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On Tunnel Number One Alternating Knots and Links

By Koya Shimokawa

Abstract. In this article we consider tunnel number one alternating knots and links. We characterize tunnel number one alternating knots and links which have unknotting tunnels contained in regions of reduced alternating diagrams. We show that such a knot or link is a 2-bridge or a Montesinos knot or link or the connected sum of the Hopf link and a 2-bridge knot.

§1. Introduction

C. C. Adams [1] and M. Sakuma and J. Weeks [17] studied unknotting tunnels for hyperbolic knots and links in S^3 by analyzing vertical geodesics and canonical decompositions of their complements. For tunnel number one hyperbolic alternating knots and links, the relation among unknotting tunnels, canonical decompositions of their complements and their reduced alternating diagrams is studied in [17]. Furthermore Sakuma proposed the following conjecture,

Conjecture. *Let L be a tunnel number one hyperbolic alternating knot or link. Then some unknotting tunnel for L is isotopic to the polar axis of some crossing ball of a reduced alternating diagram of L.*

Note that prime alternating knots other than $(2, p)$-torus knots are hyperbolic. See [9], [10] and [14].

There are many knots and links satisfying the above conjecture. For example, upper and lower tunnels for 2-bridge knots and links satisfy the condition. See Figure 1.3(1).

In this article, we consider tunnel number one alternating knots and links with unknotting tunnels satisfying a little weaker condition. We characterize an alternating diagram of a tunnel number one alternating knot

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or link which has an unknotting tunnel contained in a region of a reduced alternating diagram and show that such a knot or link is a 2-bridge or a Montesinos knot or link or the connected sum of the Hopf link and a 2-bridge knot.

Let E be a diagram of L on S^2. As in [9], we place a 3-ball at each crossing of E and isotope L so that the overstrand at the crossing runs on the upper hemisphere and the understrand runs on the lower hemisphere as shown in Figure 1.1. The 3-ball above is called a *crossing ball*. The *polar axis* of a crossing ball is an arc γ as in Figure 1.1.

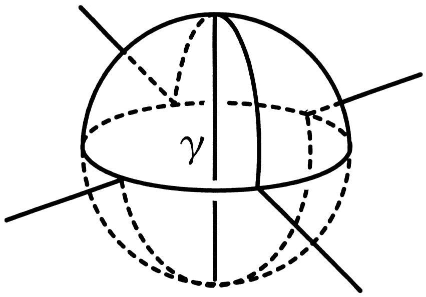

Fig. 1.1

A *region* of a diagram E on the 2-sphere S^2 of a link is a component of $S^2 - (\Delta \cup E)$, where Δ denotes the disjoint union of all crossing balls. We say an unknotting tunnel t is *contained in a regeion* R of E if t has a projection without crossings which is contained in R and meets ∂R in only its endpoints. See Figure 1.2. We remark that after a slight isotopy an unknotting tunnel in Conjecture is contained in a region of a reduced alternating diagram.

Note that tunnel number one composite alternating links do not admit unknotting tunnels satisfying the condition in the above conjecture. K. Morimoto showed that a composite tunnel number one link is the connected sum of the Hopf link and a 2-bridge knot and characterized its unknotting tunnels in [11]. This link is alternating and its unknotting tunnel is not isotopic to the polar axis of a crossing ball of a reduced alternating diagram but contained in a region.

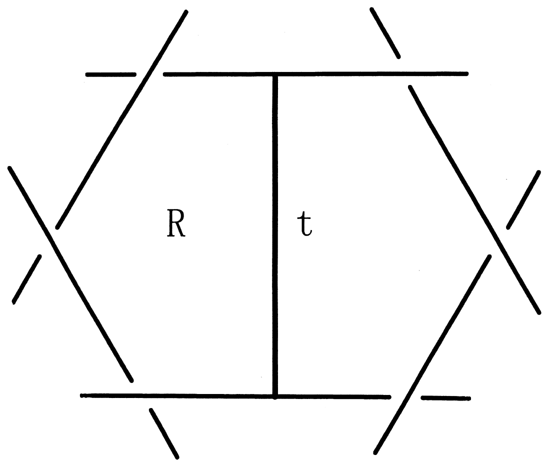

Fig. 1.2

Let L be a knot or link in S^3. Hereafter we consider a knot is a link with one component. Let $E(L)$ denote the exterior of L, i.e. $S^3 - \mathrm{int}\, N(L)$, where $N(\cdot)$ denotes the regular neighborhood. A set $\{t_1, \cdots, t_n\}$ of mutually disjoint properly embedded arcs in $E(L)$ is called an *unknotting tunnel system for* L if $E(L) - \mathrm{int}\, N(t_1 \cup \cdots \cup t_n)$ is a genus $n+1$ handlebody. The *tunnel number* of L, denoted by $t(L)$, is the minimal number of arcs among all unknotting tunnel systems for L. In case $t(L) = 1$, the arc is called an *unknotting tunnel* for L.

Note that the tunnel number of every 2-bridge link is one. There are six tunnels for 2-bridge knots as in Figure 1.3(1) (See [5]). The tunnel τ_1(resp. τ_2) is called an *upper tunnel* (resp. a *lower tunnel*) and ρ_1 and ρ_1'(resp. ρ_2 and ρ_2') *dual tunnels* of τ_1 (resp. τ_2) (See [12]). In the trivial tangles of the 2-bridge decomposition, these six tunnels are as in Figure 1.3(2). Recently T. Kobayashi proved that an unknotting tunnel for a 2-bridge knot is isotopic to one of these six in [6]. Unknotting tunnels for 2-component 2-bridge links are classified independently by Adams and A. Reid in [3] and M. Kuhn in [7].

From a dual tunnel, say t, of an upper or a lower tunnel for a 2-bridge knot, we can obtain an unknotting tunnel, say t', for the connected sum of the Hopf link and the 2-bridge knot as in Figure 1.4. In this case, we

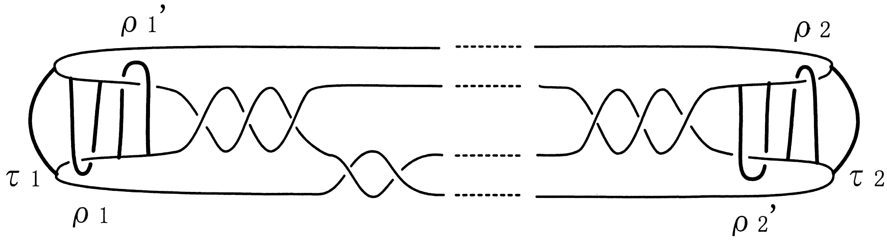

Fig. 1.3(1)

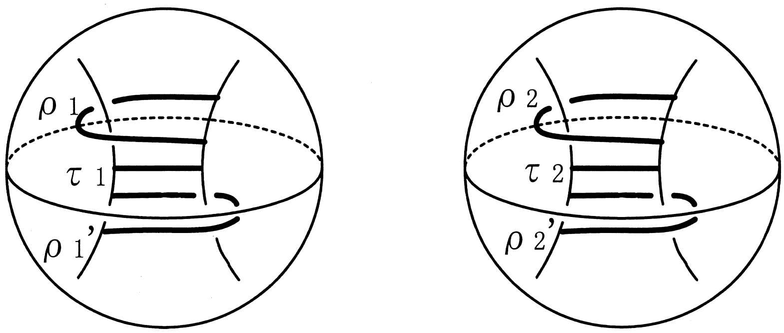

Fig. 1.3(2)

say t' *is obtained from* t. Morimoto showed that each unknotting tunnel for the connected sum of the Hopf link and a 2-bridge knot is isotopic to one obtained from a dual tunnel of an upper or a lower tunnel for the 2-bridge knot in [11].

A link diagram is *reduced* if it has no nugatory crossings as in Figure 1.5. A diagram is *trivial* if it contains no crossings.

Let $M(b;(\alpha_1,\beta_1),\cdots,(\alpha_r,\beta_r))$ stand for a Montesinos link in S^3 with b half twists and r rational tangles of slopes $\frac{\beta_1}{\alpha_1},\cdots,\frac{\beta_r}{\alpha_r}$, where b, r, α_i and β_i are integers such that $r \geq 2$, $\alpha_i \geq 2$ and $g.c.d.(\alpha_i,\beta_i)=1$. See [4, Chapter 12].

Let (B,T) be the trivial 2-string tangle. An embedded arc τ in B is the *core* of (B,T) if $\tau \cap T = \partial\tau$, τ connects two strings of T and there is a disc properly embedded in B which contains $T \cup \tau$. Upper and lower tunnels of a 2-bridge link is the cores of the two rational tangles of the 2-bridge

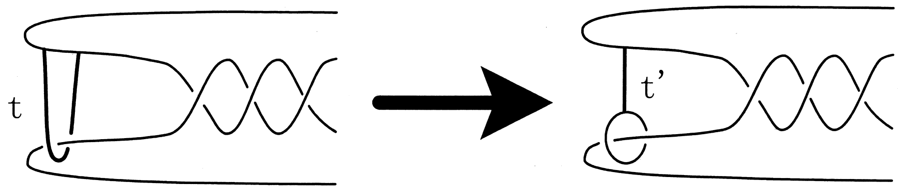

Fig. 1.4

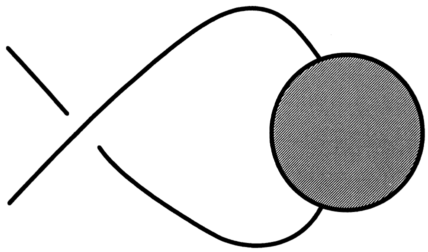

Fig. 1.5

decomposition. A rational tangle is the trivial tangle with a chart on the boundary of the 3-ball.

THEOREM 1.1. *Let L be a tunnel number one alternating link. Suppose an unknotting tunnel for L is contained in a region of a reduced alternating diagram E of L. Then one of the following holds,*

(1) *L is a 2-bridge link and t is an upper or a lower tunnel or their dual tunnel, or*
(2) *after operating flypes if necessary, E is exemplified as in Figure 1.6, where E_a and E_b are non-trivial diagrams of rational tangles. L is a Montesinos link $M(b;(\alpha_1,\beta_1),(\alpha_2,\beta_3),(\alpha_3,\beta_3))$, where $\alpha_1=2$ up to cyclic permutation of the indices and t is the core of a rational tangle of slope $\frac{\beta_2}{\alpha_2}$ or $\frac{\beta_3}{\alpha_3}$, or*
(3) *E and t are exemplified as in Figure 1.7. L is the connected sum of the Hopf link and a 2-bridge knot and t is obtained from a dual tunnel of an upper or a lower tunnel for the 2-bridge knot.*

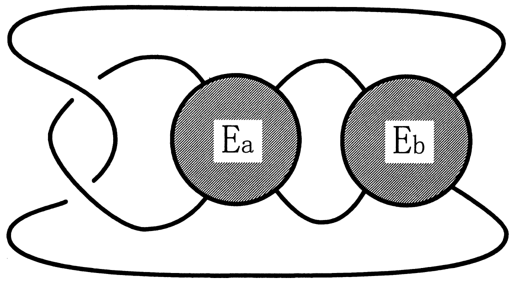

Fig. 1.6

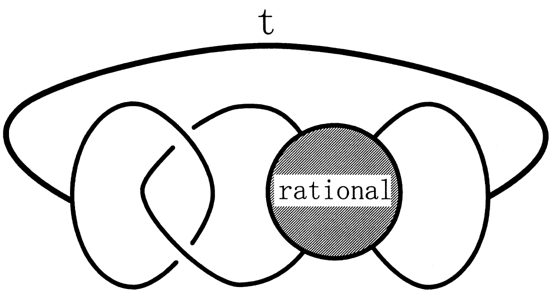

Fig. 1.7

Note that in the case (3), by [9], there is a decomposing sphere meeting S^2 in a simple closed curve, where a decomposing sphere is a 2-sphere meeting L in two points and not bounding a pair consisting of the 3-ball and a trivial arc.

Now we consider a special case of Theorem 1.1, where a link has an unknotting tunnel isotopic to the polar axis of a crossing ball.

COROLLARY 1.2. *Let L be a tunnel number one alternating link. Suppose an unknotting tunnel for L is isotopic to the polar axis of a crossing ball of a reduced alternating diagram E of L. Then either*

(1) *L is a 2-bridge link and t is an upper or a lower tunnel or their dual tunnel, or*
(2) *after operating flypes if necessary, E is exemplified as in Figure 1.6, where E_a and E_b are non-trivial diagrams of rational tangles and L*

is a Montesinos link $M(b; (\alpha_1, \beta_1), (\alpha_2, \beta_3), (\alpha_3, \beta_3))$*, where* $\alpha_1 = 2$ *up to cyclic permutation of the indices and* t *is the core of a rational tangle of slope* $\frac{\beta_2}{\alpha_2}$ *or* $\frac{\beta_3}{\alpha_3}$.

Morimoto, Sakuma and Y. Yokota determined tunnel number one Montesinos knots and show that either it is 2-bridged or has at most three branches. For details, see Theorem 2.2 and Added in Proof in [13]. Y. Nakagawa studied tunnel number one 2-component Montesinos links and proved that it is either 2-bridged or $M(b; (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3))$ where $\alpha_1 = 2$ in [15].

We can show similar results to Corollary 1.2 for a tunnel number one link admitting a locally trivial almost alternating diagram which has an unknotting tunnel isotopic to the polar axis of the crossing ball at the dealternator, i.e. the crossing which makes the diagram non-alternating. See [2] for the definition of almost alternating links.

§2. Proof

We prove Theorem 1.1 by using a result of [19] on ∂-reducibility of alternating tangles.

First we give definitions. A 2-*string tangle*, or a *tangle* in brief, is a pair (B, T), where B is a 3-ball and T is a union of two properly embedded arcs. If there is no ambiguity, we write T instead of (B, T) for short. The *trivial tangle* is a tangle homeomorphic as pair to $(D \times I, \{x_1, x_2\} \times I)$, where D is the disc and x_i's are points lying in the interior of D. We draw a tangle diagram E of T on an equatorial disc D of B. An *alternating tangle* is a tangle which admits an alternating tangle diagram. Alternating tangles are studied in [8], [20], [18] and [19].

A tangle diagram E on a disc $D \subset B$ is *connected* if each properly embedded arc in D disjoint from E cobounds a subdisc which does not meet E together with a subarc of ∂D and any properly embedded circle in D disjoint from E bounds a disc which does not meet E. Otherwise we say E is *disconnected*. A link diagram is *connected* if there is no circle as above. A tangle diagram is *reduced* if it has no nugatory crossings as in Figure 2.1. The left diagram in Figure 2.1 has an arc properly embedded in D meeting the diagram only at the nugatory crossing. A tangle diagram

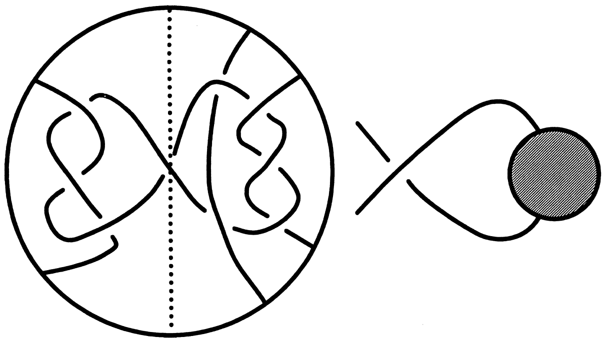

Fig. 2.1

E is *locally trivial* if each circle in D disjoint from crossings and meeting E in two points bounds a pair of a subdisc and a trivial arc.

Next we define tangle sums [21]. A *marked tangle* is a triple (B, T, Δ), where (B, T) is a tangle, and Δ is a disc on ∂B containing two endpoints of T. We call Δ a *gluing disc*. Given two marked tangles (B_1, T_1, Δ_1) and (B_2, T_2, Δ_2), we can obtain a new tangle (B, T) as follows. Take a map $\phi : \Delta_1 \to \Delta_2$ with $\phi(\Delta \cap T_1) = \Delta \cap T_2$, and use it to glue two tangles to get (B, T). This operation is called *tangle sum* and we write $T = T_1 + T_2$. A tangle sum is *non-trivial* if neither (B_i, T_i, Δ_i) is $M[0]$ nor $M[\infty]$ (See [21]).

A *marked tangle diagram* is a triple (D, E, α), where E is a tangle diagram on a disc D and α a subarc in ∂D which contains two endpoints of E. The arc α is called a *gluing arc*. Given two marked tangle diagrams (D_1, E_1, α_1) and (D_2, E_2, α_2), we can obtain a new tangle diagram E on $D = D_1 \cup D_2$ as follows. Take a homeomorphism $f : \alpha_1 \to \alpha_2$ with $f(\alpha_1 \cap E_1) = \alpha_2 \cap E_2$. Then we glue two tangle diagrams using f and get a new tangle diagram E. This operation is called the *partial sum*. See [8]. A partial sum is *non-trivial* if both E_1 and E_2 are connected and contain crossings.

Let $E(T) = B - \text{int}\, N(T)$ be the exterior of T. A tangle is *∂-reducible* if $\partial E(T)$ is compressible in $E(T)$. Otherwise it is *∂-irreducible*. In case an alternating diagram E of T is a partial sum of two tangle diagrams E_1 and E_2, let T_1 and T_2 denote the subtangles of T whose diagrams are E_1 and E_2 respectively. The author showed the following result in [19]. There, more general cases are dealt with. Terminologies used below can be found in [21].

PROPOSITION 2.1. [19] *Suppose a 2-string tangle T is ∂-reducible and has a reduced connected locally trivial alternating diagram E. Then E is a non-trivial partial sum of two alternating tangle diagrams E_1 and E_2 and T_1 is a 2-twist tangle and T_2 is a rational tangle. (See Figure 2.2.) Moreover $E(T)$ is a handlebody.*

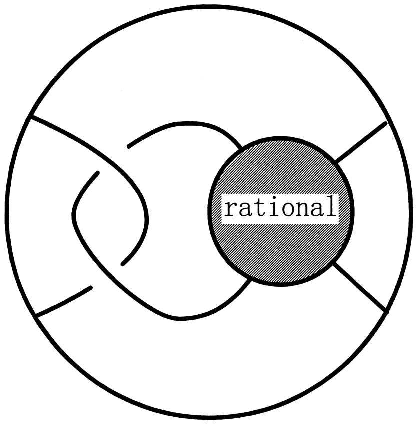

Fig. 2.2

Here we will examine E_1. An *n-twist marked tangle diagram* is a marked tangle diagram of an n-twist tangle with n half twists. For example, a 2-twist marked tangle diagram is as in Figure 2.3. The dotted line in the figure is the gluing arc.

ADDENDUM 2.2 TO PROPOSITION 2.1. *E_1 is 2 or -2-twist marked tangle diagram.*

PROOF. By Theorem 3.1 in [20], E has the minimal crossing number among all diagrams of E. Suppose, for a contradiction, that E_1 is not a 2 or -2-twist marked tangle diagram. Then, by replacing E_1 with a 2-twist marked tangle diagram, we have a diagram of T with a fewer crossing number, which is a contradiction. □

PROOF OF THEOREM 1.1. Note that E is connected. Let t be an unknotting tunnel for L which is contained in a region of E. Let $B_2 = N(t)$,

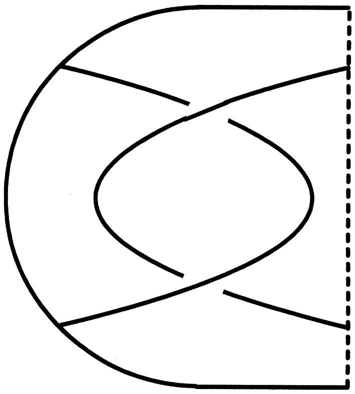

Fig. 2.3

$B_1 = S^3 - \operatorname{int} B_2$ and $T_i = L \cap B_i$, where $i = 1$ or 2. See Figure 2.4. Then L is decomposed into two 2-string tangles (B_1, T_1) and (B_2, T_2), where T_2 is the trivial tangle, and t is the core of T_2. Note that $S^3 - \operatorname{int} N(L \cup t)$ is isotopic to $E(T_1)$. Since t is an unknotting tunnel, $E(T_1)$ is a handlebody. Hence it is ∂-reducible. For $i = 1$ and 2, let E_i be the tangle diagram of T_i, which is a subdiagram of E, such that E_2 contains no crossing and E_1 is the alternating tangle diagram $E - E_2$.

Suppose, for a contradiction, that E_1 is disconnected. That is, there is an arc α on the disc on which we draw E_1 such that α misses and separates E_1. Then there is a circle β on S^2 meeing E in two points which contains α and the projection of t. Then either (a)β bounds a pair consisting of a disc and the trivial arc or (b)β does not bound such a pair. In the case (a), t can be isotoped into $\partial E(L)$. Hence L is the trivial knot, which is a contradiction. In the case (b), since alternating knots are non-trivial, L is composite and the decomposing sphere meets S^2 in β. Then, by [16], L is a 2-component link. Let K_1 and K_2 be the components of L. Since t is an unknotting tunnel, one point of $E \cap t = E \cap \beta$ is contained in the projection of K_1 and the other in that of K_2. It follows that each of K_1 and K_2 meets the decomposing sphere just once, which is a contradiction. Hence E_1 is connected.

Suppose, for a contradiction, that E_1 is not locally trivial. Then L is composite. By [16], L is a 2-component link. In this case, we can isotope

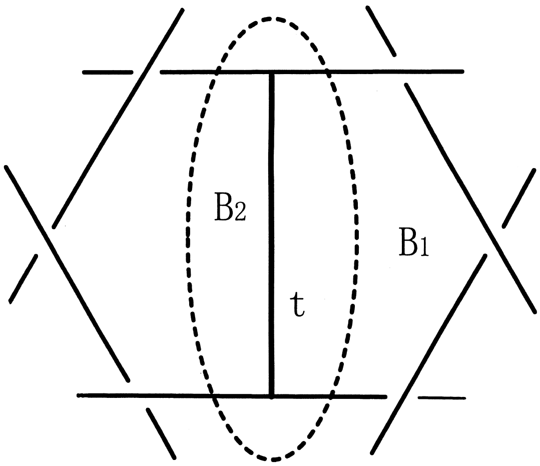

Fig. 2.4

the decomposing sphere into B_1. It follows that t does not meet the decomposing sphere, which contradicts Lemma 3.1 in [11]. Hence E_1 is locally trivial.

Suppose E_1 is reduced. Then, by Proposition 2.1 and Addendum 2.2, it follows either (a)E is a diagram of the connected sum of the Hopf link and a 2-bridge knot as in Figure 1.7, or (b)E is a diagram of 2-bridge link as in Figure 2.5. In the case (a), t is obtained from a dual tunnel of an upper or a lower tunnel for the 2-bridge knot. In the case (b), L is a 2-bridge knot and t is a dual tunnel of an upper or a lower tunnel.

If E_1 is not reduced, we operate flypes to E so as to eliminate nugatory crossings in E_1. Let E' be the resultant link diagram. Then E_1 becomes either (a)the trivial diagram or (b)a reduced alternating diagram E_1'. In the case (a), L is a 2-bridge link and t is the core of T_2. Hence t is an upper or a lower tunnel. In the case (b), E_1' is the partial sum of the 2 or -2-twist marked tangle diagram and an alternating diagram of a rational tangle and E_2' is an alternating diagram of a rational tangle, where $E_2' = E' - E_1'$. There are two cases. If the partial sum of T_2 and the rational tangle in T_1 is not a rational tangle (i.e. T_2 is a non-integral tangle), then $L = M(b; (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3))$, where $\alpha_1 = 2$, and t is the core of a rational tangle of slope $\frac{\beta_2}{\alpha_2}$ or $\frac{\beta_3}{\alpha_3}$. Hence (2) holds. If the partial sum of

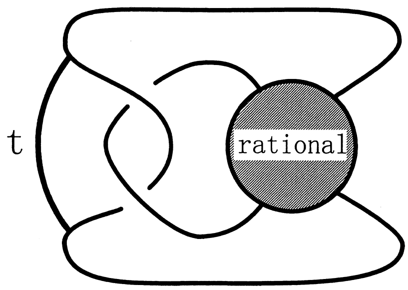

Fig. 2.5

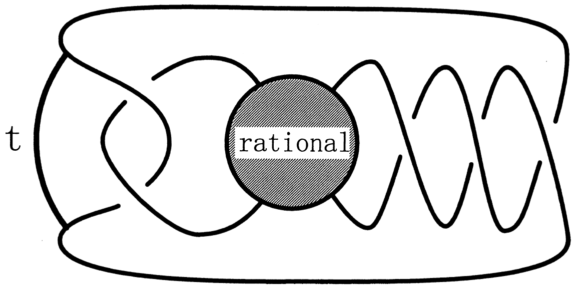

Fig. 2.6

T_2 and the rational tangle in T_1 is a rational tangle (i.e. T_2 is an integral tangle), then L is a 2-bridge link and t is a dual tunnel of an upper or a lower tunnel. See Figure 2.6. This completes the proof. □

PROOF OF COROLLARY 1.2. We repeat the argument in the proof of Theorem 1.1. In this case, E_1 always has the nugatory crossing. Hence, by the argument in the fifth paragraph in the proof of Theorem 1.1, Corollary 1.2 follows. □

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