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Lifts of Analytic Discs from X to T^*X

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Abstract. We state a general criterion for existence of analytic discs attached to conormal bundles of CR manifolds. In particular let S be a CR (non–generic) submanifold of $X = \mathbb{C}^n$ and E^* a CR subbundle of the complex conormal bundle $T^*_S X \cap \sqrt{-1}T^*_S X$ such that $E^* + \sqrt{-1}E^* = T^*_S X \cap \sqrt{-1}T^*_S X$ (where sum and multiplication by $\sqrt{-1}$ are understood in the sense of the fibers). We then show that for any small disc A attached to S through z_o , and for any point $p_o \in (E^*)_{z_o}$, there is an analytic lift A^* attached to E^* through p_o. In particular we regain the theorem by Trepreau and Tumanov [T 3] on existence of lifts for discs attached to non–minimal manifolds. Our criterion also applies to discs attached to manifolds with a constant number of negative Levi–eigenvalues. We finally state the uniqueness of small discs attached to (non–necessarily CR) manifolds M through a given point z_o and with prescribed components in $T^{\mathbb{C}}_{z_o}M$. This is a slight, but perhaps interesting, generalization of the classical result (often used all through this paper), on uniqueness of lifts of small discs attached to generic manifolds.

§1. Sufficient Conditions for Lifting Analytic Discs

Let X be a complex manifold of dimension n, M a real C^2 submanifold of codimension l, z_o a point of M. We assume that M is *generic* at z_o in the sense that it is defined by l real equations $r^j = 0$ $j = 1, \dots, l$ whose differentials $\partial r^j(z_o)$ $j = 1, \dots, l$ are $\mathbb{C}$-independent. It is easy to see that we can then choose complex coordinates $z = (z', z'')$, $z = x + \sqrt{-1}y$ in $\mathbb{C}^n = \mathbb{C}^l \times C^{n-l}$ such that $\partial r^j = dx'_j$ $j = 1, \dots, l$. Also, by the implicit function theorem, the equations of M can be put in the form:

$$(1.1) \qquad (r^j =) x_j - g_j(y', z'', \bar{z}'') = 0 \quad \text{with} \quad \mathrm{d}g_j(0,0,0) = 0 \, \forall j = 1, \dots, l.$$

We shall denote in the sequel by r the column vector ${}^t(r^1, \dots, r^l)$.

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Let T^*X be the cotangent bundle to X, $\dot{T}^*X$ the bundle T^*X with the 0–section removed, T^*_MX the conormal bundle to M in X, p_o a point of $\dot{T}^*_MX$ with $\pi_M(p_o) = z_o$. Recall that T^*_MX is an $\mathbb{R}$–Lagrangian submanifold of T^*X i.e. Lagrangian for $\sigma^{\mathbb{R}}$ the real part of the symplectic 2–form σ. T^*_MX is said to be $\mathbb{I}$ symplectic when $\sigma^{\mathbb{I}}$, the imaginary part of σ, is non–degenerate over T^*_MX. This is equivalent to the fact that the *Levi form* of M (cf. the definition which follows Lemma 1.2) is non–degenerate. (Either of the above conditions turn out to coincide with the assumption that T^*_MX is *totally real*.)

Let π (resp. π_M) denote the projection $T^*X \to X$ (resp. $T^*_MX \to M$). Let $\mathbb{C} \otimes_{\mathbb{R}} T^*_MX = T^*_MX \oplus \sqrt{-1}T^*_MX$ be the complexification of T^*_MX in the (totally real) fibers.

We shall consider an analytic disc in X attached to M through z_o, that is an analytic mapping $A = A(\tau)$ from the unit disc $\Delta \subset \mathbb{C}$ into X, C^1 up to the boundary, with $A(\partial\Delta) \subset M$ and $A(1) = z_o$. We shall also consider a *lift* A^* of A to $\dot{T}^*X$ attached to T^*_MX through p_o (i.e. an analytic section of $\dot{T}^*X$ over A with $A^*(\partial\Delta) \subset T^*_MX$ and $A^*(1) = p_o$). Special attention will be voted to the A^*'s attached to T^*_MX and contained in $\mathbb{C} \otimes_{\mathbb{R}} T^*_MX$ i.e. in the form $A^* = (A; \sum_{i=1}^{l} \theta^i \partial r^i \circ A)$ with $A(\Delta) \subset M$ and $(\theta^i)|_{\partial\Delta} \in \mathbb{R}^l$. In the sequel we shall denote by A (or A^*) both the disc $A(\Delta)$ (or $A^*(\Delta)$) and its parametrization $A(\tau)$ (or $A^*(\tau)$).

Recall (1.1) and observe that it implies $\partial_{z'} r(z_o) = \frac{1}{2}\mathrm{id}_{l\times l}$. It follows that $\partial_{z'} r|_A$ will be close to $\frac{1}{2}\mathrm{id}_{l\times l}$ if A is small. In this situation we may find an $l \times l$ invertible matrix $\lambda(\tau) = (\lambda_{ij})(\tau)$ such that

$$\text{(1.2)} \qquad \begin{cases} \lambda|_{\partial A} \text{ real }, \lambda(z_o) = \frac{1}{2}\mathrm{id}_{l\times l} \\ \lambda\partial_{z'} r \text{ extends holomorphically to } A. \end{cases}$$

In fact the solution λ to (1.2) can be found by solving in the Hölder spaces $C^\alpha(\partial A, \mathbb{R}^{l\times l})$ $\alpha < 1$, the functional system

$$\text{(1.3)} \qquad (F(\lambda \circ A) =) \lambda \Im\mathrm{m}\partial_{z'} r \circ A - T_1(\lambda \Re\mathrm{e}\partial_{z'} r \circ A) = 0 \quad \text{in } \partial\Delta,$$

where T_1 is the Hilbert transform normalized by the condition $T_1(u)|_{\tau=1} = 0$ for $u \in C^\alpha(\partial\Delta, \mathbb{R}^{l\times l})$. Since F is linear in λ and close to $-T_1(\lambda \circ A)$ which is invertible, then the implicit function theorem for $F : C^\alpha(\partial\Delta, \mathbb{R}^{l\times l}) \to C^\alpha(\partial\Delta, \mathbb{R}^{l\times l})$ provides the conclusion.

Note that since $\partial_{z'} r$ is invertible, then the extension is still in the form $\lambda \partial_{z'} r$ for an extension of λ from ∂A to A that we still denote by λ. Let $\theta = (\theta_1, \ldots, \theta_l)$ be a real valued function on ∂A such that $\theta \cdot \partial_{z'} r$ extends holomorphically to A. Then

$$\theta \lambda^{-1} = (\theta \cdot \partial_{z'} r)(\lambda \partial_{z'} r)^{-1} \text{ is holomorphic in } A.$$

Since

$$\theta \lambda^{-1}|_{\partial A} \text{ is real,}$$

then

$$\theta \lambda^{-1} \text{ is constant.}$$

Finally, from $\lambda^{-1}(z_o) = 2\mathrm{id}_{l \times l}$, we get

$$\theta(A(\tau)) = \theta(z_o) 2\lambda(A(\tau)) = \theta_o 2\lambda(A(\tau)).$$

Let λ be the (extension to $\overline{\Delta}$ of the) $l \times l$ matrix which satisfies (1.2).

LEMMA 1.1. *Let A be a small disc with $A(1) = z_o$, and assume, for some θ_o and for $\theta = \theta_o 2\lambda$:*

$$\partial_\tau A(\tau) \in \mathrm{Ker}\bar{\partial}(\theta \partial r)(z)|_{\{u | <\partial r, u> = 0\}} \ \forall \tau \in \Delta \text{ and with } z = A(\tau). \tag{1.4}$$

Then

$$A^* = (A; \theta \partial r) \text{ is holomorphic.} \tag{1.5}$$

PROOF. Fix an $(n-l) \times l$ matrix U'' and solve:

$$\partial_{z'} r U' = \partial_{z''} r U'', \tag{1.6}$$

in the unknown $l \times l$ matrix U'. Choose coordinates such that A is a disc in the z_n–plane. By assumption, for any U', U'' verifying (1.6), we must have by (1.4)

$$\partial_{\bar{z}_n}(\theta \partial_{z'} r) U' = \partial_{\bar{z}_n}(\theta \partial_{z''} r) U''.$$

Since $\lambda\partial_{z'}r$ is holomorphic and $\theta = \theta_0 2\lambda$, then $\partial_{\bar{z}_n}\theta\partial_{z'}r = 0$ and therefore

$$\partial_{\bar{z}_n}(\theta\partial_{z''}r)U'' = 0.$$

Since this holds for any U'', it follows that

$$\theta\partial_{z''}r|_A \text{ is holomorphic. } \square$$

Clearly (1.4) is also a necessary condition for $A^* = (A; \theta\partial r)$ to be analytic.

Similarly to Lemma 1.1, we can prove:

Lemma 1.2. *Suppose*

$$\partial_\tau A(\tau) \in \mathrm{Ker}\bar{\partial}\partial r_j(z)|_{\{u|<\partial r,u>=0\}}\ \forall j, \forall\tau\in\Delta \text{ and with } z = A(\tau). \tag{1.7}$$

Then for any $p_o \in \dot{T}^*_M X_{z_o}$, $p_o = \theta_o\partial r(z_o)$, there exists a holomorphic lift $A^* = (A; \theta\partial r)$ with $\theta(z_o) = \theta_o$.

Proof. Let λ be the complex invertible $l\times l$ matrix of above. By (1.7):

$$\partial(\theta_o 2\lambda r)|_A \text{ is holomorphic for any } \theta_o. \ \square$$

Denote by $T^{\mathbb{C}}M = TM \cap \sqrt{-1}TM$ the complex tangent bundle to M. Let $\theta\partial r$ be a section of $T^*_M X$; we have for any $z_o \in M$:

$$\theta(z_o)\bar{\partial}\partial r(z_o)|_{T^{\mathbb{C}}_{z_o}M} = \bar{\partial}(\theta\partial r)(z_o)|_{T^{\mathbb{C}}_{z_o}M}.$$

Thus the above form only depends on $p_o = (z_o; \theta(z_o)\partial r(z_o))$ and not on the choice of the section $\theta\partial r$ in $T^*_M X$ through p_o; we shall denote it by $L_M(p_o)$. We shall also use the notation $\theta\bar{\partial}\partial r = \sum \theta_j\bar{\partial}\partial r^j$.

Proposition 1.3. (i) *Let A be an analytic disc attached to M, λ a solution of (1.2), set $\theta = \theta_o 2\lambda$, $A^* = (A; \theta\partial r)$, and suppose*

$$\partial_\tau A(\tau) \in \mathrm{Ker}\,\theta\bar{\partial}\partial r(z)|_{\{w;<w,\partial r>=0\}}\ \forall\tau\in\Delta \text{ and with } z = A(\tau). \tag{1.8}$$

Then A^* is analytic.

(ii) *In particular let A be contained in M and assume (with $A^* = (A; \theta\partial r)$):*

$$\partial_\tau A(\tau) \subset \mathrm{Ker} L_M(A^*(\tau)) \quad \forall \tau \in \Delta. \tag{1.9}$$

Then A^* is analytic. (Note that in this case A^* is a lift (attached to $T^*_M X$ and) contained in $\mathbb{C} \otimes_{\mathbb{R}} T^*_M X$.

Proof. (i): Immediate consequence of Lemma 1.1.

(ii): Since $A \subset M$, then $\forall z \in A$ we have $\{u :< \partial r(z), u >= 0\} = T^{\mathbb{C}}_z M$, and moreover $\bar\partial(\theta\partial r)(z)|_{T^{\mathbb{C}}_z M} = L_M(z; \theta\partial r(z))(= L_M(A^*(z))$. □

Remark 1.4. We can similarly prove (i) (resp. (ii)) $\forall \theta_o$ above by requiring

$$\begin{aligned} &\partial_\tau A(\tau) \in \mathrm{Ker}\bar\partial\partial r^j(z)|_{\{w;<w,\partial r>=0\}} \\ &\qquad (\text{resp. } \partial_\tau A(\tau) \in \mathrm{Ker} L_M(z; \partial r^j(z))) \quad \forall z \in A,\ \forall j. \end{aligned} \tag{1.10}$$

Remark 1.5. In the situation of Proposition 1.3, there cannot be analytic lifts of A attached to $T^*_M X$ through $p_o = \theta_o \partial r(z_o)$ other than $A^* = (A; \theta\partial r)$ (cf. subsequent §3).

Let S be now a CR submanifold of X i.e. such that $T^{\mathbb{C}}M$ has constant rank, and define

$$\delta = \dim(T^*_S X \cap \sqrt{-1} T^*_S X).$$

We shall assume $\delta > 0$ i.e. S non–generic. The following proposition deals with the problem of finding δ *quasi–analytic* and *Levi–flat* equations for S.

Proposition 1.6. *There are δ complex functions $h_1, \ldots, h_\delta$ vanishing on S such that ∂h_j are $\mathbb{C}$–independent, and verify $\bar\partial h_j|_S = 0$ and $\bar\partial\partial h_j|_S \equiv 0\, \forall j$. Moreover on any small disc A attached to S the differentials ∂h_j have holomorphic extensions $\widetilde{\partial h_j}$.*

Proof. We set $T_{z_o} S + \sqrt{-1} T_{z_o} S = \mathbb{C}^m, (m = n - \delta)$ with coordinate z'. Then the projection $\tau : z \mapsto z'$ induces a bijection between S and τS. Moreover $g \overset{\text{def.}}{:=} (\tau|_S)^{-1}$ being CR on τS, it is extended to a function that

we still denote $g = (g_j)$ on $\mathbb{C}^m$ such that $\bar\partial g$ vanishes together with its derivatives on τS. If we then set $h_j = z_j - g_j$, $j = n-\delta+1, \dots, n$, we get the desired equations.

As for the second claim we notice that the ∂h_j's being CR on S, they are approximated by polynomials, and hence extended to A by the maximum principle (cf. e.g. [B]). □

THEOREM 1.7. *Let S be CR, and let E^* be a CR subbundle of $T^*_S X \cap \sqrt{-1}T^*_S X$, such that*

$$E^* \oplus \sqrt{-1}E^* = T^*_S X \cap \sqrt{-1}T^*_S X. \tag{1.11}$$

Then for any small disc A attached to S with $A(1) = z_o$, and for any $p_o \in E^*_{z_o}$ there is an unique analytic lift A^* attached to E^* with $A^*(1) = p_o$.

PROOF. According to Proposition 1.6 there is a $\mathbb{C}$ basis $\{\partial h_1, \dots, \partial h_\delta\}$ for $T^*_S X \cap \sqrt{-1}T^*_S X$ such that each ∂h_j extends holomorphically to A. Let d denote the real dimension of E^* (in the fibers). Let $\{v^j\}_j = \{\sum_k a_{jk}\partial h_k\}_{j=1,\dots,d}$ verify:

$$E^* = \mathrm{Span}_{\mathbb{R}}\{v^1, \dots, v^d\},$$

and

$$p_o = \sum_{j=1}^{d} \theta^o_j v^j, \quad \theta^o \in \mathbb{R}^d.$$

Put $V = \begin{pmatrix} v^1 \\ \vdots \\ v^d \end{pmatrix} = (V', V'')$ where V' (resp. V'') are the first d (resp. the last $n-d$) column in V. We can suppose that coordinates in $\mathbb{C}^n$ are so chosen that $V'(z_0) = id_{d\times d}$ and $V''(z_0) = 0$. For any function θ with values in $\mathbb{R}^d$ we have

$$< \bar\partial_u(\theta V), w >= 0 \quad \text{if } w \in \mathrm{Span}\{\partial h_j\}^\perp.$$

On the other hand our assumption (1.11) is equivalent to $E^{*\perp} = \{\mathrm{Span}\partial h_j\}^\perp$. It follows $\forall u \in \mathbb{C}^n$:

$$< \bar\partial\theta V, w >= 0 \quad \text{if } Vw = 0. \tag{1.12}$$

According to the remarks which precede Lemma 1.1, there is a function λ on A, real on ∂A, with $\lambda(z_0) = \frac{1}{2}id$ and such that $\lambda V'$ is holomorphic. Hence for $\theta = \theta_0 2\lambda$ and $u = \partial_\tau A$, we have $\bar\partial_u \theta V' = 0$. Together with (1.12) this gives $\bar\partial_u \theta V'' = 0$ and finally $\bar\partial_u \theta V = 0$. Hence $A^* = (A; \theta V \circ A)$ is the lift of A through $p_0 = \theta_0 V(z_0) \in E^*_{z_0}$. Finally notice that, since E^* has totally real fibers, then the lift of A through any prescribed point of $E^*_{z_o}, (z_o = A(1))$, exists and is unique (cf. §3). □

Corollary 1.8. *Let M be generic, assume there exists $S \subset M$ such that $T^{\mathbb{C}}M|_S = T^{\mathbb{C}}S$, and define $E^* = T^*_M X|_S \cap \sqrt{-1}T^*_S X$. Then for any A attached to S with $A(1) = z_o$ there exists an unique lift A^* attached to E^* through each point $p_o \in E^*_{z_o}$.*

Proof. If $T^{\mathbb{C}}M|_S = T^{\mathbb{C}}S$ then $T_S M \simeq \frac{TX|_S}{TM|S + \sqrt{-1}TS}$. Hence $E^* \simeq T^*_S M$ has constant rank ($= \mathrm{cod}_M S$). It remains to prove that (1.11) is fullfilled. In fact

$$T^{\mathbb{C}}M|_S = T^{\mathbb{C}}S$$

if and only if

$$T^*_S X \subset T^*_M X|_S + \sqrt{-1}T^*_M X|_S$$

only if

$$T^*_S X \cap \sqrt{-1}T^*_S X \subset (T^*_M X|_S \cap \sqrt{-1}T^*_S X) + \sqrt{-1}T^*_M X|_S$$

if and only if

$$T^*_S X \cap \sqrt{-1}T^*_S X \subset (E^* + \sqrt{-1}T^*_M X|_S) \cap (\sqrt{-1}E^* + T^*_M X|_S)$$

if and only if

$$T^*_S X \cap \sqrt{-1}T^*_S X \subset E^* + \sqrt{-1}E^*.$$

The converse of this inclusion is trivial and the right side is in fact a direct sum (since E^* is totally real in fibers). Thus we get (1.12). □

We are now in a position to explain the link with the results by Tumanov [T 3] and Trepreau [Tr]. We start from S CR (non–generic), consider a set of $\delta(= dim(T^*_S X_{\,z_o} \cap \sqrt{-1}T^*_S X_{\,z_o}))$ quasi–holomorphic and Levi–flat equations $h_j = 0\, j = 1, \ldots, \delta$ for S (i.e. satisfying $\bar\partial h_j|_S = 0$ and $\bar\partial\partial h_j|_S = 0$) and

complete by a set of real equations $r^j = 0\, j = 2\delta+1, \dots, l$. Let $p_o \in \dot{T}^*_S X \cap \sqrt{-1}\dot{T}^*_S X$ with $\pi(p_o) = z_o$, and assume $p_o \in \mathrm{Span}_{\mathbb{R}}\{\partial\Re e h_1, \dots, \partial\Re e h_\delta\}$ (otherwise we just need to take a complex combination of the h_j's). Define $M = \{z; \Re e h_1 = 0, \dots, \Re e h_\delta = 0, r^{2\delta+1} = 0, \dots, r^l = 0\}$. Now M is generic and, since $\partial\Re e h_j|_S = \sqrt{-1}\partial\Im m h_j|_S$ (due to $\bar\partial h_j|_S = 0$), then

$$T^{\mathbb{C}} M|_S = T^{\mathbb{C}} S. \tag{1.13}$$

In this case small analytic discs attached to M (and then in fact to S), through z_o, can be *lifted* to discs attached to $E^* := T^*_M X|_S \cap \sqrt{-1}T^*_S X$ through p_o according to [T 3], [Tr]. (This is by the way the main tool in the proof of propagation of CR–extandibility.) We point out that in [T 3], [Tr] what is given in the beginning is M (generic) with the assumption of the existence of such a $S \subset M$ which verifies (1.13). (This property is called *non–minimality* of M.) Moreover the lift A^* is obtained by showing that $E^* = T^*_M X|_S \cap \sqrt{-1}T^*_S X$ is a CR subbundle of T^*X (with the projection $TT^*X \overset{\pi'}{\to} TX$ inducing an isomorphism $T^{\mathbb{C}}E^* \overset{\pi'}{\tilde{\to}} T^{\mathbb{C}}S$) and then by attaching A^* to E^* whose CR components are the $T^{\mathbb{C}}S$–components of A via Bishop equation. On the contrary we start from S and attach discs to any subbundle $E^* \subset T^*_S X \cap \sqrt{-1}T^*_S X$ satisfying the assumptions of Theorem 1.7. We point out that if (1.11) is fullfilled and even with $E^* \cap \sqrt{-1}E^* \neq \{0\}$ we should still prove that E^* is CR (with $T^{\mathbb{C}}_{p_o}E^* \simeq T^{\mathbb{C}}_{z_o}S \oplus (E^*_{z_o} \cap \sqrt{-1}E^*_{z_o})$). However our construction of A^*, defined as a combination of differentials of Levi–flat equations of S instead of solutions of Bishop equation, is much more explicit. For instance it shows that when $A \subset S$, then $A^* \subset \mathbb{C} \otimes_{\mathbb{R}} T^*_M X|_S$. It also shows that the "size " of the discs which can be "lifted" is only subjected to the condition of finding λ such that $\lambda V'$ is holomorphic (which leads to a simple Bishop equation). This could not follow from attaching discs to the CR manifold E^*: First because E^* is never generic in T^*X (and hence the celebrated approximation argument by Baouendi–Treves is needed). And second because the Bishop's equation is in this case more complicated.

§2. Discs Attached and Discs Contained in Real Submanifolds

Let M be a generic real submanifold of codimension l in a complex manifold X of dimension n. Let $\pi : T^*X \to X$ (resp. $\pi_M : T^*_M X \to M$) be

the cotangent bundle to X (resp. the conormal bundle to M in X). Let $z_o \in M$, $p_o \in \dot{T}^*_M X (= T^*_M X \setminus \{0\})$ with $\pi(p_o) = z_o$, and recall $L_M(p_o)$ the Levi form of M at z_o in the direction p_o. If $\theta_o \partial r(z_o) = p_o$, this is defined as the Hermitian form $L_M(p_o) := \theta_o \bar{\partial}\partial r(z_o)|_{T^{\mathbb{C}}_{z_o} M}$ where $T^{\mathbb{C}}M = TM \cap \sqrt{-1}TM$ is the complex tangent plane to M. Let $s^{+,-,0}_M(p_o)$ be the numbers of respectively positive, negative, and null eigenvalues of $L_M(p_o)$. We shall consider analytic discs A (resp. A^*), C^1 up to the boundary, attached to M ($T^*_M X$), through z_o (p_o). We want to discuss some particular situations where A and A^* are contained in either M or $T^*_M X$. Let us choose complex coordinates $z \in X \simeq \mathbb{C}^n$, such that M is defined by $r^j = 0\, j = 1, \ldots, l$ with r^j of type (1.1). Let $A = A(\tau)$ be a small analytic disc attached to M with $A(1) = z_o$.

PROPOSITION 2.1. *Assume there exists a lift A^* through p_o attached to $T^*_M X$. Then*

$$< p_o, \partial_\tau A > |_{\tau=1} = 0. \tag{2.1}$$

(In particular A is tangent to M at z_o when M has codimension 1.*)*

PROOF. Denote by $\tau = te^{\sqrt{-1}\phi}$ the variable in Δ, and denote by $f \circ A$ the second components of the disc A^* (thus $A^* = (A, f \circ A)$). Then

$$\begin{aligned} \Im\mathrm{m} < f \circ A, \tau\partial_\tau A > |_{\partial\Delta} &= -\Re\mathrm{e} < f \circ A, \partial_\phi A > \\ &= 0, \end{aligned} \tag{2.2}$$

where the first equality follows from Cauchy–Riemann relations, and the second from the fact that $\partial A \subset M$ and $f|_{\partial A}$ is orthogonal to TM. □

In particular the Proposition applies to the case when for $\theta = \theta_o 2\lambda\partial r$ with $\theta_o \partial r(z_o) = p_o$, we have $\partial_\tau A \in \mathrm{Ker}\bar{\partial}(\theta\partial r)|_{\{u|<\partial r, u>=0\}}$. (Here $\lambda = \lambda(\tau)$ is the invertible $l \times l$ matrix which satisfies (1.2).) Let $L_M(p)$, $p \in \dot{T}^*_M X$ denote the Levi–form of M with respect to p; (if $\theta r|_M \equiv 0$, and $\theta\partial r(z_o) = p$, this is defined as $\theta\bar{\partial}\partial r(z_o)|_{T^{\mathbb{C}}_{z_o} M}$).

PROPOSITION 2.2. (Cf. [Z 3].) *Let A be small and have a lift A^* attached to $T^*_M X$ with $A^*(1) = p_o$. Assume in addition*

$$s^-_M(p) \equiv \text{const} \quad \forall p \in \dot{T}^*_M X, \text{close to } p_o. \tag{2.3}$$

Then A is contained in M and A^* in $\mathbb{C}T^*_M X$.

Proof. We shall follow the lines of [Z 3]. We can find a complex symplectic homogeneous transformation χ of $\dot{T}^*X$ from a neighborhood of p_o to one of $q_o = \chi(p_o)$ which interchanges

$$\chi(T^*_M X) = T^*_{\tilde{M}} X, \quad \operatorname{codim} \tilde{M} = 1,\ s^-_{\tilde{M}}(q_o) = 0. \tag{2.4}$$

(Cf. [Z 3, formula (1)].) Owing to (2.4) we also know that in fact $s^-_{\tilde{M}} \equiv 0$ i.e. $\tilde{M}$ is the boundary of a pseudoconvex domain. (In fact the constancy of the numbers s^-_M is invariant under symplectic complex transformation.) Then $\tilde{A} \overset{\text{def.}}{=} \pi\chi(A^*)$ is an analytic disc attached to a pseudoconvex hypersurface. The first consequence is that $\tilde{A}$ is in fact contained in the closed half space $\tilde{M}^+$ with boundary M and outward conormal q_o (otherwise pseudoconvexity would be violated). In fact let $\tilde{r}$ be a plurisubharmonic bounded exhaustion function for the interior of $\tilde{A}^+$ as in [D–F]. We have

$$\tilde{r} \circ \tilde{A}|_\Delta \leq 0 \quad \tilde{r} \circ \tilde{A}(1) = 0.$$

If then $\tilde{r} \circ \tilde{A} < 0$ at some $\tau \in \Delta$, then Hopf Lemma would imply (with the notation $\tau = te^{\sqrt{-1}\phi}$) $< \partial\tilde{r}(\tilde{A}), \partial_t \tilde{A} > |_{\tau=1} < 0$ whence $< q_o, \partial_\tau \tilde{A} > |_{\tau=1} \neq 0$. But this is impossible because $\tilde{A}$ has a lift (namely $\chi(A^*)$) through q_o and thus Proposition 2.1 applies. Therefore $\tilde{A} \subset \tilde{M}$.

We observe now that we have

$$\partial_\tau \tilde{A} \in \operatorname{Ker} L_{\tilde{M}}(\tilde{A}). \tag{2.5}$$

In fact, let $u = \partial_\tau \tilde{A}$. By applying $\bar{\partial}\partial$ to the identity $\tilde{r} \circ \tilde{A} \equiv 0$, we get $\bar{\partial}\partial\tilde{r}(\bar{u}, u) = 0$. But, $\bar{\partial}\partial\tilde{r}|_{T^{\mathbb{C}}\tilde{M}}$ being semi–definite, this immediately implies (2.5). We can apply now the results of §1 and conclude that there exists a lift $\tilde{A}^* \subset \mathbb{C} \otimes_{\mathbb{R}} T^*_{\tilde{M}} X (= \mathbb{C}T^*_{\tilde{M}} X)$ (where the last equality follows from the fact that $\tilde{M}$ is a hypersurface). By the uniqueness of the (small) lift (cf. subsequent §3), we have in fact $\tilde{A}^* = \chi(A^*)$. Thus $\chi(A^*) \subset \mathbb{C}T^*_{\tilde{M}} X$ whence $A^* \subset \mathbb{C}T^*_M X$, and in particular $A \subset M$. □

It is possible to prove (cf. [Z 3, Th. 2]) that A^* is in fact contained in $T^*_M X$.

§3. Uniqueness of Attached Discs

Let M be a C^2–submanifold of codimension l in a complex manifold X at a point z_o. We do not necessarily assume that M is generic and not even CR. We identify in complex coordinates $X \simeq T_{z_o}X \simeq \mathbb{C}^n$, $T^{\mathbb{C}}_{z_o}M \simeq \mathbb{C}^m$ and choose a complex projection g:

$$\begin{array}{ccccc} M & & \hookrightarrow & X & \\ \downarrow & g|_M & & \downarrow & g \\ T^{\mathbb{C}}_{z_o}M & & \simeq & \mathbb{C}^m & . \end{array} \tag{3.1}$$

Proposition 3.1. *The small analytic disc A attached to M with $A(1) = z_o$ and with prescribed $T^{\mathbb{C}}_{z_o}M$–components $g \circ A$ is unique if it exists.*

Proof. Let $l' :\overset{\text{def.}}{=} \dim M - 2m$; (thus $l' \leq l$ and $l' = l$ only in case M is generic). Let $\mathrm{id}_{m\times m} \times f$:

$$\begin{array}{ccc} \mathbb{C}^m \times \mathbb{R}^{l'} & \to & M \\ (z, \rho) & \mapsto & (z, f(z, \rho)), \end{array}$$

be a parametrization of M such that $g|_M \circ (\mathrm{id} \times f)$ is the projection $\mathbb{C}^m \times \mathbb{R}^{l'} \to \mathbb{C}^m$. Since $g|_M$ has totally real fibers, then it is possible to choose l' components $f'' = (f_{m+1}, \ldots, f_{m+l'})$ of f such that

$$\partial_\rho f''(z_o, \rho_o) = \mathrm{id}_{l'\times l'}.$$

Let Δ be the unit disc of $\mathbb{C}$, and let C^α ($\alpha < 1$) be the Banach space of the α–Lipschitz–continuous functions endowed with the norm $||\cdot||_\alpha$. Consider the mapping F between Banach spaces:

$$\begin{array}{c} C^\alpha(\partial\Delta, \mathbb{C}^m) \times C^\alpha(\partial\Delta, \mathbb{R}^{l'}) \times \mathbb{R}^{l'} \to C^\alpha(\partial\Delta, \mathbb{C}^{n-m}) \\ (A, \rho, \rho_o) \mapsto \Re\mathrm{e} f''(A, \rho) + T_1 \Im\mathrm{m} f''(A, \rho) - \Re\mathrm{e} f''(A(1), \rho_o). \end{array}$$

Since f is C^2, then F is C^1 and

$$F(A, \rho, \rho_o) = 0 \text{ iff } \begin{cases} f''(A, \rho) \text{extends holomorphically to } \Delta \\ \Re\mathrm{e} f''(A, \rho)(1) = \Re\mathrm{e} f''(A(1), \rho_o) \text{ i.e. } \rho(1) = \rho_o. \end{cases} .$$

Let A_o denote the constant disc $\{z_o\}$. We have $F(A_o, \rho_o, \rho_o) = 0$ and:

$$\frac{\partial F}{\partial \rho}(A_o, \rho_o, \rho_o) = \Re\mathrm{e}\partial_\rho f''(A_o, \rho_o) + T_1 \Im\mathrm{m}\partial_\rho f''(A_o, \rho_o) = \mathrm{id}.$$

If then $A \in C^\alpha(\partial\Delta, \mathbb{C}^m)$ with $||A - A_o||_\alpha$ small, then, by the implicit function theorem, there exists an unique ρ such that $\rho(1) = \rho_o$ and $f''(A, \rho)$ extends holomorphically to Δ.

Thus let $\tilde{A}$ be a disc of $\mathbb{C}^n$ attached to M with $\tilde{A}(1) = z_o$ and $g(\tilde{A}) = A$. This means that $f(A, \rho)$ extends holomorphically. Thus in particular $f''(A, \rho)$ extends, and therefore ρ is unique. □

Let M (resp. M^*) be a submanifold of $\mathbb{C}^m$ (resp. $\mathbb{C}^n$ $(n > m)$), and let $g : \mathbb{C}^n \to \mathbb{C}^m$ be a complex projection which induces a submersion $g|_{M^*} : M^* \to M$. We thus have a commuting diagram:

$$\begin{array}{ccccc} M^* & & \hookrightarrow & \mathbb{C}^n & \\ \downarrow & g|_{M^*} & & \downarrow & g \\ M & & \hookrightarrow & \mathbb{C}^m & . \end{array}$$

Corollary 3.2. *Let $g|_{M^*}$ have totally real fibers. Then for any small disc A attached to M there exists at most one lift A^* attached to M^* through any fixed point of M^*.*

Proof. Fix $z^* \in M^*$ with $g(z^*) = z$. Then

$$T^{\mathbb{C}}_{z^*} g|^{-1}_{M^*}(z) = \{0\} \text{ if and only if } T^{\mathbb{C}}_{z^*} M^* \overset{g'}{\hookrightarrow} T^{\mathbb{C}}_z M \text{ is injective.}$$

Let us consider the projections:

$$\begin{array}{ccccccccc} \mathbb{C}^n & \overset{g}{\to} & \mathbb{C}^m & \overset{g_1}{\to} & \mathbb{C}^{m_1} & \overset{g_2}{\to} & \mathbb{C}^{m_2} \\ & & & & || & & || \\ & & & & T^{\mathbb{C}}_z M & & T^{\mathbb{C}}_{z^*} M^*, \end{array}$$

and define $g_3 = g_2 \circ g_1 \circ g : \mathbb{C}^n \to \mathbb{C}^{m_2} \simeq T^{\mathbb{C}}_{z_o} M^*$. Let A^*_1 and A^*_2 be two lifts of A. Then

$$g_3(A^*_1) = g_2 \circ g_1 \circ g(A^*_1) = g_2 \circ g_1(A) = g_2 \circ g_1 \circ g(A^*_2) = g_3(A^*_2).$$

If therefore A^*_1 and A^*_2 are attached to M^* and have a common point z^*, we must have $A^*_1 = A^*_2$ due to Proposition 3.1. □

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