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Strict Convexity of Hypersurfaces in Spheres, II

By Naoya Ando

Abstract. This paper introduces and investigates the notion of r-strict convexity for hypersurfaces in the unit sphere. Particularly the r-strict convexity of hypersurfaces with the definite second fundamental form is studied.

1. Introduction

Let M be a connected, compact, n-dimensional differentiable manifold ($n \geqq 1$) and $\iota : M \to S^{n+1}$ an immersion of M into the unit $(n+1)$-sphere S^{n+1}. For any $p \in M$, we denote by S_p the totally geodesic hypersphere in S^{n+1} with $\iota(p) \in S_p$ and with $T_{\iota(p)}(S_p) = d\iota_p(T_p(M))$, and by H_p either of the two open hemispheres determined by S_p. An immersion ι is said to be *locally strictly convex at a point* $p \in M$ if there exists a neighborhood U_p of p in M such that $\iota(U_p \setminus \{p\})$ is contained in H_p. Moreover, ι is said to be *strictly convex at* p if $\iota(M \setminus \{p\})$ is contained in H_p. In [1], we have studied strict convexity of ι, particularly proved that if ι is an embedding, then ι is strictly convex at each point of M if and only if ι is locally strictly convex at each point of M. In this paper, we introduce and discuss the notion of r-strict convexity($r \in (0, \pi)$) for an immersion ι. For any $p \in M$, we denote by $B_p(r)$ either of the two geodesic balls in S^{n+1} with radius r such that $\iota(p) \in \partial B_p(r)$ and $T_{\iota(p)}(\partial B_p(r)) = d\iota_p(T_p(M))$, and say that M *is tangent to* $B_p(r)$ *at* p *by* ι. An immersion ι is said to be *locally* r*-strictly convex at a point* $p \in M$ if there exists a neighborhood U_p of p in M such that $\iota(U_p \setminus \{p\})$ is contained in $B_p(r)$. Moreover, ι is said to be r*-strictly convex at* p if $\iota(M \setminus \{p\})$ is contained in $B_p(r)$. Notice that the (local) $\frac{\pi}{2}$-strict convexity is just the (local) strict convexity defined as above. As an analogy of our result with respect to strict convexity, we have the following.

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THEOREM 1.1. *Let M be a connected, compact, n-dimensional differentiable manifold* $(n \geqq 1)$ *and* $\iota : M \to S^{n+1}$ *an embedding of* M *into* S^{n+1}. *Then for* $r \leqq \pi/2$, *the following are equivalent*:

(1) ι *is locally r-strictly convex at each point of* M;

(2) ι *is r-strictly convex at each point of* M.

In addition to the above definitions, we say that an immersion ι is *r-strictly convex*$(r \in (\pi/2, \pi))$ *on both sides at* $p \in M$ if $\iota(M \backslash \{p\}) \subset B_p^{(1)}(r) \cap B_p^{(2)}(r)$, where $B_p^{(1)}(r)$ and $B_p^{(2)}(r)$ are the geodesic balls in S^{n+1} with radius r to which M is tangent at p by ι (see Figure 1). Then we have the following.

THEOREM 1.2. *Let M be a connected, compact, n-dimensional differentiable manifold* $(n \geqq 1)$ *and* $\iota : M \to S^{n+1}$ *an embedding of* M *into* S^{n+1} *with positive principal curvatures* $\{\kappa_i\}_{i=1}^{n} (\kappa_1 \leqq \ldots \leqq \kappa_n)$. *Set*

$$\mu_1 := \min_{p \in M}\{\kappa_1(p)\}, \qquad \mu_2 := \max_{p \in M}\{\kappa_n(p)\},$$

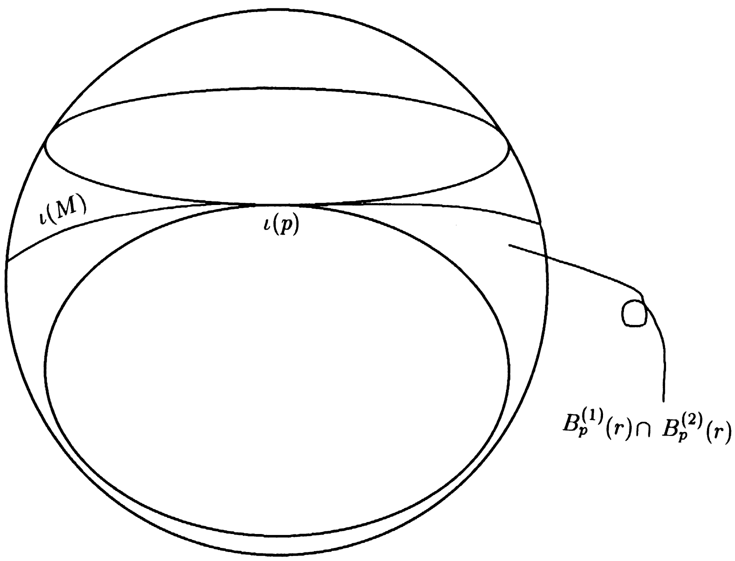

Figure 1. r-strict convexity on both sides$(r \in (\pi/2, \pi))$.

and let r_1, r_2 be the two constants satisfying

$$0 < r_1 < \pi/2 < r_2 < \pi,$$

$$\mu_i = |\cot r_i| \quad (i = 1, 2).$$

Then for any $\rho_1 \in (r_1, \pi/2)$, ι is ρ_1-strictly convex at each point of M and for any $\rho_2 \in (r_2, \pi)$, ι is ρ_2-strictly convex on both sides at each point of M.

Moreover, we have

THEOREM 1.3. *If $n = 1$ or 2, and if $\mu_1 \neq \mu_2$, then there exists a point of M at which ι is r_1-strictly convex and r_2-strictly convex on both sides.*

This paper is organized as follows. In Section 2, we prepare for the following sections, particularly define the Gauss map G_0 for an immersion ι carefully and discuss the properties of G_0. In Section 3, in Section 4 and in Section 5, we prove Theorem 1.1, Theorem 1.2 and Theorem 1.3, respectively.

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2. Preliminaries

Let M be an n-dimensional orientable differentiable manifold($n \geqq 1$) and ι an immersion of M into S^{n+1}, and g the metric of M induced from the standard metric g' of S^{n+1}. We denote by ∇, ∇' the Levi-Civita connections of M, S^{n+1} respectively. In the following, we shall denote by X, Y, Z and W differentiable vector fields tangent to M and by ξ a differentiable unit vector field normal to M. Then Gauss' formula and Weingarten's formula are the following:

$$\begin{aligned} \nabla'_X Y &= \nabla_X Y + \alpha(X, Y), \\ \nabla'_X \xi &= -A_\xi(X), \end{aligned}$$

where α is the second fundamental form of M, which is symmetric in X, Y, and A_ξ is the Weingarten map with respect to ξ. The following relation between α and A_ξ holds:

$$g(A_\xi(X), Y) = g'(\alpha(X, Y), \xi).$$

Consequently, at each $p \in M$, A_ξ is a symmetric linear mapping of the tangent space $T_p(M)$ with respect to g. Therefore A_ξ is diagonalizable and has the real eigenvalues $\kappa_1 \leqq \ldots \leqq \kappa_n$. These are called *the principal curvatures* and the directions of the corresponding eigenvectors are called *the principal directions*. Two principal directions corresponding to two distinct principal curvatures are perpendicular to each other. *The Gauss-Cronecker curvature* is defined by $K_n = \kappa_1 \cdots \kappa_n$. Particularly, if $n = 1$, i.e., M is a smooth curve, then $K_1 = \kappa_1$ is called *the geodesic curvature*, denoted by κ_g. The equations of Gauss and of Codazzi are

$$\begin{aligned} g(R(X,Y)Z,W) &= \{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\} \\ &\quad + \{g'(\alpha(X,W), \alpha(Y,Z)) - g'(\alpha(X,Z), \alpha(Y,W))\} \end{aligned}$$

and

$$(\nabla_X \alpha)(Y, Z) = (\nabla_Y \alpha)(X, Z),$$

where R denotes the curvature tensor of M and

$$(\nabla_X \alpha)(Y, Z) = D_X(\alpha(Y, Z)) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z)$$

(D is the normal connection of ι).

Suppose that M is oriented. We shall define the Gauss map for an immersion ι of M into S^{n+1}. A map $G : M \to S^{n+1}$ is said to be *a Gauss map for an immersion ι of M into S^{n+1}* if for any $p \in M$, $G(p)$ is identified in $\mathbf{R}^{n+2}$ with a unit vector tangent to S^{n+1} and normal to $(d\iota)_p(T_p(M))$ at $\iota(p)$. If M is connected and compact, and if ι is strictly convex at each point of M, then a Gauss map is one-to-one. We want to pay attension to a special Gauss map. We particularly denote by ι_M an immersion ι of M into S^{n+1} and by $\iota_{S^{n+1}}$ the standard embedding of S^{n+1} into R^{n+2}. However we shall also denote by ι_M the composite map of ι and $\iota_{S^{n+1}}$. Firstly, determine the positive orientation of $\mathbf{R}^{n+2}$. Next, we determine the positive orientation of S^{n+1} as follows: At each $q \in S^{n+1}$, an ordered base $\{E'_1, \ldots, E'_{n+1}\}$ of $T_q S^{n+1}$ is positive if

$$\{(d\iota_{S^{n+1}})_q(E'_1), \ldots, (d\iota_{S^{n+1}})_q(E'_{n+1}),\ \iota_{S^{n+1}}(q)\}$$

is a positive ordered base of $T_{\iota_{S^{n+1}}(q)}\mathbf{R}^{n+2}$, where we denote by $\iota_{S^{n+1}}(q)$ not only the image of q by $\iota_{S^{n+1}}$ but also the unit vector tangent to $\mathbf{R}^{n+2}$ at the image of q identified with the position vector of the image of q. Then we can uniquely determine the continuous unit vector field ξ normal to M in S^{n+1} as follows: For $p \in M$ and for a positive ordered base $\{E_1, \ldots, E_n\}$ of T_pM,

$$\{(d\iota_M)_p(E_1), \ldots, (d\iota_M)_p(E_n),\ \xi(p)\}$$

is a positive ordered base of $T_{\iota_M(p)}S^{n+1}$. We call $\xi(p)$ *the unit normal vector for an immersion ι_M compatible with the orientation of M*. So we define *the Gauss map $G_0 : M \to S^{n+1}$ for an immersion ι_M* so that for any $p \in M$, $G_0(p)$ is identified with $\xi(p)$ in $\mathbf{R}^{n+2}$. The map G_0 is differentiable. The following holds:

$$(dG_0)_p(X) = -A_\xi(X).$$

This implies that

$$(dG_0)_p(e_i) = -\kappa_i(p)e_i,$$

where $e_i (i = 1, \ldots, n) \in T_p(M)$ are unit vectors such that

(1) $g(e_i, e_j) = 0$ if $i \neq j$;

(2) e_i is in the principal directions corresponding to κ_i.

Then it follows that at $p \in M$, the map G_0 is non-degenerate if and only if the Gauss-Cronecker curvature K_n is nonzero at p. Therefore if K_n is non-vanishing everywhere, then G_0 is also an immersion of M into S^{n+1}. If G_0 is an immersion, then the Gauss map for G_0 can be considered and it is seen that the Gauss map for G_0 is $-\iota_M$, which is the map defined by $(-\iota_M)(p) = -\iota_M(p)$(the antipodal point of $\iota_M(p)$ in S^{n+1}). We denote by η the unit normal vector field for G_0 compatible with the orientation of M. The following holds:

PROPOSITION 2.1. *Let ι be an immersion of M into S^{n+1} with $K_n(p) \neq 0$ at each $p \in M$. Then the principal curvatures for G_0 with respect to $\eta(p)$ are given by*

$$-\frac{1}{\kappa_1(p)}, \ldots, -\frac{1}{\kappa_n(p)}.$$

Proof. Avoiding confusion, we don't identify X and $(d\iota)_p(X)$, or X and $(dG_0)_p(X)$ for $X \in T_p(M)$. However we still identify two vectors parallel to each other in $\mathbf{R}^{n+2}$. Then we find that

$$(dG_0)_p(X) = -A_{\iota,\xi}((d\iota)_p(X))$$

and that

$$(d\iota)_p(X) = A_{G_0,\eta}((dG_0)_p(X)),$$

where $A_{\iota,\xi}$(resp. $A_{G_0,\eta}$) is the Weingarten map for ι with respect to ξ(resp. for G_0 with respect to η). Particularly, setting $X = e_i$, we obtain

$$A_{G_0,\eta}((d\iota)_p(e_i)) = -\frac{1}{\kappa_i}(d\iota)_p(e_i).$$

This implies that the principal curvatures for G_0 with respect to $\eta(p)$ are given by

$$-\frac{1}{\kappa_1(p)}, \ldots, -\frac{1}{\kappa_n(p)}. \quad \square$$

3. Proof of Theorem 1.1

If there is not any danger of confusion, then we shall treat properties of an embedding ι as those of M and denote merely by M the image $\iota(M)$. Since (2) implies (1), we need to show that the converse holds.

Case 1. $n = 1$.

Let C be a simply closed curve in S^2 satisfying (1). For any $p \in C$, let $D_p(r)$ be the geodesic disc in S^2 with radius r such that

(1) C is tangent to $D_p(r)$ at p;

(2) $D_p(r)$ contains $U_p \setminus \{p\}$, where U_p is a neighborhood of p in C.

Suppose that there exists a point $q \in C \setminus \{p\}$ contained in $C_p(r)$, the boundary of $D_p(r)$. It follows that $q \notin U_p$. Moreover suppose that L, one of the two subarcs of C joining p and q, does not contain any point of $C_p(r)$ except p and q. Let s, t be the arc-length parameters of $C, C_p(r)$ respectively satisfying

$$C(0) = C_p(r)(0) = p, \quad C'(0) = [C_p(r)]'(0),$$

and $L = C([0, s_0])$ for a positive number $s_0 > 0$. We find that $C(s_0) = q$ and that $C_p(r)(t_0) = q$ for a number $t_0 \in (0, 2\pi \sin r)$ (notice that $\sin r$ is the radius of $C_p(r)$ in the plane in $\mathbf{R}^3$ containing $C_p(r)$). Then, with respect to t_0, exactly one of the following happens:

(a) $t_0 \in (0, \pi \sin r)$;

(b) $t_0 = \pi \sin r$;

(c) $t_0 \in (\pi \sin r,\ 2\pi \sin r)$.

Firstly, in case (a), we set Euclidean coordinates (x_1, x_2, x_3) of $\mathbf{R}^3$ such that

$$C_p(r) = S^2 \cap \{x_3 = \cos r\},$$

$$p, q \in \{x_2 > 0\}.$$

Then it follows that

$$L \setminus \{p, q\} \subset \{x_3 > \cos r\}.$$

Now, set

$$\pi(\alpha) := (0, \sin\alpha, \cos\alpha)$$

for $\alpha \in (-\pi, 0)$ and denote by $D(\pi(\alpha); r)$ the geodesic disc in S^2 centered at $\pi(\alpha)$ with radius r, and set

$$\alpha_0 := \inf\{\alpha \in (-\pi, 0)\ ;\ D(\pi(\alpha); r) \cap L \neq \emptyset\}.$$

Then we see that L is tangent to $D(\pi(\alpha_0); r)$ at any point of $\partial D(\pi(\alpha_0); r) \cap L (\neq \emptyset)$ and that $D(\pi(\alpha_0); r) \cap L = \emptyset$. These contradict local r-strict convexity of C. If q is as in case (b) or (c), then we can also obtain the contradiction.

Case 2. $n \geqq 2$.
For any $p \in M$ and for any integer m with $2 \leqq m \leqq n$, set

$$X^m(p) := \{\, \Pi^m\ ;\ \text{an } m\text{-dimensional totally geodesic sphere perpendicular to } M \text{ at } p \text{ in } S^{n+1}\}.$$

Then the following lemma is obtained.

LEMMA 3.1. *If M satisfies (1), then for any $\Pi^m \in X^m(p)$, the connected component $M_{\Pi^m}(p)$ of $M \cap \Pi^m$ containing p is a compact hypersurface embedded in Π^m and locally r-strictly convex at each point of $M_{\Pi^m}(p)$.*

On the other hand, the following holds:

LEMMA 3.2 ([1]). *For any $p \in M$,*

$$M = \sqcup_{\Pi \in X^2(p)} M_\Pi(p).$$

That is, for any $p \in M$ and for any $\Pi \in X^2(p)$,

$$M_\Pi(p) = M \cap \Pi.$$

Using Theorem 1.1 for $n = 1$, Lemma 3.1 and Lemma 3.2, we can prove Theorem 1.1 for $n \geqq 2$.

4. Proof of Theorem 1.2

Notice that for a geodesic ball $B(r_i)$ in S^{n+1} with radius r_i, all the principal curvatures of its boundary $\partial B(r_i)$ are equal to $\cot r_i$ or $-\cot r_i$. The embedding ι is locally ρ_1-strictly convex ($\rho_1 \in (r_1, \pi/2)$) and locally ρ_2-strictly convex($\rho_2 \in (r_2, \pi)$) on both sides at each point. Therefore it follows from Theorem 1.1 that ι is ρ_1-strictly convex at each point. We need to show that ι is ρ_2-strictly convex on both sides at each point.

For any $p_0 \in M$, we set Euclidean coordinates $(x_1, \ldots, x_{n+2})$ of $\mathbf{R}^{n+2}$ such that

$$\partial B_{\iota(p_0)}(\rho_2) = S^{n+1} \cap \{x_{n+2} = \cos \rho_2\},$$

where $B_{\iota(p_0)}(\rho_2)$ is a geodesic ball with radius ρ_2 to which M is tangent at p_0 by ι, and determine the orientation of $\mathbf{R}^{n+2}$ such that

$$G_0(p_0) \in \{x_{n+2} = -\sin \rho_2\},$$

where G_0 is the Gauss map for the embedding ι. To prove that ι is ρ_2-strictly convex on both sides at each point of M, we have only to show that at p_0,

$$\iota(M \setminus \{p_0\}) \subset S^{n+1} \cap \{x_{n+2} > \cos \rho_2\}.$$

Since ι is an embedding with positive principal curvatures, it follows that G_0 is an embedding. It follows from Proposition 2.1 that the principal curvatures $\{\kappa_i'\}_{i=1}^n$ for G_0 with respect to η(see Section 2) are negative and satisfy

$$-\frac{1}{\mu_1} \leqq \kappa_1' \leqq \ldots \leqq \kappa_n' \leqq -\frac{1}{\mu_2}.$$

Then it is seen that G_0 is locally ρ_2'-strictly convex($\rho_2' \in (0, \pi/2),\ \cos\rho_2' = \sin\rho_2$), so it follows from Theorem 1.1 that G_0 is ρ_2'-strictly convex at each point. Particularly we obtain

$$G_0(M \setminus \{p_0\}) \subset B_{G_0(p_0)}(\rho_2'), \tag{4.1}$$

where $B_{G_0(p_0)}(\rho_2')$ is a geodesic ball with radius ρ_2' to which M is tangent at p_0 by G_0. The following holds:

$$B_{G_0(p_0)}(\rho_2') = \{x_{n+2} < -\sin\rho_2\}. \tag{4.2}$$

If there exists a point $p \in M \setminus \{p_0\}$ such that

$$\iota(p) \in S^{n+1} \cap \{x_{n+2} \leqq \cos\rho_2\},$$

then as p, we take a point at which a function x_{n+2} on M attains its minimum. Then we find that

$$G_0(p) \in \{x_{n+2} \geqq -\sin\rho_2\},$$

which contradicts (4.1) and (4.2). Since p_0 is any point of M, it follows that ι is ρ_2-strictly convex on both sides at each point of M.

5. Proof of Theorem 1.3

By Theorem 1.2, we find that M is r_1-convex at each point, i.e., $M \subset \overline{B_p(r_1)}$ for each $p \in M$, and that M is r_2-convex on both sides at each point. To prove Theorem 1.3, firstly suppose that $n = 1$ and that $\mu_1 \neq \mu_2$, and let C be a closed curve as in Theorem 1.2. Then there exists a point $p \in C$ such that $\mu_1 < \kappa_g(p) < \mu_2$. It is seen that at p, C is locally r_1-strictly convex and locally r_2-strictly convex on both sides. Then as in Section 3 and as in Section 4, we find that C is r_1-strictly convex and r_2-strictly convex on both sides at p.

Next, suppose that $n = 2$ and that $\mu_1 \neq \mu_2$. We need lemmas.

Lemma 5.1 (Hilbert). *Let D be a domain of a surface in S^3 with positive Gaussian curvature and suppose that a point $p \in D$ is not umbilical, i.e., $\kappa_1(p) < \kappa_2(p)$. Then it is not possible that κ_2 has a relative maximum at p and κ_1 has a relative minimum at p.*

Proof. Noticing the equation of Codazzi for a surface in S^3 and the discussion in [3, pp. 124–125] or in [4, pp. 344–345], we can prove Lemma 5.1. □

Lemma 5.2. *For a compact surface M in S^3 with positive Gaussian curvature, just one of the following two holds*:

(1) *M is a geodesic sphere*;

(2) *There exists a point $p \in M$ such that $\kappa_1(p) > \mu_1$ and $\kappa_2(p) < \mu_2$.*

In addition, (1) *corresponds to the case $\mu_1 = \mu_2$ and* (2) *corresponds to the case $\mu_1 < \mu_2$.*

Proof. It follows from Lemma 5.1 that M is a constant curvature sphere if and only if $\mu_1 = \mu_2$. Suppose that $\mu_1 < \mu_2$. Then it follows from Lemma 5.1 that there exist two points $p_1, p_2 \in M$ such that $\kappa_1(p_1) > \mu_1$ and $\kappa_2(p_2) < \mu_2$. So set

$$X_{\mu_1} := \{m \in M \ ; \ \kappa_1(m) = \mu_1\},$$

$$X_{<\mu_2} := \{m \in M \ ; \ \kappa_2(m) < \mu_2\}.$$

Notice that $X_{\mu_1} \neq \emptyset$ and M, and that $X_{<\mu_2} \neq \emptyset$ and M. It follows from Lemma 5.1 that $X_{\mu_1} \subset X_{<\mu_2}$. Since X_{μ_1} is compact in M and since $X_{<\mu_2}$ is open in M, it follows that $X_{\mu_1} \subsetneqq X_{<\mu_2}$. For any $p \in X_{<\mu_2} \setminus X_{\mu_1}$, we find that $\kappa_1(p) > \mu_1$ and that $\kappa_2(p) < \mu_2$. □

By Lemma 5.2, we find that there exists a point $p \in M$ satisfying (2) of Lemma 5.2. In relation to p, we find that for any $\Pi \in X^2(p)$, the geodesic curvature of $M_\Pi(p)$ in Π is not less than μ_1 at any point of $M_\Pi(p)$ and more than μ_1 at p, which implies that in Π, $M_\Pi(p)$ is r_1-strictly convex at p. Therefore it follows from Lemma 3.2 that M is r_1-strictly convex at

p. Using the Gauss map G_0 as in Section 4, we find that M is r_2-strictly convex on both sides at p. Thus we have proved Theorem 1.3.

Noticing discussion appeared already, we obtain the following.

THEOREM 5.3. *Let M be a connected, compact, n-dimensional differentiable manifold $(n \geqq 1)$ and $\iota : M \to S^{n+1}$ an embedding of M into S^{n+1}. Suppose that the principal curvatures for ι are positive and that μ_1 and μ_2 are as in Theorem 1.2, and that r'_1, r'_2 are the two constants satisfying*

$$0 < r'_2 < \frac{\pi}{2} < r'_1 < \pi,$$

$$\mu_i = |\tan r'_i| \quad (i = 1, 2).$$

Then for any $\rho'_2 \in (r'_2, \pi/2)$, the Gauss map G_0 for ι is ρ'_2-strictly convex at each point, and for any $\rho'_1 \in (r'_1, \pi)$, G_0 is ρ'_1-strictly convex on both sides at each point. Moreover, if $n = 1$ or 2, and if $\mu_1 \neq \mu_2$, then there exists a point of M at which G_0 is r'_2-strictly convex and r'_1-strictly convex on both sides.

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Graduate School of Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914, Japan
E-mail: andou@ms.u-tokyo.ac.jp