplace transformations to the above convergent sum with respect to q in this order, then we can obtain the following representation:

$$K(z,t;q) = c^S \int_0^\infty e^{-qp} \left[\sum_{j=1}^\infty \frac{S_j^0(z,t)}{\Gamma(f_j+1)} p^{f_j} \right] dp, \tag{1.11}$$

which is equal to the above convergent sum. Note that the Borel sum in the above integral absolutely converges for $p \geq 0$ (see §5.7). Now we regard q as a complex variable. Since the sum in the integral has a good estimate (Proposition 3.2 in §3), the above integral can be analytically continued to $q = 1$. Then we obtain the integral representation (1.2) in the theorem.

The estimate (1.4) of the Borel sum can be obtained by using the idea of Wright [28]. The method of Wright is useful to obtain the asymptotic expansion of entire functions which are expressed by Taylor series.

2. Construction of Singular Solutions

In this section we recall Christ's construction [5],[6] of singular solutions to $\bar\partial_b u = 0$ that are in the range of $\bar\partial_b^*$.

We show that there exist the function F such that $\bar\partial_b^* F$ is not real analytic and $\bar\partial_b \bar\partial_b^* F \equiv 0$. We identify M_m with $\mathbb{C} \times \mathbb{R}$ as in the Introduction. Let $\bar\partial_b^*$ be the formal adjoint of $\bar\partial_b$ with respect to the Lebesgue measure on $\mathbb{C} \times \mathbb{R}$.

In the case of $M_m = \{\Im z_2 = [\Re z_1]^{2m}\}$ $(m \in \mathbb{N})$, $\bar\partial_b = X + iY$ and $\bar\partial_b^* = -X + iY$ where

$$X = \frac{\partial}{\partial x}, \quad Y = \frac{\partial}{\partial y} - 2mx^{2m-1}\frac{\partial}{\partial t}.$$

Applying the partial Fourier transformation in the y and t variables, we seek solutions to $\bar{\partial}_b\bar{\partial}_b^* u \equiv 0$ of the form $u(x,y,t) = e^{it\tau}e^{i\eta y}f(x)$. Then $\bar{\partial}_b\bar{\partial}_b^* u = 0$ reduces to the ordinary differential equation:

$$\left[-\frac{d}{dx} + \left(\eta - 2m\tau x^{2m-1}\right)\right] \circ \left[\frac{d}{dx} + \left(\eta - 2m\tau x^{2m-1}\right)\right] f(x) = 0. \tag{2.1}$$

We suppose that $\tau > 0$ and change the variables by $u = \tau^{1/(2m)}x$, $\eta = \tau^{1/(2m)}\xi$, with $\xi \in \mathbb{C}$. If we set $f(x) = g(\tau^{1/(2m)}x)$, then (2.1) reduces to

$$L_\xi g(u) := \left[-\frac{d}{du} + (\xi - 2mu^{2m-1})\right] \circ \left[\frac{d}{du} + (\xi - 2mu^{2m-1})\right] g(u) = 0$$

Since L_ξ factors as a product of two first-order operators, one finds an explicit solution of $L_\xi g_\xi = 0$:

$$g_\xi(u) = e^{-\xi u + u^{2m}} \int_{-\infty}^{u} e^{-2(s^{2m} - \xi s)} ds.$$

Here we fix $\xi \in \mathbb{C}$ and define formally

$$F_\xi(z,t) = \int_0^\infty e^{it\tau} e^{i\xi y \tau^{1/(2m)}} g_\xi(\tau^{1/(2m)}x) d\tau.$$

Then F_ξ satisfies $\bar{\partial}_b\bar{\partial}_b^* F_\xi \equiv 0$ formally and

$$\bar{\partial}_b^* F_\xi(z,t) = \int_0^\infty e^{it\tau} e^{-x^{2m}\tau} e^{z\xi\tau^{1/(2m)}} d\tau.$$

If the imaginary part σ of ξ is positive (resp. negative) and if g_ξ is a bounded function on $\mathbb{R}$, then the above two integrals converge absolutely for $y > 0$ (resp. $y < 0$). Moreover

$$\left|\frac{\partial^k}{\partial t^k}\bar{\partial}_b^* F_\xi(0+iy,0)\right| = \left|\int_0^\infty \tau^k e^{-\tau^{1/(2m)}\sigma y} d\tau\right|$$

$$= 2m(\sigma y)^{-2mk-2m}\Gamma(2mk+2m).$$

Thus $\bar{\partial}_b^* F_\xi$ would not be real analytic and moreover $\bar{\partial}_b^* F_\xi$ would belong to s-th order Gevrey class G^s for all $s \geq 2m$, but no better, where $G^s = \{f\,;\, \exists C > 0 \, s.t. \, |\partial^\alpha f| \leq C^{|\alpha|}\Gamma(s|\alpha|) \ \forall \alpha\}$.

From the above, if there exists $\xi \in \mathbb{C}$, with $\Im\xi \neq 0$, such that g_ξ is bounded on $\mathbb{R}$, we obtain singular solutions of $\bar{\partial}_b u = 0$. The following lemma gives the condition for g_ξ to be bounded.

LEMMA 2.1 ([4]). *The function g_ξ is bounded on $\mathbb{R}$ if and only if $\xi \in \mathbb{C}$ satisfies $\varphi(\xi) = 0$, where the function φ is as in the Introduction* (0.2).

Here we remark that the function φ appears in the integral formula of the Szegö kernel of M_m (1.8), which has been computed by Nagel [23].

As was mentioned in Section 1, the function φ has infinitely many zeros and all of them exist on the imaginary axis for $m \in \{2, 3, \ldots\}$. Hence we obtain non-real analytic solutions, in the range of $\bar{\partial}_b^*$, to $\bar{\partial}_b u = 0$ for $m \in \{2, 3, \ldots\}$. However when $m = 1$, φ is a Gaussian function and it has no zeros, so we can not construct the singular solutions.

3. Proofs of Theorems 1.1 and 1.2

In this section we give the proofs of Theorems 1.1 and 1.2. Suppose that m is an integer with $m \geq 2$.

3.1. The function φ

In this subsection, let us recall important properties of φ, which are given in [25],[6],[8],[9],[18],[19].

From the integral formula (1.8) due to Nagel, in order to investigate the properties of $K(z, t)$, the analysis of the function φ seems to be important. Actually the following properties of φ in the two lemmas below are very important in our computations. From now on we regard the function φ as an entire function on the complex plane. Note that φ is an even function (i.e. $\varphi(-x) = \varphi(x)$ for $x \in \mathbb{C}$) and $\varphi(0) > 0$.

The first lemma is concerned with the asymptotic behavior of φ at infinity. Set $b = (1/2m)^{1/(2m-1)} - (1/2m)^{2m/(2m-1)}$.

LEMMA 3.1. *Let $A(x) = c_1 x^{(1-m)/(2m-1)} \exp\{c_2 x^{2m/(2m-1)}\}$, where $c_1 = \sqrt{\pi}(2m-1)^{-1/2}(2m)^{-m/(4m-2)}$ and $c_2 = 2b$. If $-\pi/2 + \epsilon \leq \arg x \leq \pi/2 - \epsilon$, then*

$$\varphi(x) = A(x)\{1 + O(x^{-2m/(2m-1)})\},$$

and if $\pi/2 + \epsilon \le \arg x \le 3\pi/2 - \epsilon$, *then*

$$\varphi(x) = A(xe^{-\pi i})\{1 + O(x^{-2m/(2m-1)})\},$$

as $x \to \infty$, *where* ϵ *is an arbitrary constant with* $0 < \epsilon < \pi/4$. *On the line* $\arg x = \pi/2$,

$$\varphi(x) = c_3 x^{(1-m)/(2m-1)} e^{-c_4 x^{2m/(2m-1)}} \times \\ \cos\left(c_5 x^{2m/(2m-1)} + \frac{1-m}{2m-1}\frac{\pi}{2}\right)\{1 + O(x^{2m/(2m-1)})\},$$

as $x \to \infty$, *where* $c_3 = 2^{(m+1)/(2m)}(2m)^{(m-1)/(2m-1)}$, $c_4 = 2b\sin(\pi/(4m-2))$ *and* $c_5 = 2b\cos(\pi/(4m-2))$.

The above lemma gives the exponential order of φ and the distribution of its zeros by Rouché's theorem. Moreover, it is known in [25] that all zeros of φ exist on the imaginary axis. The following lemma implies more detailed properties of zeros of φ.

LEMMA 3.2. *All zeros of* φ *are simple. Let* $\{\pm i a_j; 0 < a_j < a_{j+1}, j \in \mathbb{N}\}$ *be the set of zeros of* φ, *then we have*

$$j = \frac{1}{\pi} c_5 a_j^{2m/(2m-1)} + \frac{m}{2(2m-1)} + O(j^{-1}) \text{ as } j \to \infty.$$

The simpleness of the zeros is shown in [19].

3.2. Proof of Theorem 1.1

The proofs of the lemmas and propositions below are given in Sections 4,5.

For the computation below, we prepare the integral contours $\Gamma_\pm$, $L_\pm$ in the v-plane. Let $\epsilon_0 \ll 1$, $0 < \epsilon_1 < a_1$ be arbitrary positive numbers. Let $D_\pm$ be domains defined by $D_\pm = \{v \in \mathbb{C}; |\Re v| < \epsilon_0 \text{ and } \pm\Im v > \epsilon_1\}$, respectively. The integral contours Γ_+, L_+ are defined in the following way. L_+ follows the boundary of D_+ from $-\epsilon_0 + i\infty$ to $+\epsilon_0 + i\infty$. Next the contour Γ_+ is defined in the following way: Γ_+ follows the half-line $v = re^{i(m+1)\pi/(2m)}$ from $r = \infty$ to $r = \epsilon_1$, the circle $v = \epsilon_1 e^{+i\theta}$ from $\theta = (m+1)\pi/(2m)$ to $\theta = (m-1)\pi/(2m)$ and the half-line $v = re^{i(m-1)\pi/(2m)}$ from $r = \epsilon_1$ to

$r = \infty$. Reflecting the contours Γ_+, L_+ with respect to the real axis, we obtain Γ_-, L_-, respectively. (See Figure 1.)

First as mentioned in the outline of the proof in §1, Christ [6] obtains the formula (1.9). The function $K(z,t)$ is expressed as follows:

$$K(z,t) = c^S \int_0^\infty e^{it\tau} e^{-x^{2m}\tau} \left[\int_{-\infty}^\infty e^{z\tau^{1/(2m)}v} \frac{1}{\varphi(v)} dv \right] \tau^{1/m} d\tau, \tag{3.1}$$

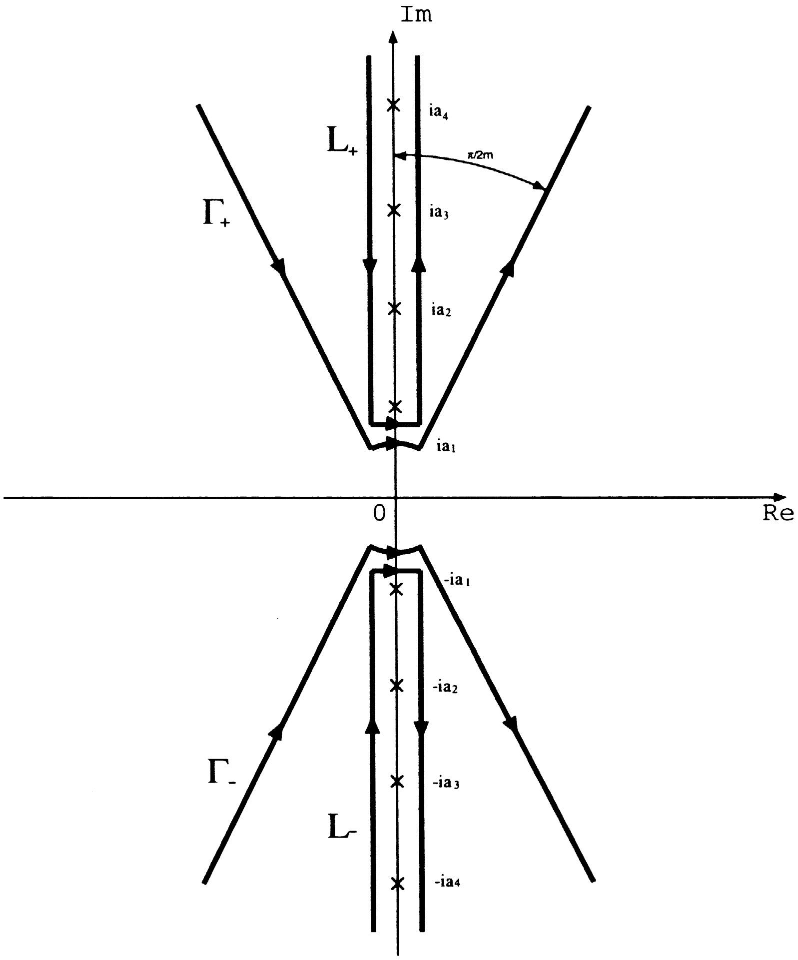

Fig. 1. Integral contours $\Gamma_\pm$, $L_\pm$.

where $\sigma(y)$ is the sign of y $(=\Im z)$. Next we define $K(z,t;q)$ as follows:

$$K(z,t;q)=c^S\int_0^\infty e^{it\tau}e^{-x^{2m}\tau}P(z\tau^{1/(2m)};q)\tau^{1/m}d\tau, \tag{3.2}$$

with

$$P(u;q)=\int_{\Gamma_{\sigma(y)}}e^{uv}\frac{q^{-F_{\sigma(y)}(v)-1}}{\varphi(v)}dv, \tag{3.3}$$

$$\begin{aligned} F_\pm(v) &= c_2e^{\mp i(2m)\pi/(4m-2)}v^{2m/(2m-1)}+\frac{m}{2(2m-1)} \\ &= c_2\left[\pm\frac{1}{i}v\right]^{2m/(2m-1)}+\frac{m}{2(2m-1)}, \end{aligned} \tag{3.4}$$

where q is a complex parameter belonging to a region on which the integral in (3.3) makes sense. Such a region is given by the following lemma.

LEMMA 3.3. *There exist positive constants $\delta<1$ and $\alpha_0<\pi/2$ such that $P(u;q)$ is a holomorphic function of q in the domain $V:=\{q\in\mathbb{C};|q|>1-\delta$ and $|\arg q|<\alpha_0\}$.*

From now on we suppose that q belongs to V. We remark that the point $\{1\}$ is contained in V and $K(z,t;1)=K(z,t)$. In fact we can deform the integral contour $\mathbb{R}$ into Γ_+ or Γ_- in (3.3) by Lemma 3.1.

First we give the proof of the theorem in the case where z is in the sector $|\arg z-\pi/2|<\pi/(4m-2)$. By Lemma 3.1, there exists a positive number $q_0>1$ such that if $|\arg v-\pi/2|<\pi/(2m)$ and $v\notin D_+$, then

$$\left|e^{uv}\frac{q_0^{-F_+(v)-1}}{\varphi(v)}\right|\le C\exp\left\{-c|v|^{2m/(2m-1)}\cos\frac{\pi}{2m-1}\right\},$$

where C,c are positive constants independent of v. Thus we have

$$P(u;q_0)=\int_{\Gamma_+}e^{uv}\frac{q_0^{-F_+(v)-1}}{\varphi(v)}dv=\int_{L_+}e^{uv}\frac{q_0^{-F_+(v)-1}}{\varphi(v)}dv,$$

by Cauchy's theorem. Moreover by the residue formula and Lemma 3.2, we have

$$P(u;q_0)=2\pi i\sum_{j=1}^\infty\frac{1}{\varphi'(a_ji)}e^{ia_ju}q_0^{-f_j-1}, \tag{3.5}$$

for some sequence $\{f_j\}_j$ satisfying that $f_j = j + O(j^{-1}) > 0$ as $j \to \infty$. Substituting (3.5) into (3.2) and changing the order of the sum and the integral, we have

$$K(z,t;q_0) = c^S \sum_{j=1}^{\infty} S_j^0(z,t) q_0^{-f_j - 1}, \tag{3.6}$$

with

$$S_j^0(z,t) = \frac{2\pi i}{\varphi'(a_j i)} \int_0^\infty e^{it\tau} e^{-x^{2m}\tau} e^{\sigma(y) i a_j z \tau^{1/(2m)}} \tau^{\frac{1}{m}} d\tau. \tag{3.7}$$

Now we define by $H(z,t;p)$ the Borel transformation of (3.6) with respect to q, where

$$H(z,t;p) = \sum_{j=1}^{\infty} \frac{S_j^0(z,t)}{\Gamma(f_j+1)} p^{f_j}. \tag{3.8}$$

Proposition 3.1. *If* $|\arg z \pm \pi/2| < \pi/(4m-2)$, *then the Borel sum* (3.8) *absolutely converges with respect to* $p \geq 0$ *; moreover there is a positive constant* $C(z)$ *depending* z *such that*

$$\sum_{j=1}^{\infty} \left| \frac{S_j^0(z,t)}{\Gamma(f_j+1)} p^{f_j} \right| < C(z) p^2 e^p \qquad p \geq 1.$$

The above proof will be given in Subsection 5.7.

By Proposition 3.1, Fubini's theorem implies that

$$K(z,t;q_0) = c^S \int_0^\infty e^{-q_0 p} H(z,t;p) dp.$$

In the same fashion, we have

$$K(z,t;q) = c^S \int_0^\infty e^{-qp} H(z,t;p) dp \quad \text{for } q \geq q_0. \tag{3.9}$$

In fact if $q \geq q_0$, then

$$\left| e^{uv} \frac{q^{-F_+(v)-1}}{\varphi(v)} \right| \leq \left| e^{uv} \frac{q_0^{-F_+(v)-1}}{\varphi(v)} \right|,$$

in $|\arg v - \pi/2| < \pi/(2m)$. Moreover, since the right-hand side of (3.9) extends analytically with respect to q to the region $\{\Re q > q_0\} \cap V$, (3.9) is satisfied there. If we admit Proposition 3.2 below, then (3.9) is satisfied on the region V in the same fashion.

Proposition 3.2. *If $|\arg z \pm \pi/2| < \pi/(4m-2)$, then there is a positive constant $C^S(z)$ depending on z such that*

$$|H(z,t;p)| \leq C^S(z) p^{m/(4m-2)} \quad \text{for } p \geq 1.$$

In particular, (3.9) is satisfied when $q = 1$. Hence we have

$$K(z,t) = K(z,t;1) = c^S \int_0^\infty e^{-p} H(z,t;p)dp. \tag{3.10}$$

On the other hand, if z is in the sector $|\arg z + \pi/2| < \pi/(4m-2)$, then we can also obtain (3.3) by replacing Γ_+ with Γ_-, $F_+(v)$ with $F_-(v)$ and $+ia_j$ with $-ia_j$ in the above argument.

This completes the proof of Theorem 1.1. □

3.3. Proof of Theorem 1.2

The relation between the Szegö kernel of M_m and the Bergman kernel $B((z_1,z_2);(w_1,w_2))$ of the domain $D_m = \{(z_1,z_2) \in \mathbb{C}^2; \Im z_2 > [\Re z_1]^{2m}\}$ was obtained in [24], §7 . We define $K^B(z,t) = B((z, t+ix^{2m});(0,0))$. This relation and the integral formula (1.8) due to Nagel yield

$$K^B(z,t) = c^B \int_0^\infty e^{it\tau} e^{-x^{2m}\tau} \left[\int_{-\infty}^\infty e^{z\tau^{1/(2m)}v} \frac{1}{\varphi(v)} dv\right] \tau^{1+1/m} d\tau, \tag{3.11}$$

The difference between (1.8) and (3.11) does not give any essential influence on the argument in the proof of Theorem 1.1. If we admit Proposition 4.1 in the next section, we can obtain Theorem 1.2 in a similar fashion. □

4. Proposition 4.1

In this section, we prove Proposition 4.1 below, which is a slight generalization of Proposition 3.2. Now we define the function H^v by

$$H^v(z,t;p) = \sum_{j=1}^\infty \frac{S_j^v(z,t)}{\Gamma(f_j+1)} p^{f_j}, \tag{4.1}$$

where S_j^v is as in (3.7).

PROPOSITION 4.1. *If* $|\arg z \pm \pi/2| < \pi/(4m-2)$, *then there is a positive constant* $C(z,v)$ *depending on* z *and* v *such that*

$$|H^v(z,t;p)| \leq C(z,v)p^{m/(4m-2)} \quad \text{for } p \geq 1. \tag{4.2}$$

PROOF OF PROPOSITION 4.1. First we consider the case where $z\,(= x+iy)$ is in the sector $|\arg z - \pi/2| < \pi/(4m-2)$.

By Cauchy's theorem and the residue formula, we have

$$H^v(z,t;p) = \int_{L_+} \frac{1}{\Gamma(F_+(\xi)+1)} \frac{1}{\varphi(\xi)} S^v(\xi;z,t)p^{F_+(\xi)}d\xi, \tag{4.3}$$

with

$$S^v(\xi;z,t) = \int_0^\infty e^{it\tau}e^{-x^{2m}\tau}e^{\xi z\tau^{1/(2m)}}\tau^{v+1/m}d\tau,$$

where the function $F_+(\xi)$ and the integral contour L_+ is as in Section 3. Define the domains $D_\xi^\pm, D_\zeta$ in $\mathbb{C}$ by

$$\begin{aligned} D_\xi^\pm &= \left\{\xi \in \mathbb{C}; \left|\arg \xi \mp \frac{\pi}{2}\right| < \frac{2m-2}{2m-1}\frac{\pi}{2}\right\}, \\ D_\zeta &= \{\zeta \in \mathbb{C}; \Re\zeta > A_0\}. \end{aligned}$$

Then $F_\pm : D_\xi^\pm \to D_\zeta$ are biholomorphic functions. Let $G_\pm : D_\zeta \to D_\xi^\pm$ be the inverse functions of $F_\pm$. Here $G_\pm(\zeta) = \pm i[c_2^{-1}(\zeta - A_0)]^{(2m-1)/(2m)}$.

Let A_j be the values of $F_+(ia_j) = F_-(-ia_j) > 0$ for $j \in \mathbb{N}$ ($A_0 := m/(4m-2)$). Let $\epsilon_1 \ll 1$ and $\epsilon < A_1 - A_0$ be arbitrary positive numbers. Let $\mathcal{D}$ be the domain defined by $\mathcal{D} = \{\zeta; |\Im\zeta| < \epsilon_2 \text{ and } \Re\zeta > A_0 + \epsilon\}$. We define the integral contours L, M_1, M_2, N in $\overline{D_\zeta}$ in the following way. L follows the boundary of $\mathcal{D}$ from $+\infty + i\epsilon_2$ to $+\infty - i\epsilon_2$. M_1 follows the circle $\zeta = Re^{i\theta}$ from $\theta = -\theta_1$ to $\theta = \theta_1$, where $\cos\theta_1 = A_0/R$ ($\theta_1 > 0$) and $R > 0$ is a large number, with $R \neq A_j$. M_2 follows the circle $\zeta = r[e^{i\theta} + A_0]$ from $\theta = -\pi/2$ to $\theta = \pi/2$, where r is a small positive number. N follows the line $\Re\zeta = A_0$ downwards. (See Figure 2.)

By changing the integral variable in (4.3), we have

$$H^v(z,t;p) = \int_L P^v(\zeta;z,t;p)d\zeta, \tag{4.4}$$

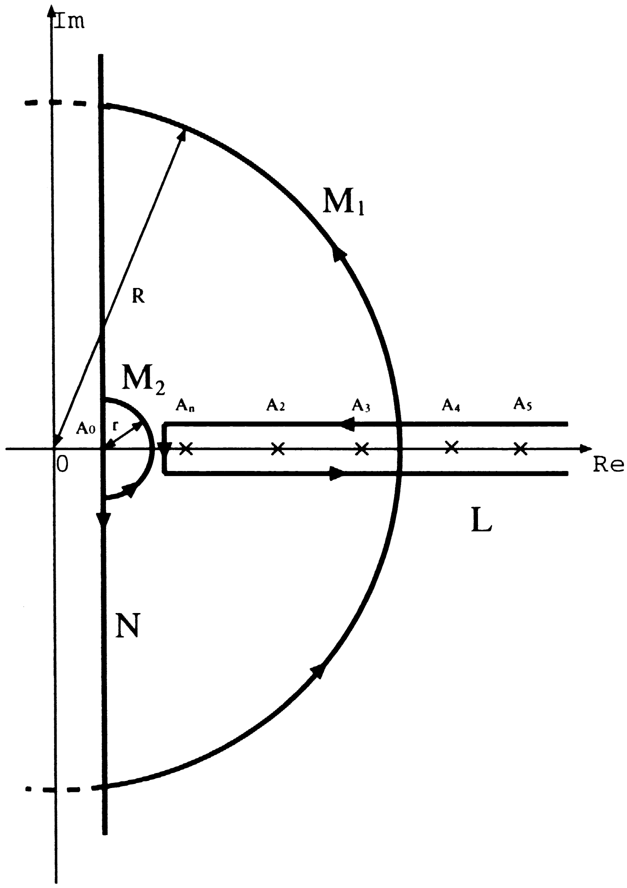

Fig. 2. Integral contours L, M_1, M_2, N.

with

$$P^v(\zeta;z,t;p)\ (=P(\zeta;p))=\frac{p^\zeta}{\Gamma(\zeta+1)}\Phi^v(\zeta;z,t),$$

$$\Phi^v(\zeta;z,t)=\frac{1}{\varphi(G_+(\zeta))}S^v(G_+(\zeta);z,t)G'_+(\zeta).$$

Note that $P(\zeta;p)$ extends analytically with respect to ζ to the region $\{\Re\zeta>0\}\setminus(\{A_j\}_{j\in\mathbb{N}}\cup\{\zeta\leq A_0\})$ for fixed z and p. Set $\zeta=re^{i\theta}$.

LEMMA 4.1. *For any $h\geq A_0(=m/(4m-2))$, there are positive num-*

bers $a, \rho_0, \alpha_0, \kappa_0$ *and* δ *such that,* (i) *if* $\Re\zeta \geq h, |\zeta| \geq \max\{\rho_0, ap\}$ *and* $|\Im\zeta| \geq \epsilon_2$*, then*

$$|P(\zeta;p)| \leq C(z,v,h) r^{-\alpha_0} e^{-\kappa_0 r} p^h, \tag{4.5}$$

(ii) *if* $\Re\zeta \geq A_0$ *and* $|\zeta - A_0| \leq \epsilon_3$*, then*

$$|P(\zeta;p)| \leq C(z)\, |\zeta - A_0|^{-1/(2m)} \cdot p^{A_0}. \tag{4.6}$$

Lemma 4.1 (i) implies

$$\int_{M_1} P(\zeta;p) d\zeta \ \text{ tends to zero as } R \to \infty, \tag{4.7}$$

whereas Lemma 4.1 (iii) implies

$$\int_{M_2} P(\zeta;p) d\zeta \ \text{ tends to zero as } r \to 0. \tag{4.8}$$

Cauchy's theorem applied to (4.7) yields

$$\int_L P(\zeta;p) d\zeta = \int_N P(\zeta;p) d\zeta. \tag{4.9}$$

Lemma 4.1 implies

$$\begin{aligned} \left| \int_N P(\zeta;p) d\zeta \right| &\leq \int_{-\infty}^{\infty} |P(A_0 + it; p)| dt \\ &\leq C(z,v) p^{A_0} \end{aligned} \tag{4.10}$$

Therefore we obtain the estimate (4.2) in the proposition by putting (4.4),(4.9),(4.10) together.

On the other hand, we consider the case where z is in the sector $|\arg z + \pi/2| < \pi/(4m-2)$. Replacing $+ia_j$ with $-ia_j$, $F_+(\xi)$ with $F_-(\xi)$ and $G_+(\zeta)$ with $G_-(\zeta)$ in the above argument, we can obtain (4.2) in the same fashion.

This completes the proof of Proposition 4.1. □

5. Proof of Lemmas

In this section we establish the lemmas mentioned previously. We suppose that z is in the sector $|\arg z \pm \pi/2| < \theta_0$, where $\theta_0 := \pi/(4m-2)$. Set $\zeta = re^{i\theta}$.

5.1. Proof of Lemma 3.3

We only give the proof for the case where the integral contour in (3.3) is Γ_+. The proof in the case of Γ_- can be done in the same fashion. By Lemma 4.1, we obtain

$$\left|\frac{e^{uv}}{\varphi(v)}\right| \leq C(u)\exp\left\{-c_1 r^{2m/(2m-1)}\sin\theta_0\right\},$$

and

$$\begin{aligned}&\left|q^{-F_+(v)-1}\right| \leq c|q|^{-3/4}\times\\ &\exp\left\{-c_2 r^{2m/(2m-1)}[\cos 2\theta_0\cdot\log|q| - \sin 2\theta_0\cdot|\arg q|]\right\},\end{aligned}$$

for $v\in\Gamma_+$, where c_1 is as in Lemma 3.1, c_2 is as in Lemma 3.2. Then

$$\begin{aligned}&\left|e^{uv}\frac{q^{-F_+(v)-1}}{\varphi(v)}\right| \leq C(u)|q|^{-1}\times\\ &\quad\exp\left\{-r^{2m/(2m-1)}\left[c_1\sin\theta_0 + c_2\cos 2\theta_0\cdot\log|q| - c_2\sin 2\theta_0\cdot|\arg q|\right]\right\}.\end{aligned}$$

Now set

$$\begin{aligned}V = \Big\{q\in\mathbb{C}\,;\ |q| \geq \exp\left\{-\frac{1}{3}\frac{\sin\theta_0}{\cos 2\theta_0}\frac{c_1}{c_2}\right\} \text{ and}\\ |\arg q| \leq \min\left\{\frac{\pi}{2}, \frac{1}{3}\frac{\sin\theta_0}{\sin 2\theta_0}\frac{c_1}{c_2}\right\}\Big\},\end{aligned}$$

then we obtain

$$\left|e^{uv}\frac{q^{-F_+(v)-1}}{\varphi(v)}\right| \leq C(u)e^{-cr^{2m/(2m-1)}} \quad \text{on } V.$$

Note that $0 < \exp\{-(c_1\sin\theta_0)/(3c_2\cos 2\theta_0)\} < 1$. Since the integrand in (3.3) satisfies the above inequality and is a holomorphic function of q on V, we can obtain Lemma 3.3. $\square$

5.2. Proof of Lemma 4.1

In order to prove Lemma 4.1, we prepare the following two lemmas. We write $P(\zeta;p) = p^{\zeta}\Gamma(\zeta+1)^{-1}\Phi^v(\zeta;z,t)$, where $\Phi^v(\zeta;z,t) = \varphi(G_\pm(\zeta))^{-1}S^v(G_\pm(\zeta);z,t)G'_\pm(\zeta)$.

LEMMA 5.1. *Let h be a real number. If either* (i) $\Re\zeta = h$ *or* (ii) $\Re\zeta \geq h$ *and* $|\zeta| \geq bep$ *with* $b \geq 1$, *then*

$$\left|\frac{p^\zeta}{\Gamma(\zeta+1)}\right| \leq cr^{-1/2-h} b^{h-r\cos\theta} p^h \exp\{r\sin\theta\cdot\theta\}. \tag{5.1}$$

LEMMA 5.2. *There are positive constants* $\rho_0, \beta_0, \epsilon_3$ *such that,* (i) *if* $\Re\zeta \geq A_0, |\zeta| \geq \rho_0$ *and* $|\Im\zeta| \geq \epsilon_2$, *then*

$$|\Phi^v(\zeta;z,t)| \leq C(z,v)r^{-\beta_0}\exp\left\{-\pi r\frac{\cos(\theta_0+\pi/2-|\theta|)}{\cos\theta_0}\right\}, \tag{5.2}$$

and (ii) *if* $\Re\zeta \geq A_0$ *and* $|\zeta - A_0| \leq \epsilon_3$, *then*

$$|\Phi^v(\zeta;z,t)| \leq C(z)|\zeta - A_0|^{-1/(2m)}. \tag{5.3}$$

By using the above two lemmas, we obtain Lemma 4.1 as follows.

First we consider the case (i). By Lemma 5.1 (ii) and Lemma 5.2 (i), if $\Re\zeta \geq h, |\zeta| \geq \max\{\rho_0, bep\}$ $(b \geq 1)$ and $|\Im\zeta| \geq \epsilon_2$, then we have

$$\begin{aligned}
|P(\zeta;p)| &= \left|\frac{p^\zeta}{\Gamma(\zeta+1)}\right|\cdot|\Phi^v(\zeta;z,t)| \\
&\leq C(z,v)r^{-\beta_0-\frac{1}{2}-h}b^{h-r\cos\theta}\times \\
&\quad \exp\left\{-\pi r\frac{\cos(\theta_0+\pi/2-|\theta|)}{\cos\theta_0}+r\sin\theta\cdot\theta\right\}\cdot p^h \\
&\leq C(z,v,h)r^{-\beta_0-\frac{1}{2}-h}\times \\
&\quad \exp\left\{-r\left[\log b\cdot\cos\theta+\pi\frac{\cos(\theta_0+\pi/2-|\theta|)}{\cos\theta_0}+\sin\theta\cdot\theta\right]\right\}\cdot p^h.
\end{aligned}$$

We consider the case $\theta \geq 0$. We put $b = \exp\{\pi/(\cos\theta_0)\}$, then we can obtain

$$\log b\cdot\cos\theta + \pi\frac{\cos(\theta_0+\pi/2-\theta)}{\cos\theta_0} \geq \frac{1}{\cos\theta_0}\frac{\pi}{2},$$

for $0 \leq \theta \leq \pi/2$. Therefore if $0 \leq \theta \leq \pi/2$ and $|\Im\zeta| \geq \epsilon_2$, then we have

$$\begin{aligned}
|P(\zeta;p)| &\leq C(z,v,h)r^{-\beta_0-1/2-h}\exp\left\{-r\left[\frac{1}{\cos\theta_0}\frac{\pi}{2}-\sin\theta\cdot\theta\right]\right\} \\
&\leq C(z,v,h)r^{-\beta_0-1/2-h}\exp\left\{-r\left[\frac{\pi}{4}\left(\frac{1}{\cos\theta_0}-1\right)\right]\right\}. \qquad (5.4)
\end{aligned}$$

If $-\pi/2 \le \theta \le 0$ and $\Im\zeta \ge \epsilon_2$, then we obtain the inequality (5.4) in the same fashion. Here if we put $a = \exp\{\pi/(\cos\theta_0) + 1\}, \alpha_0 = \beta_0 + 1/2 + h$ and $\kappa_0 = (\pi/4)(1/(\cos\theta_0) - 1)$, then we have (4.5) under the condition (i).

Last we can easily obtain the estimate (4.6) by Lemma 5.2 (ii). □

5.3. Proof of Lemma 5.1

When $\Re\zeta \ge h$, we have

$$\frac{1}{\Gamma(\zeta+1)} = \frac{1}{(2\pi)^{1/2}} \frac{e^{\zeta}}{\zeta^{\zeta+1/2}} \left\{1 + O(\zeta^{-1})\right\}.$$

by Stirling's formula. Hence

$$\begin{aligned} \left|\frac{p^{\zeta}}{\Gamma(\zeta+1)}\right| &< cr^{-1/2}\exp\{r\cos\theta\log(epr^{-1}) + r\sin\theta\cdot\theta\} \\ &= cr^{-1/2}b^{-r\cos\theta}\cdot\exp\{r\cos\theta\cdot\log(bepr^{-1}) + r\sin\theta\cdot\theta\}. \end{aligned}$$

If the condition (i) is satisfied, then

$$\begin{aligned} \left|\frac{p^{\zeta}}{\Gamma(\zeta+1)}\right| &< cr^{-1/2}b^{-r\cos\theta}\cdot\exp\{h\log(bepr^{-1}) + r\sin\theta\cdot\theta\} \\ &< cr^{-1/2-h}b^{h-r\cos\theta}p^{h}\cdot\exp\{r\sin\theta\cdot\theta\}. \end{aligned}$$

If the condition (ii) is satisfied, then

$$r\cos\theta\log(bepr^{-1}) \le h\log(bepr^{-1}),$$

and the result follows as before. □

5.4. Proof of Lemma 5.2

First we consider the case (i). We define the sectors $V_{\pm} \subset \mathbb{C}\times\mathbb{C}\times\mathbb{R}$ by

$$V_{\pm} = \left\{(\xi; z, t);\ \left|\arg\xi \mp \frac{\pi}{2}\right| \le \frac{2m-2}{2m-1}\frac{\pi}{2} \text{ and } \left|\arg z \mp \frac{\pi}{2}\right| < \frac{1}{2m-1}\frac{\pi}{2}\right\}.$$

We need the following lemma about the behavior of $S^v(\cdot; z, t)$ at infinity on $V_{\pm}$:

LEMMA 5.3. *If $(\xi; z, t)$ is in the sectors $V_{\pm}$, then there is a non-zero constant $A^v(z) \in \mathbb{C}$ depending on z and v such that*

$$\lim_{|\xi|\to\infty} \xi^{2mv+2m} S^v(\xi; z, t) = A^v(z). \tag{5.5}$$

By the above lemma, we have

$$|S^v(\xi; z, t)| \leq C(z, v)|\xi|^{-2mv-2m}. \tag{5.6}$$

By Lemma 3.1, we have the following: If $|\arg \zeta| \leq \pi/2$ and $|\Im\zeta| \geq \epsilon_1$, then

$$\left|\frac{1}{\varphi(G_\pm(\zeta))}\right| \leq Cr^{\frac{m-1}{2m}} \exp\left\{-\pi r \frac{\cos(\theta_0 + \pi/2 - |\theta|)}{\cos\theta_0}\right\}. \tag{5.7}$$

Since $G'_\pm(\zeta) = \pm i(2m-1)/(2m)[(1/c_2)(\zeta - A_0)]^{-1/(2m)}$ $(A_0 = m/(4m-2))$, we have

$$|G'_\pm(\zeta)| \leq C|\zeta - A_0|^{-1/(2m)}. \tag{5.8}$$

If we put $\beta_0 = (2m-1)(v+1)-(m-1)/(2m)+1/(2m)$, we have (5.2) under the condition (i) by (5.6), (5.7), (5.8). We remark that the above value of β_0 is best possible.

Last we can easily obtain the estimate (5.3) by (5.8). □

5.5. Proof of Lemma 5.3

First we consider the case $(\xi; z, t) \in V_+$. Writing $\xi = \rho e^{i\alpha}$, with $\rho > 0$ and $|\alpha - \pi/2| \leq (2m-2)\pi/(4m-2)$ and changing the integral variable, we have

$$\begin{aligned} S^v(\xi; z, t) &= \int_0^\infty e^{it\tau} e^{-x^{2m}\tau} e^{\xi z \tau^{1/(2m)}} \tau^v d\tau \\ &= \frac{2m}{\rho^{2mv+2m}} \int_0^\infty \exp\left\{it\frac{u^{2m}}{\rho^{2m}} - x^{2m}\frac{u^{2m}}{\rho^{2m}} + e^{i\alpha} z u\right\} u^{2mv+2m-1} du. \end{aligned}$$

Since $\Re(e^{i\alpha} z) < 0$ on V_+,

$$\int_0^\infty \exp\left\{it\frac{u^{2m}}{\rho^{2m}} - x^{2m}\frac{u^{2m}}{\rho^{2m}} + e^{i\alpha} z u\right\} u^{2mv+2m-1} du$$

$$\to \int_0^\infty e^{e^{i\alpha} z u} u^{2mv+2m-1} du = \frac{\Gamma(2mv+2m)}{[e^{i(\alpha-\pi)} z]^{2mv+2m}} \neq 0,$$

as $\rho \to \infty$.

In the case $(\xi; z, t) \in V_-$, we have (5.5) in the same fashion as above. □

5.6. Divergence of the formal sum in Remark 3

If we change the order of the integral and the sum in (1.2) (resp. (1.5)), we obtain the formal sum:

$$c\sum_{j=1}^{\infty} S_j^v(z,t), \tag{5.9}$$

where $v=0$ (resp. $v=1$). Then we have

Proposition 5.1. *If* $|\arg z \pm \pi/2| < \pi/(4m-2)$*, the formal sum* (*5.9*) *is not convergent in the usual sense.*

Proof. Lemma 3.1 implies that there are two constants a_1, a_2 $(1 < a_1 < a_2)$ such that

$$a_1^j \leq |\varphi'(ia_j)|^{-1} \leq a_2^j \quad \text{for } j \in \mathbb{N}. \tag{5.10}$$

Moreover by Lemma 5.3, there are positive constants $j_0 \in \mathbb{N}$ and $b_1(z)$, $b_2(z)$ depending on z such that

$$\begin{aligned} b_1(z) j^{-2mv-2m} &\leq \left| \int_0^\infty e^{it\tau} e^{-x^{2m}\tau} e^{\sigma(y) i a_j z \tau^{1/(2m)}} \tau^{v+1/m} d\tau \right| \\ &\leq b_2(z) j^{-2mv-2m} \end{aligned} \tag{5.11}$$

on $|\arg z \pm \pi/2| \leq \pi/(4m-2)$ for $j \geq j_0$. The inequalities (5.10),(5.11) imply that the formal sum (5.9) is not convergent in the usual sense. □

5.7. Absolute convergence of the Borel sums H and H^B

In this subsection, we give the proof of Proposition 3.1.

From the previous subsection, there is an integer j_1 greater than j_0 such that

$$\begin{aligned} &\Gamma(f_j+1) \geq \Gamma(j), \ f_j \leq j+1 \text{ and} \\ &|S_j^v(z,t)| \leq a_2 b_2(z) j^{-2mv-2m} \leq a_2 b_2(z), \end{aligned}$$

for $j \geq j_1$, $v \geq 0$ and $|\arg z \pm \pi/2| < \pi/(4m-2)$. The above three inequalities imply

$$\sum_{j=j_1}^{\infty} \left| \frac{S_j^v(z,t)}{\Gamma(f_j+1)} p^{f_j} \right| \leq a_2 b_2(z) \sum_{j=j_1}^{\infty} \frac{p^{j+1}}{\Gamma(j)} \leq a_2 b_2(z) p^2 e^p,$$

for $p \geq 1$. This completes the proof of Proposition 3.1.

6. Direct Proof of Theorem 0.2

In this section, we can directly show that the Szegö kernel of M_m, off the diagonal, fails to be real analytic and moreover it belongs to s-th order Gevrey class G^s for all $s \geq 2m$, but no better on certain set. In the case of the Bergman kernel of D_m, we can obtain the same result in a similar fashion.

We show that the singularity of the Szegö kernel of M_m is almost equal to that of the function S_1^0, where S_1^0 is the singular solution involving the first eigenfunction for $L_\xi g = 0$ (see §2). Now we suppose that z is in the sector $|\arg z - \pi/2| < \pi/(4m-2)$. Recall the expansion due to Christ :

$$K(z,t) = c^S \sum_{j=1}^{n-1} S_j^0(z,t) + E_n(z,t). \tag{6.1}$$

The following lemma is also obtained by Christ [6].

Lemma 6.1. *There is a positive constant C independent of k such that*

$$\left| \frac{\partial^k}{\partial t^k} E_n(0+iy,0) \right| \leq C \frac{\Gamma(2mk+2m+2)}{[|y|(a_{n-1}+\epsilon)]^{2mk+2m+2}},$$

where ϵ is an arbitrary constant with $0 < \epsilon < a_n - a_{n-1}$.

By the above lemma and (6.1), we have

$$\frac{\partial^k}{\partial t^k} K(0+iy,0) = c \frac{\partial^k}{\partial t^k} S_1^0(0+iy,0) \left\{ 1 + O\left(a^{-k}\right) \right\}$$

as $k \to \infty$, where $a > 1$ is a constant. Note that

$$\frac{\partial^k}{\partial t^k} S_j^0(0+iy,0) = 2mi^k \frac{\Gamma(2mk+2m+2)}{(|y|a_j)^{2mk+2m+2}}$$

(see §2). Since S_1^0 does not belong to s-th order Gevrey class G^s for $s < 2m$, we can easily obtain

$$\left| \frac{\partial^k}{\partial t^k} K(0+iy,0) \right| \geq c \frac{\Gamma(2mk+2m+2)}{(|y|a_1)^{2mk+2m+2}}.$$

for sufficiently large $k \in \mathbb{N}$. We can obtain the same result in the case where $|\arg z + \pi/2| < \pi/(4m-2)$ in the same fashion.

PROOF OF LEMMA 6.1. It is easy to obtain the following equation:

$$\frac{\partial^k}{\partial t^k} E_n(0+iy, 0) = c^S i^{(2m+1)k+2m+2} 2m\Gamma(2mk+2m+2)|y|^{-2mk-2m-2} I_{k,n},$$

with

$$I_{k,n} = \int_{\gamma_{\sigma(y)}} h(v)dv,$$

where $h(v) = [\varphi(v)]^{-1} v^{-2mk-2m-2}$ and integral contour $\gamma_{\sigma(y)}$ is as in Section 3. Therefore in order to prove Lemma 6.1, it is sufficient to show that

$$|I_{k,n}| \leq c(a_n + \epsilon)^{-2mk}, \tag{6.2}$$

where ϵ is an arbitrary constant with $0 < \epsilon < a_n - a_{n-1}$.

We shall prove the inequality (6.2). By changing the integral contour, we have

$$I_{k,n} = \int_{C^n} h(v)dv,$$

where the integral contour C^n consists of three parts L_1, L_2, L_3: L_1 follows the half-line from $-\infty$ to $-a_{n-1} - \epsilon$, L_2 follows the circle $v = (a_{n-1} + \epsilon)e^{i\theta}$ from $\theta = \pi$ to $\theta = 0$ and L_3 follows the half-line from $a_{n-1} + \epsilon$ to ∞. (See Figure 3.)

Since h is a positive function on L_3, we have by Schwarz's inequality

$$\begin{aligned} \int_{L_3} h(v)dv &\leq \left\{ \int_{a_{n-1}+\epsilon}^{\infty} \varphi(v)^{-2} dv \right\}^{1/2} \cdot \left\{ \int_{a_{n-1}+\epsilon}^{\infty} v^{-4mk-4m-4} dv \right\}^{1/2} \\ &\leq c(a_{n-1} + \epsilon)^{-2mk}. \end{aligned} \tag{6.3}$$

Since h is an even function, we have

$$\int_{L_1} h(v)dv \leq (a_{n-1} + \epsilon)^{-2mk}. \tag{6.4}$$

Moreover we have

$$\begin{aligned} \left| \int_{L_2} h(v)dv \right| &\leq (a_{n-1} + \epsilon)^{-2mk-2m-3} \left| \int_0^{\pi} \frac{e^{-i(2mk+2m+1)\theta}}{\varphi((a_{n-1} + \epsilon)e^{i\theta})} d\theta \right| \\ &\leq c(a_{n-1} + \epsilon)^{-2mk}. \end{aligned} \tag{6.5}$$

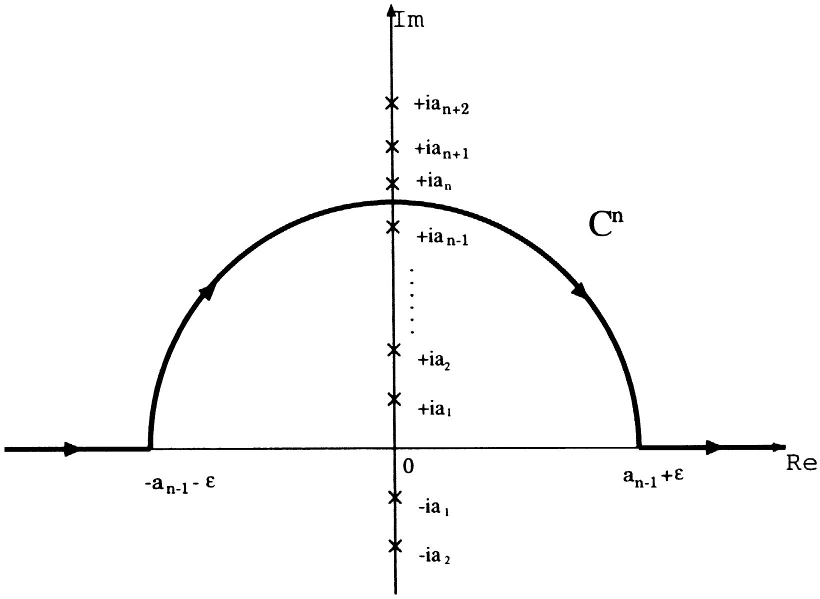

Fig. 3. Integral contour C^n.

Therefore we have

$$\left|\int_C h(v)dv\right| < c(a_{n-1}+\epsilon)^{-2mk},$$

by (6.3),(6.4),(6.5), so we obtain (6.2).

This completes the proof of Lemma 6.1. □

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