J. Math. Sci. Univ. Tokyo
5 (1998), 747–785.

Pricing of Passport Option

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Abstract. Passport options are derivatives on actively managed funds. There is no known explicit formula for the price of passport options in general. We estimate and analyze the value function, and give the condition that the optimal strategy becomes trivial. We also show an approximation theorem which enables us to compute the value function numerically. Finally we give the explicit formula for the value function in the case that the interest rate is zero by using stochastic analytic approach. This is somehow done in symmetric case by Hyer, Lipton-Lifschitz, and Pugachevsky using partial differential equation approach.

1. Introduction

Many traded securities are managed in the trading account. The value of the trading account is the wealth accumulated by trading the securities following to some trading strategies. The derivative security called "Passport option" is the contract that an option holder has the right to gain the positive part of trading account at maturity. The option holder executes a trading strategy (fictitiously) which is chosen by himself in the class of strategies specified in the contract, and whenever the option holder changes the position, he has to report the amount of security to the option writer.

In this paper we suppose that there is one traded security whose value process S_t follows Black & Scholes model [1], i.e. under the risk neutral measure P, S_t satisfies

$$dS_t = S_t(rdt + \sigma dB_t), \tag{1.1}$$

where r and σ are constants and B is a P-Brownian motion. The value $X_t^{\theta,x}$ of trading account at time t under the strategy θ with initial value x

1991 *Mathematics Subject Classification*. Primary 90A09; Secondary 60J55.
Key words: passport option, Black & Scholes model, local time, Skorohod's Theorem, stochastic control.

is expressed as

$$X_t^{\theta,x} = x + \int_0^t \theta_s dS_s. \tag{1.2}$$

Let $\kappa_1 < \kappa_2$. We denote by $\Theta(\kappa_1, \kappa_2)$ the set of predictable processes θ satisfying $\kappa_1 \le \theta_t \le \kappa_2$ for any $t \ge 0$. The passport option whose pay-off is a positive part of trading account at maturity T is evaluated by

$$C(T,x) = \sup_{\theta \in \Theta(\kappa_1,\kappa_2)} E[e^{-rT}(X_T^{\theta,x})^+]. \tag{1.3}$$

We define Z_t as a process which satisfies

$$dZ_t = Z_t(rdt + \sigma dB_t), \quad Z_0 = 1$$

and define

$$\psi(t,y) = \psi^{r,\sigma}_{\kappa_1,\kappa_2}(t,y) \stackrel{\mathrm{def}}{=} \sup_{\theta \in \Theta(\kappa_1,\kappa_2)} E[(y + \int_0^t \theta_u dZ_u)^+].$$

Then, we have $C(T,x) = S_0 e^{-rT}\psi(T, \frac{x}{S_0})$.

The optimal strategy which attain the supremum in (1.3) does not always exist in the strategy class $\Theta(\kappa_1, \kappa_2)$, and the explicit formula of (1.3) is not known in general.

In Section 2 we give the definition of passport option and show that its price is given by (1.3).

In Section 3 we assume that $r \ge 0$, $\kappa_1 = -1$, and $\kappa_2 = 1$. And letting

$$\psi_0(t,y) = \psi_0^{r,\sigma}(t,y) \stackrel{\mathrm{def}}{=} E[(y + \int_0^t dZ_u)^+] = E[(y + Z_t - 1)^+],$$

we estimate $\psi(t,y)$ and examine the conditions for $\psi = \psi_0$. Let $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ and $\Phi(x) = \int_{-\infty}^x \phi(y)dy$. We have the following theorem as the estimation of $\psi(t,y)$.

THEOREM 1.1. *For any $c > 0$, $t > 0$, and $y \in \mathbf{R}$,*

$$\begin{aligned} &\psi(t,y) - \psi_0(t,y) \\ &\quad \le (e^{\sigma^2 t} - 1)^{\frac{1}{2}} \left\{ 2\Phi\left(-\frac{c}{\sqrt{t}}\right) + 2\Phi\left(-\frac{|y|}{e^{c\sigma}\sqrt{(1-e^{-\sigma^2 t})}}\right)\right\}^{\frac{1}{4}} e^{\frac{3r^2}{2\sigma^2}t}. \end{aligned}$$

For each $T > 0$, we define

$$R^\sigma(T) \stackrel{\text{def}}{=} \inf\{r^* > 0 : \psi^{r,\sigma}(t,y) = \psi_0^{r,\sigma}(t,y) \\ \text{for all } r \geq r^*, t \in [0,T], \text{ and } y \in \mathbf{R}\}.$$

We show the following results.

THEOREM 1.2. *Let c be the solution of $c\Phi(\sqrt{1-c}) = \sqrt{1-c} \times \phi(\sqrt{1-c})$. Then*

$$\begin{aligned} R^\sigma(T) &= \sigma^2 c + \frac{\sigma^3 c(1-\frac{3}{2}c)}{\sqrt{1-c}}\sqrt{T} \\ &\quad -\sigma^4 c\left\{\frac{c^2\sigma(1-2c)}{16(1-c)^{5/2}} + \frac{131c^3 - 270c^2 + 168c - 32}{48(1-c)^2}\right\}T + o(T), \\ &\qquad\qquad \textit{as } T \downarrow 0. \end{aligned}$$

THEOREM 1.3. $R^\sigma(T) = \sigma^2 - \dfrac{\sigma^2 \log(2\pi\sigma^2 T)}{2(\sigma^2 T + 1)} + o\left(\dfrac{1}{T^2}\right), \quad$ *as* $T \uparrow \infty$.

THEOREM 1.4. *For any r, there exists $T_0 = T_0(r,\sigma^2) > 0$ such that*

$$\psi_0(t,y) = \psi(t,y), \quad \textit{for any } t \in [0,T_0] \textit{ and } y \geq 1.$$

In Section 4, we prove that the value function of passport option can be derived as the limit of discrete time approximation.

In Section 5, we consider the case that $r = 0$. Hyer, Lipton-Lifschitz and Pugachevsky [2] derived the explicit formula in the case that $\kappa_1 = -\kappa_2$ using partial differential equation approach. We derive the explicit formula in general case that $\kappa_1 \leq \kappa_2$ using stochastic analytic approach. By introducing the Equivalent Martingale Measure $\tilde{P}$ with respect to the numeraire S_t, we show that the problem of passport option valuation comes down to the problem of calculation of an expected value $\tilde{E}[(\bar{Y}_T^y)^+]$ where $\bar{Y}_t^y$ is the solution of

$$d\bar{Y}_t = \sigma(1 + |\bar{Y}_t|)d\tilde{B}_t, \qquad \bar{Y}_0 = y.$$

Using Skorohod's Theorem, we derive the explicit formula for valuation of passport options (see Theorem 5.6).

In Section 6, by numerical calculations we illustrate the functions $r^*(t)$ and $\eta(r^*(t), t)$ introduced in Section 3. Furthermore numerical experiences in the discrete time framework discussed at Section 4 are reported.

The author expresses her sincere thanks to Professors Shigeo Kusuoka and Freddy Delbaen for their kind advice and discussion.

2. Notation and Problems

Let $(B_t)_{0\le t<\infty}$ be a Brownian motion defined in a probability space $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t)_{0\le t<\infty}$ be the Brownian filtration generated by $(B_t)_{0\le t<\infty}$. Let S_t be the price at time t of the security. We assume that S_t satisfies the following S.D.E.

$$dS_t = S_t(rdt + \sigma dB_t), \tag{2.1}$$

where r and σ are constants. To simplify the notation, we assume that the probability measure P is risk neutral with respect to the numeraire e^{rt}.

A trading strategy is defined as a predictable stochastic process. For fixed $\kappa_1 < \kappa_2$, we define $\Theta = \Theta(\kappa_1, \kappa_2)$ by the set of predictable processes θ satisfying $\kappa_1 \le \theta_t \le \kappa_2$ for all $0 \le t < \infty$. For any $\theta \in \Theta$ and $x \in \mathbf{R}$, let

$$X_t^{\theta,x} = x + \int_0^t \theta_s dS_s. \tag{2.2}$$

The processes $\{X_t^{\theta,x}\}_{0\le t<\infty}$ are called *trading accounts*. For any $\theta \in \Theta$, $x \in \mathbf{R}$, and $t \ge 0$, we have $E[|X_t^{\theta,x}|^2] < \infty$.

In this paper, *a passport call option with maturity T* is defined as a contract that an option holder has the right to gain the positive part of the trading account at time T under the trading strategy in Θ chosen arbitrarily by him. The option holder has to report the current value of the strategy θ to the option writer at every moment.

Proposition 2.1. *Let $C(T, x, S, \kappa_1, \kappa_2)$ be the value of the passport call option of maturity T, where x is current value of the trading account and S is the current price of the security. Then*

$$C(T, x, S, \kappa_1, \kappa_2) = \sup_{\theta\in\Theta(\kappa_1,\kappa_2)} E\left[e^{-rT} (X_T^{\theta,x})^+ \Big|_{S_0=S} \right].$$

To prove this proposition, we need some preparations. Let Z_t be a process given by the following S.D.E.

$$dZ_t = Z_t(rdt + \sigma dB_t), \quad Z_0 = 1,$$

i.e. $Z_t = \exp(\sigma B_t - \frac{1}{2}\sigma^2 t + rt)$. Let $U_t^{\theta,x} = \sup_{\theta' \in \Theta} E\left[(X_t^{\theta,x} + \int_t^T \theta'_u dS_u)^+ | \mathcal{F}_t\right]$ for $\theta \in \Theta$, $x \in \mathbf{R}$, and $t \in [0,T]$. Let

$$\psi(t,y) = \psi_{\kappa_1,\kappa_2}^{r,\sigma}(t,y) \stackrel{\text{def}}{=} \sup_{\theta \in \Theta(\kappa_1,\kappa_2)} E[(y + \int_0^t \theta_u dZ_u)^+].$$

Then it is easy to see that $0 \leq \psi(t,y_2) - \psi(t,y_1) \leq y_2 - y_1$ for $y_1 \leq y_2$ and that $U_t^{\theta,x} = S_t \psi(T-t, \frac{X_t^{\theta,x}}{S_t})$. Then we have the following.

LEMMA 2.2. *$\{U_t^{\theta,x}\}_{0 \leq t \leq T}$ is a non-negative supermartingale.*

PROOF. $E\left[|U_T^{\theta,x}|\right] = E\left[|X_T^{\theta,x}|\right] < \infty$. So the assertion follows from

$$\begin{aligned} E[U_t^{\theta,x}|\mathcal{F}_s] &\leq \sup_{\theta' \in \Theta} E\left[(x + \int_0^t \theta_u dS_u + \int_t^T \theta'_u dS_u)^+ \middle| \mathcal{F}_s\right] \\ &\leq \sup_{\theta' \in \Theta} E\left[(x + \int_0^s \theta_u dS_u + \int_s^T \theta'_u dS_u)^+ \middle| \mathcal{F}_s\right] = U_s^{\theta,x}, \quad s < t. \end{aligned}$$

Here the first inequality is derived as follows. For any $y \in \mathbf{R}$ and $n \in \mathbf{N}$ there exists a $\theta^{y,n} \in \Theta$ such that $0 \leq \psi(T-t,y) - E[(y + \int_0^{T-t} \theta_u^{y,n} dZ_u)^+] < \dfrac{1}{n}$. Then there is a predictable measurable function $F^{y,n} : [0,\infty) \times C([0,\infty)) \to \mathbf{R}$ such that $\theta^{y,n} = F^{y,n}(\cdot, B_\cdot)$. Define $\tilde{\theta}^n$ such as $\tilde{\theta}_s^n = \dfrac{\kappa_1 + \kappa_2}{2}$, $s \leq t$ and $\tilde{\theta}_{t+u}^n(\omega) = F^{k2^{-n},n}(u, B_{\cdot+t}(\omega) - B_t(\omega))$, if $\dfrac{k}{2^n} \leq \dfrac{X_t^{\theta,x}(\omega)}{S_t(\omega)} < \dfrac{k+1}{2^n}$. Then $\tilde{\theta}^n \in \Theta$, and

$$-\frac{1}{2^n} < \psi(T-t, \frac{X_t^{\theta,x}}{S_t}) - E\left[\left(\frac{X_t^{\theta,x}}{S_t} + \int_t^T \tilde{\theta}_s^n dZ_s\right)^+ \middle| \mathcal{F}_t\right] < \frac{1}{n} + \frac{1}{2^n} \qquad \text{a.s.}$$

Therefore $\psi(T-t, \frac{X_t^{\theta,x}}{S_t}) = \lim_{n\to\infty} E\left[\left(\frac{X_t^{\theta,x}}{S_t} + \int_t^T \tilde{\theta}_s^n dZ_s\right)^+ \middle| \mathcal{F}_t\right]$, a.s. Then by the dominated convergence theorem we have

$$\begin{aligned} E[U_t^{\theta,x}|\mathcal{F}_s] &= \lim_{n\to\infty} E\left[(X_t^{\theta,x} + \int_t^T \tilde{\theta}_u^n dS_u)^+ \middle| \mathcal{F}_s\right] \\ &\le \sup_{\theta'\in\Theta} E\left[(X_t^{\theta,x} + \int_t^T \theta'_u dS_u)^+ \middle| \mathcal{F}_s\right]. \square \end{aligned}$$

PROOF OF PROPOSITION 2.1. Suppose that $C(T,x,S,\kappa_1,\kappa_2) < E\left[e^{-rT}(x+\int_0^T \theta_s dS_s)^+\right]$ for some $\theta \in \Theta$. The representation theorem of Brownian martingale implies that the European call option of the maturity T with pay-off $(x+\int_0^T \theta_s dS_s)^+$ is replicable with initial cost $E\left[e^{-rT}(x+\int_0^T \theta_s dS_s)^+\right]$. If we think of the strategy to buy a passport option by value $C(T,x,S,\kappa_1,\kappa_2)$ and to take trading strategy θ while replicate the short position of that European call option with initial gain $E\left[e^{-rT}(x+\int_0^T \theta_s dS_s)^+\right]$, we see that it is an arbitrage strategy. Therefore $C(T,x,S,\kappa_1,\kappa_2) \ge \sup_{\theta\in\Theta} E\left[e^{-rT}(x+\int_0^T \theta_s dS_s)^+\right]$.

Suppose that $C(T,x,S,\kappa_1,\kappa_2) > \sup_{\theta\in\Theta} E\left[e^{-rT}(X_T^{\theta,x})^+\Big|_{S_0=S}\right]$. Then we sell the passport option by value $C(T,x,S,\kappa_1,\kappa_2)$. Suppose the option holder takes a trading strategy θ. From Lemma 2.2, $\{e^{-rT}U_t^{\theta,x}\}_{0\le t\le T}$ is a non-negative supermartingale, so it has a unique decomposition $e^{-rT}U_t^{\theta,x} = M_t^{\theta,x} - A_t^{\theta,x}$, where $\{M_t^{\theta,x}\}$ is a square integrable martingale and $\{A_t^{\theta,x}\}$ is a non-decreasing predictable process with $A_0 = 0$. Let $\tilde{S}_t = e^{-rt}S_t$, then we have $d\tilde{S}_t = \sigma\tilde{S}_t dB_t$ and $\tilde{S}$ is a P-martingale. From the representation theorem of Brownian martingales, there exists an adapted process $\{H_t\}$ such that $E[\int_0^T H_s^2\tilde{S}_s^2 ds] < \infty$ and

$$M_t^{\theta,x} = M_0^{\theta,x} + \int_0^t H_s d\tilde{S}_s.$$

H_t can be determined at time t from $\mathcal{F}_t$ and $\{\theta_s : s \le t\}$ which has been reported by the option holder. Then by taking the position of H_t units of S_t and $e^{rt}M_t^{\theta,x} - H_tS_t$ amount of saving account, we can replicate $e^{rT}M_T^{\theta,x} =$

$U_T^{\theta,x} + e^{rT}A_T^{\theta,x} = (X_T^{\theta,x})^+ + e^{rT}A_T^{\theta,x}$. Then we will have the gain $e^{rT}A_T^{\theta,x} \geq 0$ at time T. Since the initial gain was $C(T, x, S, \kappa_1, \kappa_2) - M_0^{\theta,x} > 0$, it is an arbitrage. $\square$

For any positive constant λ, by considering the passport option whose underlying is λ times the original underlying, we have that $C(T, \lambda x, S, \lambda\kappa_1, \lambda\kappa_2) = \lambda C(T, x, S, \kappa_1, \kappa_2)$. And by changing time, we have $C^{r,\sigma}(T, x, S, \kappa_1, \kappa_2) = C^{r/\sigma^2,1}(\sigma^2 T, x, S, \kappa_1, \kappa_2)$.

Proposition 2.3. *$\psi(t, y)$ is convex, increasing, and Lipschitz with respect to y. Moreover if $\kappa_1 \leq 0 \leq \kappa_2$, $\psi(t, y)$ is increasing in t.*

Proof. For any convex function g and $0 \leq \lambda \leq 1$, we have

$$\begin{aligned} &\sup_{\theta\in\Theta} E[g(\lambda x + (1-\lambda)y + \int_0^t \theta_u dZ_u)] \\ &\quad \leq \sup_{\theta\in\Theta}\left[\lambda E[g(x + \int_0^t \theta_u dZ_u)] + (1-\lambda)E[g(y + \int_0^t \theta_u dZ_u)]\right] \\ &\quad \leq \lambda \sup_{\theta\in\Theta} E[g(x + \int_0^t \theta_u dZ_u)] + (1-\lambda)\sup_{\theta\in\Theta} E[g(y + \int_0^t \theta_u dZ_u)]. \end{aligned}$$

Therefore $\psi(t, \cdot)$ is convex. We can easily see that $\psi(t, y)$ is increasing and Lipschitz with respect to y. Let $\kappa_1 \leq 0 \leq \kappa_2$ and $s < t$, then

$$\psi(t, y) = \sup_{\theta\in\Theta} E[(y + \int_0^t \theta_u dZ_u)^+] \geq \sup_{\theta\in\Theta} E[(y + \int_0^s \theta_u dZ_u)^+] \ = \ \psi(s, y). \ \square$$

3. The Estimation of the Value Function

In this section, we assume $r \geq 0$, $\kappa_1 = -1$, and $\kappa_2 = 1$. By Proposition 2.1, $C(t, x, S) = Se^{-rt}\psi(t, \frac{x}{S})$. Let

$$\psi_0(t, y) = \psi_0^{r,\sigma}(t, y) \stackrel{\text{def}}{=} E[(y + \int_0^t dZ_u)^+] = E[(y + Z_t - 1)^+]\,, \quad t \geq 0,\ y \in \mathbf{R}.$$

Then obviously we have $\psi(t, y) \geq \psi_0(t, y)$. Let

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad \text{and} \quad \Phi(x) = \int_{-\infty}^x \phi(y)dy.$$

THEOREM 3.1. *For any $c > 0$,*

$$\psi(t,y) - \psi_0(t,y) \\ \leq (e^{\sigma^2 t} - 1)^{\frac{1}{2}} \left\{ 2\Phi\left(-\frac{c}{\sqrt{t}}\right) + 2\Phi\left(-\frac{|y|}{e^{c\sigma}\sqrt{(1-e^{-\sigma^2 t})}}\right) \right\}^{\frac{1}{4}} e^{\frac{3r^2}{2\sigma^2}t}.$$

PROOF. For each $\theta \in \Theta$, let $\tau^\theta = \inf\{t > 0 : y + \int_0^t \theta_s dZ_s = 0\}$. First, let us assume that $y \geq 0$. Then we have

$$\begin{aligned} \psi(t,y) &= \sup_{\theta\in\Theta} \left(E[y + \int_0^t \theta_s dZ_s] + E[(y + \int_0^t \theta_s dZ_s)^-, t > \tau^\theta] \right) \\ &\leq \psi_0(t,y) + \sup_{\theta\in\Theta} E[(y + \int_0^t \theta_s dZ_s)^-, t > \tau^\theta]. \end{aligned}$$

Recall that $dZ_t = Z_t(rdt + \sigma dB_t)$. Define a P-equivalent probability measure Q by $(\frac{dQ}{dP})_t = \rho_t \stackrel{\text{def}}{=} \exp(-\frac{r}{\sigma}B_t - \frac{r^2}{2\sigma^2}t)$. Since $W_t \stackrel{\text{def}}{=} \frac{r}{\sigma}t + B_t$ is a Q-Brownian motion, Z is a Q-square integrable martingale. Then, for each $\theta \in \Theta$, we have

$$\begin{aligned} E\left[(y + \int_0^t \theta_s dZ_s)^-, t > \tau^\theta\right] &\leq E\left[\left|\int_0^t \theta_s dZ_s\right| I_{\{t>\tau^\theta\}}\right] \\ &= E^Q\left[\left|\int_0^t \theta_s dZ_s\right| I_{\{t>\tau^\theta\}} \frac{1}{\rho_t}\right] \\ &\leq E^Q\left[\left|\int_0^t \theta_s dZ_s\right|^2\right]^{\frac{1}{2}} E^Q[I_{\{t>\tau^\theta\}}]^{\frac{1}{4}} E^Q\left[\frac{1}{\rho_t^4}\right]^{\frac{1}{4}}. \end{aligned}$$

Here we have $E^Q\left[\frac{1}{\rho_t^4}\right] = E\left[\frac{1}{\rho_t^3}\right] = E\left[\exp\left(\frac{3r}{\sigma}B_t + \frac{3r^2}{2\sigma^2}t\right)\right] = e^{\frac{6r^2}{\sigma^2}t}$. Let $M_t = \int_0^t \theta_s dZ_s$, then we have

$$[M,M]_t = \int_0^t \theta_s^2 d[Z,Z]_s \leq \int_0^t \sigma^2 Z_s^2 ds = \sigma^2 \int_0^t \exp(2\sigma W_s - \sigma^2 s) ds.$$

Therefore we have

$$E^Q\left[\left|\int_0^t \theta_s dZ_s\right|^2\right] \leq \sigma^2 \int_0^t e^{\sigma^2 s} ds = (e^{\sigma^2 t} - 1).$$

By Knight's theorem (see Ikeda and Watanabe [3]) there exists a Q-Brownian motion $\{\hat{W}_t\}_{t\in[0,\infty)}$ such that $M_s = \hat{W}_{[M,M]_s}$. Therefore we have for any $c > 0$

$$\begin{aligned} E^Q[I_{\{t>\tau^\theta\}}] &= Q\left[\inf_{0\le s\le t} M_s < -y\right] = Q\left[\inf_{0\le s\le [M,M]_t} \hat{W}_s < -y\right] \\ &\le Q\left[\inf\{\hat{W}_s : 0 \le s \le (1-e^{-\sigma^2 t})\exp(2\sigma \max_{0\le u\le t} W_u)\} < -y\right] \\ &\le Q\left[\max_{0\le u\le t} W_u \ge c\right] + Q\left[\inf_{0\le s\le (1-e^{-\sigma^2 t})e^{2c\sigma}} \hat{W}_s < -y\right]. \end{aligned}$$

It is well known that $Q[\max_{0\le u\le t} W_u \ge a] = 2\Phi\left(-\frac{a}{\sqrt{t}}\right)$ for $a > 0$. So we have $E^Q[I_{\{t>\tau^\theta\}}] \le 2\Phi\left(-\frac{c}{\sqrt{t}}\right) + 2\Phi\left(-\frac{y}{e^{c\sigma}\sqrt{(1-e^{-\sigma^2 t})}}\right)$. Therefore we have our assertion in the case that $y \ge 0$.

We can prove the case of $y < 0$ similarly. □

By Theorem 3.1, we can easily prove that for any $p > 0$, there exists $c_p > 0$ such that $\psi(t,y) \le \psi_0(t,y) + c_p|y|^{-p/2}$, $|y| \ge 1$. Therefore we have

$$\lim_{y\to-\infty} \psi(t,y) = 0, \quad \lim_{y\to\infty} \frac{1}{y}\psi(t,y) = 1, \quad \text{and}$$
$$\lim_{y\to\infty} \frac{\partial^+}{\partial y^+}\psi(t,y) = \lim_{y\to\infty} \frac{\partial^-}{\partial y^-}\psi(t,y) = 1.$$

In the rest of this section, we examine the condition such that $\psi = \psi_0$ is satisfied.

Remark 3.2. We can easily see the following.

$$\psi_0(t,y) = \begin{cases} \displaystyle\int_{-\alpha(r,y,t,\sigma)}^{\infty} (y + e^{\sigma\sqrt{t}x - \frac{1}{2}\sigma^2 t + rt} - 1)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx & \text{if } y < 1 \\ y + e^{rt} - 1 & \text{if } y \ge 1 \end{cases}$$

where $\alpha(r,y,t,\sigma) = \dfrac{-\log(1-y) - \frac{1}{2}\sigma^2 t + rt}{\sigma\sqrt{t}}$ for $t > 0$, $y < 1$.

For each $T > 0$ and $\sigma > 0$, define

$$R^\sigma(T) = \inf\{r^* > 0 \quad : \quad \psi^{r,\sigma}(t,y) = \psi_0^{r,\sigma}(t,y) \\ \text{for all } r \geq r^*, t \in [0,T], \text{ and } y \in \mathbf{R}\}.$$

Let $V_t^{\theta,x} = V_t^{\theta,x,r,\sigma} \stackrel{\text{def}}{=} S_t \psi_0^{r,\sigma}\left(T - t, \dfrac{X_t^{\theta,x}}{S_t}\right)$, for $\theta \in \Theta$, $x \in \mathbf{R}$, $0 \leq t \leq T$. Then $\{V_t^{1,x}\}_{0 \leq t \leq T}$ is a martingale. Let

$$F(r,y,t,\sigma) = r\frac{\partial \psi_0}{\partial y}(t,y) - y\sigma^2 \frac{\partial^2 \psi_0}{\partial y^2}(t,y).$$

LEMMA 3.3. *For any $T > 0$ and $\sigma > 0$,*

$$R^\sigma(T) = \inf\left\{r : F(r,y,t,\sigma) \geq 0, \text{ for any } y \in (0,1),\ t \in (0,T]\right\}.$$

PROOF. Noting that $V_T^{\theta,x} = \left(x + \int_0^T \theta_t dS_t\right)^+$, we have the following from the proof of Proposition 2.1.

$$R^\sigma(T) = \inf\{r^* > 0 \quad : \quad \{V_t^{\theta,x,r,\sigma}\}_{0 \leq t \leq T} \text{ is a } P\text{-supermartingale}, \\ \text{for all } r \geq r^*, x \in \mathbf{R}, \text{ and } \theta \in \Theta\}.$$

So it is sufficient to show that $\{V_t^{\theta,x,r,\sigma}\}_{0 \leq t \leq T}$ is a P-supermartingale for all $\theta \in \Theta$ and $x \in \mathbf{R}$ if and only if $F(r,y,t,\sigma) \geq 0$ for all $y \in (0,1)$ and $t \in (0,T]$. Let

$$Y_t^{\theta,y} \stackrel{\text{def}}{=} \frac{X_t^{\theta,x}}{S_t}, \qquad x = yS_0. \tag{3.1}$$

Then by Ito's formula we have

$$dY_t^{\theta,y} = (\theta_t - Y_t^{\theta,y})(rdt + \sigma dB_t - \sigma^2 dt)$$

and

$$dV_t^{\theta,x,r,\sigma} = \sigma^V(S_t, Y_t^{\theta,y}, \theta_t, T-t)dB_t + \mu^V(S_t, Y_t^{\theta,y}, \theta_t, T-t)dt,$$

where

$$\begin{aligned}\sigma^V(S,y,\theta,t) &= \sigma S(\theta-y)\frac{\partial\psi_0}{\partial y}(t,y)+\sigma S\psi_0(t,y)\\ \mu^V(S,y,\theta,t) &= rS(\theta-y)\frac{\partial\psi_0}{\partial y}(t,y)-S\frac{\partial\psi_0}{\partial t}(t,y)\\ &\quad+\frac{1}{2}S(\theta-y)^2\sigma^2\frac{\partial^2\psi_0}{\partial y^2}(t,y)+rS\psi_0(t,y).\end{aligned}$$

Since $\{V_t^{1,x,r,\sigma}\}_{0\le t\le T}$ is a martingale, we have $\mu^V(S,y,1,t)=0$ for all $t\in(0,T]$. Note that $\{V_t^{\theta,x,r,\sigma}\}_{0\le t\le T}$ is supermartingale if and only if $\mu^V(S,y,\theta,t)\le 0=\mu^V(S,y,1,t)$ for any $y\in\mathbf{R}$, $S>0$, $t\in(0,T]$, $|\theta|\le 1$. This is equivalent to

$$r\frac{\partial\psi_0}{\partial y}(t,y)+\frac{\sigma^2}{2}(\theta+1-2y)\frac{\partial^2\psi_0}{\partial y^2}(t,y)\ge 0,\quad \text{for } y\in\mathbf{R}, t\in(0,T], |\theta|\le 1. \tag{3.2}$$

Since $\psi_0(t,y)$ is convex in y, we have $\frac{\partial^2\psi_0}{\partial y^2}\ge 0$, and so the condition (3.2) is equivalent to that

$$F(r,y,t,\sigma)=r\frac{\partial\psi_0}{\partial y}(t,y)-y\sigma^2\frac{\partial^2\psi_0}{\partial y^2}(t,y)\ge 0,\quad t\in(0,T] \tag{3.3}$$

for all $y\in\mathbf{R}$. Here we have

$$\begin{aligned}\frac{\partial\psi_0}{\partial y}(t,y) &= \begin{cases}1 & \text{if } y\ge 1\\ \Phi(\alpha(r,y,t,\sigma)) & \text{if } y<1\end{cases}\\ \frac{\partial^2\psi_0}{\partial y^2}(t,y) &= \begin{cases}0 & \text{if } y\ge 1\\ \frac{1}{\sigma(1-y)\sqrt{t}}\phi(\alpha(r,y,t,\sigma)) & \text{if } y<1.\end{cases}\end{aligned}$$

So in the case that $y\ge 1$ or $y\le 0$, condition (3.3) is satisfied for all $t\in(0,T]$. This proves our assertion. □

REMARK 3.4. From Remark 3.2, we have the following.

$$F(r,y,t,\sigma) = r\Phi(\alpha(r,y,t,\sigma))-\frac{y\sigma}{(1-y)\sqrt{t}}\phi(\alpha(r,y,t,\sigma)) \tag{3.4}$$

$$\frac{\partial F}{\partial r}(r,y,t,\sigma) = \Phi(\alpha(r,y,t,\sigma)) \tag{3.5}$$

$$+\phi(\alpha(r,y,t,\sigma))\left\{\frac{\sqrt{t}}{\sigma}r+\frac{y}{1-y}\alpha(r,y,t,\sigma)\right\}$$

$$(3.6)\quad \frac{\partial F}{\partial y}(r,y,t,\sigma) = \frac{\phi(\alpha(r,y,t,\sigma))}{\sigma(1-y)^2\sqrt{t^3}}\left\{(r-\sigma^2)t-\frac{1}{2}\sigma^2 ty - y\log(1-y)\right\}$$

$$(3.7)\quad \frac{\partial F}{\partial t}(r,y,t,\sigma) = \phi(\alpha(r,y,t,\sigma))\left\{\frac{y\sigma}{2(1-y)\sqrt{t^3}} + \left(r+\frac{y\sigma\alpha(r,y,t,\sigma)}{(1-y)\sqrt{t}}\right)\frac{\log(1-y)-\frac{1}{2}\sigma^2 t+rt}{2\sigma\sqrt{t^3}}\right\}.$$

LEMMA 3.5. $R^\sigma(T) \le \sigma^2$.

PROOF. From (3.5), we have $\frac{\partial F}{\partial r}(r,y,t,\sigma) \ge 0$ for all $r \ge \sigma^2$, $y \in (0,1)$ and $t \in (0,T]$. So it is sufficient to show that $F(\sigma^2,y,t,\sigma) \ge 0$ for all $y \in (0,1)$ and $t \in (0,T]$. From (3.6), we have

$$\frac{\partial F}{\partial y}(\sigma^2,y,t,\sigma)\begin{cases}\le 0 & \text{if } 0<y\le 1-e^{-\frac{1}{2}\sigma^2 t}\\ \ge 0 & \text{if } 1-e^{-\frac{1}{2}\sigma^2 t}\le y<1.\end{cases}$$

It is sufficient to show that $F_0(t) \overset{\text{def}}{=} F(\sigma^2, 1-e^{-\frac{1}{2}\sigma^2 t}, t, \sigma) \ge 0$ for all $t>0$. Since $F_0'(t) = \dfrac{(e^{\frac{1}{2}\sigma^2 t}-1)\sigma}{2\sqrt{t^3}}\phi(\sigma\sqrt{t}) \ge 0$, $F_0(t)$ is an increasing function. Also,

$$\lim_{t\downarrow 0}F_0(t) = \lim_{t\downarrow 0}\sigma^2\Phi(\sigma\sqrt{t}) - \lim_{t\downarrow 0}\frac{\sigma(e^{\frac{1}{2}\sigma^2 t}-1)}{\sqrt{t}}\phi(\sigma\sqrt{t}) = \frac{\sigma^2}{2}.$$

Therefore $F(\sigma^2, 1-e^{-\frac{1}{2}\sigma^2 t}, t, \sigma) \ge \frac{\sigma^2}{2}$ for all $t>0$. □

LEMMA 3.6. *Let $r<\sigma^2$. Then for each $t>0$, there exists a unique $\eta(r,t)\in(0,1)$ such that $F(r,\eta(r,t),t,\sigma) = \min_{y\in(0,1)} F(r,y,t,\sigma)$.*

PROOF. Note that $\dfrac{\phi(\alpha(r,y,t,\sigma))}{\sigma(1-y)^2\sqrt{t^3}} > 0$. So from (3.6), $\dfrac{\partial F}{\partial y}(r,y,t,\sigma) > 0$ (<0) if and only if $f(y) = (r-\sigma^2)t - \dfrac{1}{2}\sigma^2 ty - y\log(1-y) > 0$ $(<0,$

respectively). Note that $f'(y) = -\frac{1}{2}\sigma^2 t - \log(1-y) + \frac{y}{1-y}$. So $f'(y)$ is increasing in $y \in (0,1)$. Also we have $f'(0) = -\frac{1}{2}\sigma^2 t < 0$, $\lim_{y \uparrow 1} f'(y) = \infty$, $f(0) = rt - \sigma^2 t < 0$, and $\lim_{y \uparrow 1} f(y) = \infty$. So we see that the equation $\frac{\partial F}{\partial y}(r, y, t, \sigma) = 0$ in y has a unique solution and it gives the minimum of $F(r, \cdot, t, \sigma)$. □

Remark 3.7. The proof of Lemma 3.6 shows that $\eta(r,t)$ is a solution of $\frac{\partial F}{\partial y}(r, y, t, \sigma) = 0$. Therefore $\eta(r,t)$ is characterized as a solution of

$$\log(1 - \eta(r,t)) = \frac{rt - \sigma^2 t}{\eta(r,t)} - \frac{1}{2}\sigma^2 t, \qquad 0 < \eta(r,t) < 1. \tag{3.8}$$

Then we have

$$\alpha(r, \eta(r,t), t, \sigma) = \frac{\sigma^2 - r}{\sigma \eta(r,t)}\sqrt{t} + \frac{r\sqrt{t}}{\sigma}. \tag{3.9}$$

By (3.8) we see that $\lim_{t \downarrow 0}\{\eta(r,t)\log(1 - \eta(r,t))\}$ and $\lim_{t \uparrow \infty}\{\eta(r,t) \cdot \log(1 - \eta(r,t))\} = -\infty$. This implies

$$\lim_{t \downarrow 0} \eta(r,t) = 0 \text{ and } \lim_{t \uparrow \infty} \eta(r,t) = 1. \tag{3.10}$$

Lemma 3.8. *For $r \in (0, \sigma^2)$, $\eta(r,t) = \sqrt{t(\sigma^2 - r)} + \frac{r}{4}t + o(t)$ as $t \downarrow 0$.*

Proof. Note that we have $\log(1-x) = -x(1 + R(x))$, where $R(x) = \sum_{i=1}^{\infty} \frac{1}{i+1} x^i$, for $|x| < 1$. From Equation (3.8), we have

$$\eta(r,t)^2(1 + R(\eta(r,t))) - \frac{1}{2}\sigma^2 t \eta(r,t) - (\sigma^2 - r)t = 0.$$

Since $\eta(r,t) \in (0,1)$, we have

$$\eta(r,t) = \sqrt{t}\frac{\sigma^2\sqrt{t} + \sqrt{\sigma^4 t + 16(\sigma^2 - r)(1 + R(\eta(r,t)))}}{4(1 + R(\eta(r,t)))}. \tag{3.11}$$

So $\lim_{t\to 0}\dfrac{\eta(r,t)}{\sqrt{t}} = \sqrt{\sigma^2-r}$. Let $f(t)=\eta(r,t)-\sqrt{\sigma^2-r}\sqrt{t}$. Then from (3.8) we have

$$\begin{aligned}&(\sqrt{t(\sigma^2-r)}+f(t))\log(1-\sqrt{t(\sigma^2-r)}-f(t))\\ &\quad+\frac{1}{2}\sigma^2 t(\sqrt{t(\sigma^2-r)}+f(t))+(\sigma^2-r)t=0.\end{aligned}$$

Since we have

$$\log(1-\sqrt{t(\sigma^2-r)}-f(t))=\log(1-\sqrt{t(\sigma^2-r)})-\frac{f(t)}{1-\sqrt{t(\sigma^2-r)}}(1+\hat{R}(t))$$

where $\hat{R}(t) = R\left(\dfrac{f(t)}{1-\sqrt{t(\sigma^2-r)}}\right)$, $f(t)$ is the solution of the following equation.

$$\begin{aligned}A_t f(t)^2+B_t f(t)+C_t &= 0,\\ \text{where } A_t &= -\frac{1+\hat{R}(t)}{1-\sqrt{t(\sigma^2-r)}}\\ B_t &= \log(1-\sqrt{t(\sigma^2-r)})\\ &\quad -\frac{\sqrt{t(\sigma^2-r)}(1+\hat{R}(t))}{1-\sqrt{t(\sigma^2-r)}}+\frac{1}{2}\sigma^2 t\\ C_t &= \sqrt{t(\sigma^2-r)}\log(1-\sqrt{t(\sigma^2-r)})\\ &\quad +\frac{1}{2}\sigma^2 t\sqrt{t(\sigma^2-r)}+(\sigma^2-r)t.\end{aligned}$$

Since $\lim_{t\to 0}A_t=-1$, $\lim_{t\to 0}\dfrac{B_t}{\sqrt{t}}=-2\sqrt{\sigma^2-r}$, and $\lim_{t\to 0}\dfrac{C_t}{\sqrt{t^3}}=-\dfrac{1}{2}(\sigma^2-r)^{3/2}+\dfrac{1}{2}\sigma^2\sqrt{\sigma^2-r}$, we have $\lim_{t\downarrow 0}\dfrac{f(t)}{t}=\dfrac{r}{4}$. □

We can show the following lemma similarly to Lemma 3.8.

LEMMA 3.9. *For $r\in(0,\sigma^2)$,*

$$\begin{aligned}\eta(r,t) &= \sqrt{t(\sigma^2-r)}+\frac{r}{4}t+\frac{3r^2-12r(\sigma^2-r)-16(\sigma^2-r)^2}{96\sqrt{\sigma^2-r}}t^{3/2}\\ &\quad +o(t\sqrt{t}) \text{ as } t\downarrow 0.\end{aligned}$$

Let $G(r,t) = F(r,\eta(r,t),t,\sigma)$. Then we have the following.

PROPOSITION 3.10. *For each $t > 0$, the equation $G(r,t) = 0$ in r has a unique solution $r^*(t)$ in $(0,\sigma^2)$. Moreover $r^*(t)$ is continuous in $t \in (0,\infty)$.*

PROOF. Note that by (3.5) we have

$$\begin{aligned}
\frac{\partial G}{\partial r}(r,t) &= \Phi(\alpha(r,\eta(r,t),t,\sigma)) \\
&\quad + \phi(\alpha(r,\eta(r,t),t,\sigma)) \left\{ \frac{\sqrt{t}}{\sigma} r + \frac{\eta(r,t)}{1-\eta(r,t)} \frac{-\frac{r-\sigma^2}{\eta(r,t)} t + rt}{\sigma\sqrt{t}} \right\} \\
&> 0 \quad \text{for } 0 \le r < \sigma^2.
\end{aligned}$$

From (3.8) we have $\eta(r,t) = 1 - \exp\left\{ \frac{(r-\sigma^2)t}{\eta(r,t)} - \frac{1}{2}\sigma^2 t \right\}$. Since $0 < \eta(r,t) < 1$ we have $0 < \liminf_{r\downarrow 0} \eta(r,t) \le \limsup_{r\downarrow 0} \eta(r,t) < 1$. Therefore $\lim_{r\downarrow 0} G(r,t) < 0$. We also have $\liminf_{r\uparrow\sigma^2} \eta(r,t) > 0$ from (3.8), so $\lim_{r\uparrow\sigma^2} \eta(r,t) = 1 - e^{-\frac{1}{2}\sigma^2 t}$. This implies that $\lim_{r\uparrow\sigma^2} G(r,t) = F(\sigma^2, 1 - e^{-\frac{1}{2}\sigma^2 t}, t, \sigma) \ge \frac{\sigma^2}{2}$ (see the proof of Lemma 3.5). So we have the first assertion and we complete the proof by implicit function theorem. $\square$

COROLLARY 3.11. *$R^\sigma(T) = \max_{0\le t\le T} r^*(t)$. In particular, $R^\sigma(T) < \sigma^2$.*

PROPOSITION 3.12. *$\lim_{t\downarrow 0} r^*(t) = c\sigma^2$, where c is the solution of $c\Phi(\sqrt{1-c}) = \sqrt{1-c}\,\phi(\sqrt{1-c})$.*

PROOF. Let $G_0(r) = \lim_{t\downarrow 0} G(r,t)$. Then we have

$$G_0(r) = r\Phi\left(\frac{1}{\sigma}\sqrt{\sigma^2 - r}\right) - \sigma\sqrt{\sigma^2 - r}\,\phi\left(\frac{1}{\sigma}\sqrt{\sigma^2 - r}\right),$$

and

$$G_0(c\sigma^2) = c\sigma^2\Phi(\sqrt{1-c}) - \sigma^2\sqrt{1-c}\,\phi(\sqrt{1-c}) = 0.$$

Since $G_0'(r) = \Phi\left(\frac{1}{\sigma}\sqrt{\sigma^2 - r}\right) > 0$, we have $G_0(c\sigma^2 - \epsilon) < 0 < G_0(c\sigma^2 + \epsilon)$, for any $\epsilon > 0$. So by Proposition 3.10 $\lim_{t\downarrow 0} r^*(t) = c\sigma^2$. □

We get $c = 0.2945...$ by numerical calculation.

PROPOSITION 3.13. $R^\sigma(T) = \sigma^2 c + \frac{\sigma^3 c(1 - \frac{3}{2}c)}{\sqrt{1-c}}\sqrt{T} + o(\sqrt{T})$, *as* $T \downarrow 0$.

PROOF. Let $H(t, a) = G(c\sigma^2 + a\sqrt{t}, t)$, where c is the value defined in Proposition 3.12. By Lemma 3.8 we have

$$\eta(c\sigma^2 + a\sqrt{t}, t) = \sigma\sqrt{1-c}\sqrt{t} - \left(\frac{a}{2\sigma\sqrt{1-c}} - \frac{c\sigma^2}{4}\right) t + o(t).$$

Therefore we have from (3.9)

$$\begin{aligned}
&\alpha(c\sigma^2 + a\sqrt{t}, \eta(c\sigma^2 + a\sqrt{t}, t), t, \sigma) \\
&\quad = \frac{\sigma^2(1-c) - a\sqrt{t}}{\sigma^2\sqrt{1-c} - \left(\frac{a}{2\sqrt{1-c}} - \frac{c\sigma^3}{4}\right)\sqrt{t}} + c\sigma\sqrt{t} + o(\sqrt{t}) \\
&\quad = \sqrt{1-c} + h(a)\sqrt{t} + o(\sqrt{t}),
\end{aligned}$$

where $h(a) = -\frac{a}{2\sigma^2\sqrt{1-c}} + \frac{3}{4}c\sigma$. And we have

$$\begin{aligned}
\frac{\sigma\eta(c\sigma^2 + a\sqrt{t}, t)}{(1 - \eta(c\sigma^2 + a\sqrt{t}, t))\sqrt{t}} &= \frac{\sigma^2\sqrt{1-c} - \left(\frac{a}{2\sqrt{1-c}} - \frac{c\sigma^3}{4}\right)\sqrt{t}}{1 - \sigma\sqrt{1-c}\sqrt{t} + \left(\frac{a}{2\sigma\sqrt{1-c}} - \frac{c\sigma^2}{4}\right)t} + o(\sqrt{t}) \\
&= \sigma^2\sqrt{1-c} + k(a)\sqrt{t} + o(\sqrt{t}),
\end{aligned}$$

where $k(a) = -\frac{a}{2\sqrt{1-c}} + \sigma^3 - \frac{3c\sigma^3}{4}$. Since $\Phi(x+z) = \Phi(x) + \phi(x)z + o(z)$ as $z \downarrow 0$ and $\phi(x+z) = \phi(x) - x\phi(x)z + o(z)$ as $z \downarrow 0$, we have

$$\begin{aligned}
H(t, a) &= (c\sigma^2 + a\sqrt{t})\{\Phi(\sqrt{1-c}) + \phi(\sqrt{1-c})h(a)\sqrt{t} + o(\sqrt{t})\} \\
&\quad -(\sigma^2\sqrt{1-c} + k(a)\sqrt{t}) \\
&\quad \cdot\{\phi(\sqrt{1-c}) - \sqrt{1-c}\phi(\sqrt{1-c})h(a)\sqrt{t} + o(\sqrt{t})\}.
\end{aligned}$$

Therefore from the definitions of c, we have

$$H_0(a) = \lim_{t\downarrow 0} \frac{H(t,a)}{\sqrt{t}} = \phi(\sqrt{1-c})\{c\sigma^2 h(a) + \frac{a\sqrt{1-c}}{c} + \sigma^2(1-c)h(a) - k(a)\}.$$

Noting that $H_0'(a) = \phi(\sqrt{1-c})\dfrac{1-2c}{c\sqrt{1-c}} > 0$ and $H_0\left(\dfrac{\sigma^3 c(1-\frac{3}{2}c)}{\sqrt{1-c}}\right) = 0$, we see $H_0(\frac{\sigma^3 c(1-\frac{3}{2}c)}{\sqrt{1-c}} - \epsilon) < 0 < H_0(\frac{\sigma^3 c(1-\frac{3}{2}c)}{\sqrt{1-c}} + \epsilon)$ for any $\epsilon > 0$. So $r^*(t) = c\sigma^2 + \dfrac{\sigma^3 c(1-\frac{3}{2}c)}{\sqrt{1-c}}\sqrt{t} + o(\sqrt{t})$. By Corollary 3.11, we have our assertion. □

We can also show the following. Although we need finer computation, the idea of the proof is similar and so we omit the proof.

Theorem 3.14.

$$\begin{aligned} R^\sigma(T) \ &= \ \sigma^2 c + \frac{\sigma^3 c(1-\frac{3}{2}c)}{\sqrt{1-c}}\sqrt{T} \\ &\quad -\sigma^4 c\left\{\frac{c^2\sigma(1-2c)}{16(1-c)^{5/2}} + \frac{131c^3 - 270c^2 + 168c - 32}{48(1-c)^2}\right\}T \\ &\quad +o(T), \quad \text{as } T \downarrow 0. \end{aligned}$$

We also have the following theorem.

Theorem 3.15. $R^\sigma(T) = \sigma^2 - \dfrac{\sigma^2 \log(2\pi\sigma^2 T)}{2(\sigma^2 T + 1)} + o\left(\dfrac{1}{T^2}\right)$, *as* $T \uparrow \infty$.

Proof. For each $h \in \mathbf{R}$, set $r_0(h,t) = \sigma^2 - \dfrac{\sigma^2 \log(2\pi\sigma^2 t)}{2(\sigma^2 t + 1)} + \dfrac{h}{t^2}$. Then, (3.8) yields

$$\begin{aligned} \eta_0(h,t) = \eta(r_0(h,t),t)) \ &= 1 - e^{-\frac{\sigma^2}{2}t}\exp\left(-\frac{\sigma^2 t\log(2\pi\sigma^2 t)}{2(\sigma^2 t+1)\eta_0(h,t)} + \frac{h}{t\eta_0(h,t)}\right) \\ &= 1 - o(e^{-\frac{\sigma^2}{2}t}) \quad \text{ as } t \uparrow \infty, \end{aligned}$$

and we have from (3.9)

$$\begin{aligned}\alpha_0(h,t) &\stackrel{\text{def}}{=} \alpha(r_0(h,t),\eta_0(h,t),t,\sigma)\\ &= \sigma\sqrt{t}+\left(\frac{\sigma^2\sqrt{t}\log(2\pi\sigma^2 t)}{2\sigma(\sigma^2 t+1)}-\frac{h}{\sigma t\sqrt{t}}\right)\left(\frac{1}{\eta_0(h,t)}-1\right)\\ &\to \infty \quad \text{as } t\uparrow\infty.\end{aligned}$$

Then we have

$$\begin{aligned}&\frac{\phi(\alpha_0(h,t))}{\sqrt{t}(1-\eta_0(h,t))}\\ &\quad= \sigma\exp\left(-\frac{\log(2\pi\sigma^2 t)}{2(\sigma^2 t+1)}-\frac{h}{t}\right)\\ &\qquad\cdot\exp\left\{-\frac{1}{2}\left(\frac{\sigma^2\sqrt{t}\log(2\pi\sigma^2 t)}{2\sigma(\sigma^2 t+1)}-\frac{h}{\sigma t\sqrt{t}}\right)^2\left(\frac{1-\eta_0(h,t)}{\eta_0(h,t)}\right)^2\right\}.\end{aligned}$$

Therefore we have

$$\begin{aligned}&G(r_0(h,t),t)=r_0(h,t)\Phi(\alpha_0(h,t))-\sigma\eta_0(h,t)\frac{\phi(\alpha_0(h,t))}{\sqrt{t}(1-\eta_0(h,t))}\\ &= \left\{\sigma^2-\frac{\sigma^2\log(2\pi\sigma^2 t)}{2(\sigma^2 t+1)}+\frac{h}{t^2}\right\}\left(1-\int_{\alpha_0(h,t)}^{\infty}\phi(x)dx\right)\\ &\quad -\sigma^2\eta_0(h,t)\exp\left(-\frac{\log(2\pi\sigma^2 t)}{2(\sigma^2 t+1)}-\frac{h}{t}\right)\\ &\qquad\cdot\exp\left\{-\frac{1}{2}\left(\frac{\sigma^2\sqrt{t}\log(2\pi\sigma^2 t)}{2\sigma(\sigma^2 t+1)}-\frac{h}{\sigma t\sqrt{t}}\right)^2\left(\frac{1-\eta_0(h,t)}{\eta_0(h,t)}\right)^2\right\}\\ &= \frac{h}{t^2}+\frac{\sigma^2 h}{t}+o\left(\frac{1}{t}\right)\qquad \text{as } t\uparrow\infty.\end{aligned}$$

So we have $tG(r_0(h,t),t)\to\sigma^2 h$ as $t\uparrow\infty$. Consequently, we have $r^*(t)=\sigma^2-\dfrac{\sigma^2\log(2\pi\sigma^2 t)}{2(\sigma^2 t+1)}+o\left(\dfrac{1}{t^2}\right)$ as $t\uparrow\infty$, since G is increasing in r by Proposition 3.10. So Corollary 3.11 implies our assertion. □

THEOREM 3.16. *For any $r>0$, there exists $T_0=T_0(r,\sigma^2)>0$ such that*

$$\psi_0(t,y)=\psi(t,y),\quad 0\le t\le T_0,\ y\ge 1.$$

In order to prove this theorem, we show some propositions. Let $Y_t^{\theta,y}$ be the process given by (3.1). Note that we have the following by Ito's formula.

$$d(e^{rt}S_t^{-1}) = -e^{rt}S_t^{-1}(\sigma dB_t - \sigma^2 dt).$$

Let $\rho_t \overset{\mathrm{def}}{=} \exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t\right)$ and define the probability measure $\tilde{P}$ by $\left(\dfrac{d\tilde{P}}{dP}\right)_t = \rho_t$. Then $\tilde{B}_t \overset{\mathrm{def}}{=} -\sigma t + B_t$ is a $\tilde{P}$ - Brownian motion. Since $e^{rt}S_t^{-1}$ is a martingale under the measure $\tilde{P}$, $\tilde{P}$ is EMM with respect to the numeraire S_t. By applying Ito's formula to (3.1), we have

$$dY_t^{\theta,y} = (\theta_t - Y_t^{\theta,y})(\sigma d\tilde{B}_t + rdt), \qquad Y_0^{\theta,y} = y. \tag{3.12}$$

Moreover by definition of $\tilde{P}$, we have $E[e^{-rT}X] = S_0\tilde{E}[S_T^{-1}X]$ for any $\mathcal{F}_T$ measurable random variable X, where $\tilde{E}[\ \ ]$ denotes the expectation under $\tilde{P}$. So we have

$$C(T,x,S) = S\sup_{\theta\in\Theta}\tilde{E}\left[(Y_T^{\theta,x/S})^+\right]. \tag{3.13}$$

Let $\tilde{\psi}(t,y) = \sup_{\theta\in\Theta}\tilde{E}\left[(Y_T^{\theta,y})^+\right]$ and $\tilde{\psi}_0(t,y) = \tilde{E}\left[(Y_T^{1,y})^+\right]$. Then $\psi(t,y) = e^{rt}\tilde{\psi}(t,y)$ and $\psi_0(t,y) = e^{rt}\tilde{\psi}_0(t,y)$, so $\psi(t,y) = \psi_0(t,y)$ if and only if $\tilde{\psi}(t,y) = \tilde{\psi}_0(t,y)$.

PROPOSITION 3.17. *$Y_t^{1,y} \geq 1$, P-a.s, for all $y \geq 1$.*

PROOF. From (3.12) we have that $Y_t^{1,y} = 1 + (y-1)e^{-\sigma\tilde{B}_t - rt - \frac{1}{2}\sigma^2 t} \geq 1$ if $y \geq 1$. □

PROPOSITION 3.18. *(1) $\tilde{E}[Y_t^{1,y} - Y_t^{\theta,y}] = r\int_0^t e^{r(s-t)}\tilde{E}[(1-\theta_s)]ds$, for all $\theta \in \Theta$ and $t > 0$.*
(2) $\tilde{E}[(Y_t^{1,y} - Y_t^{\theta,y})^2] \leq 2\sigma^2 e^{2\sigma^2 t}\int_0^t \tilde{E}[(1-\theta_s)^2]ds$, for all $\theta \in \Theta$ and $t > 0$.

PROOF. (1) Let $U_t = Y_t^{1,y} - Y_t^{\theta,y}$. Then

$$\begin{cases} dU_t = \{(1-\theta_t) - U_t\}(\sigma d\tilde{B}_t + rdt) \\ U_0 = 0. \end{cases}$$

Therefore $\frac{d}{dt}\tilde{E}[U_t] = r\tilde{E}[1-\theta_t] - r\tilde{E}[U_t]$. So $\tilde{E}[U_t] = r\int_0^t e^{r(s-t)}\tilde{E}[(1-\theta_s)]ds$.

(2) Similarly we have by Ito's formula,

$$
\begin{aligned}
\frac{d}{dt}\tilde{E}[U_t^2] &= 2r\tilde{E}[\{(1-\theta_t)-U_t\}U_t] + \sigma^2\tilde{E}[\{(1-\theta_t)-U_t\}^2] \\
&= (\sigma^2-2r)\tilde{E}[U_t^2] + \sigma^2\tilde{E}[(1-\theta_t)^2] + 2(r-\sigma^2)\tilde{E}[U_t(1-\theta_t)] \\
&\leq \{(\sigma^2-2r)+|\sigma^2-r|\}\tilde{E}[U_t^2] + (\sigma^2+|\sigma^2-r|)\tilde{E}[(1-\theta_t)^2].
\end{aligned}
$$

Therefore we have from Gronwall's inequality

$$
\begin{aligned}
\tilde{E}[U_t^2] &\leq (\sigma^2+|\sigma^2-r|)\exp(\{(\sigma^2-2r)+|\sigma^2-r|\}t)\int_0^t \tilde{E}[(1-\theta_s)^2]ds \\
&\leq 2\sigma^2 e^{2\sigma^2 t}\int_0^t \tilde{E}[(1-\theta_s)^2]ds. \ \square
\end{aligned}
$$

PROPOSITION 3.19. *There exists a constant $K = K(r,\sigma^2)$ such that*

$$
\tilde{P}(\sup_{0\leq s\leq t}|Y_s^{1,y} - Y_s^{\theta,y}| > 1) \leq Kt\tilde{E}[\int_0^t (1-\theta_s)^2 ds],
$$

for any $\theta \in \Theta$ and $t \in [0,1]$.

PROOF. Letting $U_t = Y_t^{1,y} - Y_t^{\theta,y}$, we have

$$
U_t = \sigma\int_0^t (1-\theta_s)d\tilde{B}_s - \sigma\int_0^t U_s d\tilde{B}_s + r\int_0^t (1-\theta_s)ds - r\int_0^t U_s ds.
$$

So we have

$$
\begin{aligned}
\tilde{P}(\sup_{0\leq s\leq t}|U_s| > 1) \ \leq\ & \tilde{P}\left(\sigma\sup_{0\leq s\leq t}|\int_0^s (1-\theta_u)d\tilde{B}_u| > \frac{1}{4}\right) \\
&+\tilde{P}\left(\sigma\sup_{0\leq s\leq t}|\int_0^s U_u d\tilde{B}_u| > \frac{1}{4}\right) \\
&+\tilde{P}\left(r\int_0^t |1-\theta_s|ds > \frac{1}{4}\right) + \tilde{P}\left(r\int_0^t |U_s|ds > \frac{1}{4}\right).
\end{aligned}
$$

Note that

$$\begin{aligned}\tilde{P}\left(r\int_0^t |1-\theta_s|ds > \frac{1}{4}\right) &\le 16r^2\tilde{E}\left[\left(\int_0^t |1-\theta_s|ds\right)^2\right] \\ &\le 16r^2 t\tilde{E}\left[\int_0^t (1-\theta_s)^2 ds\right].\end{aligned}$$

Also by Proposition 3.18 (2)

$$\begin{aligned}\tilde{P}\left(r\int_0^t |U_s|ds > \frac{1}{4}\right) &\le 16r^2 t\tilde{E}\left[\int_0^t U_s^2 ds\right] \\ &\le 32r^2\sigma^2 e^{2\sigma^2 t} t\tilde{E}\left[\int_0^t (1-\theta_s)^2 ds\right].\end{aligned}$$

By Doob's inequality

$$\begin{aligned}\tilde{P}\left(\sigma \sup_{0\le s\le t} |\int_0^s U_u d\tilde{B}_u| > \frac{1}{4}\right) &\le 16\sigma^2\tilde{E}\left[\sup_{0\le s\le t}|\int_0^s U_u d\tilde{B}_u|^2\right] \\ &\le 64\sigma^2\tilde{E}\left[|\int_0^t U_s d\tilde{B}_s|^2\right] \\ &= 64\sigma^2\tilde{E}\left[\int_0^t U_s^2 ds\right] \\ &\le 128\sigma^4 e^{2\sigma^2 t} t\tilde{E}\left[\int_0^t (1-\theta_s)^2 ds\right].\end{aligned}$$

By Burkholder's inequality there exists some constant C_4 such that

$$\begin{aligned}&\tilde{P}\left(\sigma \sup_{0\le s\le t} |\int_0^s (1-\theta_u) d\tilde{B}_u| > \frac{1}{4}\right) \le 4^4\sigma^4\tilde{E}\left[\sup_{0\le s\le t}|\int_0^s (1-\theta_u) d\tilde{B}_u|^4\right] \\ &\le C_4\sigma^4\tilde{E}\left[(\int_0^t (1-\theta_s)^2 ds)^2\right] \le 4C_4\sigma^4 t\tilde{E}\left[\int_0^t (1-\theta_s)^2 ds\right]. \ \square\end{aligned}$$

PROOF OF THEOREM 3.16. If $x \ge 1$ and $y \in \mathbf{R}$, then

$$x^+ - y^+ = x - y - 1_{\{y\le 0\}}|y| \ge x - y - 1_{\{|x-y|\ge 1\}}|x-y|.$$

Therefore for each $\theta \in \Theta$, if $e^{-rt}r - (8\sigma^2 K e^{2\sigma^2 t})^{1/2} t^{1/2} \ge 0$

$\tilde{E}[(Y_t^{1,y})^+] - \tilde{E}[(Y_t^{\theta,y})^+]$

$$
\begin{aligned}
&\geq \tilde{E}[Y_t^{1,y} - Y_t^{\theta,y}] - \tilde{E}[|Y_t^{1,y} - Y_t^{\theta,y}|, |Y_t^{1,y} - Y_t^{\theta,y}| \geq 1] \\
&\geq \tilde{E}[Y_t^{1,y} - Y_t^{\theta,y}] - \tilde{E}[|Y_t^{1,y} - Y_t^{\theta,y}|^2]^{1/2} \tilde{P}(|Y_t^{1,y} - Y_t^{\theta,y}| \geq 1)^{1/2} \\
&\geq r\tilde{E}[\int_0^t e^{r(s-t)}(1-\theta_s)ds] \\
&\quad - \left\{2\sigma^2 \tilde{E}[e^{2\sigma^2 t} \int_0^t (1-\theta_s)^2 ds]\right\}^{1/2} \left\{Kt\tilde{E}[\int_0^t (1-\theta_s)^2 ds]\right\}^{1/2} \\
&\geq (e^{-rt}r - (8\sigma^2 K e^{2\sigma^2 t})^{1/2} t^{1/2}) \tilde{E}[\int_0^t (1-\theta_s)ds] \geq 0. \ \square
\end{aligned}
$$

We can obtain similar results in the case of $r < 0$ similarly.

4. Discrete Time Approximation

Let us fix $T > 0$ and assume $r \geq 0$. We introduce the discrete time framework. Let N be any positive integer, and let

$$
\begin{aligned}
\Delta = \Delta_N &= \frac{T}{N}, \\
\Theta^N &= \{\theta \in \Theta : \theta_t = \theta_{n\Delta}, \text{ for } t \in [n\Delta, (n+1)\Delta), n = 0, 1, \cdots, N-1\}, \\
\hat{Y}_t^{N,\theta^N,y} &= y + \sum_{n=0}^{N-1} (\theta_{n\Delta}^N - \hat{Y}_{n\Delta}^{N,\theta^N,y}) \\
&\qquad \cdot \{\sigma(\tilde{B}_{t\wedge(n+1)\Delta} - \tilde{B}_{t\wedge n\Delta}) + r(t\wedge(n+1)\Delta - t\wedge n\Delta)\},
\end{aligned}
$$

and

$$
\tilde{\psi}^N(m, y) = \sup_{\theta^N \in \Theta^N} \tilde{E}[(\hat{Y}_{m\Delta}^{N,\theta^N,y})^+].
$$

We will consider the optimal trading strategy in the case of discrete time framework.

PROPOSITION 4.1. *$\tilde{\psi}^N(m, \cdot)$ is convex and*

$$
\begin{aligned}
\tilde{\psi}^N(m, y) &= \max\Big\{\tilde{E}\left[\tilde{\psi}^N\left(m-1, y + (\kappa_2 - y)(\sigma\tilde{B}_\Delta + r\Delta)\right)\right], \\
&\qquad \tilde{E}\left[\tilde{\psi}^N\left(m-1, y + (\kappa_1 - y)(\sigma\tilde{B}_\Delta + r\Delta)\right)\right]\Big\}.
\end{aligned}
$$

PROOF. This can be shown inductively. Let us assume that $\tilde{\psi}^N(m-1, y)$ is convex in y. Then we have

$$\begin{aligned}\tilde{\psi}^N(m,y) = & \sup_{\kappa_1 \leq k \leq \kappa_2} \tilde{E}[\sup_{\theta \in \Theta^N, \theta_0 = k} \tilde{E}[(y + (k-y)(\sigma \tilde{B}_\Delta + r\Delta) \\ & + \sum_{n=1}^{m-1} (\theta_{n\Delta} - Y_{n\Delta}^{N,\theta,y})(\sigma(\tilde{B}_{(n+1)\Delta} - \tilde{B}_{n\Delta}) + r\Delta))^+ | \mathcal{F}_\Delta]] \\ = & \sup_{\kappa_1 \leq k \leq \kappa_2} \tilde{E}[\tilde{\psi}^N(m-1, y + (k-y)(\sigma \tilde{B}_\Delta + r\Delta))] \\ \leq & \max \Big\{ \tilde{E}[\tilde{\psi}^N(m-1, y + (\kappa_1 - y)(\sigma \tilde{B}_\Delta + r\Delta))], \\ & \tilde{E}[\tilde{\psi}^N(m-1, y + (\kappa_2 - y)(\sigma \tilde{B}_\Delta + r\Delta))] \Big\}.\end{aligned}$$

The opposite side of the inequality is trivial. And so $\tilde{\psi}^N(m, y)$ is convex in y. $\square$

LEMMA 4.2. $\sup_{\theta \in \Theta} \tilde{E}[(Y_t^{\theta,y})^2] \leq (y^2 + 2\sigma^2 \max\{\kappa_1^2, \kappa_2^2\} t) e^{2\sigma^2 t}$.

PROOF. Since $d(Y_t^{\theta,y})^2 = 2Y_t^{\theta,y}(\theta_t - Y_t^{\theta,y})(\sigma d\tilde{B}_t + r dt) + \sigma^2(\theta_t - Y_t^{\theta,y})^2 dt$ for any $\theta \in \Theta$, we have the following similarly to Proposition 3.18 (2).

$$\frac{d}{dt} \tilde{E}[|Y_t^{\theta,y}|^2] \leq \{(\sigma^2 - 2r) + |\sigma^2 - r|\} \tilde{E}[(Y_t^{\theta,y})^2] + (\sigma^2 + |\sigma^2 - r|) \tilde{E}[\theta_t^2].$$

Therefore we have

$$\begin{aligned}\tilde{E}[|Y_t^{\theta,y}|^2] & \leq (y^2 + 2\sigma^2 \int_0^t \tilde{E}[\theta_s^2] ds) \exp(2\sigma^2 t) \\ & \leq (y^2 + 2\sigma^2 \max(\kappa_1^2, \kappa_2^2) t) e^{2\sigma^2 t}. \ \square\end{aligned}$$

LEMMA 4.3. *Let $b \geq 0$, $c \geq 0$ and let $\{a_n\}_{n \geq 0}$ be a sequence of numbers such that $a_0 = 0$ and*

$$a_n \leq b + c \sum_{k=0}^{n-1} a_k, \qquad n \geq 1.$$

Then $a_n \leq b(1+c)^{n-1}$.

PROOF. We show our assertion by induction. It is easy to verify it in case of $n=1$. Let $a_n \le b + c\sum_{k=0}^{n-1} a_k \le b(1+c)^{n-1}$. Then we have

$$a_{n+1} \le b + c\sum_{k=0}^{n} a_k \le b(1+c)^{n-1} + ca_n \le b(1+c)^n. \square$$

LEMMA 4.4. *There exists a sequence $\{C_N\}$ such that $C_N \to 0$ as $N \to \infty$ such that*

$$\tilde{E}\left[|Y_T^{\theta,y} - \hat{Y}_T^{N,\theta,y}|^2\right] \le \left(1 + 4(\sigma^2 + r^2 T)\frac{T}{N}\right)^{N-1} C_N,$$

for any $\theta \in \Theta^N$.

PROOF. Let $\theta \in \Theta^N$. From the definitions of $\hat{Y}_T^{N,\theta,y}$ and $Y_T^{\theta,y}$ we have

$$\begin{aligned}
\tilde{E}\left[|Y_{N\Delta}^{\theta,y} - \hat{Y}_{N\Delta}^{N,\theta,y}|^2\right] &\le 2(\sigma^2 + r^2 T)\tilde{E}\left[\sum_{n=0}^{N-1}\int_{n\Delta}^{(n+1)\Delta} |Y_t^{\theta,y} - \hat{Y}_{n\Delta}^{N,\theta,y}|^2 dt\right] \\
&= 4(\sigma^2 + r^2 T)\sum_{n=0}^{N-1}\int_{n\Delta}^{(n+1)\Delta} \tilde{E}\left[|Y_t^{\theta,y} - Y_{n\Delta}^{\theta,y}|^2\right] dt \\
&\quad +4(\sigma^2 + r^2 T)\sum_{n=0}^{N-1} \Delta \tilde{E}\left[|Y_{n\Delta}^{\theta,y} - \hat{Y}_{n\Delta}^{N,\theta,y}|^2\right].
\end{aligned}$$

Since we have $Y_t^{\theta,y} = Y_{n\Delta}^{\theta,y} + (\theta_{n\Delta}^N - Y_{n\Delta}^{\theta,y})(1 - e^{-\sigma(\tilde{B}_t - \tilde{B}_{n\Delta}) - (\frac{\sigma^2}{2} + r)(t - n\Delta)})$ for $n\Delta \le t < (n+1)\Delta$, we have

$$\begin{aligned}
&\int_{n\Delta}^{(n+1)\Delta} \tilde{E}\left[|Y_t^{\theta,y} - Y_{n\Delta}^{\theta,y}|^2\right] dt \\
&= \int_{n\Delta}^{(n+1)\Delta} \tilde{E}\left[(\theta_{n\Delta}^N - Y_{n\Delta}^{\theta,y})^2 (1 - e^{-\sigma(\tilde{B}_t - \tilde{B}_{n\Delta}) - (\frac{\sigma^2}{2} + r)(t - n\Delta)})^2\right] dt \\
&\le K\left(\Delta - \frac{2}{r}(1 - e^{-r\Delta}) + \frac{1}{-2r + \sigma^2}(e^{(-2r + \sigma^2)\Delta} - 1)\right),
\end{aligned}$$

where $K = 2(y^2 + 2\sigma^2 \max\{\kappa_1^2, \kappa_2^2\}T)e^{2\sigma^2 T} + 2 \ge \sup_{\theta \in \Theta^N} \tilde{E}[(Y_{n\Delta}^{\theta,y} - \theta_{n\Delta}^N)^2]$. Therefore

$$\tilde{E}\left[|Y_{N\Delta}^{\theta,y} - \hat{Y}_{N\Delta}^{N,\theta,y}|^2\right]$$

$$\leq 4(\sigma^2 + r^2T)KN\left(\Delta - \frac{2(1-e^{-r\Delta})}{r} + \frac{e^{(-2r+\sigma^2)\Delta} - 1}{-2r+\sigma^2}\right)$$
$$+4(\sigma^2 + r^2T)\sum_{n=0}^{N-1}\Delta\tilde{E}\left[|Y_{n\Delta}^{\theta,y} - \hat{Y}_{n\Delta}^{N,\theta,y}|^2\right].$$

So by Lemma 4.3

$$\tilde{E}\left[|Y_{N\Delta}^{\theta,y} - \hat{Y}_{N\Delta}^{N,\theta,y}|^2\right]$$
$$\leq 4(\sigma^2 + r^2T)KN\left(\Delta - \frac{2(1-e^{-r\Delta})}{r} + \frac{e^{(-2r+\sigma^2)\Delta} - 1}{-2r+\sigma^2}\right)$$
$$\cdot\Big(1 + 4(\sigma^2 + r^2T)\Delta\Big)^{N-1}.$$

Since $\Delta - \frac{2(1-e^{-r\Delta})}{r} + \frac{e^{(-2r+\sigma^2)\Delta} - 1}{-2r+\sigma^2} = O\left(\Delta^2\right)$, we have our assertion. $\square$

Proposition 4.5. $\lim_{N\to\infty}\tilde{\psi}^N(t,y) = \tilde{\psi}(t,y)$.

Proof. Lemma 4.4 and $\Theta^N \subset \Theta$ imply that

$$\limsup_{N\to\infty}\sup_{\theta\in\Theta^N}\tilde{E}[(\hat{Y}_t^{N,\theta,y})^+] = \limsup_{N\to\infty}\sup_{\theta\in\Theta^N}\tilde{E}[(Y_t^{\theta,y})^+] \leq \tilde{\psi}(t,y).$$

On the other hand, for each $\theta \in \Theta$, there exists a sequence of strategies $\{\theta^N\}_{N=1}^{\infty}$ such that $\theta^N \in \Theta^N$ and $\lim_{N\to\infty}\tilde{E}\left[\int_0^T(\theta_t - \theta_t^N)^2dt\right] = 0$. Then,

$$\begin{aligned}\tilde{E}\left[|Y_t^{\theta,y} - Y_t^{\theta^N,y}|^2\right] &= 2(\sigma^2 + r^2T)\tilde{E}\left[\int_0^t|(\theta_s - \theta_s^N) - (Y_s^{\theta,y} - Y_s^{\theta^N,y})|^2ds\right]\\ &\leq 4(\sigma^2 + r^2t)\int_0^t\tilde{E}\left[(\theta_s - \theta_s^N)^2\right]ds\\ &\quad +4(\sigma^2 + r^2t)\int_0^t\tilde{E}\left[|Y_s^{\theta,y} - Y_s^{\theta^N,y}|^2\right]ds.\end{aligned}$$

By Gronwall's inequality

$$\tilde{E}\left[|Y_t^{\theta,y} - Y_t^{\theta^N,y}|^2\right] \leq 4(\sigma^2 + r^2t)\tilde{E}\left[\int_0^t(\theta_s - \theta_s^N)^2ds\right]e^{4(\sigma^2+r^2t)t} \to 0,$$
$$\text{as } N \to \infty.$$

Lemma 4.2 implies that for each N there exists a sequence $\{\theta^{(N,n)} \in \Theta^n\}_n$ such that $\tilde{E}\left[|Y_T^{\theta(N),y} - \hat{Y}_T^{n,\theta^{(N,n)},y}|^2\right] \to 0$ as $n \to \infty$ uniformly with respect to N. Therefore we have subsequence $\{\theta^{(N,n(N))}\}_N$ such that $\tilde{E}\Big[|Y_T^{\theta,y} - \hat{Y}_T^{n(N),\theta^{(N,n(N))},y}|^2\Big] \to 0$ as $N \to \infty$. Therefore $\lim_{N\to\infty} \tilde{E}[(\hat{Y}_T^{n(N),\theta^{(N,n(N))},y})^+] = \tilde{E}[(Y_T^{\theta,y})^+]$. Therefore $\liminf_{N\to\infty} \tilde{\psi}^N(T,y) \geq \tilde{\psi}(T,y)$. □

5. Closed Form of Valuation in the Case of $r = 0$

In this section we assume that $r = 0$. Then, under the measure $\tilde{P}$, we have

$$dY_t^{\theta,y} = (\theta_t - Y_t^{\theta,y})\sigma d\tilde{B}_t.$$

Letting $\kappa_1 < \kappa_2$, we derive the closed form for valuation of passport call option. Let $\bar{\kappa} = \dfrac{\kappa_2 + \kappa_1}{2}$, $\kappa = \dfrac{\kappa_2 - \kappa_1}{2}$ and fix $T > 0$. We consider the discrete time approximation defined in the previous section.

LEMMA 5.1. *For any convex function g, we have*

$$\sup_{\theta\in\Theta(\kappa_1,\kappa_2)} E\left[g\left(\int_0^t \theta_s dB_s\right)\right] = E[g(\kappa_i B_t)],$$

$$\text{where } \kappa_i = \begin{cases} \kappa_2 & \text{if } |\kappa_1| \leq |\kappa_2| \\ \kappa_1 & \text{if } |\kappa_1| > |\kappa_2|. \end{cases}$$

PROOF. Let $\acute{B}$ be a Brownian motion independent of B. Then for any $\theta \in \Theta$

$$\begin{aligned} E\left[g\left(\int_0^t \theta_s dB_s\right)\right] &= E\left[g\left(E\left[\int_0^t (\theta_s dB_s + (\kappa_i^2 - \theta_s^2)^{1/2} d\acute{B}_s)|\mathcal{F}_t\right]\right)\right] \\ &\leq E\left[g\left(\int_0^t (\theta_s dB_s + (\kappa_i^2 - \theta_s^2)^{1/2} d\acute{B}_s)\right)\right] \\ &= E\left[g\left(\kappa_i B_t\right)\right]. \ \square \end{aligned}$$

COROLLARY 5.2.

$$\tilde{\psi}^N(m,y) = \begin{cases} \tilde{E}\left[\tilde{\psi}^N\left(m-1, y + (\kappa_2 - y)\sigma\tilde{B}_\Delta\right)\right] & \text{if } y \leq \bar{\kappa} \\ \tilde{E}\left[\tilde{\psi}^N\left(m-1, y + (\kappa_1 - y)\sigma\tilde{B}_\Delta\right)\right] & \text{if } y > \bar{\kappa}. \end{cases}$$

PROOF. This follows from Lemma 5.1 and the following.

$$\begin{aligned}\tilde{\psi}^N(m,y) &= \sup_{\kappa_1 \le \theta' \le \kappa_2} \tilde{E}[\tilde{\psi}^N(m-1, y+(\theta'-y)\sigma\tilde{B}_\Delta)] \\ &= \sup_{\kappa_1 - y \le \theta' \le \kappa_2 - y} \tilde{E}[\tilde{\psi}^N(m-1, y+\theta'\sigma\tilde{B}_\Delta)]. \ \square\end{aligned}$$

By Corollary 5.2 the optimal strategy θ^{N*} in the discrete framework is inductively given by the following.

$$\theta_t^{N*} = \begin{cases} \kappa_1 & \text{if } \hat{Y}_{n\Delta}^{N,\theta^{N*},y} > \bar{\kappa} \\ \kappa_2 & \text{if } \hat{Y}_{n\Delta}^{N,\theta^{N*},y} \le \bar{\kappa}, \end{cases} \qquad n\Delta \le t < (n+1)\Delta,$$

where

$$\begin{aligned}\hat{Y}_t^{N,\theta^{N*},y} &= y + \sum_{n=0}^{N-1} (\theta_{n\Delta}^{N*} - \hat{Y}_{n\Delta}^{N,\theta^{N*},y})\sigma(\tilde{B}_{t\wedge(n+1)\Delta} - \tilde{B}_{t\wedge n\Delta}) \\ &= y - \sum_{n=0}^{N-1} \Big[\operatorname{sign}\left(\hat{Y}_{n\Delta}^{N,\theta^{N*},y} - \bar{\kappa}\right)\left(\kappa + \left|\hat{Y}_{n\Delta}^{N,\theta^{N*},y} - \bar{\kappa}\right|\right) \\ &\qquad \cdot\sigma(\tilde{B}_{t\wedge(n+1)\Delta} - \tilde{B}_{t\wedge n\Delta})\Big].\end{aligned}$$

Let $\tilde{B}_t^{N,\sharp}$ be given inductively by

$$\tilde{B}_t^{N,\sharp} = \tilde{B}_{n\Delta}^{N,\sharp} - \operatorname{sign}(\hat{Y}_{n\Delta}^{N,\theta^{N*},y} - \bar{\kappa})(\tilde{B}_t - \tilde{B}_{n\Delta}), \qquad n\Delta < t \le (n+1)\Delta.$$

Then $\{\tilde{B}_t^{N,\sharp}\}$ is another $\tilde{P}$-Brownian motion and we have

$$\hat{Y}_t^{N,\theta^{N*},y} = y + \sum_{n=0}^{N-1}\left(\left(\kappa + \left|\hat{Y}_{n\Delta}^{N,\theta^{N*},y} - \bar{\kappa}\right|\right)\sigma(\tilde{B}_{t\wedge(n+1)\Delta}^{N,\sharp} - \tilde{B}_{t\wedge n\Delta}^{N,\sharp})\right).$$

Now let

$$\hat{Y}_t^{N,\sharp,y} = y + \sum_{n=0}^{N-1}\left(\left(\kappa + \left|\hat{Y}_{n\Delta}^{N,\sharp,y} - \bar{\kappa}\right|\right)\sigma(\tilde{B}_{t\wedge(n+1)\Delta} - \tilde{B}_{t\wedge n\Delta})\right).$$

Then $\hat{Y}_t^{N,\theta^{N*},y}$ and $\hat{Y}_t^{N,\sharp,y}$ have the same distribution and we have $E[(\hat{Y}_t^{N,\theta^{N*},y})^+] = E[(\hat{Y}_t^{N,\sharp,y})^+]$. And let $Y_t^{\sharp,y}$ to be the solution of the following S.D.E.

$$dY_t^{\sharp,y} = \left(\kappa + \left|Y_t^{\sharp,y} - \bar{\kappa}\right|\right)\sigma d\tilde{B}_t, \qquad Y_0^{\sharp,y} = y. \tag{5.1}$$

Then we have $\lim_{N\to\infty} E[|\hat{Y}_T^{N,\sharp,y} - Y_T^{\sharp,y}|^2] = 0$ by Euler-Maruyama approximation (see Kloeden and Platen [5]), and so $\lim_{N\to\infty} E[(\hat{Y}_T^{N,\sharp,y})^+] = E[(Y_T^{\sharp,y})^+]$. Therefore by Proposition 4.5 we have $\tilde{\psi}(T,y) = \tilde{E}[(Y_T^{\sharp,y})^+]$.

Remark 5.3. From the form of θ^{N*}, one may guess that the optimal strategy θ^* is given by the following.

$$\theta_t^* = \begin{cases} \kappa_1 & \text{if } Y_t > \bar{\kappa} \\ \kappa_2 & \text{if } Y_t \le \bar{\kappa}, \end{cases} \qquad n\Delta \le t < (n+1)\Delta$$

where Y is given by

$$dY_t = -\operatorname{sign}(Y_t - \bar{\kappa})(\kappa + |Y_t - \bar{\kappa}|)\sigma d\tilde{B}_t, \qquad Y_0 = y.$$

For simplicity, let $\kappa_1 = -1$ and $\kappa_2 = 1$. Then recalling that $B_t = \tilde{B}_t + \sigma t$ is P-Brownian motion, we see that $X_t = Y_t S_t$ satisfies

$$dX_t = -\operatorname{sign}(X_t) dS_t, \qquad X_0 = y, \tag{5.2}$$

under the measure P. It is well known as Tanaka's counter-example that S.D.E. (5.2) does not have any strong solutions. Therefore such an adapted process θ^* does not exist.

Now we derive the formula for pricing the passport option. Let $\bar{Y}_t^{\bar{y}} \stackrel{\text{def}}{=} \frac{1}{\kappa}\left(Y_t^{\sharp,\kappa\bar{y}+\bar{\kappa}} - \bar{\kappa}\right)$. Then we have

$$d\bar{Y}_t^{\bar{y}} = \sigma(1 + |\bar{Y}_t^{\bar{y}}|)d\tilde{B}_t.$$

Let $f^{\bar{Y}^0}(y : t)dy = \tilde{P}[\bar{Y}_t^0 \in dy]$ and $q(a,b,t;\mu)dadb = \tilde{P}[\tilde{B}_t + \mu t \in da, \min_{0\le s\le t}(\tilde{B}_s + \mu s) \in db]$. It is well known(see Karatzas & Shreve [4]) that

$$q(a,b,t;\mu) = \frac{2(a-2b)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(a-2b-\mu t)^2}{2t} + 2b\mu\right).$$

Skorohod's Theorem implies the following lemma.

LEMMA 5.4.

$$\begin{aligned} f^{\bar{Y}^0}(y:t) &= \frac{1}{\sigma(1+|y|)^2}\left\{\frac{1}{\sigma\sqrt{t}}\phi\left(-\frac{\log(1+|y|)}{\sigma\sqrt{t}}+\frac{1}{2}\sigma\sqrt{t}\right)\right. \\ &\qquad \left. +\frac{1}{2}\Phi\left(-\frac{\log(1+|y|)}{\sigma\sqrt{t}}+\frac{1}{2}\sigma\sqrt{t}\right)\right\}. \end{aligned}$$

PROOF. We have

$$d(\log(1+|\bar{Y}_t^0|)) = \sigma\,\mathrm{sign}(\bar{Y}_t^0)d\tilde{B}_t + dL_t^0 - \frac{1}{2}\sigma^2 dt = \sigma d\tilde{B}'_t + dL_t^0 - \frac{1}{2}\sigma^2 dt,$$

where L_t^0 is the local time of $\bar{Y}^0$ at 0, and $\tilde{B}'_t = \int_0^t \mathrm{sign}(\bar{Y}_s^0)d\tilde{B}_s$. Define a $\tilde{P}$-equivalent probability measure Q by $(\frac{dQ}{d\tilde{P}})_t = \rho_t \stackrel{\text{def}}{=} \exp(\frac{\sigma}{2}\tilde{B}'_t - \frac{\sigma^2}{8}t)$. Then $B_t^Q \stackrel{\text{def}}{=} \tilde{B}'_t - \frac{\sigma}{2}t$ is Q-Brownian motion, and we have $d\left(\frac{1}{\sigma}\log(1+|\bar{Y}_t^0|)\right) = dB_t^Q + dL_t^0$. Let $W_t \stackrel{\text{def}}{=} \frac{1}{\sigma}\log(1+|\bar{Y}_t^0|)$. From Skorohod's Theorem, we have $W_t = B_t^Q - \min_{0\le u\le t} B_u^Q$ (See Ikeda and Watanabe [3]). Since

$$\begin{aligned} Q[W_t\in dw, B_t^Q\in db] &= q(b, b-w, t, 0)dwdb \\ &= \frac{2(2w-b)}{\sqrt{2\pi t^3}}\exp\left(-\frac{(2w-b)^2}{2t}\right)dwdb, \end{aligned}$$

we have

$$\begin{aligned} g(w:t) &\stackrel{\text{def}}{=} \frac{d}{dw}\tilde{P}[W_t\le w] = \frac{d}{dw}E^Q[I_{\{W_t\le w\}}\rho_t^{-1}] \\ &= \int_{-\infty}^{w} e^{-\frac{1}{8}\sigma^2 t-\frac{1}{2}\sigma b}\frac{d^2}{dwdb}Q[W_t\le w, B_t^Q\le b]db \\ &= 2\sigma e^{-\sigma w}\left\{\frac{1}{\sigma\sqrt{t}}\phi\left(-\frac{w}{\sqrt{t}}+\frac{1}{2}\sigma\sqrt{t}\right)+\frac{1}{2}\Phi\left(-\frac{w}{\sqrt{t}}+\frac{1}{2}\sigma\sqrt{t}\right)\right\}, \\ &\qquad\qquad t, w\ge 0. \end{aligned}$$

Then,

$$f^{\bar{Y}^0}(y:t) = \frac{1}{2\sigma(1+|y|)}g(\frac{1}{\sigma}\log(1+|y|):t)$$

$$
\begin{aligned}
= \quad & \frac{1}{(1+|y|)^2}\left\{\frac{1}{\sigma\sqrt{t}}\phi\left(-\frac{\log(1+|y|)}{\sigma\sqrt{t}}+\frac{1}{2}\sigma\sqrt{t}\right)\right.\\
& \left. +\frac{1}{2}\Phi\left(-\frac{\log(1+|y|)}{\sigma\sqrt{t}}+\frac{1}{2}\sigma\sqrt{t}\right)\right\}. \square
\end{aligned}
$$

LEMMA 5.5.

$$
\begin{aligned}
& C(T, S\bar{\kappa}, S, \kappa_1, \kappa_2) \\
& \quad = S\left[\frac{\kappa}{2}\left\{(1+\sigma\sqrt{T}d_1(T))\Phi(d_1(T))+\sigma\sqrt{T}\phi(d_1(T))\right\}\right.\\
& \qquad \left. +\bar{\kappa}^+ - \frac{\kappa+|\bar{\kappa}|}{2}\Phi(d_2(T))\right]
\end{aligned}
$$

where $d_1(T) = -\dfrac{1}{\sigma\sqrt{T}}\log\dfrac{\kappa+|\bar{\kappa}|}{\kappa}+\dfrac{\sigma\sqrt{T}}{2}$ *and* $d_2(T) = d_1(T)-\sigma\sqrt{T}$.

PROOF. Note that $C(T, S\bar{\kappa}, S, \kappa_1, \kappa_2) = S\tilde{E}\left[(Y_T^{\sharp,\bar{\kappa}})^+\right]$. Then we have

$$
\begin{aligned}
& \tilde{E}\left[(Y_T^{\sharp,\bar{\kappa}})^+\right] = \kappa\tilde{E}\left[\left(\bar{Y}_T^0+\frac{\bar{\kappa}}{\kappa}\right)^+\right] = \int_{-\frac{\bar{\kappa}}{\kappa}}^{\infty}(\kappa y+\bar{\kappa})\, f^{\bar{Y}^0}(y:T)dy \\
= \quad & \int_{\frac{|\bar{\kappa}|}{\kappa}}^{\infty}(\kappa y+\bar{\kappa})\, f^{\bar{Y}^0}(y:T)dy + 2\bar{\kappa}^+\int_0^{\frac{|\bar{\kappa}|}{\kappa}} f^{\bar{Y}^0}(y:T)dy \\
= \quad & \int_{-\infty}^{d_1(T)}\left(\kappa e^{-\sigma\sqrt{T}z+\frac{1}{2}\sigma^2 T}-\kappa+\bar{\kappa}\right)e^{\sigma\sqrt{T}z-\frac{1}{2}\sigma^2 T}\left(\phi(z)+\frac{\sigma\sqrt{T}}{2}\Phi(z)\right)dz \\
& +2\bar{\kappa}^+\int_{d_1(T)}^{\frac{1}{2}\sigma\sqrt{T}} e^{\sigma\sqrt{T}z-\frac{1}{2}\sigma^2 T}\left(\phi(z)+\frac{\sigma\sqrt{T}}{2}\Phi(z)\right)dz \\
= \quad & \frac{\kappa}{2}\left\{(1+\sigma\sqrt{T}d_1(T))\Phi(d_1(T))+\sigma\sqrt{T}\phi(d_1(T))\right\} \\
& +\bar{\kappa}^+ - \frac{\kappa+|\bar{\kappa}|}{2}\Phi(d_2(T)). \square
\end{aligned}
$$

THEOREM 5.6.

$C(T, x, S, \kappa_1, \kappa_2)$

$$
\begin{aligned}
&= S\kappa e^{-\sigma\sqrt{T}f_2(x,T)}\Phi\left(-f_2(x,T)-f_3(x,T)^+ + \frac{\sigma}{2}\sqrt{T}\right)\\
&\quad -S(\kappa-\operatorname{sign}(x-\bar{\kappa}S)\bar{\kappa})\Phi\left(-f_2(x,T)-f_3(x,T)^+ - \frac{\sigma}{2}\sqrt{T}\right)\\
&\quad +Se^{-\sigma\sqrt{T}f_2(x,T)}(\kappa-\operatorname{sign}(x-\bar{\kappa}S)\bar{\kappa})\\
&\quad \cdot\Phi\left(f_2(x,T)-f_3(x,T)^+ - \frac{\sigma}{2}\sqrt{T}\right)\\
&\quad -S\kappa\Phi\left(f_2(x,T)-f_3(x,T)^+ + \frac{\sigma}{2}\sqrt{T}\right) + x1_{\{x<\bar{\kappa}S\}}\\
&\quad +S\bar{\kappa}(1_{\{\bar{\kappa}>0\}}-1_{\{x<\bar{\kappa}S\}})\\
&\quad \cdot\left[\Phi\left(f_2(x,T)+\frac{\sigma}{2}\sqrt{T}\right)+e^{-\sigma\sqrt{T}f_2(x,T)}\Phi\left(f_2(x,T)-\frac{\sigma}{2}\sqrt{T}\right)\right]\\
&\quad +S\frac{\kappa}{2}\left(1-\sigma\sqrt{T}f_1(T)+\frac{\sigma^2T}{2}\right)\Phi\left(f_1(T)+f_2(x,T)+\frac{\sigma}{2}\sqrt{T}\right)\\
&\quad -S\frac{\kappa^2}{4}e^{-\sigma\sqrt{T}(f_1(T)+f_2(x,T))}\Phi\left(f_1(T)+f_2(x,T)-\frac{\sigma}{2}\sqrt{T}\right)\\
&\quad +S\frac{\kappa}{2}\int_0^T g^x(T-t)\left\{\frac{\sigma^2 t}{2}\Phi(f_1(t)+\frac{\sigma}{2}\sqrt{T})\right.\\
&\qquad\qquad \left.+\sigma\sqrt{t}\phi(f_1(t)+\frac{\sigma}{2}\sqrt{T})\right\}dt
\end{aligned}
$$

where

$$
\begin{aligned}
f_1(t) &= -\frac{\log\frac{\kappa+|\bar{\kappa}|}{\kappa}}{\sigma\sqrt{t}}\\
f_2(x,t) &= -\frac{\log\left(1+\frac{1}{\kappa}|\frac{x}{S}-\bar{\kappa}|\right)}{\sigma\sqrt{t}}\\
f_3(x,t) &= \begin{cases}\frac{1}{\sigma\sqrt{t}}\log\frac{\kappa-\operatorname{sign}(x-\bar{\kappa}S)\bar{\kappa}}{\kappa} & \text{if } \kappa-\operatorname{sign}(x-\bar{\kappa}S)\bar{\kappa}>0\\ -\infty & \text{if } \kappa-\operatorname{sign}(x-\bar{\kappa}S)\bar{\kappa}\le 0\end{cases}\\
g^x(t) &= -\frac{f_2(x,t)}{t}\phi\left(f_2(x,t)+\frac{1}{2}\sigma\sqrt{t}\right).
\end{aligned}
$$

PROOF. Note that $C(T,x,S,\kappa_1,\kappa_2) = S\tilde{E}\left[(Y_T^{\sharp,x/S})^+\right]$ and let $\tau^y = \inf\left\{t>0 : Y_t^{\sharp,y}=\bar{\kappa}\right\} = \inf\left\{t>0 : \bar{Y}_t^{\bar{y}}=0\right\}$, where $\bar{y} = \dfrac{y-\bar{\kappa}}{\kappa}$. Then, we have

$$\tilde{E}\left[(Y_T^{\sharp,y})^+\right] = \tilde{E}\left[(Y_T^{\sharp,y})^+, T \geq \tau^y\right] + \tilde{E}\left[(Y_T^{\sharp,y})^+, T < \tau^y\right].$$

And

$$d\bar{Y}_{t\wedge\tau^y}^{\bar{y}} = \begin{cases} \sigma(1+\bar{Y}_{t\wedge\tau^y}^{\bar{y}})dB_{t\wedge\tau^y} & \text{if } \bar{y} \geq 0 \\ \sigma(1-\bar{Y}_{t\wedge\tau^y}^{\bar{y}})dB_{t\wedge\tau^y} & \text{if } \bar{y} < 0. \end{cases}$$

Therefore

$$\bar{Y}_{t\wedge\tau^y}^{\bar{y}} = \mathrm{sign}(\bar{y})\left[(1+|\bar{y}|)\exp\left\{\mathrm{sign}(\bar{y})\sigma B_{t\wedge\tau^y} - \frac{1}{2}\sigma^2(t\wedge\tau^y)\right\} - 1\right].$$

Then, we have

$$\begin{aligned} \tau^y &= \begin{cases} \inf\{t>0 : B_t = \frac{1}{\sigma}\left(-\log(1+\bar{y}) + \frac{1}{2}\sigma^2 t\right)\} & \text{if } \bar{y} \geq 0 \\ \inf\{t>0 : B_t = -\frac{1}{\sigma}\left(-\log(1-\bar{y}) + \frac{1}{2}\sigma^2 t\right)\} & \text{if } \bar{y} < 0 \end{cases} \\ &= \inf\left\{t>0 : B_t = \mathrm{sign}(\bar{y})\frac{1}{\sigma}\left(-\log(1+|\bar{y}|) + \frac{1}{2}\sigma^2 t\right)\right\}. \end{aligned}$$

Then

$$\begin{aligned} &\tilde{E}\left[(Y_T^{\sharp,y})^+, T < \tau^y\right] = \kappa\tilde{E}\left[\left(\bar{Y}_T^{\bar{y}} + \frac{\bar{\kappa}}{\kappa}\right)^+, T < \tau^y\right] \\ &= \kappa\int_{-\frac{1}{\sigma}\log(1+|\bar{y}|)}^{0}\Bigg(\int_b^\infty q(a,b,T;-\frac{\sigma}{2}) \\ &\qquad\qquad \cdot\{\mathrm{sign}(\bar{y})((1+|\bar{y}|)e^{\sigma a}-1) + \frac{\bar{\kappa}}{\kappa}\}^+ da\Bigg)db \\ &= \begin{cases} \kappa\displaystyle\int_f^0\left(\int_{b\vee(f+h)}^\infty q(a,b,T;-\frac{\sigma}{2})\left\{(1+\bar{y})e^{\sigma a} - \frac{\kappa-\bar{\kappa}}{\kappa}\right\}da\right)db \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } \bar{y} > 0 \\ \kappa\displaystyle\int_f^0\left(\int_b^{b\vee(f+h)} q(a,b,T;-\frac{\sigma}{2})\left\{(\bar{y}-1)e^{\sigma a} + \frac{\kappa+\bar{\kappa}}{\kappa}\right\}da\right)db \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } \bar{y} < 0 \end{cases} \end{aligned}$$

where

$$\begin{aligned} f &= -\frac{1}{\sigma}\log(1+|\bar{y}|) \\ h &= \begin{cases} \frac{1}{\sigma}\log\frac{\kappa-\mathrm{sign}(\bar{y})\bar{\kappa}}{\kappa} & \text{if } \kappa-\mathrm{sign}(\bar{y})\bar{\kappa} > 0 \\ -\infty & \text{if } \kappa-\mathrm{sign}(\bar{y})\bar{\kappa} \leq 0. \end{cases} \end{aligned}$$

Since

$$\begin{aligned}
&\int_\alpha^\beta \left(\int_\gamma^\delta e^{\sigma a} q(a,b,T;-\frac{\sigma}{2})da\right)db \\
&= e^{\sigma\beta}\left[\Phi\left(\frac{2\beta-\gamma}{\sqrt{T}}+\frac{\sigma}{2}\sqrt{T}\right)-\Phi\left(\frac{2\beta-\delta}{\sqrt{T}}+\frac{\sigma}{2}\sqrt{T}\right)\right] \\
&\quad -e^{\sigma\alpha}\left[\Phi\left(\frac{2\alpha-\gamma}{\sqrt{T}}+\frac{\sigma}{2}\sqrt{T}\right)-\Phi\left(\frac{2\alpha-\delta}{\sqrt{T}}+\frac{\sigma}{2}\sqrt{T}\right)\right], \\
&\int_\alpha^\beta \left(\int_b^\delta e^{\sigma a} q(a,b,T;-\frac{\sigma}{2})da\right)db \\
&= \Phi\left(\frac{\beta}{\sqrt{T}}-\frac{\sigma}{2}\sqrt{T}\right)-\Phi\left(\frac{\alpha}{\sqrt{T}}-\frac{\sigma}{2}\sqrt{T}\right) \\
&\quad +e^{\sigma\alpha}\left[\Phi\left(\frac{2\alpha-\delta}{\sqrt{T}}+\frac{\sigma}{2}\sqrt{T}\right)-\Phi\left(\frac{\alpha}{\sqrt{T}}+\frac{\sigma}{2}\sqrt{T}\right)\right] \\
&\quad +e^{\sigma\beta}\left[\Phi\left(\frac{\beta}{\sqrt{T}}+\frac{\sigma}{2}\sqrt{T}\right)-\Phi\left(\frac{2\beta-\delta}{\sqrt{T}}+\frac{\sigma}{2}\sqrt{T}\right)\right], \quad \text{for } \beta\le\delta,
\end{aligned}$$

and

$$q(a,b,T;-\frac{\sigma}{2}) = e^{-\sigma a} q(a,b,T;\frac{\sigma}{2}),$$

we have

$$\begin{aligned}
&\tilde{E}\left[(Y_T^{\sharp,y})^+, T<\tau^y\right] \\
&= (\kappa+|y-\bar{\kappa}|)\Phi\left(-\frac{f+h^+}{\sqrt{T}}+\frac{\sigma}{2}\sqrt{T}\right) \\
&\quad -(\kappa-\mathrm{sign}(\bar{y})\bar{\kappa})\Phi\left(-\frac{f+h^+}{\sqrt{T}}-\frac{\sigma}{2}\sqrt{T}\right) \\
&\quad +\frac{\kappa-\mathrm{sign}(\bar{y})\bar{\kappa}}{\kappa}(\kappa+|y-\bar{\kappa}|)\Phi\left(\frac{f-h^+}{\sqrt{T}}-\frac{\sigma}{2}\sqrt{T}\right) \\
&\quad -\kappa\Phi\left(\frac{f-h^+}{\sqrt{T}}+\frac{\sigma}{2}\sqrt{T}\right) \\
&\quad -1_{\{y<\bar{\kappa}\}}\left[-y+\frac{\bar{\kappa}}{\kappa}(\kappa+\bar{\kappa}-y)\Phi\left(\frac{f}{\sqrt{T}}-\frac{\sigma}{2}\sqrt{T}\right)\right. \\
&\qquad\qquad\qquad \left.+\bar{\kappa}\Phi\left(\frac{f}{\sqrt{T}}+\frac{\sigma}{2}\sqrt{T}\right)\right].
\end{aligned}$$

Next, we have

$$
\begin{aligned}
&\tilde{E}\left[(Y_T^{\sharp,y})^+, T \geq \tau^y\right] \\
&= \int_0^T g^{\tau^y}(T-t)\tilde{E}\left[(Y_t^{\sharp,\bar{\kappa}})^+\right] dt \\
&= \int_0^T g^{\tau^y}(T-t)\left[\frac{\kappa}{2}\left\{(1+\sigma\sqrt{t}d_1(t))\Phi(d_1(t)) + \sigma\sqrt{t}\phi(d_1(t))\right\}\right. \\
&\qquad\qquad \left. + \bar{\kappa}^+ - \frac{\kappa+|\bar{\kappa}|}{2}\Phi(d_2(t))\right] dt \\
&= \int_0^T g^{\tau^y}(T-t)\left[\frac{\kappa}{2}\left\{(1-\log\frac{\kappa+|\bar{\kappa}|}{\kappa} + \frac{\sigma^2 t}{2})\Phi(d_1(t)) + \sigma\sqrt{t}\phi(d_1(t))\right\}\right. \\
&\qquad\qquad \left. + \bar{\kappa}^+ - \frac{\kappa+|\bar{\kappa}|}{2}\Phi(d_2(t))\right] dt
\end{aligned}
$$

where

$$
\begin{aligned}
g^{\tau^y}(t) \stackrel{\text{def}}{=} \frac{d}{dt}P[\tau^y \leq t] &= \frac{\log(1+|\bar{y}|)}{\sigma\sqrt{t^3}}\phi\left(-\frac{\log(1+|\bar{y}|)}{\sigma\sqrt{t}} + \frac{1}{2}\sigma\sqrt{t}\right) \\
&= -\frac{f}{\sqrt{t^3}}\phi\left(\frac{f}{\sqrt{t}} + \frac{1}{2}\sigma\sqrt{t}\right).
\end{aligned}
$$

Since

$$
\begin{aligned}
\int_0^T g^{\tau^y}(T-t)dt &= \Phi\left(\frac{f}{\sqrt{T}} + \frac{\sigma}{2}\sqrt{T}\right) + (1+|\bar{y}|)\Phi\left(\frac{f}{\sqrt{T}} - \frac{\sigma}{2}\sqrt{T}\right) \\
\int_0^T g^{\tau^y}(T-t)\Phi(d_1(t))dt &= \Phi\left(d_1(T) + \frac{f}{\sqrt{T}}\right) \\
\int_0^T g^{\tau^y}(T-t)\Phi(d_2(t))dt &= (1+|\bar{y}|)\Phi\left(d_2(T) + \frac{f}{\sqrt{T}}\right),
\end{aligned}
$$

we have

$$
\begin{aligned}
&\tilde{E}\left[(Y_T^{\sharp,y})^+, T \geq \tau^y\right] \\
&\quad = \bar{\kappa}^+\left[\Phi\left(\frac{f}{\sqrt{T}} + \frac{\sigma}{2}\sqrt{T}\right) + (1+|\bar{y}|)\Phi\left(\frac{f}{\sqrt{T}} - \frac{\sigma}{2}\sqrt{T}\right)\right] \\
&\quad\quad + \frac{\kappa}{2}\left(1 - \log\frac{\kappa+|\bar{\kappa}|}{\kappa} + \frac{\sigma^2 t}{2}\right)\Phi\left(d_1(T) + \frac{f}{\sqrt{T}}\right)
\end{aligned}
$$

$$-\frac{\kappa}{2}\frac{\kappa+|\bar{\kappa}|}{2}(1+|\bar{y}|)\Phi\left(d_2(T)+\frac{f}{\sqrt{T}}\right)$$
$$+\frac{\kappa}{2}\int_0^T g^{\tau^y}(T-t)\left\{\frac{\sigma^2 t}{2}\Phi(d_1(t))+\sigma\sqrt{t}\phi(d_1(t))\right\}dt. \square$$

6. Remarks and Numerical Calculations

We examined the functions $r^*(t)$ and $\eta(r^*(t),t)$ with $\sigma = 1$ by numerical calculations based on Equation (3.8) and the definition of $r^*(t)$ in Proposition 3.10. Figure 1 depicts the function $r^*(t)$ and its approximation functions $r_0^*(t) = c + \frac{c(1-\frac{3}{2}c)}{\sqrt{1-c}}\sqrt{t} - c\left\{\frac{c^2(1-2c)}{16(1-c)^{5/2}} + \frac{131c^3-270c^2+168c-32}{48(1-c)^2}\right\}t$ and $r_\infty^*(t) = 1 - \frac{\log(2\pi)}{2(t+1)}$ which are examined in Theorem 3.14 and 3.15. Figures 2 and 3 show these approximation errors respectively. In Figure 4, we illustrate the function $\eta(r^*(t),t)$ with it's approximation function $\eta_0(r^*(t),t) = \sqrt{t(1-r^*(t))} + \frac{r^*(t)}{4}t + \left\{\frac{r^*(t)^2}{32\sqrt{1-r^*(t)}} - \right.$

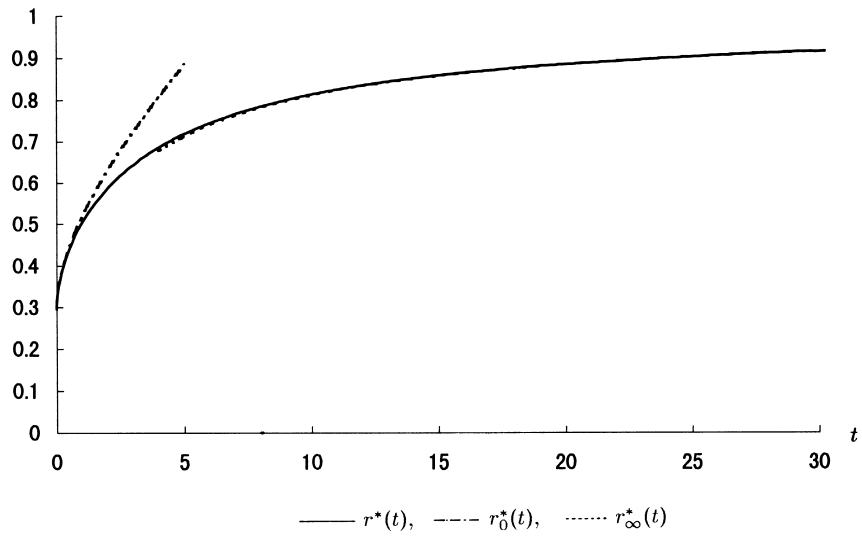

Figure 1.

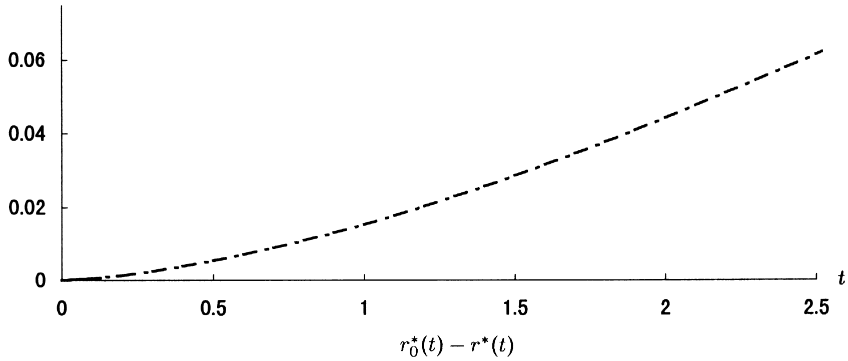

Figure 2.

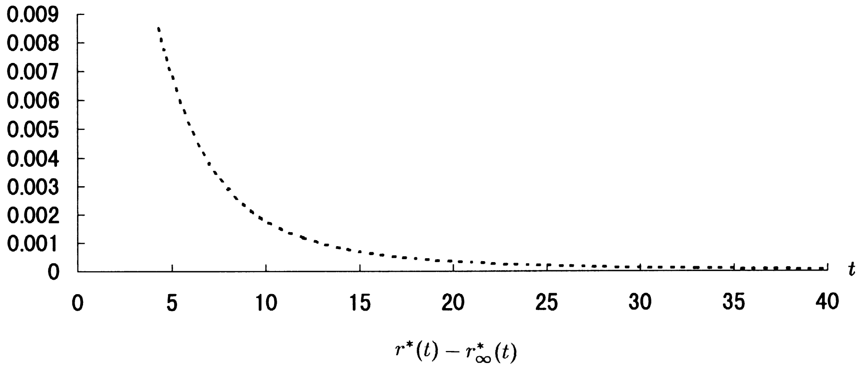

Figure 3.

$\frac{r^*(t)}{8}\sqrt{1-r^*(t)} - \frac{1}{6}(1-r^*(t))^{3/2}\Big\}t\sqrt{t}$ which are examined in Lemma 3.8. Figure 5 shows approximation error.

As numerical experiences we also examined the area where the optimal trading strategy is -1 in the discrete time framework discussed in Proposition 4.1, when $\kappa_1 = -1$, $\kappa_2 = 1$, and $\sigma = 1$. We calculate the value of ψ^N by using numerical integration technique inductively. Figures 6 and 7 show the case of $r = 0.35(> c)$ and $r = 0.25(< c)$ respectively. The symbol "$\bullet$" in these figures indicates the points where the optimal strategy is

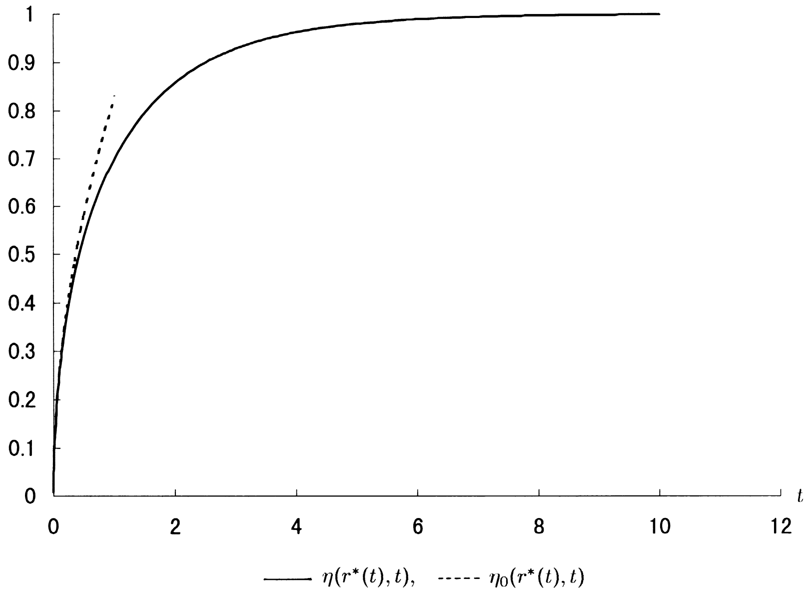

Figure 4.

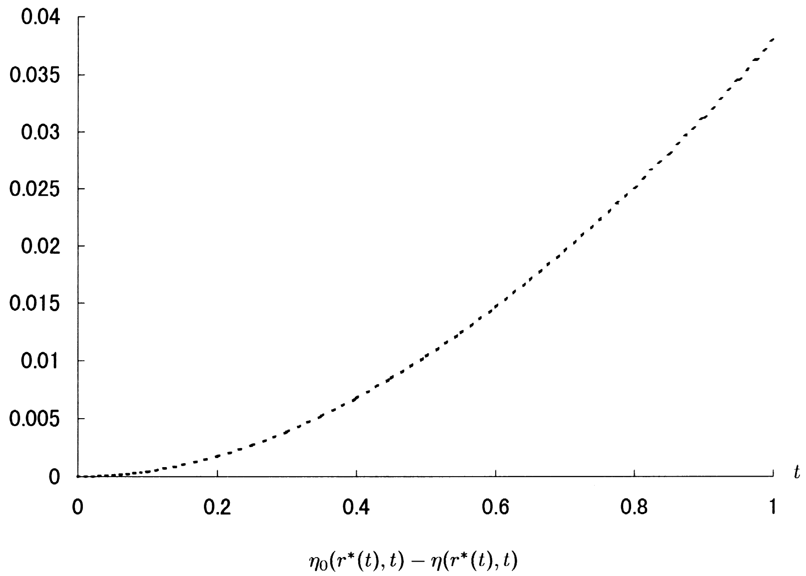

Figure 5.

Figure 6.

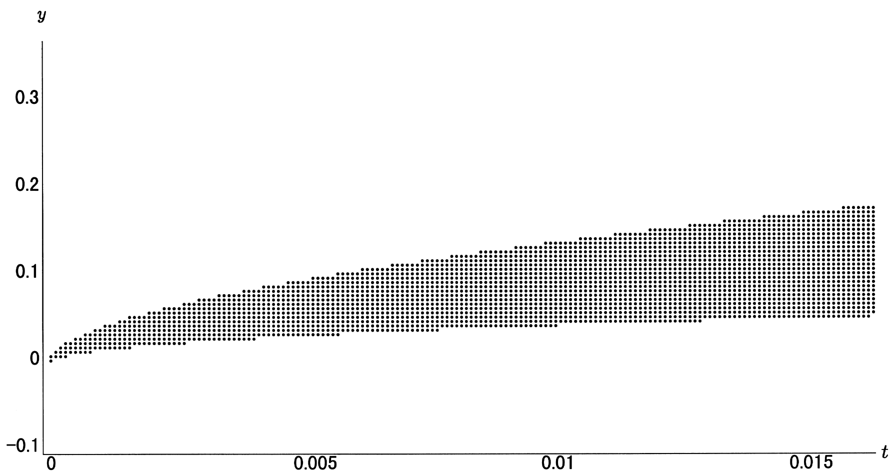

Figure 7.

-1, in the coordinate (y, t) where y is the value of trading strategy divided by the current value of the security, and t is the remaining life time of passport option. By numerical calculation we have $r^*(0.075) \approx 0.35$ and $\eta(0.35, 0.075) \approx 0.225$. The symbol "$\times$" in Figure 6 indicates the point where $y = 0.225$ and $t = 0.075$.

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(Received May 21, 1998)
(Revised July 21, 1998)

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