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Confluence Procedures in the Generalized Hypergeometric Family

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Abstract. We work out two confluence procedures in the family of generalized hypergeometric differential equations allowing to compute the Stokes matrices of some confluent equations as limits of well chosen matrices attached to the associated regular equations.

1. Introduction

The family of generalized hypergeometric equations is given by

$$D_{q,p}(\underline{\alpha};\underline{\beta}) = (-1)^{q-p} z \prod_{j=1}^{p} (z\frac{d}{dz} + \alpha_j) - \prod_{j=1}^{q} (z\frac{d}{dz} + \beta_j - 1)$$

with $\underline{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_p) \in \mathbf{C}^p$ and $\underline{\beta} = (\beta_1, \beta_2, \cdots, \beta_q) \in \mathbf{C}^q$.
The variable z lies in the Riemann sphere $\mathbf{P}^1(\mathbf{C})$.
As the change of variable $z \mapsto 1/z$ transforms $D_{q,p}(\underline{\alpha}, \underline{\beta})$ in $D_{p,q}(\underline{\beta}', \underline{\alpha}')$ where $\underline{\beta}' = (1 - \beta_1, \cdots, 1 - \beta_q)$ and $\underline{\alpha}' = (1 - \alpha_1, \cdots, 1 - \alpha_p)$, we may suppose without loss of generality that $p \leq q$.
The equation $D_{q,p}$ then has

- three singular points, namely $0, 1, \infty$ if $p = q$. All of them are *regular* singularities.
- two singular points, 0 and ∞ if $p < q$. The point 0 is a regular singularity and the point ∞ is an irregular one.

The irregular singularity gives rise to a so-called Stokes phenomenon which has been explicitely described for these equations in [1]. In the same paper and in a subsequent paper from C. Mitschi alone, [6], it is shown how to use this result to compute, with the help of J.-P. Ramis' fondamental theorem (see [4]), the differential Galois group of some equations in the family.

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We are interested here in the case $q = p + 1$ with $p \geq 2$. This case is the "less irregular" one, in the sense that the Newton polygon of $D_{p+1,p}$ has two sides, one with slope 0 and length p and the other one with slope 1 and length 1. In this case there are, up to conjugacy by the formal monodromy matrix, two Stokes matrices, S_0 and S_π.

A confluence procedure is obtained by allowing two singularities to collapse and at the same time one parameter to "disappear". Both goals are achieved by making the change of variable $z = t/b$ in the equation $D_{p+1,p+1}(\underline{\alpha}; \underline{\beta})$ where one of the components of $\underline{\alpha}$ is $-b$ and then by making $b \to \infty$.
In a first procedure $b \to \infty$ in a *non real* direction. We show how the two Stokes matrices can be obtained as limit values of connection matrices linking well chosen fundamental sets of solutions around b and ∞ of the Fuchsian equation. We can get both Stokes matrices as limits of the same connection matrix but one with $b \to \infty$ in one of the half-plane of $\mathbf{C}$ delimited by the real line and the other matrix as $b \to \infty$ in the opposite half-plane. Up to conjugacy by the formal monodromy matrix for one of the Stokes matrices, both matrices can also be obtained with $b \to \infty$ in one and the same half-plane. As a by-product of this study one gets sectorial limits for some fundamental sets of solutions.
In a second procedure $b \to \infty$ in one of the two real directions. More precisely $b = b_0 + n$ with $n \in \mathbf{Z}$. Thus the *local monodromy* around each one of the three singularities is fixed. This time one gets the Stokes matrices, up to some convenient part of the formal monodromy, as limits $n \to +\infty$ (resp. $n \to -\infty$) of the monodromy matrices around b and around ∞ expressed in a "mixed" basis combining solutions which are eigenvectors of the local monodromy around b or around ∞.

These two procedures have been suggested by J.-P. Ramis in [8] where the first procedure is developped for the hypergeometric case ($p = 1$). The second procedure for $p = 1$ is studied by Zhang in [9].

The paper is organised as follows. The rest of this section is devoted to some notation used throughout this paper. We then recall for the Fuchsian equation and for the confluent one some more or less classical properties, putting them in a form convenient for our purpose. In the two final sections the confluence procedures explained above are successively studied.

- We denote by $\widetilde{\mathbf{C}}$ the universal covering of $\mathbf{C}^*$.
  For $z \in \widetilde{\mathbf{C}}$ and $\lambda \in \mathbf{C}$, z^λ means $e^{\lambda \ln z}$ where $\ln z = \ln |z| + i \arg z$.
  For $\theta \in \mathbf{R}$, $\arg z e^{i\theta} = \arg z + \theta$.

- For $\underline{\mu} = (\mu_1, \cdots, \mu_p) \in \mathbf{C}^p$ and $\lambda \in \mathbf{C}$,

$$
\begin{aligned}
&|\underline{\mu}| = \mu_1 + \cdots + \mu_p,\\
&\underline{\mu} + \lambda = \lambda + \underline{\mu} = (\mu_1 + \lambda, \cdots, \mu_p + \lambda),\\
&\lambda\underline{\mu} = (\lambda\mu_1, \cdots, \lambda\mu_p)\\
&\underline{\mu}_j = (\mu_1, \cdots, \mu_{j-1}, \mu_{j+1}, \cdots, \mu_p) \in \mathbf{C}^{p-1},\\
&\underline{\mu}_j^* = (\underline{\mu} - \mu_j)_j = (\mu_1 - \mu_j, \cdots, \mu_{j-1} - \mu_j, \mu_{j+1} - \mu_j, \cdots, \mu_p - \mu_j),\\
&(\underline{\mu}, \lambda) = (\mu_1, \cdots, \mu_p, \lambda) \text{ and } (\lambda, \underline{\mu}) = (\lambda, \mu_1, \cdots, \mu_p)\ (\in \mathbf{C}^{p+1}).
\end{aligned}
$$

- For $j \in \mathbf{N}$, $\underline{\mu} \in \mathbf{C}^p$ and $\lambda \in \mathbf{C}$,

$$
\langle \lambda \rangle_j = \begin{cases} 1 & \text{if} \quad j = 0 \\ \lambda(\lambda+1)\cdots(\lambda+j-1) & \text{if} \quad j \neq 0 \end{cases}
$$
$$
\langle \underline{\mu} \rangle_j = \begin{cases} 1 & \text{if} \quad j = 0 \\ \prod_{k=1}^{p} \langle \mu_k \rangle_j & \text{if} \quad j \neq 0 \end{cases}
$$

- For $\underline{x} \in \mathbf{C}^n$, if f is some function of one complex variable such that for $j = 1, \cdots, n$, $f(x_j)$ is defined, then $f(\underline{x}) = \prod_{j=1}^{n} f(x_j)$.

- For $p, q \in \mathbf{N}$ and $\underline{\alpha} \in \mathbf{C}^p, \underline{\beta} \in (\mathbf{C} \setminus \mathbf{Z}^-)^q$,

$$
{}_pF_q(\underline{\alpha}; \underline{\beta}; z) = \sum_{j \geq 0} \frac{\langle \underline{\alpha} \rangle_j}{\langle \underline{\beta} \rangle_j} \frac{z^j}{j!}.
$$

  If moreover $\underline{\alpha} \in (\mathbf{C} \setminus \mathbf{Z})^p$, for $0 \leq m \leq q$, $0 \leq n \leq p$,

$$
G_{p,q}^{m,n}(\underline{\alpha}; \underline{\beta}; z) = \frac{1}{2i\pi} \int_\gamma \frac{\prod_{j=1}^{m} \Gamma(\beta_j - s) \prod_{j=1}^{n} \Gamma(1 - \alpha_j + s)}{\prod_{j=m+1}^{q} \Gamma(1 - \beta_j + s) \prod_{j=n+1}^{p} \Gamma(\alpha_j - s)} z^s \, ds
$$

  for a suitable path γ.

- For $\underline{\alpha} \in \mathbf{C}^n$,

$$
diag\,(\underline{\alpha}) = diag\,(\alpha_1, \cdots, \alpha_n)
$$

  denotes the diagonal matrix with entries $\alpha_1, \cdots, \alpha_n$ on the main diagonal.

2. The Fuchsian Equation

Let b denote the complex parameter which will be used in the confluence procedure. The starting point is the Fuchsian hypergeometric equation $D_{p+1,p+1}((\underline{\mu},-b);\underline{\nu})$ with $\underline{\mu} \in \mathbf{C}^p$, $\underline{\nu} \in \mathbf{C}^{p+1}$ and $b \in \mathbf{C}$ satisfying the genericity hypotheses: for $i \neq j$, $\nu_i - \nu_j \notin \mathbf{Z}$, $\mu_i - \mu_j \notin \mathbf{Z}$ and $\mu_i + b \notin \mathbf{Z}$. Moreover we suppose that the complex number

$$\lambda = |\underline{\mu}| - |\underline{\nu}| + 1$$

is not an integer.

2.1. Fundamental sets of solutions

Under the previous conditions the following facts are well-known and can be found for example in [7]. In the punctured neighborhood of 0 in $\widetilde{\mathbf{C}}$ given by $\{0 < |z| < 1\}$, a fundamental set of solutions is provided by the line vector

$$\Sigma_0(z) = \Big(w_1(z), \cdots, w_{p+1}(z)\Big)$$

where for $j = 1, \cdots, p+1$,

$$w_j(z) = z^{1-\nu_j}\,{}_{p+1}F_p((1+\underline{\mu}-\nu_j, 1-b-\nu_j); 1+\underline{\nu}_j^*; z)$$

In the neighborhood $\{|z| > 1\}$ of ∞, a fundamental set of solutions is

$$\Sigma_\infty(z) = \Big(h_1(z), \cdots, h_{p+1}(z)\Big)$$

where for $j = 1, \cdots, p$,

$$h_j(z) = z^{-\mu_j}\,{}_{p+1}F_p(1+\mu_j-\underline{\nu}; (1-\underline{\mu}_j^*, 1+b+\mu_j); 1/z)$$

and

$$h_{p+1}(z) = z^b\,{}_{p+1}F_p(1-b-\underline{\nu}; 1-b-\underline{\mu}; 1/z).$$

Note that this set can be obtained from the set of solutions near 0 using the change of variable $z \mapsto 1/z$ and the remark in the introduction.

Near the singular point 1, there is a fundamental set of solutions with p functions holomorphic in a full neighborhood of 1, the $p+1$-th one being

$O((1-z)^{-\lambda+b})$. According to [7] one can choose for the p holomorphic functions the Meijer G-functions given for $j = 1, \cdots, p$ by :

$$\varphi_j(z) = \frac{\Gamma(1-\underline{\mu}_j^*)}{\Gamma(-b-\mu_j)\Gamma(1+\mu_j-\underline{\nu})} G_{p+1,p+1}^{p+1,2}((1+b, 1-\mu_j, 1-\underline{\mu}_j); 1-\underline{\nu}; z)$$

where the path γ in the definition of the G-function goes from $-i\infty$ to $+i\infty$ so that all poles of $\Gamma(1-\nu_j-s)$ $(j = 1, \cdots, p+1)$ lie to the right of γ and all poles of $\Gamma(-b+s)$ and $\Gamma(\mu_j+s)$ $(j = 1, \cdots, p)$ lie to the left of γ.
The $\varphi_j(z)$ are analytic functions of z with a branch point at the origin. The given integral definition is valid for $z \in \widetilde{\mathbf{C}}$ with $|\arg z| < 2\pi$.
For the last function, we will choose (see [7])

$$\varphi_{p+1}(z) = z^\lambda (1-z)^{-\lambda+b} \sum_{k\geq 0} \frac{c_k}{\langle 1-\lambda+b\rangle_k} \left(1-\frac{1}{z}\right)^k$$

where the c_k are *independant of* b and given explicitely by the following formulas ([7] p. 330) :

$$\begin{aligned} c_0 &= 1 \\ c_k &= \sum_{0\leq i_1 \leq i_2 \cdots \leq i_{p-1} \leq k} \frac{B_{i_1,\cdots,i_{p-1}} N_{i_1,\cdots,i_{p-1}}}{i_1!(i_2-i_1)!\cdots(k-i_{p-1})!} \end{aligned}$$

with, if for $n = 1, \cdots, p$, $\beta_n = \sum_{i=1}^n \nu_i - \mu_i$,

$$B_{i_1,\cdots,i_{p-1}} = \langle\beta_1\rangle_{i_1}\langle\beta_2+i_1\rangle_{i_2-i_1}\cdots\langle\beta_p+i_{p-1}\rangle_{k-i_{p-1}}$$

and

$$N_{i_1,\cdots,i_{p-1}} = \langle\nu_2-\mu_1\rangle_{i_1}\langle\nu_3-\mu_2\rangle_{i_2-i_1}\cdots\langle\nu_{p+1}-\mu_p\rangle_{k-i_{p-1}}.$$

The function φ_{p+1} is holomorphic for $\Re z > 1/2$.

Note that for $p = 1$, one gets

$$c_k = \frac{\langle\nu_1-\mu_1\rangle_k\langle\nu_2-\mu_1\rangle_k}{k!}$$

in accordance with the usual hypergeometric case.
For $p = 2$, one has

$$c_k = \frac{\langle 1-\lambda-\nu_1\rangle_k\langle 1-\lambda-\nu_2\rangle_k}{k!} {}_3F_2(-k, \nu_3-\mu_1, \nu_3-\mu_2; 1-\lambda-\nu_1, 1-\lambda-\nu_2; 1).$$

We will denote this fundamental set of solutions by

$$\Sigma_1(z) = \Big(\varphi_1(z), \cdots, \varphi_{p+1}(z)\Big).$$

2.2. Connection matrices

The following formulas give linear relations between the analytic continuation (denoted by the same name) of the functions belonging to the previous sets of solutions. They can be found for example in [3] or [7].

For $j = 1, \cdots, p$,

$$\varphi_j(z) = \frac{\Gamma(1-\underline{\mu}_j^*)}{\Gamma(-b-\mu_j)\Gamma(1+\mu_j-\underline{\nu})} \sum_{i=1}^{p+1} \frac{\Gamma(1+\mu_j-\nu_i)\Gamma(1-b-\nu_i)\Gamma(-\underline{\nu}_i^*)}{\Gamma(\nu_i-\underline{\mu}_j)} w_i(z)$$

and

$$\varphi_{p+1}(z) = \Gamma(1+b-\lambda) \sum_{i=1}^{p+1} \frac{\Gamma(-\underline{\nu}_i^*)}{\Gamma(\nu_i-\underline{\mu})\Gamma(\nu_i+b)} w_i(z)$$

In the next formulas, $\arg z \in]0, 2\pi[$ and if necessary $\arg 1/z = -\arg z$.

For $j = 1, \cdots, p+1$,

$$\begin{aligned} w_j(z) = \frac{\Gamma(1+\underline{\nu}_j^*)}{\Gamma(1+\underline{\mu}-\nu_j)\Gamma(1-b-\nu_j)} & \textstyle\sum_{i=1}^{p} & \frac{\Gamma(1+\mu_i-\nu_j)\Gamma(-b-\mu_i)\Gamma(\underline{\mu}_i^*)}{\Gamma(\underline{\nu}_j-\mu_i)} e^{i\pi(1+\mu_i-\nu_j)} h_i(z) \\ & + & \frac{\Gamma(1-b-\nu_j)\Gamma(\underline{\mu}+b)}{\Gamma(\underline{\nu}_j+b)} e^{i\pi(1-b-\nu_j)} h_{p+1}(z). \end{aligned}$$

For $j = 1, \cdots, p$ and under the same condition for the argument,

$$h_j(z) = \frac{\Gamma(1-\underline{\mu}_j^*)\Gamma(1+\mu_j+b)}{\Gamma(1+\mu_j-\underline{\nu})} \sum_{i=1}^{p+1} \frac{\Gamma(1+\mu_j-\nu_i)\Gamma(-\underline{\nu}_i^*)}{\Gamma(\nu_i+b)\Gamma(\nu_i-\underline{\mu}_j)} e^{-i\pi(1+\mu_j-\nu_i)} w_i(z)$$

and

$$h_{p+1}(z) = \frac{\Gamma(1-b-\underline{\mu})}{\Gamma(1-b-\underline{\nu})} \sum_{i=1}^{p+1} \frac{\Gamma(1-b-\nu_i)\Gamma(-\underline{\nu}_i^*)}{\Gamma(\nu_i-\underline{\mu})} e^{-i\pi(1-b-\nu_i)} w_i(z)$$

For $j = 1, \cdots, p$,

$$\varphi_j(z) = h_j(z) - \frac{\Gamma(1-\underline{\mu}_j^*)\Gamma(1+b+\mu_j)\Gamma(1-b-\underline{\nu})}{\Gamma(1+\mu_j-\underline{\nu})\Gamma(1-b-\underline{\mu})} h_{p+1}(z)$$

and

$$\varphi_{p+1}(z) = e^{i\pi(\lambda-b)}\Gamma(1+b-\lambda)\Big[\sum_{i=1}^{p} \frac{\Gamma(\underline{\mu}_i^*)\Gamma(-b-\mu_i)}{\Gamma(\underline{\nu}-\mu_i)} h_i(z) + \frac{\Gamma(\underline{\mu}+b)}{\Gamma(\underline{\nu}+b)} h_{p+1}(z)\Big].$$

DEFINITION 2.1. For $c, d \in \{0, 1, \infty\}$ with $c \neq d$, the connection matrix $M_{c,d}$ is defined by the relation

$$\Sigma_c(z) = \Sigma_d(z) M_{c,d}.$$

The six matrices so defined belong to $GL(p+1, \mathbf{C})$. They satisfy the relations $M_{c,d} = M_{d,c}^{-1}$ and if $\{c, d, e\} = \{0, 1, \infty\}$, $M_{c,e} = M_{d,e} M_{c,d}$.

The previous connection formulas give the entries of the four matrices $M_{0,\infty}$, $M_{\infty,0}$, $M_{1,0}$ and $M_{1,\infty}$.

For $c, d \in \{0, 1, \infty\}$, let $M_{c,d} = (m_{i,j}^{c,d})$. Then

- for $1 \leq i \leq p$ and $1 \leq j \leq p+1$,

$$m_{i,j}^{0,\infty} = \frac{\Gamma(1+\underline{\nu}_j^*)\Gamma(-b-\mu_i)\Gamma(\underline{\mu}_i^*)}{\Gamma(1+\underline{\mu}_i-\nu_j)\Gamma(\underline{\nu}_j-\mu_i)\Gamma(1-b-\nu_j)} e^{i\pi(1+\mu_i-\nu_j)}$$

- for $1 \leq j \leq p+1$,

$$m_{p+1,j}^{0,\infty} = \frac{\Gamma(1+\underline{\nu}_j^*)\Gamma(\underline{\mu}+b)}{\Gamma(\underline{\nu}_j+b)\Gamma(1+\underline{\mu}-\nu_j)} e^{i\pi(1-b-\nu_j)}$$

- for $1 \leq i \leq p+1$ and $1 \leq j \leq p$,

$$m_{i,j}^{\infty,0} = \frac{\Gamma(1-\underline{\mu}_j^*)\Gamma(1+b+\mu_j)\Gamma(-\underline{\nu}_i^*)}{\Gamma(\nu_i-\underline{\mu}_j)\Gamma(1+\mu_j-\underline{\nu}_i)\Gamma(\nu_i+b)} e^{-i\pi(1+\mu_j-\nu_i)}$$

- for $1 \leq i \leq p+1$,

$$m_{i,p+1}^{\infty,0} = \frac{\Gamma(-\underline{\nu}_i^*)\Gamma(1-b-\underline{\mu})}{\Gamma(1-b-\underline{\nu}_i)\Gamma(\nu_i-\underline{\mu})} e^{-i\pi(1-b-\nu_i)}$$

- for $1 \leq i \leq p+1$ and $1 \leq j \leq p$,

$$m_{i,j}^{1,0} = \frac{\Gamma(1-\underline{\mu}_j^*)\Gamma(1-b-\nu_i)\Gamma(-\underline{\nu}_i^*)}{\Gamma(\nu_i-\underline{\mu}_j)\Gamma(1+\mu_j-\underline{\nu}_i)\Gamma(-b-\mu_j)}$$

- for $1 \leq i \leq p+1$,

$$m_{i,p+1}^{1,0} = \frac{\Gamma(1-\lambda+b)\Gamma(-\underline{\nu}_i^*)}{\Gamma(\nu_i-\underline{\mu})\Gamma(\nu_i+b)}$$

- for $1 \leq i \leq p$, $1 \leq j \leq p$ and $i \neq j$,

$$m_{i,j}^{1,\infty} = 0$$

- for $1 \leq j \leq p$,

$$m_{j,j}^{1,\infty} = 1$$

- for $1 \leq j \leq p$,

$$m_{p+1,j}^{1,\infty} = -\frac{\Gamma(1-\underline{\mu}_j^*)\Gamma(1+b+\mu_j)\Gamma(-b+1-\underline{\nu})}{\Gamma(-b+1-\underline{\mu})\Gamma(1+\mu_j-\underline{\nu})}$$

- for $1 \leq i \leq p$,

$$m_{i,p+1}^{1,\infty} = \frac{\Gamma(\underline{\mu}_i^*)\Gamma(-b-\mu_i)\Gamma(1-\lambda+b)}{\Gamma(\underline{\nu}-\mu_i)} e^{i\pi(\lambda-b)}$$

- finally

$$m_{p+1,p+1}^{1,\infty} = \frac{\Gamma(\underline{\mu}+b)\Gamma(1-\lambda+b)}{\Gamma(\underline{\nu}+b)} e^{i\pi(\lambda-b)}.$$

3. The Confluent Equation

The equation

$$D_{p+1,p}(\underline{\mu};\underline{\nu}) = -t\prod_{j=1}^{p}(t\frac{d}{dt}+\mu_j) - \prod_{j=1}^{p+1}(t\frac{d}{dt}+\nu_j-1)$$

is studied in [1] where the Stokes phenomenon is described. We will recall here the results in the form we need and give some other useful formulas. We make for the coefficients μ_i and ν_i the same genericity hypotheses as in the previous section.

3.1. Solutions of the confluent equation

A fundamental set of solutions analytic in $\widetilde{\mathbf{C}}$ is given by

$$\widetilde{\Sigma}_0(t) = \Big(\tilde{w}_1(t), \cdots, \tilde{w}_{p+1}(t)\Big)$$

where for $1 \leq j \leq p+1$,

$$\tilde{w}_j(t) = t^{1-\nu_j}{}_pF_p(1-\nu_j+\underline{\mu}; 1+\underline{\nu}_j^*; -t)$$

A fundamental set of *formal* solutions at ∞ is given by:

$$\widehat{\Sigma}_\infty = \Big(\Phi_1, \cdots, \Phi_{p+1}\Big)$$

where for $1 \le j \le p$,

$$\Phi_j = t^{-\mu_j}\,{}_{p+1}F_{p-1}(1+\mu_j-\underline{\nu}; 1-\underline{\mu}_j^*; 1/t)$$

and

$$\Phi_{p+1} = e^{-t}t^{\lambda}\Theta$$

where Θ is a formal series in $1/t$ normalized by choosing the constant term equal to 1. This series is *not* hypergeometric when $p \ge 2$.
For $1 \le k \le p$ we set

$$G_k(t) = G^{p+1,1}_{p,p+1}((1-\mu_k, 1-\underline{\mu}_k); 1-\underline{\nu}; te^{i\pi})$$

where the path of integration joins $-i\infty$ to $+i\infty$, leaving to its right the points $-\nu_j + n$ ($j = 1, \cdots, p+1; n \in \mathbf{N}$) and to its left the points $-\mu_k - n$ ($n \in \mathbf{N}$).
We set

$$G_0(t) = G^{p+1,0}_{p,p+1}(1-\underline{\mu}; 1-\underline{\nu}; t)$$

where the path of integration leaves to its right the points $-\nu_j + n$ ($j = 1, \cdots, p+1; n \in \mathbf{N}$).

The following facts are asserted in [3].

Proposition 3.1. *For $k = 1, \cdots, p$ the function $G_k(t)$ is a solution of $D_{p+1,p}(\underline{\mu}; \underline{\nu})$.*
For $-\frac{5\pi}{2} < \arg t < \frac{\pi}{2}$, the function $t^{\mu_k}G_k(t)$ admits the series :

$$e^{-i\pi\mu_k}\frac{\Gamma(1+\mu_k-\underline{\nu})}{\Gamma(1-\underline{\mu}_k^*)}\,{}_{p+1}F_{p-1}(1+\mu_k-\underline{\nu}; 1-\underline{\mu}_k^*; 1/t)$$

as asymptotic expansion near ∞.
The function $G_0(t)$ is also a solution of this equation.
For $-\frac{3\pi}{2} < \arg t < \frac{3\pi}{2}$, the function $e^t t^{-\lambda}G_0(t)$ admits the series Θ as asymptotic expansion near ∞.

According to these results it is natural to introduce the following fundamental sets of solutions which are in fact "sums" in the sense of 1-summability ([5]) of $\widehat{\Sigma}_\infty$ in the given sectors.

- For $-\frac{3\pi}{2} < \arg t < \frac{\pi}{2}$,

$$\widetilde{\Sigma}_{-\pi/2} = \left(e^{i\pi\mu_1}\tfrac{\Gamma(1-\underline{\mu}_1^*)}{\Gamma(1+\mu_1-\underline{\nu})}G_1(t), \cdots, e^{i\pi\mu_p}\tfrac{\Gamma(1-\underline{\mu}_p^*)}{\Gamma(1+\mu_p-\underline{\nu})}G_p(t), G_0(t)\right)$$

which we also write

$$\widetilde{\Sigma}_{-\pi/2} = \left(e^{i\pi\mu_j}\tfrac{\Gamma(1-\underline{\mu}_j^*)}{\Gamma(1+\mu_j-\underline{\nu})}G_j(t), (j = 1\cdots, p), G_0(t)\right)$$

- For $-\frac{\pi}{2} < \arg t < \frac{3\pi}{2}$,

$$\widetilde{\Sigma}_{\pi/2} = \left(e^{-i\pi\mu_j}\tfrac{\Gamma(1-\underline{\mu}_j^*)}{\Gamma(1+\mu_j-\underline{\nu})}G_j(te^{-2i\pi}), (j = 1, \cdots, p), G_0(t)\right)$$

- For $\frac{\pi}{2} < \arg t < \frac{5\pi}{2}$,

$$\widetilde{\Sigma}_{3\pi/2} = \left(e^{-i\pi\mu_j}\tfrac{\Gamma(1-\underline{\mu}_j^*)}{\Gamma(1+\mu_j-\underline{\nu})}G_j(te^{-2i\pi}), (j = 1, \cdots p), e^{2i\pi\lambda}G_0(te^{-2i\pi})\right)$$

3.2. Stokes and connection matrices

In [3] one can find the following formulas connecting the functions G_j and $\tilde{w}_j$.

For $1 \leq j \leq p$ and $-\frac{5\pi}{2} < \arg t < \frac{\pi}{2}$,

$$G_j(t) = \sum_{i=1}^{p+1} \frac{\Gamma(-\underline{\nu}_i^*)\Gamma(1+\mu_j-\nu_i)}{\Gamma(\nu_i-\underline{\mu}_j)} e^{i\pi(1-\nu_i)}\tilde{w}_i(t).$$

For $-\frac{3\pi}{2} < \arg t < \frac{3\pi}{2}$,

$$G_0(t) = \sum_{i=1}^{p+1} \frac{\Gamma(-\underline{\nu}_i^*)}{\Gamma(\nu_i-\underline{\mu})}\tilde{w}_i(t).$$

Conversely one can deduce from classical formulas ([3]) the two relations for $-\frac{3\pi}{2} < \arg t < \frac{\pi}{2}$, $1 \leq j \leq p+1$,

$$\tilde{w}_j(t) = \tfrac{\Gamma(1+\underline{\nu}_j^*)}{\Gamma(1-\nu_j+\underline{\mu})}\left[e^{i\pi(\lambda+\nu_j-1)}G_0(t) + \sum_{i=1}^{p}\tfrac{\Gamma(\underline{\mu}_i^*)\Gamma(1-\underline{\mu}_i^*)}{\Gamma(\underline{\nu}_j-\mu_i)\Gamma(1+\mu_i-\underline{\nu}_j)}e^{i\pi\mu_i}G_i(t)\right]$$

and for $-\frac{\pi}{2} < \arg t < \frac{3\pi}{2}$,

$$\tilde{w}_j(t) = \tfrac{\Gamma(1+\underline{\nu}_j^*)}{\Gamma(1-\nu_j+\underline{\mu})}\left[e^{-i\pi(\lambda+\nu_j-1)}G_0(t) + \sum_{i=1}^{p}\tfrac{\Gamma(\underline{\mu}_i^*)\Gamma(1-\underline{\mu}_i^*)e^{-i\pi\mu_i}}{\Gamma(\underline{\nu}_j-\mu_i)\Gamma(1+\mu_i-\underline{\nu}_j)}G_i(te^{-2i\pi})\right]$$

For $c \in \{-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}\}$, let us define the connection matrix $\widetilde{M_c}$ by the equality

$$\widetilde{\Sigma}_c(t) = \widetilde{\Sigma}_0(t)\widetilde{M_c}$$

with $c - \pi < \arg t < c + \pi$.
We will denote by $\widetilde{P}_c$ the inverse matrix $\widetilde{M_c}^{-1}$. The previous formulas give explicit values for the entries of these matrices.

If $\widetilde{M_c} = (\tilde{m}^c_{i,j})$ and $\widetilde{P}_c = (\tilde{p}^c_{i,j})$ then

- for $1 \le j \le p$ and $1 \le i \le p+1$,
$$\tilde{m}^{-\pi/2}_{i,j} = \frac{\Gamma(1 - \underline{\mu}^*_j)\Gamma(-\underline{\nu}^*_i)}{\Gamma(1 + \mu_j - \underline{\nu}_i)\Gamma(\nu_i - \underline{\mu}_j)} e^{i\pi(1+\mu_j-\nu_i)}$$

- for $1 \le i \le p+1$,
$$\tilde{m}^{-\pi/2}_{i,p+1} = \frac{\Gamma(-\underline{\nu}^*_i)}{\Gamma(\nu_i - \underline{\mu})}$$

- for $1 \le j \le p$ and $1 \le i \le p+1$,
$$\tilde{m}^{\pi/2}_{i,j} = \tilde{m}^{-\pi/2}_{i,j} e^{-2i\pi(1+\mu_j-\nu_i)}$$

- for $1 \le i \le p+1$,
$$\tilde{m}^{\pi/2}_{i,p+1} = \tilde{m}^{-\pi/2}_{i,p+1}$$

- $1 \le j \le p$ and $1 \le i \le p+1$,
$$\tilde{m}^{3\pi/2}_{i,j} = \tilde{m}^{\pi/2}_{i,j}$$

- for $1 \le i \le p+1$,
$$\tilde{m}^{3\pi/2}_{i,p+1} = \tilde{m}^{-\pi/2}_{i,p+1} e^{2i\pi(\lambda+\nu_i)}$$

- for $1 \le i \le p$ and $1 \le j \le p+1$,
$$\tilde{p}^{-\pi/2}_{i,j} = \frac{\Gamma(1 + \underline{\nu}^*_j)\Gamma(\underline{\mu}^*_i)}{\Gamma(1 - \nu_j + \underline{\mu}_i)\Gamma(\underline{\nu}_j - \mu_i)}$$

- for $1 \le j \le p+1$,
$$\tilde{p}^{-\pi/2}_{p+1,j} = \frac{\Gamma(1 + \underline{\nu}^*_j)}{\Gamma(1 - \nu_j + \underline{\mu})} e^{i\pi(\lambda+\nu_j-1)}$$

- for $1 \leq i \leq p$ and $1 \leq j \leq p+1$,

$$\tilde{p}_{i,j}^{\pi/2} = \tilde{p}_{i,j}^{-\pi/2}$$

- for $1 \leq j \leq p+1$,

$$\tilde{p}_{p+1,j}^{\pi/2} = \tilde{p}_{p+1,j}^{-\pi/2} e^{-2i\pi(\lambda+\nu_j-1)}$$

- for $1 \leq i \leq p$ and $1 \leq j \leq p+1$,

$$\tilde{p}_{i,j}^{3\pi/2} = \tilde{p}_{i,j}^{-\pi/2} e^{2i\pi(\mu_i-\nu_j)}$$

- for $1 \leq j \leq p+1$,

$$\tilde{p}_{p+1,j}^{3\pi/2} = \tilde{p}_{p+1,j}^{\frac{\pi}{2}}$$

In accordance with [1], one defines the Stokes matrices S_0 and S_π by the equalities

$$\widetilde{\Sigma}_{-\frac{\pi}{2}}(t) = \widetilde{\Sigma}_{\frac{\pi}{2}}(t) S_0$$

with $-\frac{\pi}{2} < \arg t < \frac{\pi}{2}$ and

$$\widetilde{\Sigma}_{\frac{\pi}{2}}(t) = \widetilde{\Sigma}_{\frac{3\pi}{2}}(t) S_\pi$$

with $\frac{\pi}{2} < \arg t < \frac{3\pi}{2}$.

We will denote by $E_{i,j}$ the matrix with all entries equal to 0 except the (i,j)-th one equal to 1 and by I the identity matrix of order $p+1$. Then

$$S_0 = I + \sum_{j=1}^{p} 2i\pi \frac{\Gamma(1-\underline{\mu}_j^*)}{\Gamma(1+\mu_j-\underline{\nu})} E_{p+1,j}$$

and

$$S_\pi = I + \sum_{j=1}^{p} 2i\pi e^{i\pi(\lambda+\mu_j)} \frac{\Gamma(\underline{\mu}_j^*)}{\Gamma(-\mu_j+\underline{\nu})} E_{j,p+1}.$$

These constant invertible matrices satisfy the relations : $\widetilde{M}_{-\frac{\pi}{2}} = \widetilde{M}_{\frac{\pi}{2}} S_0$, $\widetilde{M}_{\frac{\pi}{2}} = \widetilde{M}_{\frac{3\pi}{2}} S_\pi$, $S_0 = \widetilde{P}_{\frac{\pi}{2}} \widetilde{M}_{-\frac{\pi}{2}}$, $S_\pi = \widetilde{P}_{\frac{3\pi}{2}} \widetilde{M}_{\frac{\pi}{2}}$, $\widetilde{P}_{\frac{\pi}{2}} = S_0 \widetilde{P}_{-\frac{\pi}{2}}$ and $\widetilde{P}_{\frac{3\pi}{2}} = S_\pi \widetilde{P}_{\frac{\pi}{2}}$.

4. A First Confluence Procedure

As explained in the introduction one makes the change of variable $z = t/b$ in the equation $D_{p+1,p+1}((\underline{\mu}, -b); \underline{\nu})$ and then allows b to tend to ∞ in such a way that the conditions $\mu_i + b \notin \mathbf{Z}$ and $-\lambda + b \notin \mathbf{Z}$ still remain valid. This change of variable leads to the equation

$$D^b_{p+1,p+1} = \frac{t}{b}(t\frac{d}{dt} - b)\prod_{j=1}^{p}(t\frac{d}{dt} + \mu_j) - \prod_{j=1}^{p+1}(t\frac{d}{dt} + \nu_j - 1)$$

which has the three regular singular points 0, b and ∞. Formally $D^b_{p+1,p+1}$ tends to $D_{p+1,p}(\underline{\mu}, \underline{\nu})$ as $|b| \to \infty$.

4.1. Limit properties

We recall some well-known confluence properties. If necessary the proofs can be found in [3].

LEMMA 4.1. *For $z, \sigma \in \mathbf{C}$,*

$$\lim_{|\sigma|\to\infty}(1 - \frac{z}{\sigma})^{-\sigma} = e^z$$

uniformly on compact sets.

LEMMA 4.2. *Let $\underline{a} \in \mathbf{C}^p$, $\underline{c} \in \mathbf{C}^q$ and $-\pi < \arg b < \pi$, then*

$$\lim_{|b|\to\infty} \frac{\Gamma(b + \underline{a})}{\Gamma(b + \underline{c})} b^{|\underline{c}| - |\underline{a}|} = 1.$$

LEMMA 4.3. *Let $\underline{A} \in \mathbf{C}^p$, $\underline{B} \in (\mathbf{C} \setminus \mathbf{Z}^-)^q$, $\beta \in \mathbf{C}$ and $\varepsilon = \pm$,*

$$\lim_{|b|\to\infty} {}_{p+1}F_q((\underline{A}, \varepsilon b + \beta); \underline{B}; z/b) = {}_pF_q(\underline{A}; \underline{B}; \varepsilon z)$$

uniformly on compact sets if $q \geq p$ and term by term if $q < p$.

LEMMA 4.4. *Let $\underline{A} \in \mathbf{C}^{p+1}$, $\underline{B} \in (\mathbf{C} \setminus \mathbf{Z}^-)^{p-1}$, $\beta \in \mathbf{C}$ and $\varepsilon = \pm$, then term by term*

$$\lim_{|b|\to\infty} {}_{p+1}F_p(\underline{A}; (\underline{B}, \varepsilon b + \beta); b/t) = {}_{p+1}F_{p-1}(\underline{A}; \underline{B}; \varepsilon/t).$$

The next proposition is easily deduced from these lemmas.

PROPOSITION 4.5. *Uniformly on compact sets of* $\widetilde{\mathbf{C}}$, *for* $1 \leq j \leq p+1$,

$$\lim_{|b|\to\infty} b^{1-\nu_j} w_j(\tfrac{t}{b}) = \tilde{w}_j(t).$$

Moreover, for $1 \leq j \leq p$, *then term by term*

$$\lim_{|b|\to\infty} b^{-\mu_j} h_j(\tfrac{t}{b}) = \Phi_j.$$

The last function h_{p+1} cannot be studied with the help of the above mentionned lemmas as the parameter b appears in all the coefficients of the hypergeometric function. This kind of situation has been studied by Knottnerus [2] where one can find the proof of the following assertion also cited by [3].

LEMMA 4.6. *For* $\underline{A} \in \mathbf{C}^{p+1}$, $\underline{B} \in (\mathbf{C} \setminus \mathbf{Z}^-)^p$, *r a sufficiently large positive real number and* $|\arg(1-z)| < \pi$,

$$_{p+1}F_p(r+\underline{A}; r+\underline{B}; z) = (1-z)^{|\underline{B}|-|\underline{A}|-r}\Big[1 + \frac{d_1 z}{r} + \sum_{k=2}^{n} \frac{p_k(z)}{r^k} + O(r^{-n-1}\Big]$$

where d_1 *only depends upon* $\underline{A}$ *and* $\underline{B}$ *and* $p_k(z)$ *is a polynomial in* z *without constant term and of degree less than* k, *the coefficients of which only depends upon* $\underline{A}$ *and* $\underline{B}$.

From this lemma one "deduces" that for each $n \in \mathbf{N}$,

$$\begin{aligned} h_{p+1}(t/b) \quad \text{“}=\text{”} \quad & b^{-b} t^b (1 - \frac{b}{t})^{-\lambda+b}\Big[1 - \frac{d_1}{t} + \sum_{k=2}^{n} (-1)^k \frac{d_k}{t^k} + O(\frac{1}{b})\Big] \\ \text{“}=\text{”} \quad & e^{i\pi(b-\lambda)} t^\lambda b^{-\lambda} (1 - \frac{t}{b})^{-\lambda+b}\Big[1 - \frac{d_1}{t} + \sum_{k=2}^{n} (-1)^k \frac{d_k}{t^k} + O(\frac{1}{b})\Big] \end{aligned}$$

where d_k is the coefficient of the leading term in the polynomial p_k.

This heuristic computation will be helpful in choosing the proper normalization.

Finally, as

$$\begin{aligned}\varphi_{p+1}(t/b) &= t^{\lambda}b^{-\lambda}(1-\frac{t}{b})^{-\lambda+b}\sum_{k\geq 0}\frac{c_k}{\langle 1-\lambda+b\rangle_k}\left(1-\frac{b}{t}\right)^k\\ &= t^{\lambda}b^{-\lambda}(1-\frac{t}{b})^{-\lambda+b}\sum_{k\geq 0}\frac{c_k}{t^k}\frac{(t-b)^k}{\langle 1-\lambda+b\rangle_k}\end{aligned}$$

and taking into account the fact that the space of formal solutions of the form $t^{\lambda}e^{-t}$ times a formal series is one dimensional, one deduces the next proposition.

PROPOSITION 4.7. *Term by term,*

$$\lim_{|b|\to\infty} b^{\lambda}\varphi_{p+1}(t/b) = t^{\lambda}e^{-t}\Theta$$

where

$$\Theta = \sum_{k\geq 0}(-1)^k\frac{c_k}{t^k}.$$

Note that this last formula gives an *explicit* value for the coefficients of the formal series Θ.

4.2. Normalized solutions

According to the results of the previous section it is natural to define the following fundamental sets of solutions near the three regular singular points of $D^b_{p+1,p+1}$.

Around 0, one defines

$$\Sigma_0^b(t) = \Sigma_0(t/b)N_0$$

where

$$N_0 = diag\,(b^{1-\nu_1},\cdots,b^{1-\nu_{p+1}}).$$

Around ∞, one defines

$$\Sigma_\infty^b(t) = \Sigma_\infty(t/b)N_\infty$$

where

$$N_\infty = diag\,(b^{-\mu_1},\cdots,b^{-\mu_p},b^{\lambda}e^{i\pi(\lambda-b)}).$$

Around b, one defines

$$\Sigma_b^b(t) = \Sigma_1(t/b)N_b$$

where

$$N_b = diag\,(b^{-\mu_1}, \cdots, b^{-\mu_p}, b^{\lambda}).$$

The next proposition is a consequence of the previous section.

PROPOSITION 4.8. *Uniformly on each compact set of* $\widetilde{\mathbf{C}}$,

$$\lim_{|b|\to\infty} \Sigma_0^b(t) = \widetilde{\Sigma}_0(t).$$

For $c, d \in \{0, b, \infty\}, c \neq d$, one defines the connection matrix $M_{c,d}^b$ by the equality

$$\Sigma_c^b(t) = \Sigma_d^b(t) M_{c,d}^b.$$

The next lemma shows the relationship between these matrices and those defined for the equation $D_{p+1,p+1}((\underline{\mu}, -b); \underline{\nu})$.

LEMMA 4.9. *For* $c, d \in \{0, b, \infty\}, c \neq d$,

$$M_{c,d}^b = N_d^{-1} M_{\frac{c}{b},\frac{d}{b}} N_c.$$

In particular if for $1 \leq i, j \leq p+1$ the (i,j)-th entry of the matrix $M_{c,d}^b$ is denoted by ${}^b m_{i,j}^{c,d}$, one has :

- for $1 \leq i \leq p$ and $1 \leq j \leq p+1$,

$${}^b m_{i,j}^{0,\infty} = \frac{\Gamma(1+\underline{\nu}_j^*)\Gamma(\underline{\mu}_i^*)}{\Gamma(1-\nu_j+\underline{\mu}_i)\Gamma(\underline{\nu}_j-\mu_i)} \frac{\Gamma(-b-\mu_i)}{\Gamma(1-b-\nu_j)} b^{1+\mu_i-\nu_j} e^{i\pi(1+\mu_i-\nu_j)},$$

- for $1 \leq j \leq p+1$,

$${}^b m_{p+1,j}^{0,\infty} = \frac{\Gamma(1+\underline{\nu}_j^*)}{\Gamma(1-\nu_j+\underline{\mu})} \frac{\Gamma(\underline{\mu}+b)}{\Gamma(\underline{\nu}_j+b)} b^{-\lambda-\nu_j+1} e^{i\pi(1-\lambda-\nu_j)}.$$

Proposition 4.10. *If $-2\pi < \arg b < -\pi$ then*

$$\lim_{|b|\to\infty} M^b_{0,\infty} = \widetilde{P}_{-\frac{\pi}{2}}.$$

If $-\pi < \arg b < 0$ then

$$\lim_{|b|\to\infty} M^b_{0,\infty} = \widetilde{P}_{\frac{\pi}{2}}.$$

If $0 < \arg b < \pi$ then

$$\lim_{|b|\to\infty} M^b_{0,\infty} = \widetilde{P}_{\frac{3\pi}{2}}.$$

Proof. For the $(p+1, j)$-th entry, the result is a consequence of lemma 4.1 if $-\pi < \arg b < \pi$. If $-2\pi < \arg b < -\pi$ one writes $b = b'e^{-2i\pi}$ so that $0 < \arg b' < \pi$ and then one uses the same lemma.
For the other rows one writes $b = (-b)e^{-i\pi}$ if $-2\pi < \arg b < 0$ and $b = (-b)e^{i\pi}$ if $0 < \arg b < \pi$, so that $-\pi < \arg(-b) < \pi$ in each case. Then one uses lemma 4.1. □

The proof of the following proposition is along the same lines.

Proposition 4.11. *If $-\pi < \arg b < 0$ then*

$$\lim_{|b|\to\infty} M^b_{b,0} = \widetilde{M}_{-\frac{\pi}{2}}.$$

If $0 < \arg b < \pi$ then

$$\lim_{|b|\to\infty} M^b_{b,0} = \widetilde{M}_{\frac{\pi}{2}}.$$

If $\pi < \arg b < 2\pi$ then

$$\lim_{|b|\to\infty} M^b_{b,0} = \widetilde{M}_{\frac{3\pi}{2}}.$$

Using these two propositions together with the relation

$$M^b_{b,\infty} = M^b_{0,\infty} M^b_{b,0},$$

one gets :

Proposition 4.12. *If $-\pi < \arg b < 0$ then*

$$\lim_{|b|\to\infty} M^b_{b,\infty} = \widetilde{P}_{\frac{\pi}{2}} \widetilde{M}_{-\frac{\pi}{2}} = S_0.$$

If $0 < \arg b < \pi$ *then*

$$\lim_{|b|\to\infty} M^b_{b,\infty} = \widetilde{P}_{\frac{3\pi}{2}} \widetilde{M}_{\frac{\pi}{2}} = S_\pi.$$

It is worthwhile to notice that these limits can be obtained directly from the explicit values of the entries of the matrix $M^b_{b,\infty}$. In fact one has :

- for $1 \le i,j \le p$, ${}^b m^{b,\infty}_{i,j} = 0$ if $i \ne j$ and ${}^b m^{b,\infty}_{j,j} = 1$,
- for $1 \le j \le p$,

$$\begin{aligned} {}^b m^{b,\infty}_{p+1,j} &= -\tfrac{\Gamma(1-\underline{\mu}^*_j)}{\Gamma(1+\mu_j-\underline{\nu})} \tfrac{\Gamma(1+\mu_j+b)\Gamma(1-b-\underline{\nu})}{\Gamma(1-b-\underline{\mu})} b^{-\lambda-\mu_j} e^{-i\pi(\lambda-b)} \\ &= -\frac{\Gamma(1-\underline{\mu}^*_j)}{\Gamma(1+\mu_j-\underline{\nu})} U(b). \end{aligned}$$

If $-\pi < \arg b < 0$, one writes $b = (-b)e^{-i\pi}$ so that $0 < \arg(-b) < \pi$. Then $U(b) = U_1(b)U_2(b)$ with

$$U_1(b) = \frac{\Gamma(1-b-\underline{\nu})}{\Gamma(1-b-\underline{\mu})\Gamma(-\mu_j-b)} (-b)^{-\lambda-\mu_j} \to 1$$

when $|b| \to \infty$ as $|1-\underline{\mu}| - \mu_j - |1-\underline{\nu}| = -\lambda - \mu_j$.

$$\begin{aligned} U_2(b) &= \Gamma(-\mu_j - b)\Gamma(1+\mu_j+b) e^{i\pi(b+\mu_j)} \\ &= 2i\pi \frac{e^{i\pi(b+\mu_j)}}{e^{-i\pi(b+\mu_j)} - e^{i\pi(b+\mu_j)}} \end{aligned}$$

using the well-known relation $\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x$.
As $\Im b < 0$ when $-\pi < \arg b < 0$, $|e^{i\pi(b+\mu_j)}| \to \infty$ as $|b| \to \infty$ and thus $U_2(b) \to -2i\pi$.
If $0 < \arg b < \pi$, one writes $b = (-b)e^{i\pi}$ and, with the same notations as before, one has

$$U_2(b) = 2i\pi e^{-2i\pi\lambda} \frac{e^{i\pi(b-\mu_j)}}{e^{-i\pi(b+\mu_j)} - e^{i\pi(b+\mu_j)}}.$$

This time $\Im b > 0$ and thus $U_2(b) \to 0$ as $|b| \to \infty$. More precisely $U_2(b) = O(e^{-2\pi\Im b})$.

- for $1 \le i \le p$,

$$\begin{aligned} {}^{b}m^{b,\infty}_{i,p+1} &= \frac{\Gamma(\underline{\mu}^*_i)}{\Gamma(\underline{\nu}-\mu_i)}\Gamma(-b-\mu_i)\Gamma(1-\lambda+b)b^{\lambda+\mu_i}e^{i\pi(\lambda-b)} \\ &= \frac{\Gamma(\underline{\mu}^*_i)}{\Gamma(\underline{\nu}-\mu_i)}e^{i\pi(\lambda+\mu_i)}V(b). \end{aligned}$$

  Writing as before $b = (-b)e^{\varepsilon i\pi}$ with $\varepsilon = 1$ if $0 < \arg b < \pi$ and $\varepsilon = -1$ if $-\pi < \arg b < 0$ so that $-\pi < \arg(-b) < \pi$, one has $V(b) = V_1(b)V_2(b)$ with

$$V_1(b) = \frac{\Gamma(-b-\mu_i)}{\Gamma(\lambda-b)}(-b)^{\lambda+\mu_i} \to 1$$

  and

  - if $0 < \arg b < \pi$,

$$V_2(b) = 2i\pi\frac{e^{i\pi(\lambda-b)}}{e^{i\pi(\lambda-b)}-e^{-i\pi(\lambda-b)}} \to 2i\pi,$$

  - if $-\pi < \arg b < 0$,

$$V_2(b) = 2i\pi e^{-2i\pi\mu_i}\frac{e^{i\pi(\lambda-b)}}{e^{i\pi(\lambda-b)}-e^{-i\pi(\lambda-b)}} = O(e^{2\pi\Im b})$$

    with $\Im b < 0$.

- Finally

$${}^{b}m^{b,\infty}_{p+1,p+1} = \frac{\Gamma(\underline{\mu}+b)\Gamma(1-\lambda+b)}{\Gamma(\underline{\nu}+b)} \to 1$$

  when $|b| \to \infty$ with $-\pi < \arg b < \pi$, as it follows from the relation $\lambda - |\underline{\mu}| + |\underline{\nu}| - 1 = 0$.

These results altogether give a direct proof of Proposition 4.12 and moreover give information on the speed with which the last row or column of the matrix $M^b_{b,\infty}$ converges to 0.

As a by-product of the previous results we get the existence of *actual* limits for the fundamantal sets of solutions of $D^b_{p+1,p+1}$ around b and ∞.

Proposition 4.13. *The following limits hold uniformly on compact subsets of* $\widetilde{\mathbf{C}}$,

$$\begin{aligned}
\lim_{\substack{|b|\to\infty \\ -\pi<\arg b<0}} \Sigma_b^b(t) &= \lim_{\substack{|b|\to\infty \\ -2\pi<\arg b<-\pi}} \Sigma_\infty^b(t) = \widetilde{\Sigma}_{-\frac{\pi}{2}}(t) \\
\lim_{\substack{|b|\to\infty \\ 0<\arg b<\pi}} \Sigma_b^b(t) &= \lim_{\substack{|b|\to\infty \\ -\pi<\arg b<0}} \Sigma_\infty^b(t) = \widetilde{\Sigma}_{\frac{\pi}{2}}(t) \\
\lim_{\substack{|b|\to\infty \\ \pi<\arg b<2\pi}} \Sigma_b^b(t) &= \lim_{\substack{|b|\to\infty \\ 0<\arg b<\pi}} \Sigma_\infty^b(t) = \widetilde{\Sigma}_{\frac{3\pi}{2}}(t)
\end{aligned}$$

Proof. All proofs are along the same lines. We give one of them. We start with the relation $\widetilde{\Sigma}_{-\frac{\pi}{2}}(t) = \widetilde{\Sigma}_0(t)\widetilde{M}_{-\frac{\pi}{2}}$. For $-\pi < \arg b < 0$, Proposition 4.8 and 4.11 assert that uniformly on compact sets of $\widetilde{\mathbf{C}}$,

$$\widetilde{\Sigma}_{-\frac{\pi}{2}}(t) = \lim_{|b|\to\infty} \Sigma_0^b(t) M_{b,0}^b = \lim_{|b|\to\infty} \Sigma_b^b(t). \ \square$$

Taking into account the definition of the functions entering these fundamental sets one is led to the limit properties listed in the next proposition.

Proposition 4.14. *Let* $a, b, \sigma \in \mathbf{C}, \underline{A} \in \mathbf{C}^{p-1}, \underline{B} \in \mathbf{C}^{p+1}$. *Let* $\arg b$ *be fixed with* $-\pi < \arg b < 0$ *or* $0 < \arg b < \pi$, *then uniformly on compact sets of* $\widetilde{\mathbf{C}}$,

1.

$$\lim_{|b|\to\infty} \frac{b^{a-1}}{\Gamma(a+b)} G_{p+1,p+1}^{p+1,2}((-b, a, \underline{A}); \underline{B}; \frac{t}{b}) = G_{p,p+1}^{p+1,1}((a, \underline{A}); \underline{B}; t)$$

2.

$$\lim_{|b|\to\infty} \sum_{k\geq 0} \frac{c_k}{\langle 1-\lambda+b\rangle_k}(1-\frac{b}{t})^k = t^{-\lambda} e^t G_{p,p+1}^{p+1,0}(1-\underline{\mu}; 1-\underline{\nu}; t)$$

3.

$$\lim_{|b|\to\infty} {}_{p+1}F_p(\underline{B}; (\underline{A}, b+\sigma); \frac{b}{t}) = \frac{\Gamma(\underline{A})}{\Gamma(\underline{B})} G_{p,p+1}^{p+1,1}((1, \underline{A}); \underline{B}; te^{-i\pi})$$

4.

$$\lim_{|b|\to\infty} b^{\lambda}(\frac{t}{b})^{b}{}_{p+1}F_p(1+b-\underline{\nu};1+b-\underline{\mu};-\frac{b}{t}) = G^{p+1,0}_{p,p+1}(1-\underline{\mu};1-\underline{\nu};t)$$

By expressing the hypergeometric function ${}_{p+1}F_p$ as a G-function ([3] p. 147), one can also deduce from properties 3. and 4. the following formulas.

COROLLARY 4.15. *With the same notations and hypotheses, uniformly on compact sets of* $\widetilde{\mathbf{C}}$

1.

$$\lim_{|b|\to\infty} \Gamma(b+\sigma)G^{p+1,1}_{p+1,p+1}((1,\underline{A},b+\sigma);\underline{B};\frac{t}{b}) = G^{p+1,1}_{p,p+1}((1,\underline{A});\underline{B};t)$$

2.

$$\lim_{|b|\to\infty} \frac{1}{\Gamma(b)}G^{p+1,1}_{p+1,p+1}((1-b,1-\underline{\mu};1-\underline{\nu};\frac{t}{b}) = G^{p+1,0}_{p,p+1}(1-\underline{\mu};1-\underline{\nu};t)$$

Note that the two properties of this corollary as well as property 1. of 4.14 express the validity of "taking the limit under the $\int$ sign in the integral defining the G-functions under consideration.

As a final remark, let us note that it is also possible to get both Stokes matrices S_0 and S_π as limits when $b\to\infty$ in one and the same half-plane. We have seen that in the half-plane $-\pi < \arg b < 0$, $S_0 = \lim_{|b|\to\infty} M^b_{b,\infty}$. The next proposition shows that S_π^{-1} is obtained up to conjugation by the formal monodromy as the limit in the same half-plane of another matrix, denoted by ${M'}^b_{b,\infty}$.
The monodromy around b is given in the basis $\Sigma^b_b(t)$ by the matrix

$$T_b = diag\,(\underbrace{1,\cdots,1}_{p}, e^{-2i\pi(\lambda-b)}).$$

PROPOSITION 4.16. *Let* ${M'}^b_{b,\infty} = T_b^{-1}M^b_{b,\infty}T_b$. *Then*

$$\lim_{\substack{|b|\to\infty\\ -\pi<\arg b<0}} {M'}^b_{b,\infty} = \widehat{M}^{-1}S_\pi^{-1}\widehat{M}$$

where $\widehat{M}$ *denotes the* formal monodromy *matrix of the confluent equation given in the basis* $\widehat{\Sigma}$ *by*

$$\widehat{M} = diag\,(e^{2i\pi\mu_1}, \cdots, e^{2i\pi\mu_p}, e^{-2i\pi\lambda}).$$

Proof. Let ${}^b{m'}^{b,\infty}_{i,j}$ denote the (i,j)-th entry of the matrix ${M'}^b_{b,\infty}$. One has

- for $1 \leq i,j \leq p$ or $i = j = p+1$,

$${}^b{m'}^{b,\infty}_{i,j} = {}^b m^{b,\infty}_{i,j}$$

  which is equal or tends to 0 if $i \neq j$ and 1 if $i = j$.

- For $1 \leq j \leq p$,

$${}^b{m'}^{b,\infty}_{p+1,j} = {}^b m^{b,\infty}_{p+1,j} e^{2i\pi(\lambda-b)} = O(e^{2\pi\Im b}).$$

- For $1 \leq i \leq p$, with $b = (-b)e^{i\pi}$,

$$\begin{aligned} {}^b{m'}^{b,\infty}_{i,p+1} &= {}^b m^{b,\infty}_{i,p+1} e^{-2i\pi(\lambda-b)} \\ &= \frac{\Gamma(\underline{\mu}_i^*)}{\Gamma(\underline{\nu} - \mu_i)} e^{i\pi(\lambda+\mu_i)} V_1(b) V_2(b) e^{-2i\pi(\lambda-b)} \end{aligned}$$

  with $\lim_{|b|\to\infty} V_1(b) = 1$ and

$$V_2(b)e^{-2i\pi(\lambda-b)} = 2i\pi \frac{e^{-2i\pi(\mu_i+\lambda)}e^{i\pi(b-\lambda)}}{e^{i\pi(\lambda-b)} - e^{i\pi(b-\lambda)}} \to -2i\pi e^{-2i\pi(\mu_i+\lambda)}.$$

These formulas altogether give the result. Note that, as $S_\pi = I + N$ and $N^2 = 0$, one has $S_\pi^{-1} = I - N$. $\square$

5. A Second Confluence Procedure

For this second type of confluence, we express the monodromy around b and around ∞ in one of the two "mixed" basis Σ_m and Σ'_m defined as follows.

Let $\Sigma_b^b = (\varphi_1^b, \cdots, \varphi_{p+1}^b)$ and $\Sigma_\infty^b = (h_1^b, \cdots, h_{p+1}^b)$, then

$$\Sigma_m = (h_1^b, \cdots, h_p^b, \varphi_{p+1}^b)$$

and

$$\Sigma_m' = (\varphi_1^b, \cdots, \varphi_p^b, h_{p+1}^b).$$

If f is some solution of the differential equation $D_{p+1,p+1}$, we denote by ${}^\gamma f$ the image of f under the monodromy around b and by ${}^{\tilde{\gamma}} f$ the image of f under the monodromy around ∞.
As the basis Σ_b^b consists of eigenvectors for the monodromy around b, one has ${}^\gamma\varphi_j^b = \varphi_j^b$ for $j = 1, \cdots, p$ and ${}^\gamma\varphi_{p+1}^b = e^{-2i\pi(\lambda-b)}\varphi_{p+1}^b$.
Similarly, as the basis Σ_∞^b consists of eigenvectors for the monodromy around ∞, one has ${}^{\tilde{\gamma}}h_j^b = e^{2i\pi\mu_j}h_j^b$ for $j = 1, \cdots, p$ and ${}^{\tilde{\gamma}}h_{p+1}^b = e^{-2i\pi b}h_{p+1}^b$.

We use the following notations, for $j = 1, \cdots, p$

$$\begin{aligned}
\alpha_j &= \frac{\Gamma(1-\underline{\mu}_j^*)\Gamma(1+\mu_j+b)\Gamma(1-b-\underline{\nu})}{\Gamma(1+\mu_j-\underline{\nu})\Gamma(1-b-\underline{\mu})} b^{-\lambda-\mu_j} e^{-i\pi(\lambda-b)}, \\
\beta_j &= \frac{\Gamma(\underline{\mu}_j^*)\Gamma(-\mu_j-b)\Gamma(1-\lambda+b)}{\Gamma(\underline{\nu}-\mu_j)} b^{\lambda+\mu_j} e^{i\pi(\lambda-b)}, \\
\beta_{p+1} &= \frac{\Gamma(\underline{\mu}+b)\Gamma(1-\lambda+b)}{\Gamma(\underline{\nu}+b)}, \\
\Delta_{p+1} &= \beta_{p+1} + \sum_{j=1}^{p} \alpha_j\beta_j.
\end{aligned}$$

Proposition 5.1. *The matrix T_γ expressing the monodromy around b in the basis Σ_m is given by*

$$T_\gamma = \begin{pmatrix}
1 & 0 & \cdots & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \cdots & 0 \\
0 & \cdots & 0 & 1 & 0 \\
\frac{\alpha_1}{\Delta_{p+1}} m_b & \cdots & \cdots & \frac{\alpha_p}{\Delta_{p+1}} m_b & e^{-2i\pi(\lambda-b)}
\end{pmatrix}$$

with $m_b = e^{-2i\pi(\lambda-b)} - 1$.

In the same basis the monodromy around ∞ is given by the matrix

$$T_{\tilde{\gamma}} = \begin{pmatrix} e^{2i\pi\mu_1} & 0 & \cdots & 0 & \beta_1(e^{2i\pi\mu_1} - e^{-2i\pi b}) \\ 0 & e^{2i\pi\mu_2} & \cdots & 0 & \beta_2(e^{2i\pi\mu_2} - e^{-2i\pi b}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & e^{2i\pi\mu_p} & \beta_p(e^{2i\pi\mu_p} - e^{-2i\pi b}) \\ 0 & \cdots & \cdots & 0 & e^{-2i\pi b} \end{pmatrix}.$$

Proof. The relation $\Sigma_b^b = \Sigma_\infty^b M_{b,\infty}^b$ implies $h_j^b = \varphi_j^b + \alpha_j h_{p+1}^b$ and thus

$${}^\gamma h_j^b = \varphi_j^b + \alpha_j {}^\gamma h_{p+1}^b.$$

Moreover

$$\begin{aligned} \varphi_{p+1}^b &= \sum_{j=1}^{p+1} h_j^b \\ &= \sum_{j=1}^{p} \beta_j(\varphi_j^b + \alpha_j h_{p+1}^b) + \beta_{p+1} h_{p+1}^b \\ &= \sum_{j=1}^{p} \beta_j \varphi_j^b + \Delta_{p+1} h_{p+1}^b \end{aligned}$$

and thus

$$h_{p+1}^b = -\sum_{j=1}^{p} \frac{\beta_j}{\Delta_{p+1}} \varphi_j^b + \frac{1}{\Delta_{p+1}} \varphi_{p+1}^b$$

so that

$$\begin{aligned} {}^\gamma h_{p+1}^b &= -\sum_{j=1}^{p} \frac{\beta_j}{\Delta_{p+1}} \varphi_j^b + \frac{1}{\Delta_{p+1}} e^{-2i\pi(\lambda-b)} \varphi_{p+1}^b \\ &= h_{p+1}^b + \frac{1}{\Delta_{p+1}} (e^{-2i\pi(\lambda-b)} - 1) \varphi_{p+1}^b. \end{aligned}$$

From this relation one deduces for $j = 1, \cdots, p$

$$\begin{aligned} {}^\gamma h_j^b &= \varphi_j^b + \alpha_j h_{p+1}^b + \frac{\alpha_j}{\Delta_{p+1}} (e^{-2i\pi(\lambda-b)} - 1) \varphi_{p+1}^b \\ &= h_j^b + \frac{\alpha_j}{\Delta_{p+1}} (e^{-2i\pi(\lambda-b)} - 1) \varphi_{p+1}^b. \end{aligned}$$

To compute the last column of the second matrix one just has to notice that

$$\begin{aligned}
{}^{\tilde{\gamma}}\varphi_{p+1}^{b} &= \sum_{j=1}^{p} \beta_j e^{2i\pi\mu_j} h_j^b + \beta_{p+1} e^{-2i\pi b} h_{p+1}^b \\
&= \sum_{j=1}^{p} \beta_j e^{2i\pi\mu_j} h_j^b + e^{-2i\pi b}\Big[\varphi_{p+1} - \sum_{j=1}^{p} \beta_j h_j^b\Big] \\
&= \sum_{j=1}^{p} \beta_j (e^{2i\pi\mu_j} - e^{-2i\pi b}) h_j^b + e^{-2i\pi b}\varphi_{p+1}^b \ \square
\end{aligned}$$

We will now compute Δ_{p+1}. This computation should be classical but due to a lack of proper reference we prefer to give a proof. We first state a lemma.

LEMMA 5.2. *For $\underline{\alpha}, \underline{\nu} \in \mathbf{C}^n$ with $\alpha_j - \alpha_k \notin \mathbf{Z}$ if $j \neq k$, the following equality holds*

$$\sum_{j=1}^{n} \frac{\sin \pi(\underline{\nu} - \alpha_j)}{\sin \pi \underline{\alpha}_j^*} = \sin \pi(|\underline{\nu}| - |\underline{\alpha}|).$$

PROOF. The quotient $\frac{\sin \pi(x+\underline{\nu})}{\sin \pi(x+\underline{\alpha})}$ ($x \in \mathbf{C}$) may be viewed as a rational function in $e^{2i\pi x}$. By partial fraction decomposition one gets the equality

$$\frac{\sin \pi(x+\underline{\nu})}{\sin \pi(x+\underline{\alpha})} = e^{i\pi(|\underline{\nu}|-|\underline{\alpha}|)} + \sum_{j=1}^{n} \frac{\sin \pi(\underline{\nu} - \alpha_j)}{\sin \pi(\underline{\alpha}_j^*)} \frac{e^{-i\pi(x+\alpha_j)}}{\sin \pi(x+\alpha_j)}.$$

Taking all the coefficients and x real and writing down the equality of the imaginary parts, one gets

$$0 = \sin \pi(|\underline{\alpha}| - |\underline{\nu}|) - \sum_{j=1}^{n} \frac{\sin \pi(\underline{\nu} - \alpha_j)}{\sin \pi(\underline{\alpha}_j^*)}.$$

By analytic continuation this relation is still valid for complex values of the parameters. $\square$

PROPOSITION 5.3. *The determinant Δ_{p+1} of the connection matrix $M_{b,\infty}^b$ is given by*

$$\Delta_{p+1} = \beta_{p+1} + \sum_{j=1}^{p} \alpha_j \beta_j = \frac{\Gamma(1-b-\underline{\nu})}{\Gamma(1-b-\underline{\mu})\Gamma(\lambda - b)}.$$

PROOF. We use the lemma with $n = p+1$ and $\underline{\alpha} = (\underline{\mu}, -b)$, getting the relation

$$\frac{\sin\pi(\underline{\nu}+b)}{\sin\pi(b+\underline{\mu})} + \sum_{j=1}^{p} \frac{\sin\pi(\underline{\nu}-\mu_j)}{\sin\pi(\underline{\mu}_j^*)\sin\pi(-\mu_j-b)} = \sin\pi(\lambda-b).$$

But Δ_{p+1} can be written as

$$\begin{aligned}
\Delta_{p+1} &= \Gamma(1-\lambda+b)\Big[\frac{\Gamma(b+\underline{\mu})}{\Gamma(\underline{\nu}+b)} + \sum_{j=1}^{p} \frac{\Gamma(1-\underline{\mu}_j^*)\Gamma(1+\mu_j+b)\Gamma(1-b-\underline{\nu})\Gamma(\underline{\mu}_j^*)\Gamma(-b-\mu_j)}{\Gamma(1+\mu_j-\underline{\nu})\Gamma(1-b-\underline{\mu})\Gamma(\underline{\nu}-\mu_j)}\Big] \\
&= \frac{\Gamma(1-b-\underline{\nu})\Gamma(1-\lambda+b)}{\Gamma(1-b-\underline{\mu})}\frac{1}{\pi}\Big[\frac{\sin\pi(b+\underline{\nu})}{\sin\pi(b+\underline{\mu})} + \sum_{j=1}^{p} \frac{\sin\pi(\underline{\nu}-\mu_j)}{\sin\pi\underline{\mu}_j^*\sin\pi(-b-\mu_j)}\Big] \\
&= \frac{\Gamma(1-b-\underline{\nu})}{\Gamma(1-b-\underline{\mu})}\sin\pi(\lambda-b)\frac{\Gamma(1-\lambda+b)}{\pi} \\
&= \frac{\Gamma(1-b-\underline{\nu})}{\Gamma(1-b-\underline{\mu})\Gamma(\lambda-b)}. \quad \square
\end{aligned}$$

REMARK. With this value for Δ_{p+1} it would be easy to get in closed form the entries of the matrix $M_{\infty,b}^b = (M_{b,\infty}^b)^{-1}$ and thus of $M_{\infty,1}$.

PROPOSITION 5.4. *If $b = b_0 + n$ with $b_0 \in \mathbf{C}$ and $n \in \mathbf{N}$, then*

$$\begin{aligned}
\lim_{n\to+\infty} T_\gamma &= diag\,(\underbrace{1,\cdots,1}_{p}, e^{-2i\pi(\lambda-b_0)})\, S_0^{-1} \\
&= diag\,(\underbrace{1,\cdots,1}_{p}, e^{-2i\pi\lambda})\, diag\,(\underbrace{1,\cdots,1}_{p}, e^{2i\pi b_0})\, S_0^{-1} \\
\lim_{n\to+\infty} T_{\tilde\gamma} &= S_\pi^{-1}\, diag\,(e^{2i\pi\mu_1},\cdots,e^{2i\pi\mu_p}, e^{-2i\pi b_0}) \\
&= S_\pi^{-1}\, diag\,(e^{2i\pi\mu_1},\cdots e^{2i\pi\mu_p}, 1)\, diag\,(\underbrace{1,\cdots,1}_{p}, e^{-2i\pi b_0})
\end{aligned}$$

PROOF. With the help of Proposition 5.3, one may write

$$\frac{\alpha_j}{\Delta_{p+1}}(e^{-2i\pi(\lambda-b)}-1) = A_jU_j(b)$$

with

$$A_j = \frac{\Gamma(1-\underline{\mu}_j^*)}{\Gamma(1+\mu_j-\underline{\nu})}$$

and

$$\begin{aligned} U_j(b) &= \Gamma(1+\mu_j+b)\Gamma(\lambda-b)b^{-\lambda-\mu_j}e^{-i\pi(\lambda-b)}(e^{-2i\pi(\lambda-b)}-1) \\ &= \Gamma(1+\mu_j+b)\Gamma(\lambda-b)b^{-\lambda-\mu_j}e^{-2i\pi(\lambda-b)}2i\sin\pi(b-\lambda) \\ &= -2i\pi e^{-2i\pi(\lambda-b)}\frac{\Gamma(1+\mu_j+b)}{\Gamma(1+b-\lambda)}b^{-\lambda-\mu_j} \end{aligned}$$

so that

$$\lim_{n\to\infty} U_j(b_0+n) = -2i\pi e^{-2i\pi(\lambda-b_0)}.$$

Similarly

$$\beta_j(e^{2i\pi\mu_j}-e^{-2i\pi b}) = 2i\beta_j e^{i\pi(\mu_j-b)}\sin\pi(\mu_j+b) = B_j V_j(b)$$

with

$$B_j = \frac{\Gamma(\underline{\mu}_j^*)}{\Gamma(\underline{\nu}-\mu_j)}e^{i\pi(\mu_j+\lambda)}$$

and

$$\begin{aligned} V_j(b) &= 2i\Gamma(-b-\mu_j)\Gamma(1-\lambda+b)b^{\lambda+\mu_j}e^{-2i\pi b}\sin\pi(b+\mu_j) \\ &= -2i\pi\frac{\Gamma(1-\lambda+b)}{\Gamma(1+b+\mu_j}b^{\lambda+\mu_j)}e^{-2i\pi b} \end{aligned}$$

so that

$$\lim_{n\to\infty} V_j(b_0+n) = -2i\pi e^{-2i\pi b_0}. \ \Box$$

The diagonal matrix $diag\,(\underbrace{1,\cdots,1}_{p}, e^{\pm 2i\pi b_0})$ may be viewed as a random factor linked to the "exponential torus" of [4]. The diagonal matrices $diag\,(\underbrace{1,\cdots,1}_{p}, e^{-2i\pi\lambda})$ and $diag\,(e^{2i\pi\mu_1},\cdots e^{2i\pi\mu_p},1)$ are complementary parts of the formal monodromy matrix $\widehat{M}$ described at the end of section 4. It is then possible to give the same heuristic "explanation" of the limit procedure as the one given in the case $p=2$ in [9].

One can do the same computations in the other mixed basis Σ'_m and give the same interpretation of the results.

PROPOSITION 5.5. *The matrix T'_γ expressing the monodromy around b in the basis Σ'_m is given by*

$$T'_\gamma = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \frac{\beta_1}{\Delta_{p+1}} m_b \\ 0 & 1 & 0 & \cdots & \frac{\beta_2}{\Delta_{p+1}} m_b \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & \cdots & 0 & 1 & \frac{\beta_p}{\Delta_{p+1}} m_b \\ 0 & \cdots & \cdots & 0 & e^{-2i\pi(\lambda-b)} \end{pmatrix}$$

with $m_b = e^{-2i\pi(\lambda-b)} - 1$.
In the same basis the monodromy around ∞ is given by the matrix

$$T'_{\tilde\gamma} = \begin{pmatrix} e^{2i\pi\mu_1} & 0 & \cdots & 0 & 0 \\ 0 & e^{2i\pi\mu_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & e^{2i\pi\mu_p} & 0 \\ \alpha_1(e^{2i\pi\mu_1} - e^{-2i\pi b}) & \cdots & \cdots & \alpha_p(e^{2i\pi\mu_p} - e^{-2i\pi b}) & e^{-2i\pi b} \end{pmatrix}.$$

If $b = b_0 - n$ with $b_0 \in \mathbf{C}$ and $n \in \mathbf{N}$, then

$$\begin{aligned} \lim_{\substack{n\to+\infty \\ \arg b \to \pi}} T'_\gamma &= \mathit{diag}\,(\underbrace{1,\cdots,1}_{p}, e^{-2i\pi(\lambda-b_0)})\, S_\pi^{-1} \\ &= \mathit{diag}\,(\underbrace{1,\cdots,1}_{p}, e^{-2i\pi\lambda})\, \mathit{diag}\,(\underbrace{1,\cdots,1}_{p}, e^{2i\pi b_0})\, S_\pi^{-1} \end{aligned}$$

and

$$\begin{aligned} \lim_{\substack{n\to+\infty \\ \arg b \to -\pi}} T'_{\tilde\gamma} &= S_0^{-1}\, \mathit{diag}\,(e^{2i\pi\mu_1}, \cdots, e^{2i\pi\mu_p}, e^{-2i\pi b_0}) \\ &= S_0^{-1}\, \mathit{diag}\,(e^{2i\pi\mu_1}, \cdots e^{2i\pi\mu_p}, 1)\, \mathit{diag}\,(\underbrace{1,\cdots,1}_{p}, e^{-2i\pi b_0}). \end{aligned}$$

References

[1] Duval, A. and C. Mitschi, Matrices de Stokes et groupe de Galois des équations hypergéométriques confluentes généralisées, Pacific Journal of Math. **138** (1989), 25–56.

[2] Knottnerus, U. J., Approximation Formulae for Generalized Hypergeometric Functions for Large Values of the Parameters with Application to Expansion Theorems for the Function $G_{p,q}^{m,n}(z)$, J. B. Wolters, 1960, Groningen.

[3] Luke, Y. L., The Special Functions and their Approximations, Academic Press, 1969, New-York and London.

[4] Martinet, J. and J.-P. Ramis, *Théorie de Galois différentielle et resommation* in Computer Algebra and Differential Equations, E. Tournier Ed., Academic press, 1989, 117–214.

[5] Martinet, J. and J.-P. Ramis, Elementary acceleration and multisummability, Annales de l'I.H.P. série A, Physique théorique **54** (1991), 1–71.

[6] Mitschi, C., Differential Galois groups of confluent hypergeometric equations: an approach using Stokes multipliers, Pacific Journal of Math. **176** (1996), 365–405.

[7] Noerlund, N. E.,Hypergeometric Functions, Acta Math. **94** (1955), 289–349.

[8] Ramis, J.-P., Confluence et résurgence, J. Fac. Science, University of Tokyo **39** (1989), 703–716.

[9] Zhang, C., Confluence et phénomène de Stokes, J. Math. Sci. Univ. Tokyo **3** (1996), 91–107.

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